Another numerical method of finding critical values for the Andrews stability test

Stanislav Anatolyev* New Economic School

Grigory Kosenok New Economic School

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Abstract

We propose a method, alternative to that by Estrella (2003, *Econometric Theory*, 19, 1128–1143), of obtaining exact asymptotic p-values and critical values for the popular Andrews (1993, *Econometrica*, 61, 821–856) test for structural stability. The method is based on inverting an integral equation that determines the intensity of crossing a boundary by the asymptotic process underlying the test statistic. Further integration of the crossing intensity yields a p-value. The proposed method can potentially be applied to other stability tests that employ the supremum functional.

Key words: Structural stability; structural breaks; supremum statistic; asymptotic p-value; asymptotic critical value.

*Corresponding author. Address: Stanislav Anatolyev, New Economic School, Nakhimovsky Pr., 47, office 1721(3), Moscow, 117418 Russia. E-mail: sanatoly@nes.ru.

1 Introduction

The Andrews (1993) sup-test for a structural break whose date is unknown has become by now very popular, and is even advocated as a portmanteau testing tool (e.g., Hansen, 1999). The asymptotic distribution of the test statistic is nonstandard, and researchers typically employ critical values tabulated in Andrews (1993 or 2003). These are obtained using a big number of simulations for various combinations of a test level, degrees of freedom, and truncation parameter.

Estrella (2003) points out that the critical values may be obtained exactly using a result in DeLong (1981) on parabolic boundary crossing probabilities for the Bessel process. The crossing probability, however, does not have a closed form formula but instead is represented as an infinite series whose summands involve roots of a certain polynomial of infinite order. These complications lead to quite involved numerical computations. Having coped with all of them Estrella (2003) gives a more precise version of the tables in Andrews (1993, 2003) and makes comparisons with what various existing approximate methods deliver.

We here show that the exact critical values can be alternatively obtained using a general relationship between the stochastic properties of the asymptotic process underlying the test statistic, and the intensity of crossing a boundary by this process. This relationship has a form of an integral equation, and it is documented in the statistical literature, in particular Durbin (1971), quite a while ago. We adapt this idea to the more special setting of the Andrews test. Then we develop the numerical algorithm that inverts the integral equation and returns the crossing intensity which is then used to obtain p-values of the test. Finally, we present a table of the critical values for most popular settings, and compare them with those in Andrews (1993, 2003) and Estrella (2003).

Compared to Estrella's (2003) method, ours has a more intuitive appeal. It is also based on a more universal statistical relationship, and potentially can be used for tabulating asymptotic distributions of other stability tests that employ the supremum functional. Despite greater universality, the proposed method is equally easy to implement.

2 Boundary crossing intensity

Let r and s index time on $[0,1]$, and consider a non-negative continuous stochastic process $Q(r)$ on $[0,1]$ starting from zero, $Q(0)=0$. Denote by $p_r(y)$ the unconditional density of $Q(r)$, and by $p_{r|s}(y|x)$ the conditional density of $Q(r)$ given that $Q(s)$ takes value x.

Consider also a boundary $\Psi(r)$, a deterministic positive function of time. We are interested in counting all crossings (from below) of $\Psi(r)$ by $Q(r)$ for the first time (also called first passages in the statistical literature). Let us denote the intensity of such crossings by $\alpha(r)$. More formally,

$$\alpha(r)=\lim_{\delta r\to 0}\frac{\Pr\left\{Q(s)<\Psi(s) \text{ for all } s\in[0,r] \text{ and } Q(s)\geq\Psi(s) \text{ for some } s\in[r,r+\delta r]\right\}}{\delta r}.$$

The statistical literature, in particular Durbin (1971, sec.2), provides the following relationship between the conditional density of the process $Q(r)$ and the intensity $\alpha(r)$ of crossing the boundary $\Psi(r)$:

$$p_r(\Psi(r))=\int_0^r p_{r|s}(\Psi(r)|\Psi(s))\,\alpha(s)ds \tag{1}$$

that should hold for all $r\in[0,1]$. Intuitively, the meaning of the equality in (1) is the following: the unconditional density of $Q(r)$ at the boundary $\Psi(r)$ can be alternatively obtained via the law of total probability by counting, along the boundary from 0 to r, the total measure for those trajectories that pass through $\Psi(r)$ for the first time.

For given $Q(r)$ and $\Psi(r)$ the integral equation (1) allows one to find $\alpha(r)$. Given the intensity $\alpha(r)$, the total probability of crossing the boundary on $[0,1]$ can be found by integration:

$$\Pr\left\{Q(r)\geq\Psi(r) \text{ for some } r\in[0,1]\right\}=\int_0^1\alpha(r)dr. \tag{2}$$

3 Adaptation to Andrews test

The Andrews (1993) test for structural stability in parametric models is a class of tests where the alternative hypothesis is that of a single structural break that occurred at unknown time. The test is based on a supremum (over time) of a certain sequential (i.e. indexed by time) statistic that is a Wald, Likelihood Ratio or Lagrange Multiplier statistic for equality of

parameters before and after a possible break. The asymptotic distribution of the test is non-standard. In case the sup-statistic exceeds a critical value the null of structural stability is rejected.

The asymptotic distribution of any of the three Andrews test statistics, before the supremum functional is taken, is

$$\frac{B_p(r)'B_p(r)}{r\left(1-r\right)} \tag{3}$$

on $[0,1]$, where $B_p(r) = W_p\left(r\right) - rW_p\left(1\right)$ is the p-dimensional Brownian bridge, and $W_p\left(r\right)$ is the p-dimensional Wiener process on $[0,1]$. The supremum of (3) is taken over $[\pi_1, \pi_2]$, where $0 < \pi_1 < \pi_2 < 1$, and the resulting value is compared to the asymptotic critical value c_α corresponding to the test level α. The critical values obtained via simulations are tabulated in Andrews (1993, Table 1 and 2003, Table 1) for most often used α, several values of $\pi_0 \equiv \pi_1 = 1 - \pi_2$ and a range of p. These critical values can also be utilized for some of the tests for multiple breaks of Bai and Perron (1998).

It is clear that the decision rule of the test is equivalent to checking if the p-dimensional Bessel bridge process $Q\left(r\right) = B_p(r)'B_p(r)$ crosses the boundary $\Psi(r) = c_\alpha r\left(1-r\right)$ at least once on $[\pi_1, \pi_2]$. Thus, the theory outlined in the previous section can be applied in order to find the probability value α of the test for given c_α, π_1, π_2, and p.

The implementation of this idea is not straightforward though. The first prerequisite is existence of tractable conditional densities. Fortunately, in the case of Bessel bridge they can be easily derived. Note that, due to normality of $B_p(r)$, the conditional distribution of $B_p(r)$ given $B_p(s)$, where $s < r$, is conditionally homoskedastic normal:

$$B_p(r)|B_p(s) \sim N\left(\frac{1-r}{1-s}B_p(s), \frac{\left(1-r\right)\left(r-s\right)}{1-s}I_p\right),$$

and hence

$$Q\left(r\right)|Q\left(s\right) \sim \omega_{s,r}\,\chi^2\left(p, \omega_{s,r}\frac{Q\left(s\right)}{\left(r-s\right)^2}\right),$$

where

$$\omega_{s,r} = \frac{\left(1-r\right)\left(r-s\right)}{1-s},$$

and $\chi^2\left(p,\lambda\right)$ denotes a non-central chi-squared distribution with degrees of freedom p and non-centrality parameter λ. The unconditional distribution is then

$$Q\left(r\right) \sim \omega_{0,r}\,\chi^2\left(p,0\right).$$

Therefore, the unconditional and conditional densities entering (1) are

$$p_r(y) = (2r(1-r))^{-p/2}\,\Gamma\left(\frac{p}{2}\right)^{-1} y^{p/2-1}\exp\left(-\frac{y}{2r(1-r)}\right)$$

and

$$p_{r|s}(y|x) = \frac{1}{\omega_{s,r}}\, f_{\chi^2\left(p,\omega_{s,r}x/(r-s)^2\right)}\left(\frac{y}{\omega_{s,r}}\right),$$

where $f_{\chi^2(p,\lambda)}(z)$ denotes a density evaluated at z of a random variable distributed as $\chi^2(p,\lambda)$. This density is inconveniently represented analytically as a certain infinite summation, but can be numerically evaluated in statistical packages.

The second difficulty is that the supremum is taken over a specific interval $[\pi_1, \pi_2]$ rather than the unit interval $[0,1]$ used in the theory. The right end of the interval does not pose any difficulties and can be switched from 1 to π_2. However, the left end π_1 not being zero requires certain corrections of the formulas in the previous section. Crossing the boundary from below to the right of π_1 is not the only reason the Andrews test may reject; the other reason is that $Q(r)$ is already above $\Psi(r)$ prior to π_1. To account for this possibility, (1) and (2) are modified in the following way:

$$p_r(\Psi(r)) = \int_{\Psi(\pi_1)}^{+\infty} p_{\pi_1}(y)\, p_{r|\pi_1}(\Psi(r)|y)\, dy + \int_{\pi_1}^{r} p_{r|s}(\Psi(r)|\Psi(s))\,\alpha(s)ds \tag{4}$$

for all $r \in (\pi_1, \pi_2]$, and

$$\alpha = \Pr\{Q(r) \geq \Psi(r) \text{ for some } r \in [\pi_1, \pi_2]\} = \int_{\Psi(\pi_1)}^{+\infty} p_{\pi_1}(y)\, dy + \int_{\pi_1}^{\pi_2} \alpha(r)dr. \tag{5}$$

The first terms on the right sides of (4) and (5) account for those trajectories that are already above the boundary prior to π_1, and the second terms account for the trajectories crossing the boundary from below after π_1.

4 Numerical implementation

In this section we briefly outline the numerical procedures we run to calculate the p-values and critical values. Often these procedures are standard and can be found in sources familiar to economists (e.g., Judd, 1998), but in some instances additional measures are taken to increase the speed and account for specific features of the problem like singularity (see below).

We approximate $\alpha(r)$ by a piecewise constant function $\bar{\alpha}(r)$ on the interval $[\pi_1, \pi_2]$ divided to $N = 501$ segments of equal size Δ and indexed by $i = 1, ..., N$, so that the function $\bar{\alpha}(r)$ takes constant value $\bar{\alpha}_i$ on segment i. The values $\{\bar{\alpha}_i\}_{i=1}^N$ are solved for sequentially, starting from $\bar{\alpha}_1$ and finishing with $\bar{\alpha}_N$. The value of $\bar{\alpha}_1$ is a solution of the following linear equation which is an approximation of (4) on the segment 1:

$$\begin{aligned} p_r\left(\Psi(\pi_1+\Delta)\right) &= \int_{\Psi(\pi_1)}^{+\infty} p_{\pi_1}(y)\, p_{\pi_1+\Delta|\pi_1}\left(\Psi(\pi_1+\Delta)|y\right) dy \\ &\quad +\bar{\alpha}_1 \int_{\pi_1}^{\pi_1+\Delta} p_{\pi_1+\Delta|s}\left(\Psi(\pi_1+\Delta)|\Psi(s)\right) ds. \end{aligned} \tag{6}$$

The first integral in (6) is numerically evaluated using a grid with intervals whose lengths are increasing linearly, i.e. the grid points are $\{\Psi(\pi_1), \Psi(\pi_1)+\varepsilon, \Psi(\pi_1)+3\varepsilon, \Psi(\pi_1)+5\varepsilon, \Psi(\pi_1)+8\varepsilon, ...\}$, where $\varepsilon = 0.01$. For each such interval, we first calculate the endpoint values f_0 and f_1 of the function $f(y) \equiv p_{\pi_1}(y)\, p_{r|\pi_1}\left(\Psi(\pi_1+\Delta)|y\right)$ under the integral sign; then we approximate this function by an exponential function. This gives that the value of the integral over this interval approximately equals $k\varepsilon(f_1 - f_0)/\ln(f_1/f_0)$, where $k\varepsilon$ is the width of the interval. We do this starting from the first interval $[\Psi(\pi_1), \Psi(\pi_1)+\varepsilon]$, then moving to the next interval $[\Psi(\pi_1)+\varepsilon, \Psi(\pi_1)+3\varepsilon]$, and so on. We stop integrating when the relative increment in the numerical value of the integral is less than the tolerance level 10^{-7}.

In evaluating the second integral in (6) we take into account that for $s \in [\pi_1, \pi_1+\Delta]$ and small Δ,

$$p_{r|s}\left(\Psi(\pi_1+\Delta)|\Psi(s)\right) \approx \frac{g_0 + g_1(\pi_1+\Delta-s)}{\sqrt{\pi_1+\Delta-s}},$$

which reflects singularity of the conditional density near $\pi_1+\Delta$ (a nonzero element divided by an approximately zero element). We use this functional form to analytically compute the second integral in (6), with the constants g_0 and g_1 evaluated by calculating the function under the integral sign at points π_1 and $\pi_1+\Delta/2$.

Analogously, having found $\bar{\alpha}_1$ one can obtain the value of $\bar{\alpha}_2$ as a solution of the following

linear equation which is an approximation of (4) on the segment 2:

$$\begin{aligned} p_r\left(\Psi(\pi_1+2\Delta)\right) &= \int_{\Psi(\pi_1)}^{+\infty} p_{\pi_1}(y)\, p_{\pi_1+2\Delta|\pi_1}\left(\Psi(\pi_1+2\Delta)|y\right) dy \\ &+\bar{\alpha}_1 \int_{\pi_1}^{\pi_1+\Delta} p_{\pi_1+2\Delta|s}\left(\Psi(\pi_1+2\Delta)|\Psi(s)\right) ds \\ &+\bar{\alpha}_2 \int_{\pi_1+\Delta}^{\pi_1+2\Delta} p_{\pi_1+2\Delta|s}\left(\Psi(\pi_1+2\Delta)|\Psi(s)\right) ds. \end{aligned}$$

The first and third integrals (the latter possessing singularity) are evaluated as described above. The second integral is evaluated using Simpson's method: it is set to $(f_0 + 4f_1 + f_2)\Delta/3$, where f_0, f_1 and f_2 are values of the function under the integral sign at points $\pi_1, \pi_1 + \Delta/2$ and $\pi_1 + \Delta$ correspondingly. Values $\bar{\alpha}_3$, ..., $\bar{\alpha}_N$ can be found in a similar fashion.

When the approximation $\bar{\alpha}(r)$ of $\alpha(r)$ is computed, the p-value is numerically evaluated as the following approximation of (5), taking into account that the unconditional distribution is χ^2_p:

$$\alpha = (1 - F_{\chi^2_p}(c_\alpha)) + \Delta \sum_{i=1}^{N} w_i \bar{\alpha}_i,$$

where w_i's are Simpson's weights that average to unity: $\{\frac{5}{6}, \frac{4}{3}, \frac{2}{3}, \frac{4}{3}, \frac{2}{3}, ..., \frac{4}{3}, \frac{5}{6}\}$. The weighting is performed to correct for the convexity of $\alpha(r)$ in order to reduce bias. When the aim is to obtain the critical value c_α given α, the bisection method is applied.

The authors have created a GAUSS code which computes one p-value for an average of 150 seconds. *(Note to the editor and referees: we are working on reducing this running time in order to eventually make the code publicly available.)*

5 Critical values

Below we tabulate the critical values for some often used values of α, π_0 and p.

	$p=1$			$p=2$			$p=3$			$p=4$			$p=5$		
π_0	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
0.25	6.50	8.04	11.55	9.24	10.96	14.77	11.47	13.33	17.38	13.48	15.45	19.70	15.36	17.43	21.86
0.20	6.90	8.45	11.97	9.69	11.41	15.23	11.96	13.82	17.87	14.01	15.98	20.22	15.91	17.98	22.40
0.15	7.30	8.86	12.40	10.14	11.87	15.69	12.46	14.32	18.36	14.54	16.50	20.73	16.47	18.53	22.93
0.10	7.74	9.32	12.87	10.64	12.37	16.19	13.00	14.85	18.88	15.11	17.06	21.28	17.08	19.12	23.50
0.05	8.31	9.90	13.45	11.28	13.01	16.82	13.69	15.53	19.55	15.84	17.78	21.97	17.84	19.87	24.22

Comparison with a corresponding part of Table 1 in Andrews (1993, 2003) reveals differences, sometimes significant. The maximal discrepancy occurs in the case $\pi_0 = 0.25$, $p = 3$, $\alpha = 1\%$. The table above gives the critical value of 17.38, while the 10,000 simulations in Andrews (1993) resulted in the number 16.60, with a nearly 5% relative discrepancy. The true asymptotic rejection probability corresponding to the quantile 16.60 is in fact 1.37%. A tenfold increase in the number of simulations in Andrews (2003) leads to the number 17.13 which still falls short of the exact critical value given here and in Estrella (2003).

One reason for the low precision of the tabulated critical values in Andrews (1993, 2003) as discussed in Estrella (2003) is the discretization of the continuous time process done during simulations. Another one may be imperfections of the pseudorandom number generator. But we would like to stress that 100,000 simulations is still an insufficient number to get the claimed precision. Simple computations accounting for the low density at the tails reveal that, for example, in the case $\pi_0 = 0.15$, $p = 3$, $\alpha = 1\%$ no fewer than 25 million repetitions are required to report the numbers with such accuracy as reported in the tables.[1] We also note in passing that we reach even higher precision in order to attain which no fewer than 25 billion repetitions are required (assuming away any problems with discretization and pseudorandom numbers).

The critical values computed via our numerical method are the same as those in Estrella (2003) in more than 80% of cases. The critical values that do not coincide are always higher (by the round up value 0.01) in our case. A close inspection using calculations with higher accuracy reveals that the true values are in the middle between those reported here and Estrella's, a bit closer to ours. This is in agreement with Estrella's remark that his calculations of p-values induce an upward bias, and as a result his critical values are lower, although negligibly, than the true ones.

6 Concluding remarks

Among other things, the proposed method highlights the hidden potential of some statistical results documented in the literature but underused in contexts different from those they were initially intended for. We illustrate this here for the theory of first passage probabilities represented by Durbin (1971) which in econometrics is used only in the context of sequential testing.

As was mentioned before, the proposed method is more universal than that of Estrella (2003), and there are potentially other stability tests whose critical values may be tabulated likewise. Still, for this to happen there are several prerequisites. First of all, the aggregation

[1] The computations of the accuracy of simulated critical values in Andrews (1993, p. 841) do not account for the low density of the distribution in the tails. The given formula is in fact a standard error for the p-value but not for the estimated quantile. This is not noticed in any of subsequent articles cited here.

of the sequential statistic must be made using the supremum functional which only allows for association of rejection with crossing a boundary. Thus, our method (as well as Estrella's) does not apply to exp- or ave- tests of Andrews and Ploberger (1994). Another requirement is tractability of conditional densities for the asymptotic process which includes markovianity and easy computability. The former of these requirements does not seem to hold, for instance, for the predictive test for parameter constancy (Ghysels, Guay and Hall, 1997).

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