Liquidity Provision and Equity Funding of Banks

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Abstract

The paper uses a "warm-glow" model of liquidity benefits from bank deposits to discuss the role of equity funding for liquidity provision. Under certainty, there is no such role, so funding by equity and funding by deposits are simply substitutes. Under uncertainty, equity funding supports the liquidity of deposits by making a default of the bank and illiquidity of deposits less likely. Efficiency mandates this use of equity funding if producer's surplus from deposit provision is relatively small. Banks choose the requisite equity funding voluntarily, if they are able to commit and communicate their funding choices to investors so that any deviation from equilibrium funding choices would be properly priced. By contrast, if investors cannot observe deviations from equilibrium funding choices, laissez-faire competitive equilibrium allocations involve zero equity funding. Statutory equity requirements then serve to reduce the distortions from the banks' inability to fully commit and communicate their funding choices *ex ante*.

1 Introduction

Most theories of banking claim that banks should be funded primarily by debt, in particular short-term debt.[1] But the financial crisis has shown that high short-term indebtedness of banks can cause substantial damage.[2] This experience has

[1] The rationale for having banks funded by *debt* , is discussed by Diamond (1984), Krasa and Villamil (1991), and Hellwig (1991). Reasons for having banks funded by *short-term* debt are discussed by Bryant (1980), Diamond and Dybvig (1983), Calomiris and Kahn (1991), Dewatripont and Tirole (1994), Diamond and Rajan (1998, 2000). As discussed in Hellwig (1994, 1998), however, the entire literature is unable to explain why banks do not fund with claims that are contingent on collective risk. Such claims would improve the allocation of risks without inducing additional moral hazard.

[2] For a review of different estimates, see Admati and Hellwig (2013, p. 233, n. 19). Haldane (2010) estimates the worldwide output loss from the financial crisis to amount to \$ 60 trn.

led to calls for more equity funding of banks. Regulatory reforms since 2008 have moved in this direction albeit so slowly that the policy debate is not yet over.[3]

In this debate, many of those academics who had worked on banking before the crisis have sided with the industry in fighting restrictions on bank borrowing. Without engaging on the damage that the crisis had done, they have argued that tighter limits on banks' borrowing would destroy some of the economic benefits from banks' funding mainly with short-term debt.[4]

There are basically two lines of argument. The first line of argument suggests that debt is good for funding banks because debt imposes discipline on bank managers.[5] Because debt contracts provide financiers with well defined legal claims, funding by debt is said to prevent managers from wasting "free cash flow" or from diverting it for their own private benefits.[6] If the debt is callable or if it is short-term and needs to be constantly rolled over, bank managers must also be forever in fear of losing their funding; this is said to keep them on the straight and narrow.[7] The theoretical models that have been developed

[3] In the United States, the Brown-Vitter Bill calls for banks with more than $50 billion to have equity funding amounting to at least 15 percent of total assets. Using the discretionary power it has, the Federal Reserve has raised equity requirements for large banks to 6 percent of total assets. Also in Europe, regulators have been using discretionary powers to impose various add-ons to the requirements codified in the Basel Accord and the European Capital Requirements Directive and Regulation.

[4] The claim is sometimes presented as a fine point in theory and sometimes as a policy recommendation. Examples of the former are Allen, Carletti and Marquez (2015), Gorton et al. (2010), Gale and Özgür (2013), DeAngelo and Stulz (2015), examples of the latter the Squan Lake Report by French et al. (2010) and Rajan (2013). When the argument is presented as a fine point in theory, usually, not much of an attempt is made to match the model to the real world. Matching the model to the real world is of course unnecessary if one is only interested in proving the existence of theoretical models in which exclusive funding of banks by short-term debt is efficient. However, the exercise loses its innocence if the models are then used for policy recommendations, by the authors themselves or by industry lobbyists eager to take them up, without any intervening reality check. An example is given by DeAngelo and Stulz (2015) being eagerly taken up by Davis Polk (2013). For extended critical assessments of the different models and their limitations, see Admati et al (2010/2013), as well as Admati and Hellwig (2013 a, b). Pfleiderer (2014) discusses the methodological issues involved in going back and forth between theoretical models and policy recommendations, suggesting that it is inappropriate to treat models as chameleons, using them for policy recommendations if nobody is questioning the assumptions and withdrawing to the position that they are "only models" when someone begins to question the empirical relevance of the assumptions.

[5] See, in particular, Jensen (1986), Calomiris and Kahn (1991), Dewatripont and Tirole (1994, 2013), Diamond and Rajan (1998, 2000), French et al. (2010), Rajan (2013).

[6] This point is central to Jensen (1986), Hart and Moore (1994, 1997), and Rajan and Zingales (2003). It should be noted, however, that the free-cash-flow problem is different in nonfinancial companies and in financial institutions. In nonfinancial companies, the free-cash-flow problem arises when the activities that incumbent management are specialized in earn large cash flows but do not offer good investment opportunities. In financial institutions, the problem is not one of cash cows versus new activities but one of cash coming from all sources of funding (including debt) and going into whatever lending or investments seem profitable, with much less specialization than in nonfinancial firms.

[7] See Calomiris and Kahn (1991), Dewatripont - Tirole (1994, 2012), Diamond and Rajan

to formalize these arguments presume that financiers have perfect or close to perfect information at zero, or close to zero, cost. If information is costly or imperfect, the relevance of these models for thinking about the real world is highly questionable.[8]

The second line of argument focuses on liquidity provision by banks. By issuing short-term debt, banks are said to satisfy the desire of investors for liquid assets, i.e. assets that are easy to dispose of at short notice with little risk of a loss. Bank deposits that can be withdrawn at will are convenient for investors who do not know when they will actually need the funds.[9] Long-term bonds and shares that are traded in liquid markets may serve the same purpose but leave investors exposed to market risk. In normal circumstances market risk is smaller for bonds than for shares because debt is less "information sensitive"[10]. As long as the prospect of default is remote, information about the debtor does not much matter for debt holders, or for potential buyers of debt securities, because, unlike the returns to shareholders, their claims do not depend on how the debtor succeeds in his own ventures.[11]

(1998, 2000).

[8] For a critique of the literature on bank debt as a disciplining device, see Admati et al. (2010/2013), as well as Admati and Hellwig (2013 b). If information is noisy, false positives can generate runs and cause great harm. If information is costly, there is a free-rider problem in information acquisition because any one investor's paying for information and monitoring the bank has positive externalities for all investors who benefit if management is better-behaved. If this free-rider problem prevents or reduces information acquisition, one cannot expect discipline to be very effective. This is similar to the free-rider problem in the theory of corporate takeovers; see, e.g., Grossman and Hart (1980). The free-rider externality can be neutralized by a redistribution externality, which might arise, e.g., because the first investor who receives information that something is going wrong at the bank can withdraw his funds before others, this withdrawal reduces the payoffs to others in bankruptcy; see Hellwig (2005). As argued by Admati et al. (2010/2013) and Admati and Hellwig (2013 b), the financial crisis has shown that, in real life as opposed to the theoretical models, these concerns are highly relevant: In the run-up to the financial crisis, short-term creditors did not exert much discipline but seem to have provided whatever funding banks wanted quite readily. In fact, creditors seem to have acted as free riders on the information collected by stock market analysts and investors and reflected in share prices. When they did run, in March 2008 on Bear Stearns and in September 2008 on Lehman Brothers, the runs were accompanied by significant value destruction.

[9] The original papers by Bryant (1980) and Diamond and Dybvig (1983) focus on the scope for having investors who need the money quickly participate in the extra returns from long-term investments. Hellwig (1994) and von Thadden (1997) argue that liquidity provision may also be based on banks' diversifying their investments across nonfinancial borrowers and acting as "market makers" for investors who want to get at their funds quickly. Von Thadden (1997, 2002) actually shows that, in a continuous-time model, the Bryant-Diamond-Dybvig version of liquidity provision is not incentive compatible. In his analysis the issue of coordination among depositors that is the main focus of Bryant (1980) and Diamond and Dybvig (1983) is actually moot because withdrawing early always dominates withdrawing late. Hellwig (1994) shows that insurance against uncertainty about when the investor will need the funds need not be associated with insurance against interest rate risks (and other macro) risks, i.e. liquidity provision and maturity transformation are not necessarily linked.

[10] Dang et al. (2012).

[11] This observation is central to the costly-state-verification literature; see Townsend (1979), Diamond (1984), Gale-Hellwig (1985), Hellwig (1991, 1994). Dang et al. (2012) and Gorton (2010, 2012) stress the implications of information of insensitivity for the tradability of securities and for the liquidity of markets. It is worth noting that the "information insensitivity"

In policy debate, the view of banks as "producers" of liquid debt is often taken to imply that banks must be highly levered. Debt and equity are seen as substitutes because whatever funding comes in the form of equity, is not coming in the form of debt. In this line of argument, banks have to be highly levered in order to provide the economy with the benefits that only banks can provide.[12]

The view that liquidity provision and equity funding of banks are substitutes neglects the possibility that the liquidity of bank debt itself may depend on how much equity the bank is using. A bank that is funding with less equity has a greater probability of going bankrupt. Once the prospect of bankruptcy looms, the bank's debt is no longer information insensitive; at that point, any news about the returns on the bank's assets is likely to have a significant impact on the value of the bank's debt. Markets for the bank's bonds may then become less liquid because fears of insider trading in advance of bankruptcy will cause potential buyers to become more defensive.[13]

If the liquidity of bank debt is enhanced by additional equity, liquidity provision and equity funding will to some extent be complements rather than substitutes. Thus, Admati, Conti-Brown, and Pfleiderer (2012) argue that, if banks raise more equity and invest the proceeds in the market portfolio of stocks, then the amount of liquid debt they have issued is unchanged, but the quality of this debt is improved because, with the additional equity, the bank is better able to absorb losses without becoming distressed let alone insolvent.[14]

Until now, there has been no comprehensive formal analysis of the relation between liquidity provision and equity funding of banks. DeAngelo and Stulz (2013) provide a mixture of verbal arguments and formal analysis, but their paper is fundamentally flawed.[15] Admati, Conti-Brown, and Pfleiderer (2012), as well as Admati et al. (2010/2013) and Admati and Hellwig (2013), give verbal arguments without formal models. Their arguments presume the existence of assets outside the banking system that banks can buy with the additional equity.

view of bank debt is at odds with the notion of the discipline literature that debt holders are constantly on their toes, ready to run if they see bank managers doing things that they should not do.

[12] The term "producers of liquid debt" appears in Gorton (2012). The argument that equity funding of banks comes at the cost of liquidity provision through deposits is developed in Gorton and Winton (2014). DeAngelo and Stulz (2015) develop a model in which banks' profits are maximal if they fund only with deposits, i.e. short-term debt.

[13] Gorton (2010), as well as Hellwig (2009) and Bolton, Santos, Scheinkman (2011) stress the impact of lemons problems from asymmetric information in the breakdown of markets for mortgage-backed securities in 2007.

[14] See also Admati et al. (2011), Admati-Hellwig (2013).

[15] The paper gives no account of who are the actors, what are their preferences, endowments and technologies, and what precisely is traded in the different markets. The formal analysis is limited to a consideration of bank optimization. There is no analysis of market equilibrium and no welfare analysis. The paper is also flawed because it *assumes* that banks, i.e., the "producers" of deposits, are able to appropriate the consumers' liquidity benefits from these deposits. This assumption is in conflict with elementary microeconomics. Any intermediate textbook in microeconomics contains the message that, in the absence of price discrimination, consumer surplus stays with consumers, and producer profits result from quasi-rents in production. The price constellations that DeAngelo and Stulz (2015) assume in their analysis of bank optimization is in fact incompatible with market equilibrium.

As a statement about the real world, this presumption is unproblematic. At the level of theory, however, it is unsatisfactory because the allocation of assets itself must be treated as endogenous.

The present paper develops an intertemporal general-equilibrium model to provide a comprehensive analysis of the issues. Like DeAngelo and Stulz (2013), I assume that banks can obtain funding by issuing shares, bonds, and deposits. Investors value shares and bonds for their returns. They value deposits for their returns and, in addition, for the liquidity benefits they provide. I do not model these benefits explicitly but simply assume that people feel better if they hold "liquid" deposits. With due apologies to the public economics literature on voluntary giving, I refer to this as a "warm glow" model of liquidity benefits from deposits.[16] However, the "warm glow" utility from deposits is only obtained if the bank does not default on its debt. If a bank goes into default, deposits with this bank do not provide any liquidity benefits.

I also assume that deposit provision may involve a real cost. At small deposit levels, the marginal cost of additional deposits is assumed to be smaller than the (present value of) marginal liquidity from these deposits; however, beyond some critical level, the marginal costs of additional deposits exceed the marginal benefits and it would be inefficient to expand deposit provision beyond this critical level.

I will study two versions of the model. In the first version, there is no uncertainty and therefore no risk of a bank's defaulting on its debt. The market system is complete, and competitive equilibrium allocations are efficient. In this world, two types of allocations can arise. First, if consumers do not want to save very much, equilibrium is such that all bank funding takes the form of deposits, i.e., banks issue no shares or bonds. Second, if consumers want to save a lot, so that the equilibrium levels of their savings exceeds the critical deposit level beyond which the marginal costs of additional deposits exceed the marginal benefits, then equilibrium is such that banks obtain funding from shares or bonds as well as deposits. In this case, deposits are provided at the efficient level, where the (present value of) marginal liquidity benefits of additional deposits are equal to the marginal provisions costs; the excess of savings over the efficient deposit level is invested in shares or bonds, with a Modigliani-Miller indeterminacy property concerning the split between the two.

The second version of the model allow for uncertainty about the returns that banks earn on their investments. This uncertainty gives rise to a prospect of default. Default happens whenever the returns that a bank earns on its investments lie below the obligations of the bank to its depositors and its bond holders. For a bank in default, the liquidity benefits from deposits are zero.

Again, the market system is assumed to be (sequentially) complete. Finan-

[16] See in particular Andreoni (1988). The common denominator is the notion that people draw certain benefits from certain actions, that these benefits influence their behaviors, but the details of how the benefits are generated are not the focus of the analysis, which instead focuses on the implications of warm glow effects on people's behaviors and on the economy. The warm-glow specification of liquidity benefits from deposits is also used in van den Heuvel (2008).

cial securities are treated as bundles of state-contingent claims. Equilibrium pricing of debt will therefore depend on how the bank's funding mix affects the probability of default and therefore the debtholders' state-contingent returns. Equilibrium pricing of deposits will also depend on how the funding mix affects the probability that liquidity benefits might be zero because the bank is in default.

For the analysis of banks' funding policies, it makes a big difference whether banks are assumed to behave as price takers or whether banks are assumed to take account of the impact of their funding choices on market pricing. This is not a question of competitive versus oligopolistic behaviour, but a question about the ability of banks to commit to their funding policies and to communicate these policies to investors. If banks are able to commit to their funding choices *ex ante* and if they can credibly communicate these choices to investors, then a bank that is thinking about alternative choices will appreciate the effects of its choices on market pricing. If banks are unable to commit to their funding choices *ex ante* and to communicate these choices to investors, they will not consider the effects of their choices on market pricing. The interest rates they must pay in equilibrium will reflect the default risks that are implied by their choices, but the banks will take these interest rates as given because they have no way to credibly communicate a change in their funding mix to investors.[17]

I will study both cases. For the case where banks can fully commit to their funding policies and that these policies are observed by investors, I will show that competitive equilibrium allocations are constrained efficient.[18] I will also show that, unless producer's surplus from deposit provision is large, these allocations necessarily involve some equity funding of banks. In particular, equilibrium allocations *must* involve some equity funding of banks if deposit provision involves constant returns to scale so that the equilibrium level of producer's surplus is zero.[19] Producer's surplus is also relatively small if equilibrium deposit levels themselves are very small or very large so that marginal and average costs of deposit provision are approximately the same.

These results reflect the fact that equity funding reduces the default probability of a bank and may therefore raise the *expected* liquidity benefits from deposits. With constant returns to scale, it turns out that, without equity, the conditions for equilibrium levels of deposit funding would imply a default probability equal to one. Expected liquidity benefits from deposits would then be zero. From the bank's perspective, this outcome is equivalent to what it would get with pure equity funding and, like pure equity funding, is dominated by

[17]The fundamental difference between these two specifications is not always appreciated. Thus, Allen et al. (2015) have an implicit assumption of effective prior commitment and communication. In contrast, Brunnermeier and Oehmke (2012) are concerned with the implications of not having such commitments.

[18]The word "constrained" refers to the fact that changes in the implicit intertemporal prices have an effect on the interest rates that banks must pay and therefore on their default probabilities. In the efficiency claim, such changes are excluded. A specific tax on first-period consumption, with revenue effect neutralized by a lump-sum subsidy, would change the implicit intertemporal prices and might therefore induce a Pareto-superior allocation.

[19]This includes the case where deposit provision involves no cost at all.

a mix of equity and deposit funding in which the equity provides a buffer to protect the liquidity benefits from deposits.

If banks are not able to precommit their funding choices and to communicate these choices credibly to investors, constrained efficiency is not to be expected. For this case, I find that, except for the case where savings are large and return uncertainty is small, equilibrium allocations involve zero equity funding. From the banks' perspective, new share issues are dominated by new bond issues, which in turn may or may not be dominated by deposits. The preference for debt over equity reflects a standard debt overhang effect.[20]

In the absence of commitment and communication of funding choices, banks' decisions about additional equity finance or debt finance neglect the external effect of their decisions on the incumbent depositors' expected liquidity benefits. Instead, they take account of the fact that lowering the default probability raises expected payments to debtholders. With commitment and communication of funding choices, these external effects are taken into account through the effects of funding choices on the interest rates that banks must pay. Without commitment and communication, however, banks take interest rates as given because investors who cannot observe their funding choices cannot adapt their behaviours to the changes in default risks that come from banks' adding to the debts they take on. To be sure, the interest rates that banks must pay will reflect investors' anticipations of the banks' default probabilities and, in equilibrium, these anticipations will correspond to the actual default probabilities. However, the inability to commit and communicate their choices prevents banks from taking this relation into account. As a result, they end up with too little equity funding and too much borrowing.[21]

In the constant-returns-to-scale case, it follows that *any* form of capital regulation will raise welfare. More generally, if producer's surplus from deposit provision is small relative to the banks' investment levels, equilibrium allocations in the model without commitment and communication of banks' funding choices involve inefficiently low levels of equity funding and can be improved upon by regulation imposing minimum equity requirements.

Somewhat ironically, such regulation will also raise banks' profits and shareholder value. A major effect of the regulation is due to changes in competitive conduct. Equity requirements for banks may reduce the intensity of competition and allow banks to earn higher margins. The phrase "Why high leverage is optimal for banks" in the title of DeAngelo and Stulz (2015) is thereby turned on its head. Equity requirements that reduce leverage may be better for banks because they allow them to earn margins they would not otherwise earn.

Most of the literature on equity requirements for banks has focused on the effects of changes in a bank's funding mix on the split of the bank's asset returns between creditors and shareholders and on the implications of such changes for

[20] See Admati et al. (2012/2014).

[21] This is similar to the debt overhang or leverage ratchet effect discussed in Admati et al. (2012/2013).

incentives and bankruptcy prospects. By contrast, the analysis here shows that some effects may be due to changes in the market environment. When considering a regulation that affects all members of the industry, the presumption that market conditions are unchanged is inappropriate. The analysis of regulation must shift its focus from the individual bank to the overall functioning of the system.

In the following, Section 2 introduces the basic model and analyses the case of certainty. Section 3 introduces the case of uncertainty, with basics discussed in Section 3.1, the commitment case in Section 3.2, and the no-commitment case in Section 3.3. All proofs are given in the Appendix.

2 A Warm-Glow Model of Liquidity Provision: The Case of Certainty

I consider an economy with two periods and one good in each period. There are also two types of agents, banks and consumers. To abstract from issues of market power, I assume that the set of consumers and the set of banks are both represented by the unit interval with Lebesgue measure. For simplicity, I will also assume that all consumers have the same characteristics and so do all banks.

Banks

In the simplest version of the model, a typical bank has a single investment opportunity, "loans", which allow the bank to transform L units of the good in period 0 into φL units of the good in period 1. Nonfinancial firms, i.e., the banks' borrowers, are not modelled. As in most of the literature on liquidity provision by banks, e.g. Diamond-Dybvig (1983) or Calomiris-Kahn (1991), the banks' investments in loans are treated as if they were physical investments. For now I assume that there is no uncertainty, i.e. φ is given and commonly known. Subsequently, in Section 3, I will assume that φ is the realization of a random variable $\tilde{\varphi}$ and only becomes known in period 1.

The bank has no initial endowment. To fund its investments, it can issue bonds or deposits. If the bank issues B^s bonds and D^s deposits, it receives $B^s + D^s$ units of the good in period 0 and must pay its creditors $r_B B^s + r_D D^s$ units of the good in period 1, $r_B B^s$ to bond holders and $r_D D^s$ to depositors. Deposits involve a cost $K(D^s)$, where $K(\cdot)$ is a continuously differentiable, nondecreasing convex function satisfying $K(0) = 0$ and $0 \leq K'(D) < 1 - \Delta$ for some $\Delta > 0$ and all $D > 0$.

The bank can also obtain funding by selling shares. For this purpose, it chooses a fraction α^s of the firm's equity that it wants to sell and a purchase price E so that, if the offering succeeds, the proceeds are equal to $\alpha^s E$. For the offering to succeed, the purchase price E must not exceed what the market is willing to pay, which depends on the profit π that the bank is expected to make and on the discount factor η that the market applies to this profit. Thus E must satisfy the inequality

$$E \leq \eta\pi. \tag{2.1}$$

Given these data, a *plan for the bank* is a vector $(\alpha^s, E, B^s, D^s, L, \pi)$ specifying a *funding policy* (α^s, E, B^s, D^s), an *investment policy* L, and a *profit target* π. The plan $(\alpha^s, E, B^s, D^s, L, \pi)$ is *feasible* if it is nonnegative and satisfies the inequality (2.1) as well as the budget constraint

$$L \leq \alpha^s E + B^s + D^s - K(D^s) \tag{2.2}$$

in period 0 and the equation

$$\pi = \varphi L - r_B B^s - r_D D^s \tag{2.3}$$

for the bank's profit in period 1.

Consumers

A typical consumer has initial endowments $e_0 > 0$ and $e_1 = 0$ of the good in periods 0 and 1 and a share $\alpha_0 = 1$ in each bank. His preferences are given by a utility function of the form

$$u(c_0) + v(c_1 + \theta(D^d)), \tag{2.4}$$

where c_0 and c_1 are the levels of his consumption in periods 0 and 1, and $\theta(D^d)$ represents the liquidity benefits that he gets from holding deposits $D^d = \int_0^1 D^d(b)db$. Liquidity benefits here are modeled as benefits that the consumer obtains from the "warm glow" of feeling better by having "liquid" deposits as opposed to shares or bonds.[22] The function $\theta(\cdot)$ is assumed to be continuously differentiable, increasing and concave, with $\theta(0) = 0$ and $\theta'(0) < \infty$. The functions $u(\cdot)$ and $v(\cdot)$ are also assumed to be continuously differentiable, strictly increasing, and strictly concave, with $u(0) = v(0) = 0$. Moreover,

$$u'(0) = \infty, \text{ and } v'(0) = \infty, \tag{2.5}$$

$$u'(\infty) = 0, \text{ and } v'(\infty) = 0. \tag{2.6}$$

Condition (2.5) ensures that the model is nontrivial in the sense that any equilibrium must involve some transfer of resources from period 0 to period 1 and therefore some real investment and some funding of banks.

To provide for his consumption in period 1, the consumer can rely on the distribution of profits from the bank in which he initially has shares. He can

[22] I use the term "warm glow" with due apologies to the public economics literature, which has used the same term for the good feelings people may have if they contribute to public goods or to altruistic ventures, see, e.g. Andreoni (1988). The common denominator is the notion that people draw certain benefits from certain actions, that these benefits influence their behaviors, but the details of how the benefits are generated are not the focus of the analysis, which instead considers a reduced form expression for the benefit.

also acquire additional shares, deposits or bonds. As he is doing so, he must respect the period 0 budget constraint

$$e_0 = c_0 + \int_0^1 \alpha^d(b)E(b)db + \int_0^1 B^d(b)db + \int_0^1 D^d(b)db, \tag{2.7}$$

where, for each b, $\alpha^d(b)$, $B^d(b)$ and $D^d(b)$ represent the additional shares, the bonds and the deposits of bank b that the consumer acquires, the cost of these acquisitions is $\alpha^d(b)E(b) + B^d(b) + D^d(b)$. Consumption in period 1 is given by the consumer's returns on bank shares, bonds and deposits, as shown in the equation

$$c_1 = \int_0^1 (1 - \alpha^s(b) + \alpha^d(b))\pi(b)db + r_B \int_0^1 B^d(b)db + r_D \int_0^1 D^d(b)db; \tag{2.8}$$

in the first term on the right-hand side of this equation, the factor $(1 - \alpha^s(b) + \alpha^d(b))$ indicates the consumer's share in the profits of bank b, taking account of the consumer's new share acquisition $\alpha^d(b)$ as well as the dilution of his initial holding through the bank's new share issue $\alpha^s(b)$.

A *plan for the consumer* is a vector $(c_0, c_1, \alpha^d(\cdot), B^d(\cdot), D^d(\cdot))$ that specifies his consumption levels in the two periods as well as the investments in the securities issued by the different banks that he uses to transfer wealth from period 0 to period 1. Given the price system (η, r_B, r_D) and the banks' plans $(\alpha^s(b), B^s(b), D^s(b), L(b)), b \in [0, 1]$, a plan $(c_0, c_1, \alpha^d(\cdot), B^d(\cdot), D^d(\cdot))$ for the consumer is *admissible* if it satisfies the budget constraints (2.7) and (2.8), as well as nonnegativity of c_0, c_1, and $1 - \alpha^s(b) + \alpha^d(b), B^d(b), D^d(b)$ for all b.

Equilibrium

In this model, an *equilibrium* is given by a price system (η, r_B, r_D), a measurable mapping $b \to (\alpha^s(b), E(b), B^s(b), D^s(b), L(b), \pi(b))$ indicating a plan for each bank and a measurable mapping $a \to (c_0(a), c_1(a), \alpha^d(\cdot, a), B^d(\cdot, a), D^d(\cdot, a))$, indicating a plan for each consumer $a \in [0, 1]$; such that

(e.1) given the price system (η, r_B, r_D), for (almost) every bank b, the plan $(\alpha^s(b), B^s(b), D^s(b), L(b))$ maximizes the post-dilution value $(1 - \alpha^s)E$ of the bank's initial shareholders' equity subject to the constraints (2.2), (2.1), (2.3) and nonnegativity;

(e.2) given the price system (η, r_B, r_D) and the banks' plans, for (almost) every consumer a, the plan $(c_0(a), c_1(a), \alpha^d(\cdot, a), B^d(\cdot, a), D^d(\cdot, a))$ maximizes the consumer's utility (2.4), with $D^d = \int_0^1 D^d(b)db$, over the set of his admissible plans;

(e.3) the banks' and consumers' plans satisfy the market clearing conditions

$$\int_0^1 L(b)db = e_0 - \int_0^1 c_0(a)da \tag{2.9}$$

for the goods market in period 0 and

$$\alpha^s(b) = \int_0^1 \alpha^d(a,b)da, \quad B^s(b) = \int_0^1 B^d(a,b)da, \quad D^s(b) = \int_0^1 D^d(a,b)da, \tag{2.10}$$

for the markets for securities issued by (almost) every bank $b \in [0,1]$ in period 0.

The equilibrium concept presumes that both, banks and consumers, are price takers with respect to η, r_B, and r_D; this is the standard Walrasian approach to modelling lack of market power.[23] Banks can affect the market value of their equity but only as changes in their plans affect the profits they can earn at the prevailing price system. Consumers take prices, the banks' actions, and their profits, as given.

I will focus on *symmetric equilibria.* in which the plans of the different banks are all the same and the plans of the different consumers are also all the same. The market-clearing conditions (2.10) imply that, in a symmetric equilibrium, any one consumer's holdings of securities issued by the different banks must also be the same, i.e., there exists a vector $(\alpha, B, D, L, c_0, c_1))$ such that

$$\alpha = \alpha^s(b) = \alpha^d(a,b), B = B^s(b) = B^d(a,b), D = D^s(b) = D^d(a,b) \tag{2.11}$$

and

$$L = L(b), c_0 = c_0(a), c_1 = c_1(a) \tag{2.12}$$

for (almost) all a and b. In this case, obviously one can also write $E(b) = E$ and $\pi(b) = \pi$ for all b.

Analysis

In characterizing symmetric equilibria, it is useful to consider the net marginal social benefit of having funding go through deposits rather than shares or bonds. This is given by the difference

$$m(D, \varphi) := \theta'(D) - \varphi K'(D) \tag{2.13}$$

between the marginal liquidity benefit $\theta'(D)$ from additional deposits and the difference $\varphi K'(D)$ between the marginal return φ that is obtained if additional funding goes through shares or bonds and the marginal return $\varphi(1 - K'(D))$ that is obtained if the additional funding goes through deposits.

[23] An alternative to the Walrasian approach would be a Bertrand approach, with price-setting banks. For the simple model considered here, equilibrium outcomes would be the same as in the Walrasian approach. However, in the Bertrand approach, the treatment of the bank's assets as if they were physical assets involves a significant loss of generality. As shown by Yanelle (1997), the Bertrand approach to modeling perfect competition runs into difficulties in dealing with financial intermediaries competing on both the assets and the liabilities sides of their balance sheets.

To make the analysis interesting, I assume that

$$m(D, \varphi) > 0 \tag{2.14}$$

if $D > 0$ is very small. This assumption ensures that *any* equilibrium will involve at least some funding through deposits.

Indeed, the assumption that $m(D, \varphi) > 0$ if D is small implies that it is efficient to have all savings go into deposits if the initial endowment e_0 is small. If e_0 is small, the value $e_0 - c_0 \leq e_0$ of consumers' savings must also be small and so must be the value $D \leq e_0 - c_0$ of consumers' deposits. In this case, efficiency considerations would seem to suggest that, in equilibrium, all savings should in fact go into deposits, rather than shares or bonds.

If, instead, the period 0 endowment is very large, the value $e_0 - c_0$ of consumers' savings will also be very large. If $m(D, \varphi) > 0$ for *all* D, the argument for having all savings in deposits rather than shares or bonds remains valid. However, if $m(D, \varphi) < 0$ for large D, efficiency considerations would seem to suggest that the equilibrium value of deposits must be bounded by some critical level D^* and that any excess of savings over the critical level D^* will go into shares or bonds.

The following proposition confirms this intuition and shows that there are two distinct classes of equilibria, one where all savings go into deposits and one where some savings go into shares or bonds. In the latter case, deposits are equal to some critical level $D^*(\varphi)$, where

$$m(D^*(\varphi), \varphi) = 0, \tag{2.15}$$

and the investment in shares and bonds is equal to the excess of savings over $D^*(\varphi)$. For simplicity, I assume that equation (2.15) has at most one solution $D^*(\varphi)$. In an abuse of language, I will say that $D^*(\varphi) = \infty$ if equation (2.15) has no solution, i.e. if $m(D, \varphi) > 0$ for all D. For lack of a better term, I will refer to $D^*(\varphi)$ as the *satiation level* of deposits.[24]

Proposition 2.1 *A symmetric equilibrium exists and involves a Pareto-efficient allocation. Any symmetric equilibrium allocation is unique up to changes in $\alpha^s = \alpha^d$ and $B^s = B^d$ that leave the amount of non-deposit funding unchanged. Moreover, there exists $\hat{e}_0 \in (0, \infty]$ such that the following hold:*

a: If $e_0 \leq \hat{e}_0$, the equilibrium deposit level is the unique solution to the equation

$$u'(e_0 - D) = [\varphi + n(D, \varphi)]\, v'(\varphi D - \varphi K(D) + \theta(D))\,. \tag{2.16}$$

Moreover,

$$L = D - K(D) \tag{2.17}$$

[24] If $\theta'(D) > 0$ for $D > D^*(\varphi)$, consumers are not really satiated, but if the opportunity cost of funding by deposits rather than bonds are taken into account, then with $\theta'(D) < \varphi K'(D)$ consumers do not want to raise deposits above $D^*(\varphi)$.

and

$$\alpha^s E = \alpha^d E = B^s = B^d = 0. \tag{2.18}$$

b: If $e_0 > \hat{e}_0$, the equilibrium levels of deposits and loans satisfy

$$D = D^*(\varphi) \tag{2.19}$$

and

$$u'(e_0 - L - K(D^*(\varphi))) = \varphi\, v'(\varphi L + \theta(D^*(\varphi))). \tag{2.20}$$

Moreover,

$$\alpha^s E + B^s = \alpha^d E + B^d = L - D^*(\varphi) + K(D^*(\varphi)) > 0. \tag{2.21}$$

c: In both cases, if $e_0 \leq \hat{e}_0$ and if $e_0 > \hat{e}_0$, the equilibrium price system satisfies

$$\eta = \frac{1}{\varphi + m(D,\varphi)}, \tag{2.22}$$

$$r_B \in [\varphi, \varphi + m(D,\varphi)], \tag{2.23}$$

and

$$r_D = (1 - K'(D))\,\varphi. \tag{2.24}$$

d: In both cases, if $e_0 \leq \hat{e}_0$ and if $e_0 > \hat{e}_0$, the equilibrium value of profits satisfies

$$\pi = \frac{1}{1-\alpha}\varphi[DK'(D) - K(D)]. \tag{2.25}$$

The proof of Proposition 2.1, which is given in the appendix, starts from the observation that the conditions of the first welfare theorem are satisfied, so any equilibrium allocation must be Pareto efficient. Given the additional requirement that the allocation be symmetric, an equilibrium allocation must therefore maximize the utility (2.4) of a representative consumer over the set of nonnegative allocations satisfying the constraints

$$c_0 = e_0 - \alpha E - B - D, \tag{2.26}$$

and

$$c_1 = \varphi L, \tag{2.27}$$

$$L + K(D) \leq \alpha E + B + D. \tag{2.28}$$

Condition (2.26) follows from (2.7), condition (2.27) from (2.8), (2.10), and (2.3), and condition (2.28) from (2.2) and (2.10), all in combination with symmetry. In this maximization, the constraint (2.28) must obviously be binding.[25] If one uses the resulting equation to rewrite (2.26) and to substitute for c_0, one sees that the welfare maximization problem involves maximizing

$$u(e_0 - L - K(D)) + v(\varphi L + \theta(D)) \tag{2.29}$$

[25] Otherwise, L and c_1 could be increased to raise utility in period 1 without any cost.

subject to

$$L = \alpha E + B + D - K(D) \tag{2.30}$$

and nonnegativity. The first-order conditions for this maximization yield conditions (2.16) - (2.18) and (2.19) - (2.21) in Proposition 2.1. Given these conditions characterizing the efficient allocation, conditions (2.22) - (2.24) specify the price systems that support the efficient allocation; these conditions are derived from the first-order conditions for the banks' and the consumers' maximization problems.

The distinction between the case $e_0 \leq \hat{e}_0$ and the case $e_0 > 0$ arises naturally from the observation that, for the given specification of consumer preferences, savings are given by an increasing function of e_0, close to zero if e_0 is close to zero and very large if e_0 is very large. For small levels of the initial endowment, therefore, deposits will lie below the satiation level $D^*(\varphi)$ even if all savings are invested in deposits. At these small levels of initial endowments, investing all savings in deposits is efficient because, with $m(D, \varphi) > 0$, this maximizes the net social benefits from deposits. Above $\hat{e}_0$ however, deposits are kept at $D^*(\varphi)$ and additional savings are put into shares or bonds because putting them into deposits would diminish the net social benefits from deposits.

Discussion

Proposition 2.1 illustrates the importance of considering liquidity provision by banks in a general equilibrium setting. The proposition provides a framework for organizing the discussion and for correcting some misperceptions that have been introduced by a failure to recognize that the issues must be discussed in terms of general equilibrium rather than bank optimization. Thus, Proposition 2.1 shows that, in some circumstances, namely if $e_0 \leq \hat{e}_0$ in equilibrium, *all* bank funding takes the form of deposits. Contrary to the suggestions of Admati, Conti-Brown, and Pfleiderer (2012), Admati et al. (2010/2013), or of Admati and Hellwig (2013 a,b), in these circumstances, there is no way to increase equity funding of banks without reducing their deposit funding.

Moreover, because the equilibrium allocation is efficient, any regulation that changes the equilibrium allocation would lower welfare. In particular, in the case $e_0 \leq \hat{e}_0$, where all bank funding takes the form of deposits, any regulation that imposes minimum equity requirements on banks would lower welfare. Such regulation would lower the net social benefits from having bank funding come through deposits rather than equity or bonds.

For the case $e_0 \leq \hat{e}_0$ Proposition 2.1 confirms the view of DeAngelo and Stulz (2015) that, because of liquidity benefits, all bank funding should come from deposits. However, this result is obtained from equilibrium considerations rather than the analysis of bank optimization.

The analysis of bank optimization in DeAngelo and Stulz (2015) is actually irrelevant because it assumes a value of the deposit rate that is incompatible with the conditions for a competitive equilibrium. DeAngelo and Stulz focus on the "liquidity premium" in deposits rates, i.e. the difference between the interest rate on bonds and the interest rate on deposits, without appreciating that this "liquidity premium" is irrelevant for banks' profits if bonds (or shares) are not used for funding anyway.

The profitability of banks' funding by deposits depends on the margin between the rate of return φ on loans and the deposit rate r_D. DeAngelo and Stulz treat this margin as exogenously given. As indicated by equation (2.24), however, the equilibrium value of the margin $\varphi - r_D$ is equal to $K'(D))\varphi$, which depends on the equilibrium value of D. If $e_0 < \hat{e}_0$, the equilibrium value of this margin is actually *strictly less* than the "liquidity premium" $\theta'(D)$ on which De Angelo and Stulz focus their attention.

Equilibrium profits of banks are given by a measure of *producer's surplus* involving the difference $DK'(D) - K(D)$ between the value $DK'(D)$ of deposits at the "producer's price" $K'(D)$, and the cost $K(D)$. There is no direct effect of the "liquidity premium" $\theta'(D)$ on bank profits. There can only be an indirect effect as the functions $\theta'(\cdot)$ and $K'(\cdot)$ jointly affect the equilibrium deposit level D and thereby the equilibrium level of the producers' surplus $DK'(D) - K(D)$. However, if $e_0 < \hat{e}_0$, i.e., if $\varphi K'(D) < \theta'(D)$, any measure of banks' profits that involves the variable $\theta'(D)$, i.e., the "liquidity premium" in the deposit rate, is inappropriate. If the cost function K is linear, producer's surplus is actually zero. In this case, banks earn zero profits even though consumers find bank deposits attractive and are willing to accept a lower rate of return on deposits than on shares or bonds.[26]

The case $e_0 > \hat{e}_0$ is very different. In this case, in equilibrium, *some* bank funding must come from shares or bonds, and this is efficient. Moreover, by a version of the Modigliani-Miller theorem, it does not make a difference whether the nondeposit funding comes from shares or from bonds. As long as deposits are not reduced below the critical $D^*(\varphi)$, statutory minimum equity requirements for banks would be costless but also useless.

In DeAngelo and Stulz (2015), the case $e_0 > \hat{e}_0$ does not appear because they have no cost of deposit provision. With $K(D) \equiv 0$, the marginal social cost of funding by deposits rather than shares or bonds would be zero, and the net marginal social benefit would be equal to $\theta'(D)$; the critical $D^*(\varphi)$ is the lowest deposit level at which $\theta'(D) = 0$ or, if no such level exists, $D^*(\varphi) = \infty$. In the real world of course, banks do need resources for payments services and ATMs, which are directly related to the liquidity benefits that banks provide to depositors. Moreover, major banks in the real world use a lot of nondeposit funding. To the extent that one wants to take this kind of theoretical analysis seriously at all, therefore, the case $e_0 > \hat{e}_0$ seems more relevant than the case $e_0 \leq \hat{e}_0$.

[26] All this should be standard for someone who has taken a course in intermediate microeconomics. In competitive markets, a supplier's profits are derived from his inframarginal rents, which depend on the behaviour of his cost function. Consumer preferences affect producer's surplus only indirectly, by influencing the equilibrium price and quantity and thereby, possibly, the inframarginal rents.

DeAngelo and Stulz (2015) do allow for a cost of overall bank activity. In the present framework with only one asset, this would correspond to a cost $k(L)$ of lending. If such a cost is introduced into the present model, the equilibrium conditions would by and large be the same, except that the net marginal return to lending would be $\frac{\varphi}{1-k'(L)}$, rather than φ. Equilibrium profits of banks would then involve the difference $Lk'(L) - k(L)$, instead of $DK'(D) - K(D)$.

The model is easily extended to allow for multiple assets that banks (or consumers) might invest in. In the absence of uncertainty, however, not much is to be learnt from such an extension. If other assets have a rate of return ψ, the bank's choice between loans and other assets depends on whether ψ is greater or less than φ. In either case, they will specialize in the investment with the higher rate of return, and we are effectively back in a one-asset setting, with a critical deposit level equal to $D^*(\max(\varphi, \psi))$.

The situation is slightly different if investments in the asset with the higher rate of return are limited, say, if $\varphi > \psi$ and there is a constraint $L \leq \bar{L}$ on bank lending. In this case, equilibrium allocations will depend on how the deposit level that is required to fund bank lending at the level $\bar{L}$ compares to the critical levels $D^*(\varphi)$ and $D^*(\psi)$. For example, if $D^*(\psi) > \bar{L}$ (and, a fortiori, $D^*(\varphi) > \bar{L}$), there exists an intermediate range of values of e_0, such that, for initial endowments in this range, in equilibrium, all savings go into deposits, the constraint on bank lending is binding, the difference $D^*(\psi) - \bar{L}$ goes into the alternative investments, and the relevant marginal rate of return on investments is ψ rather than φ. The equilibrium deposit rate is then given by

$$r_D = (1 - K'(D))\psi, \tag{2.31}$$

rather than (2.24). Equilibrium profits will comprise a term $(\varphi - \psi)\bar{L}$ which represents the inframarginal profits that banks earn because loans provide a rate of return φ, more than the rate ψ, which according to (2.31) is key for determining the deposit rate.

However, this latter finding is an artefact of the assumption that the loan rate φ is exogenous. The notion that banks can earn an inframarginal rent $(\varphi - \psi)\bar{L}$ on loans is not robust to changes in the model that would endogenize φ. If loans are more attractive than other investments, competition between banks is likely to drive down loan rates, removing the inframarginal rents.

Similar results are to be expected if all funds from deposits are required to go into one particular class of assets, for example loans and there is a constraint $L \leq \bar{L}$ on bank lending. In this case, if φ is exogenous and $D^*(\varphi) - K(D^*(\varphi)) > \bar{L}$, competition will drive the deposit rate to the level $(1 - K'(D))\varphi$, as in Proposition 2.1, and the marginal rate of return ψ on alternative investments is irrelevant for the deposit rate. If e_0 is high enough so that savings exceed the amount needed to fund $\bar{L}$, the excess savings will be invested in shares and bonds, rather than deposits, not because the critical level $D^*(\varphi)$ has been reached, but because additional deposit creation is ruled out by a lack of lending opportunities. If the lending rate φ is endogenized, the equilibrium rates of return φ and r_D on loans and deposits will depend on the behaviours of borrowers and depositors jointly.[27]

3 Liquidity Provision and Equity Funding Under Uncertainty

3.1 Basics

Absence of uncertainty is a bad assumption for an analysis of liquidity provision. In the real world, people care about liquidity of their holdings precisely because they do not know in advance when they will want to dispose of their assets and they care about the ease of doing so when they want to.[28] In the remainder of the paper, I therefore extend the analysis to allow for uncertainty about the returns that banks earn on their investments.

I still abstract from the details of how people benefit from liquidity. However, the uncertainty about the returns that banks will earn on their investments implies that heavily indebted banks may end up being unable to fulfill their obligations to their creditors. If a bank defaults on its obligations, its deposits are assumed to yield no liquidity benefits.[29]

Return Uncertainty

The model is the same as before except that now the rate of return on loans, φ, is the realization of a random variable $\tilde{\varphi}$. This random variable is defined on some underlying probability space and takes value in some interval $[\varphi_1, \varphi_2]$. Its realization is revealed at $t = 1$. Before $t = 1$, agents know only the probability distribution F of the random variable $\tilde{\varphi}$. This distribution is assumed to have a density f, which is continuous and strictly positive on the interval $[\varphi_1, \varphi_2]$.

The uncertainty about $\tilde{\varphi}$ affects all banks alike. Shocks to the rate of return on loans are macroeconomic shocks, the result of risks to the overall economy, rather than individual risks that might be handled by appropriate subdivision between banks or consumers and by appropriate diversification across loans or

[27] DeAngelo and Stulz (2015, p. ???) write that, in their model with uncertainty, deposit funding of banks is limited by the requirement that investments backing the deposits must be riskless, so deposits cannot exceed the resources available in the least favorable state. This constraint, however, appears nowhere in their mathematics. If it did, they would have realized that, in equilibrium there must be a link between the deposit rate and the rate of return on those assets that can be used to back the deposits. They would also have realized that the scarcity of assets that can be used to back deposits imposes a bound on deposit funding of banks and therefore may give rise to an equilibrium of the type that is characterized in statement (b) of Proposition 2.1 even if $D^* = \infty$.

[28] According to Hicks's (1935) definition, which is still relevant today, an asset is the more liquid, the more certain an investor can be to realize it at short notice without a loss. In this definition, uncertainty plays a central role.

[29] The arguments of Dang, Gorton and Holmström (2012) suggest that, in a fully specified model of liquidity of different assets, default prospects would be a major source of illiquidity in the very sense of Hicks's original definition.

across banks. Subdivision and diversification of individual specific risks are important in banking, but are nevertheless neglected here.[30] In practice, the most serious banking problems tend to be tied to macroeconomic shocks. Therefore, I want to focus on those and assume that loan customer-specific risks play no role or are perfectly diversified at the level of each bank.[31]

From the perspective of period 0, the bank's profit is now a random variable. With limited liability, this random variable is given as

$$\tilde{\pi} = \pi(\tilde{\varphi}) = \max(0, \tilde{\varphi}L - r_B B^s - r_D D^s), \tag{3.1}$$

where, as before, L is the bank's investment in loans, B^s and D^s are the bank's supplies of bonds and deposits, and r_B and r_D are the interest rates on bonds and deposits. If $\tilde{\varphi}L \geq r_B B^s + r_D D^s$, the returns on loans are large enough for the bank to pay its debts; in this case, the bank's profit is equal to the difference between the returns on loans and the payments to bond holders and depositors, just as in the preceding section. If $\tilde{\varphi}L < r_B B^s + r_D D^s$, the bank cannot pay its debts in full. In this case, the profit distribution to shareholder is zero; because of limited liability, the shareholders do not have to put up the additional funds needed to pay the bank's debts. Instead, the gross return $\tilde{\varphi}L$ is shared among the bank's debt holders in proportion to the claims they have on the bank. Thus, if $\tilde{\varphi}L < r_B B^s + r_D D^s$, bond holders and depositors receive the amounts

$$r_B \frac{\tilde{\varphi}L}{r_B B^s + r_D D^s} \quad \text{and} \quad r_D \frac{\tilde{\varphi}L}{r_B B^s + r_D D^s},$$

respectively, for each unit of the date 0 good they put up. The rates of return to bond holders and depositors are given by the random variables

$$r_B \min(1, \frac{\tilde{\varphi}_L L}{r_B B^s + r_D D^s}) \;\; \text{and} \;\; r_D \min(1, \frac{\tilde{\varphi}L}{r_B B^s + r_D D^s}) \tag{3.2}$$

or, in more compact notation,

$$r_B \min(1, \frac{\tilde{\varphi}}{\hat{\varphi}}) \;\; \text{and} \;\; r_D \min(1, \frac{\tilde{\varphi}}{\hat{\varphi}}), \tag{3.3}$$

where

$$\hat{\varphi} := \frac{r_B B^s + r_D D^s}{L} \tag{3.4}$$

is the value of $\tilde{\varphi}$ at which the bank is just able to fulfill its obligations to its creditors, the boundary between the default and the non-default region.

Asset Valuation

In choosing its funding and investment policies the bank must take account of how the market values the securities that it issues. The price E at which it issues

[30] On the role of diversification and subdivision of individual risks in banking, see Diamond (1984) and Hellwig (2000).

[31] On the role of macroeconomic shocks in banking, see Hellwig (1994, 1995, 1998).

new equity must not exceed the value $V(\tilde{\pi})$ that the market in period 0 assigns to the bank's random period 1 profit. Similarly, the value $V(r_B \min(1, \frac{\tilde{\varphi}}{\hat{\varphi}}))$ that the market in period 0 assigns to the random return $r_B \min(1, \frac{\tilde{\varphi}}{\hat{\varphi}})$ on a bond must not be less than the one unit of the period 0 good for which the bond is offered.

How are these return random variables valued in period 0? To provide an answer this question, I assume that, between the initial period $t = 0$ and the final period $t = 1$, there is an intervening period $t = \frac{1}{2}$, in which the different participants can trade contingent claims in a complete market system. Prices in these markets are given by a function $q : [\varphi_1, \varphi_2] \to \mathbb{R}_+$, such that a claim to one unit of the consumption good in period 1 in the event $\{\tilde{\varphi} \in [\varphi', \varphi'']\}$ costs the amount

$$\int_{[\varphi', \varphi'']} q(\varphi) f(\varphi) d\varphi, \tag{3.5}$$

where, as mentioned above, f is the density function of the distribution of $\tilde{\varphi}$. The function q is normalized so that

$$\int_{[\varphi_1, \varphi_2]} q(\varphi) f(\varphi) d\varphi = 1. \tag{3.6}$$

In the markets at $t = \frac{1}{2}$, the returns on *any* security can be replicated by a bundle of contingent claims. If a security has the random return $h(\tilde{\varphi})$, then at $t = \frac{1}{2}$, the bundle of contingent claims that yields the same returns costs

$$\mathcal{E}_q h(\tilde{\varphi}) := \int_{[\varphi_1, \varphi_2]} q(\varphi) h(\varphi) f(\varphi) d\varphi. \tag{3.7}$$

The market value of this security at $t = \frac{1}{2}$ is therefore $\mathcal{E}_q h(\tilde{\varphi})$.

If the security is already traded in the markets at $t = 0$, its market value at that date is given as

$$V(h(\tilde{\varphi})) = \eta \, \mathcal{E}_q h(\tilde{\varphi}), \tag{3.8}$$

where, as before, η is a discount factor. This discount factor is not actually a market price but a common factor that underlies all asset pricing at $t = 0$. Because there is no resolution of uncertainty between $t = 0$ and $t = \frac{1}{2}$, simple arbitrage considerations, imply that any securities that are actively traded in $t = 0$ must be priced in such a way that they all provide the same rate of return from $t = 0$ to $t = \frac{1}{2}$. The discount factor is simply the inverse of this common rate of return.[32]

Because the bank's equity is a claim to the profit random variable $\tilde{\pi}$, its value at $t = 0$ is

$$V(\tilde{\pi}) = \eta \, \mathcal{E}_q \pi(\tilde{\varphi}) = \eta \int_{[\varphi_1, \varphi_2]} q(\varphi) \pi(\varphi) f(\varphi) d\varphi. \tag{3.9}$$

[32] Heuristically, η is the value at $t = 0$ of a bundle of contingent claims that promises to provide one unit of the consumption good at $t = 1$, regardless of the state. The model does not actually contain a market for this bundle, but, because the market system is sequentially complete, the value of this bundle at $t = 0$ is implicitly given by the equilibrium price system. The details of the argument are spelled out in Lemmas 3.1 and 3.2 below.

Thus, if the bank wants to issue new equity at a price E, it must respect the inequality

$$E \leq \eta\ \mathcal{E}_q \pi(\tilde{\varphi}). \tag{3.10}$$

Similarly, a bond with the return random variable $r_B \min(1, \frac{\tilde{\varphi}}{\hat{\varphi}})$ is worth

$$\mathcal{E}_q r_B \min(1, \frac{\tilde{\varphi}}{\hat{\varphi}}) = r_B \int_{[\varphi_1, \varphi_2]} q(\varphi) \min(1, \frac{\varphi}{\hat{\varphi}}) f(\varphi) d\varphi \tag{3.11}$$

in the markets at $t = \frac{1}{2}$ and

$$V(r_B \min(1, \frac{\tilde{\varphi}}{\hat{\varphi}})) = \eta\ \mathcal{E}_q r_B \min(1, \frac{\tilde{\varphi}}{\hat{\varphi}})$$

in the markets at $t = 0$. For investors to pay one unit of the good in period 0 for this bond, the interest rate r_B must be high enough so that

$$1 \leq V(r_B \min(1, \frac{\tilde{\varphi}}{\hat{\varphi}})) = \eta\ \mathcal{E}_q r_B \min(1, \frac{\tilde{\varphi}}{\hat{\varphi}}), \tag{3.12}$$

i.e. r_B must satisfy the inequality

$$r_B \geq \frac{\hat{\varphi}}{\eta\ \mathcal{E}_q \min(\hat{\varphi}, \tilde{\varphi})}. \tag{3.13}$$

Because deposits provide liquidity benefits as well as monetary returns, the analogous condition for the deposit rate r_D cannot be derived from asset pricing considerations alone. This condition takes the form

$$r_D \geq \frac{\hat{\varphi}}{\eta\ \mathcal{E}_q \min(\hat{\varphi}, \tilde{\varphi})} - \lambda(\hat{\varphi}), \tag{3.14}$$

where $\lambda(\hat{\varphi})$ is the liquidity premium in the deposit rate. This liquidity premium depends on the bank's default point $\hat{\varphi}$ because deposits provide liquidity benefits only if $\tilde{\varphi} \geq \hat{\varphi}$.

Banks

As before, a bank must choose a *funding policy* (α^s, E, B^s, D^s), and an *investment policy* L. In taking this choice, the bank takes the discount factor η, the system $q(\cdot)$ of contingent-claims prices, and the liquidity premium function $\lambda(\cdot)$ as given.

What about the interest rates r_B and r_D that it must pay on bonds and deposits? It is natural to assume that, in equilibrium, investors have rational expectations about the default prospects implied by the banks' choices and, therefore, that the equilibrium values of r_B and r_D are affected by its funding choices. Should banks be presumed to take this dependence into account, or should they be assumed to behave as price-takers with respect to r_B and r_D as well as $\eta, q(\cdot)$, and $\lambda(\cdot)$?

This is not a question about market power. In the large economy considered here, no bank has market power and no bank is able to affect consumer welfare by unilateral changes in its plans. The dependence of r_B and r_D on the bank's funding choices is only due to the fact that funding choices affect the bank's risk of default.

Nor is the question whether the bank has rational expectations about the effects of its funding choices on the interest rates it must pay. The real issue is whether the bank can commit to its funding choices ex ante and whether it can communicate these choices to investors. If it can do so, then by changing the announcement of its funding choice, the bank can affect investors' expectations of its default risk and thereby the interest rates that it has to pay. With rational expectations it will appreciate the effect and take account of the dependence of interest rates on its choice. For example, it will appreciate that, if it funds more with equity, borrowing costs will be lower because depositors and other lenders appreciate that the bank's default probability is lower.

In contrast, if investors do not know the bank's funding mix, their expectations will be independent of the bank's choice. In this case, the bank will take r_B and r_D as given. It is still true that, the equilibrium values of r_B and r_D reflect the default risk from the funding mix that the bank chooses in equilibrium. However, the bank knows that investors do not observe its choice and therefore cannot condition their expectations on the bank's funding mix. In considering an off-the-equilibrium-path deviation involving more equity funding, the bank does not expect its borrowing costs to be reduced because it knows that investors do not observe the deviation and therefore do not condition their expectations on it.

The issue is not only a matter of communication but also of credibility. If banks announce their funding policies *ex ante*, investors may not pay much attention anyway because they do not believe these announcements. If we think about period 0 as being divided into many subperiods and the bank is unable to commit its behaviour *ex ante*, investors will ask themselves what funding mix the bank will end up with. The answer to this question depends on strategic considerations à la Coase (1973), rather than any initial announcement the bank may make.

In the following, I will study both cases, the case where banks take account of the effects of their funding choices on the interest they must pay and the case where they take the interest rates r_B and r_D as given. I begin with the case where they take account of the effects of their funding choices on r_B and r_D. In this case, a *plan for a bank* is a vector $(\alpha^s, B^s, D^s, L, E, r_B, r_D)$, which specifies r_B and r_D as well as the price E at which shares are offered, the funding mix (α^s, B^s, D^s) and the investment L.

Consumers

Consumers have the same characteristics as before. However, the "warm-glow" feelings from the liquidity of deposits arise only if the bank in question is not in default. The warm-glow term in the consumer's utility function now takes the form $\theta(\int_0^1 D^d(b)(1-\beta(\varphi,\hat{\varphi}(b)))db)$, rather than $\theta(\int_0^1 D^d(b)db)$, where, for each bank b, $\hat{\varphi}(b)$ is the default point of bank b, and

$$\beta(\varphi,\hat{\varphi}(b)) = 1 \quad \text{if} \quad \varphi < \hat{\varphi}(b), \tag{3.15}$$

$$\beta(\varphi,\hat{\varphi}(b)) = 0 \quad \text{if} \quad \varphi \geq \hat{\varphi}(b). \tag{3.16}$$

The variable $\beta(\tilde{\varphi},\hat{\varphi}(b))$ is an indicator variable showing whether bank b is in default or not. The default points $\hat{\varphi}(b)$ of different banks may differ. In this case (3.13) and (3.14) suggest that the interest rates, $r_B(b)$ and $r_D(b)$ will also differ.

A *plan for a consumer* is a vector $(c_0, c_1(\cdot), \alpha^d(\cdot), B^d(\cdot), D^d(\cdot))$ that specifies a consumption level for period 0, a state-contingent consumption plan for period 1, and investment level for the various securities issued by the different banks m period 0. A plan $(c_0, c_1(\cdot), \alpha^d(\cdot), B^d(\cdot), D^d(\cdot))$ provides the consumer with the expected utility

$$u(c_0) + \int_{\varphi_1}^{\varphi_2} v(c_1(\varphi) + \theta(\bar{D}^d(\varphi)))f(\varphi)d\varphi, \tag{3.17}$$

where, for any φ,

$$\bar{D}^d(\varphi) := \int_0^1 D^d(\bar{b})(1-\beta(\varphi,\hat{\varphi}(\bar{b})))d\bar{b} \tag{3.18}$$

is the aggregate value of those deposits that are not in default when $\tilde{\varphi} = \varphi$. In choosing a plan $(c_0, c_1(\cdot), \alpha^d(\cdot), B^d(\cdot), D^d(\cdot))$, the consumer must satisfy the budget constraints

$$e_0 = c_0 + \int_0^1 \alpha^d(b)E(b)db + \int_0^1 B^d(b)db + \int_0^1 D^d(b)db \tag{3.19}$$

in period $t = 0$ and

$$\int_{[\varphi_1,\varphi_2]} q(\varphi)c(\varphi)f(\varphi)d\varphi = \int_{[\varphi_1,\varphi_2]} q(\varphi)R(\varphi)f(\varphi)d\varphi, \tag{3.20}$$

or

$$\mathcal{E}_q c_1(\tilde{\varphi}) = \mathcal{E}_q R(\tilde{\varphi}) \tag{3.21}$$

in period $t = \frac{1}{2}$, where, for any φ,

$$\begin{aligned} R(\tilde{\varphi}) \quad : \quad = & \int (1-\alpha^s(b)+\alpha^d(b))\max[0, \tilde{\varphi}L(b) - (r_B(b)B^s(b) + r_D(b)D^s(b)]db \\ & +B^d(b)\int_0^1 r_B(b)\min[1, \frac{\tilde{\varphi}}{\hat{\varphi}(b)}]\,db + D^d(b)\int_0^1 r_D(b)\min[1, \frac{\tilde{\varphi}}{\hat{\varphi}(b)}]\,db \end{aligned} \tag{3.22}$$

is the random variable indicating the return on the consumer's portfolio. The constraint for $t = 0$ is the same as in the preceding section. The constraint for $t = \frac{1}{2}$ replaces the previous condition equating c_1 with the return on the consumer's portfolio by the condition that the value of the bundle of contingent claims needed to provide for the consumption random variable $c_1(\tilde{\varphi})$ be equal to the value at $t = \frac{1}{2}$ of the bundle of contingent returns from the consumer's investments at $t = 0$.

Equilibrium

An *equilibrium of the model with full commitment and communication of banks' choices* is given by a *price system* $(\eta, \lambda(\cdot), q(\cdot))$, a measurable mapping $b \to (\alpha^s(b), B^s(b), D^s(b), L(b), E(b), r_B(b), r_D(b))$ indicating a plan for each bank and a measurable mapping $a \to (c_0(a), c_1(\cdot, a), \alpha^d(\cdot, a), B^d(\cdot, a), D^d(\cdot, a))$, indicating a plan for each consumer $a \in [0, 1]$; such that

(E.1) given the price system, the plan $(\alpha^s(b), B^s(b), D^s(b), L(b), E(b), r_B(b), r_D(b))$ of (almost) every bank b maximizes the value $(1-\alpha^s)E$ of the initial shareholders' stock in bank b subject to nonnegativity, the period 0 budget constraint

$$L \leq \alpha^s E + B^s + D^s - K(D^s), \tag{3.23}$$

and the inequalities (3.10), (3.13), and (3.14), where $\hat{\varphi}$ is given by (3.4);

(E.2) given the price system and the banks' plans, for every $a \in [0, 1]$, the plan $(c_0(a), c_1(\cdot, a), \alpha^d(\cdot, a), B^d(\cdot, a), D^d(\cdot, a))$ maximizes the consumer's expected utility (3.17) subject to the constraints (3.18), (3.19), (3.20), and to nonnegativity of $c_0, c_1(\cdot), 1 - \alpha^s(\cdot) + \alpha^d(\cdot), B^d(\cdot)$, and $D^d(\cdot)$, where, for any φ, $R(\varphi)$ is given by (3.22) and, for any b, $\hat{\varphi}(b) = [r_B(b)B^s(b) + r_D(b)D^s(b)]/L(b)$;

(E.3) the banks' and the consumers' plans satisfy the market clearing conditions

$$\int_0^1 L(b)db = e_0 - \int_0^1 c_0(a)da \tag{3.24}$$

and

$$\alpha^s(b) = \int_0^1 \alpha^d(a, b)da, \quad B^s(b) = \int_0^1 B^d(a, b)da, \quad D^s(b) = \int_0^1 D^d(a, b)da, \tag{3.25}$$

for the goods market and the markets for the securities of bank b, for (almost) every $b \in [0, 1]$ in $t = 0$; the banks' and consumers' plans also satisfy the market-clearing conditions

$$\int_0^1 c_1(\varphi, a)da = \int_0^1 \varphi L(b)db \tag{3.26}$$

for the contingent-claims markets in $t = \frac{1}{2}$, for (almost) all φ;

(E.4) for any $\hat{\varphi}$, the liquidity premium function $\lambda(\cdot)$ satisfies

$$\lambda(\hat{\varphi}) = \frac{\mathcal{E}_q(1-\beta(\tilde{\varphi},\hat{\varphi}))\theta'(\bar{D}^s(\tilde{\varphi}))}{\mathcal{E}_q \min(1, \frac{\tilde{\varphi}}{\hat{\varphi}})}, \tag{3.27}$$

where

$$\bar{D}^s(\tilde{\varphi}) := \int_0^1 (1-\beta(\tilde{\varphi},\hat{\varphi}(\bar{b}))D^s(\bar{b})d\bar{b}; \tag{3.28}$$

Conditions (E.1) - (E.3) are the analogues of conditions (e.1) - (e.3) in the certainty case. There is a small difference in that banks are free to choose the interest rates r_B and r_D, subject to (3.13) and (3.14). However, this difference is unimportant. If the equilibrium condition (e.1) for the certainty case were changed to allow banks to choose r_B and r_D, subject to the relevant version of (3.13) and (3.14), equilibrium allocations and prices in the certainty case would not be changed.

Consumer Behaviour and the Liquidity Premium Function

The equilibrium condition (E.4) relates the liquidity premium in the deposit rate for a bank with default point $\hat{\varphi}$ to the liquidity benefits that consumers draw from additional deposits with this bank. For the default points $\hat{\varphi}(b), b \in [0,1]$, that correspond to the banks' equilibrium plans, this condition is actually implied by conditions (E.1) - (E.3). The point of introducing condition (E.4) is to extend the liquidity premium function to all possible default points, even those that are not chosen by any bank in the equilibrium under consideration.

This extension of the liquidity premium function can be understood as a subgame-perfectness condition. If some bank b were to deviate from its equilibrium plan, then, in the model with full commitment and communication, consumers would observe the deviation and adapt their portfolio decisions. The liquidity premium in the bank's deposit rate would then be given by $\lambda(\hat{\varphi})$, where $\hat{\varphi}$ is the default point under the newly chosen policy, rather than $\lambda(\hat{\varphi}(b))$. Condition (E.4) ensures that this *off-the-equilibrium-path* response of consumers to the bank's deviation is governed by the same principles as, and is compatible with, the other equilibrium conditions.

For a better understanding of equation (3.27), I show why, for default points $\hat{\varphi}(b), b \in [0,1]$, this equation is implied by the other equilibrium conditions. I begin by considering the consumers' maximization problem.

Lemma 3.1 *Given a system $q(\cdot)$ of contingent-claims prices at $t=\frac{1}{2}$ and given the banks' plans $(\alpha^s(b), B^s(b), D^s(b), L(b), E(b), r_B(b), r_D(b)), b \in [0,1]$, suppose that $(c_0, c_1(\cdot), \alpha^d(\cdot), B^d(\cdot), D^d(\cdot))$ is an optimal plan for a consumer. Then there exists $\chi > 0$ such that, for almost all φ,*

$$\chi\, v'(c_1(\varphi)) + \theta(\bar{D}^d(\varphi)) = q(\varphi)\, u'(c_0) \tag{3.29}$$

and, for (almost) every $b \in [0,1]$,

$$\chi\, E(b) \geq \mathcal{E}_q \pi(\tilde{\varphi}), \tag{3.30}$$

with a strict inequality implying that $1 - \alpha^s(b) + \alpha^d(b) = 0$;

$$\chi \geq r_B(b)\mathcal{E}_q \min(1, \frac{\tilde{\varphi}}{\hat{\varphi}(b)}), \tag{3.31}$$

with a strict inequality implying that $B^d(b) = 0$;

and

$$\chi \geq r_D(b)\mathcal{E}_q \min(1, \frac{\tilde{\varphi}}{\hat{\varphi}(b)}) + \mathcal{E}_q(1 - \beta(\tilde{\varphi}, \hat{\varphi}(b))\theta'(\bar{D}^d(\tilde{\varphi})) \tag{3.32}$$

with a strict inequality implying that $D^d(b) = 0$.

With initial endowments $e_0 > 0$ and $e_1 = 0$, consumers want to shift resources from period 0 to period 1. They can do this by investing in shares, bonds, or deposits of banks in period 0. In addition, they can trade contingent claims at $t = \frac{1}{2}$ in order to shift consumption between states of nature, calibrating their state-contingent consumption plans to the prices of the different contingent claims.

Given the scope for trading at $t = \frac{1}{2}$ the investment decisions of consumers at $t = 0$ are governed by the rates of return that investments provide from $t = 0$ to $t = \frac{1}{2}$. Because information at $t = \frac{1}{2}$ is the same as at $t = 0$, these rates of return are certain. If different investments offer different rates of return, consumers acquire only investments that yield the maximum rate of return. The parameter χ in the lemma represents this benchmark rate of return that an asset must provide from $t = 0$ to $t = \frac{1}{2}$ if a consumer is to invest in it. Thus, if the inequality in (3.30) is strict, the price at which bank b is offering new shares is too high, the rate of return on these shares is too low, and no consumer will buy these shares. Similarly, if the inequality in (3.31) is strict, the interest rate on the bonds of bank b is too low for investors to be willing to acquire them.

For deposits, the matter is more complicated because the liquidity benefits from deposits are not priced in the markets at $t = \frac{1}{2}$. A rate-of-return interpretation of the condition for deposits can still be given because the consumption equivalent of the liquidity benefits from deposits can be priced. The marginal liquidity benefits from additional deposits with bank b are given by the random variable $(1 - \beta(\tilde{\varphi}, \hat{\varphi}(b)))\theta'(\bar{D}(\tilde{\varphi}))$. If the additional investment in deposits is not made, the same effect on expected utility could still be achieved by state-contingent additions to consumption equal to $(1 - \beta(\tilde{\varphi}, \hat{\varphi}(b)))\theta'(\bar{D}(\tilde{\varphi}))$. The market value at $t = \frac{1}{2}$ of this consumption equivalent of the marginal liquidity benefits from deposits is equal to $\mathcal{E}_q(1 - \beta(\tilde{\varphi}, \hat{\varphi})b)))\theta'(\bar{D}(\tilde{\varphi}))$. Condition (3.32) compares the benchmark rate χ to the sum of this market value of the equivalent of the marginal liquidity benefits from deposits and the market value at $t = \frac{1}{2}$ of the marginal monetary return $r_D(b)\ \mathcal{E}_q \min(1, \frac{\tilde{\varphi}}{\hat{\varphi}(b)})$, on deposits at bank b. The consumer invests in these deposits only if this sum meets the benchmark χ. If the it fails to do so, the interest rate $r_D(b)$ is too low for the consumer to be willing to acquire deposits at bank b.

In conditions (3.29) - (3.32), the benchmark rate of return, the consumption levels c_0 and $c_1(\varphi), \varphi \in [\varphi_1, \varphi_2]$, and the effective deposit holdings $\bar{D}^d(\varphi), \varphi \in [\varphi_1, \varphi_2]$, are consumer-specific. If different consumers face the same price system, however, the associated marginal rates of substitution must be the same same for all of them. This is the point of the following lemma. In the lemma, $\chi(a)$ denotes the parameter given by Lemma 3.1 for consumer a, and, for any φ, $\bar{D}^d(\varphi, a)$ denotes the value of aggregate deposits that consumer a.holds with banks that are not in default if $\tilde{\varphi} = \varphi$.

Lemma 3.2 *Suppose that a price system and a pair of mappings indicating the plans of banks and of consumers satisfy the equilibrium conditions (E.1) - (E.3). Then almost all banks' plans satisfy the constraints (3.23), (3.10), (3.13), and (3.14) with equality, and almost all consumers' plans satisfy the first-order conditions (3.30), (3.31), and (3.32) with equality. Moreover,*

$$\chi(a) = \frac{1}{\eta} \tag{3.33}$$

and

$$\theta'(\bar{D}^d(\tilde{\varphi}, a)) = \theta'\left(\bar{D}^s(\tilde{\varphi})\right) \tag{3.34}$$

for almost all $a \in [0, 1]$*, almost surely.*

The benchmark rate of return $\chi(a)$ for the portfolio choice problem of consumer a is the same for all a and is equal to the inverse of the discount factor η. Equivalently, the discount factor indicates the relation between the expected return at $t = \frac{1}{2}$ and the price at $t = 0$ that will induce an asset to earn the benchmark rate of return, which is needed if any one is to hold the asset.

The lemma also shows that, for any φ, the term $\theta'(\bar{D}^d(\varphi, a))$ in condition (3.32) is the same for all a. The optimality of the consumers' plans, implies that, if different consumers choose different plans, the slopes of the liquidity benefit functions must still be the same. Moreover, because the function $\theta(\cdot)$ is the same for all consumers and $\theta(\cdot)$ is concave, these slopes must also coincide with the slope of $\theta(\cdot)$ at the cross-section "average" $\int \bar{D}^d(\varphi, \bar{a})d\bar{a}$. With market clearing, this "average" is equal to the supply side average $\bar{D}^s(\varphi)$, which was defined in (3.28).[33]

[33] The reader may wonder why I did not impose consumer symmetry here. With symmetry in the specification of consumer preferences and endowments, condition (3.34) is trivially true if the consumers' plans are all the same. However, it is important to understand the logic behind equation (3.34) and condition (E.4) without appealing to special features of the exogenous data of the model. Without symmetry, it still must be the case that, in equilibrium, the relevant slopes are the same and that their common value depends on the supply side of the system.

Upon combining (3.14), (3.32), and (3.34), one obtains:

Lemma 3.3 *Suppose that a price system and a pair of mappings indicating the plans of banks and of consumers satisfy the equilibrium conditions (E.1) - (E.3). Then*

$$\lambda(\hat{\varphi}(b)) = \frac{\mathcal{E}_q(1 - \beta(\tilde{\varphi}, \hat{\varphi}(b)))\theta'(\bar{D}^s(\tilde{\varphi}))}{\mathcal{E}_q \min(1, \frac{\tilde{\varphi}}{\hat{\varphi}(b)})} \tag{3.35}$$

for almost all $b \in [0, 1]$.

Equation (3.35) relates the liquidity premium for deposits of bank b to the default point $\hat{\varphi}(b)$ that bank b chooses *in equilibrium.* As mentioned above, the equilibrium condition (E.4) extends this condition to arbitrary default points, regardless of whether or not they are chosen by any bank in the equilibrium under consideration.

3.2 Equilibria with Commitment and Communication: The Use of Equity Funding

Existence of Equilibrium

Turning to the analysis of liquity provision and equity funding in the model with full commitment and communication of banks' choices, I begin with an equilibrium existence result.

Proposition 3.4 *Under the stated assumptions, an equilibrium of the model with full commitment and communication of banks' choices exists.*

This result looks like the corresponding result in Proposition 2.1, but there are important differences. As far as I know, the result is not subsumed by known existence results, nor does it follow from standard arguments. The proof must overcome several difficulties: First, because the consumers' liquidity benefits depend on the banks' funding choices, there is an external effect in the model. Second, the possibility of default introduces a non-convexity into the banks' optimization problems. If the liquidity benefit function $\theta(\cdot)$ is strictly concave, one can in fact show that the externality and the nonconvexity preclude the existence of a symmetric equilibrium.[34] This is why, in contrast to the preceding section, I do not impose a symmetry condition.Third, the assignment of plans

[34] With symmetry, all banks would choose the same funding policy and the same default point. If an individual bank considers a deviation from the presumed symmetric equilibrium plan, it will observe that a change in the mix of equity and deposits that lowers its own bankruptcy point relative to the common bankruptcy point of the other banks will reduce its

to banks must be measurable. Fourth, there is no natural bound on any one bank's choices.

To overcome these difficulties, the proof of Proposition 3.4 exploits the special structure of the model. First, I use Lemmas 3.1 - 3.3 to deal with the pricing effects of the externalities in liquidity provision. Second, I use the continuum of banks to smooth over any discontinuities that might arise from the nonconvexities of the banks' optimization problems Third, following Hart, Hildenbrand, and Kohlberg (1974), I avoid the technical difficulties associated with measurability by working with cross-section distributions of banks' plans, rather than the functions assigning plans to banks. Finally, boundedness of banks' plans is obtained from the interplay of aggregate feasibility and individual optimization. The argument is more complicated than usual because, with a continuum of banks, the boundedness of aggregates that is implied by feasibility requirements does not have strong implications for the scale of individual banks' plans.

Equilibria With and Without Satiation in Deposits

As in the case of certainty, there is a clear distinction between equilibria in which there is "satiation" and equilibria in which there is no "satiation" in deposits. In the absence of default, placing a unit of additional savings in deposits of bank $\hat{b}$, rather than bonds, provides a liquidity benefit equal to $\theta'(\int D^s(b)db)$ at $t = 1$ while reducing investment returns by $\tilde{\varphi}K'(D^s(\hat{b}))$. The impact on expected utility is equal to

$$\int v'(\varphi L + \theta) \left[\theta'(\int D^s(b)db) - \varphi K'(D^s(\hat{b}))\right] f(\varphi)d\varphi,$$

which is proportional to

$$\theta'(\int D^s(b)db) - \mathcal{E}_q\tilde{\varphi}\ K'(D^s(\hat{b})), \tag{3.36}$$

where, for any φ, $q(\varphi) := v'(\varphi L + \theta)/\int v'(\varphi' L + \theta)f(\varphi')d\varphi'$. If the deposit supplies of different banks are all the same, this difference is just $m(D^s, \mathcal{E}_q\tilde{\varphi})$, where D^s is the common value of the banks' deposit supplies and m is again the function that was defined in equation (2.13) in Section 2.

Let

$$\bar{\varphi} := \int \varphi f(\varphi)d\varphi \tag{3.37}$$

cost of deposit finance by a term that is proportional to $\theta'(0)$, rather than $\theta'(\bar{D}^s)$ because for values of $\tilde{\varphi}_L$ below the common bankruptcy point, the other banks do not provide consumers with any liquidity benefits; if instead, the change in the mix raises its own bankruptcy point relative to the common bankruptcy point of the other banks, the change will raise the bank's by a term that is proportional to $\theta'(\bar{D}^s) < \theta'(0)$; where $\bar{D}^s$ is the average effective deposit supply of the other banks. Either one or the other of these two changes would increase the value of the bank's objective function, which implies that the presumed symmetric equilibrium is not an equilibrium at all.

be the mean rate of return on loans. Along the same lines as before, I assume that

$$m(D, \bar{\varphi}) > 0 \tag{3.38}$$

if D is close to zero. This condition again ensures that any equilibrium must involve *some* deposit funding of banks. Also along the same lines as before, I assume that, for any φ, there is at most one $D^*(\varphi) \in \mathbb{R}_+$ so that

$$m(D^*(\varphi), \varphi) = 0. \tag{3.39}$$

Because $m(D, \bar{\varphi}) > 0$ for D close to zero, it follows that, for any $\varphi \leq \bar{\varphi}$, the solution $D^*(\varphi)$ to equation (3.39) is strictly positive. Because $v(\cdot)$ is concave, $\mathcal{E}_q\tilde{\varphi} \leq \bar{\varphi}$, so in particular,

$$m(D, \mathcal{E}_q\tilde{\varphi}) > 0 \tag{3.40}$$

if D is close to zero. Thus $D^*(\mathcal{E}_q\tilde{\varphi})$ is strictly positive.[35] As before, I write $D^*(\mathcal{E}_q\tilde{\varphi}) = \infty$ if $m(D, \mathcal{E}_q\tilde{\varphi}) > 0$ for all D, i.e., if the equation $m(D, \mathcal{E}_q\tilde{\varphi}) = 0$ has no solution.

Proposition 3.5 *Under the stated assumptions, there are two distinct classes of equilibria, equilibria in which the equilibrium price system satisfies $\eta\ \mathcal{E}_q\tilde{\varphi} < 1$ and equilibria in which the equilibrium price system satisfies $\eta\ \mathcal{E}_q\tilde{\varphi} = 1$. The following statements hold.*

(a) If the initial endowment e_0 is sufficiently small or if $\varphi_1 = 0$, any equilibrium price system satisfies $\eta\ \mathcal{E}_q\tilde{\varphi} < 1$. If $\varphi_1 > 0$ and $D^*(\varphi_1) < \infty$, and if the initial endowment e_0 is sufficiently large, there exists an equilibrium with a price system satisfying $\eta\ \mathcal{E}_q\tilde{\varphi} = 1$.

(b) In any equilibrium with a price system satisfying $\eta\ \mathcal{E}_q\tilde{\varphi} < 1$, $\int D^s(b)db > 0$. Moreover, for almost all b. $D^s(b) > 0$ implies

$$\mathcal{E}_q\tilde{\varphi}\ K'(D^s(b)) < \mathcal{E}_q(1 - \beta(\tilde{\varphi}, \hat{\varphi}(b)))\theta'(\bar{D}^s(\tilde{\varphi})), \tag{3.41}$$

and $\hat{\varphi}(b) \geq \varphi_1$ for (almost) all b . The latter inequality is strict, i.e., $\hat{\varphi}(b) > \varphi_1$, if $\varphi_1 = 0$. Finally,$B^s(b) = 0$ for almost all b.

(c) In any equilibrium with a price system satisfying $\eta\ \mathcal{E}_q\tilde{\varphi} = 1$, $\int D^s(b)db = D^*(\mathcal{E}_q\tilde{\varphi})$ and $\hat{\varphi}(b) \leq \varphi_1$ for almost all b with $D^s(b) > 0$ In the range where equity is sufficiently large so that the default probability is zero, the mix of equity and bond finance in such an equilibrium is indeterminate.

(d) Regardless of whether $\eta\ \mathcal{E}_q\tilde{\varphi} < 1$ or $\eta\ \mathcal{E}_q\tilde{\varphi} = 1$, $D^s(b) > 0$ implies $\alpha^s(b)E(b) > 0$ if the producer's surplus from deposit provision, $D^s(b)K'(D^s(b)) - K(D^s(b))$, is small relative to the endowment e_0. In particular, $D^s(b) > 0$ implies $\alpha^s(b)E(b) > 0$ if the marginal cost of deposit provision is constant, i.e., if $K"(D) = 0$ for all D; $\alpha^s(b)E(b)$ is also positive if e_0 is sufficiently close to zero, or if e_0 is sufficiently large. If $\eta\ \mathcal{E}_q\tilde{\varphi} = 1$, then $\int \alpha^s(b)E(b)db > 0$ if

$$\frac{\varphi_1}{\mathcal{E}_q\tilde{\varphi}} < \frac{D^*(\mathcal{E}_q\tilde{\varphi}) - K'(D^*(\mathcal{E}_q\tilde{\varphi}))D^*(\mathcal{E}_q\tilde{\varphi})}{D^*(\mathcal{E}_q\tilde{\varphi}) - K(D^*(\mathcal{E}_q\tilde{\varphi}))}. \tag{3.42}$$

[35] In fact, the inequality $\mathcal{E}_q\tilde{\varphi} \leq \bar{\varphi}$ implies that $D^*(\mathcal{E}_q\tilde{\varphi}) \geq D^*(\bar{\varphi})$.

Again, there is a clear difference between equilibria in which there is satiation in deposits and equilibria in which there is no satiation. If initial endowments and consumer savings are very high and if a finite satiation level for deposits exists at all,[36] there exist equilibria in which the equilibrium value of the aggregate deposit supply is equal to the satiation level $D^*(\mathcal{E}_q\tilde{\varphi})$, and the equilibrium default probability is zero. In these equilibria, the excess of savings over $D^*(\mathcal{E}_q\tilde{\varphi})$ are put into shares and bonds, with some (local) indeterminacy of the Modigliani-Miller type. By contrast, if initial endowments and consumer savings are very low, the equilibrium value of the aggregate deposit supply is below the satiation level. In this case, bonds play no role in bank funding but, in contrast to the certainty case, equity may play a role. The equilibrium default probability of each bank is positive unless the value $f(\varphi_1)$ of the density function at φ_1 is very large.

In contrast to the certainty case, the transition between the two kinds of equilibria need not be monotonic in e_0. The satiation level $D^*(\mathcal{E}_q\tilde{\varphi})$ itself depends on the equilibrium price system q. As e_0 goes up, it is quite possible for $D^*(\mathcal{E}_q\tilde{\varphi})$ to go up as well and indeed to go up so quickly that equilibrium deposit levels shifts from satiation to non-satiation even though they are actually increasing in e_0.[37]

I also cannot rule out the possibility that, for large values of the initial endowment, even if $\varphi_1 > 0$ and $D^*(\varphi_1) < \infty$, there might be a non-satiation equilibrium as well as the satiation equilibrium that is given by statement (a). Whereas, for small values of e_0, the inequality $\int D^s(b)db < D^*(\mathcal{E}_q\tilde{\varphi})$ is trivially satisfied, for large values of e_0, the matter is unclear. With default, $\bar{D}^s(\tilde{\varphi}) < \int D^s(b)db$ so (3.41) need not impose a restriction on $\int D^s(b)db$.[38]

The Role of Equity

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[36] As in the certainty case, a finite satiation level for deposits will not exist if $\varphi_2 K'(\infty) < \theta'(\infty)$. A finite satiation level also fails to exist if $\varphi_1 = 0$.

[37] Given the conditions for an equilibrium price system, the equation $m(D, \mathcal{E}_q\tilde{\varphi}) = 0$ is actually equivalent to the equation

$$\int v'(\varphi L + \theta(D))\,[\varphi K'(D) - \theta'(D)]\, f(\varphi)d\varphi = 0.$$

Using the implicit-function theorem, one easily finds that, if the utility function v exhibits non-increasing absolute risk aversion and non-decreasing relative risk aversion, the solutions to this equation take the form $L = L^*(D)$, where L^* is a nondecreasing function. Hence, if e_0 goes up and the equilibrium level of savings goes up, the satiation level of deposits goes up as well. If the increase absorbs a sufficiently large part of the increase in savings, it might well be the case that, even though the endowment e_0 is higher, there is again no equilibrium with satiation in deposits.

[38] Exceptions are (i) the case where θ' is constant and (ii) the case where K' is constant. If θ' is constant, $\theta'(\bar{D}^s(\tilde{\varphi})) = \theta'(\int D^s(b)db)$ almost surely, so (3.41) implies $\mathcal{E}_q\tilde{\varphi}\ K'(\max_b D^s(b)) < \theta'(\int D^s(b)db)$ and hence $\mathcal{E}_q\tilde{\varphi}\ K'(\int D^s(b)db) < \theta'(\int D^s(b)db)$. If K' is constant, (3.41) implies

$$\mathcal{E}_q\tilde{\varphi}\ K'(\int D^s(b)db) < \mathcal{E}_q(1 - \beta(\tilde{\varphi}, \hat{\varphi}(b^*)))\theta'(\bar{D}^s(\tilde{\varphi})),$$

where b^* is the bank with the highest default point. By definition, $\beta(\tilde{\varphi}, \hat{\varphi}(b^*)) = 0$ implies $\beta(\tilde{\varphi}, \hat{\varphi}(b)) = 0$ for all b, and therefore $\bar{D}^s(\tilde{\varphi}) = \int D^s(b)db$, and one again obtains $\mathcal{E}_q\tilde{\varphi}\ K'(\int D^s(b)db) < \theta'(\int D^s(b)db)$.

The role of equity in Proposition 3.4 is quite different from what it was before. Because equity funding improves the safety of deposits, equilibrium levels of equity funding can be positive even when e_0 is too small for deposits to reach the level $D^*(q)$ at which the net marginal benefits of having additional funding go through deposits rather than bonds are zero. Indeed, if e_0 is very close to zero, the equilibrium level of bank equity funding is always positive.

In contrast to the case of certainty, the roles of shares and bonds are now very different. In Proposition 2.1, both shares and bonds appeared merely as substitutes for deposits, undesirable until deposits had reached the efficient level, and desirable once that level had been reached. Moreover, by a version of the Modigliani-Miller Theorem, the mix of shares and bonds was irrelevant. Now the mix of shares and bonds matters, and there is an important asymmetry between them. Whereas shares are a means of improving the safety and the liquidity of deposits, bonds not only substitute for deposits as a source of funds but they may also be harmful to the safety and the liquidity of deposits.

In equilibrium, therefore, bonds are never used when there is a positive probability of default. If they were used, replacing bond finance by share finance would enhance deposit liquidity (and lower deposit rates) without any need to change deposit levels. The assumption that liquidity benefits of deposits are tied to non-default states introduces a breakdown of Modigliani-Miller arguments in the treatment of shares and bonds but, contrary to what critics of Modigliani-Miller analysis in banking often claim, the bias is one that favours equity rather than bonds.[39]

To understand the underlying logic, it is useful to consider the typical bank's objective $(1-\alpha^s)E$, taking account of the bank's constraints. By (3.10), (3.9), and (3.1), in combination with Lemma 3.2, this can be rewritten in the form

$$\eta\, \mathcal{E}_q \tilde{\varphi}\; L - \eta\, \mathcal{E}_q \min(1, \frac{\tilde{\varphi}}{\hat{\varphi}})\; (r_B B^s + r_D D^s) - A,$$

where $A := \alpha^s E$. By (3.23), (3.13), and (3.14), in combination with Lemma 3.2 and (3.27), this expression in turn can be rewritten as

$$\eta\, \mathcal{E}_q \tilde{\varphi}\; (A + B^s + D^s - K(D^s)) - B^s - D^s + \eta\, \mathcal{E}_q (1-\beta(\tilde{\varphi},\hat{\varphi}))\theta'(\bar{D}^s(\tilde{\varphi}))\; D^s - A. \tag{3.43}$$

Similarly, the equation $\hat{\varphi} L = r_B B^s + r_D D^s$ can be rewritten as

$$\eta\, \mathcal{E}_q \min(\tilde{\varphi},\hat{\varphi})\; (A+B^s+D^s-K(D^s)) = B^s+D^s-\eta\, \mathcal{E}_q(1-\beta(\tilde{\varphi},\hat{\varphi}))\theta'(\bar{D}^s(\tilde{\varphi}))\; D^s. \tag{3.44}$$

The bank's problem in the equilibrium condition (E.1) is equivalent to the problem of choosing nonnegative A, B^s, D^s, and $\hat{\varphi}$ so as to maximize (3.43) under the constraint (3.44).

[39] The bias can be understood in terms of the traditional "tradeoff" between bankruptcy costs and tax effects in Modigliani and Miller (1963). There are no taxes here, but the loss of liquidity benefits of depositors is a bankruptcy cost. Whereas the immediate impact of this cost affects depositors rather than banks, the endogeneity of liquidity premia in deposit rates implies that banks are also affected.

The value of the objective (3.43) is equal to the difference between the sum of the values at $t = 0$ of the returns on the bank's loans and the liquidity benefits from deposits and the total funds $A+B^s+D^s$ that the bank raises. The amounts raised by issuing shares and bonds appear only in the form $A + B^s$, so, at this level, the mix of shares and bonds is irrelevant. In the constraint (3.44), the amounts raised by issuing shares and bonds enter asymmetrically. Whereas the level of bank loans $L = A + B^s + D^s - K(D^s)$ on the left-hand side of (3.44) depends on the sum $A+B^s$, the debt service on the right-hand side depends on the bond issue B^s but is independent of A. If $B^s > 0$, a one-for-one replacement of bond finance by share finance introduces some slack into the constraint (3.44) while leaving the objective (3.43) unaffected. The slack can be used to reduce the default point $\hat{\varphi}$, which raises the value of the objective (3.43) unless $\hat{\varphi}$ is already so low that $\beta(\tilde{\varphi}, \hat{\varphi}) = 0$ with probability one. Thus, for $B^s > 0$ to be part of an optimal plan of the bank, the bank's default probability must be zero.

What about shares and deposits? A unit increase in A raises the objective (3.43) by $\eta\, \mathcal{E}_q\tilde{\varphi} - 1$ while relaxing the constraint (3.44) and facilitating a reduction in the default point $\hat{\varphi}$. If $\eta\, \mathcal{E}_q\tilde{\varphi}$ were greater than one, it would be desirable to expand A without bounds, i.e. an optimal plan for the bank would fail to exist. If $\eta\, \mathcal{E}_q\tilde{\varphi} = 1$, the objective (3.43) is independent of the amount of equity funding but an increase in A introduces slack into the constraint and facilitates a reduction in $\hat{\varphi}$. This will raise the bank's expected payoff unless $\hat{\varphi} \leq \varphi_1$ and hence $\beta(\tilde{\varphi}, \hat{\varphi}) = 0$ with probability one. Finally, if $\eta\, \mathcal{E}_q\tilde{\varphi} < 1$, the bank's choice of A must balance the negative direct effect of an increase in A on the objective against the positive indirect effect from having a lower default point. This tradeoff will sometimes result in positive equity funding and sometimes in zero equity funding.

The Role of Technology

Whether for $\eta\, \mathcal{E}_q\tilde{\varphi} < 1$ the bank uses equity funding or not depends on the deposit provision cost function. As an example, consider the case of constant returns to scale, i.e. suppose that $K'(D^s) = k \geq 0$, a constant, regardless of D^s. In this case, maximization of (3.43) at $D^s > 0$ requires that

$$\eta\, \mathcal{E}_q\tilde{\varphi}\, (1-k) - 1 + \eta\, \mathcal{E}_q(1 - \beta(\tilde{\varphi}, \hat{\varphi}))\theta'(\bar{D}^s(\tilde{\varphi})) = 0. \tag{3.45}$$

At the same time, (3.44) implies

$$\eta\, \mathcal{E}_q \min(\tilde{\varphi}, \hat{\varphi})\, A \geq -D^s[\eta\, \mathcal{E}_q \min(\tilde{\varphi}, \hat{\varphi})\, (1-k) + 1 - \eta\, \mathcal{E}_q(1-\beta(\tilde{\varphi}, \hat{\varphi}))\theta'(\bar{D}^s(\tilde{\varphi}))], \tag{3.46}$$

and hence

$$\begin{aligned} \eta\, \mathcal{E}_q \min(\tilde{\varphi}, \hat{\varphi})\, A &\geq D^s(1-k)\, \eta\, (\mathcal{E}_q\tilde{\varphi} - \mathcal{E}_q \min(\tilde{\varphi}, \hat{\varphi})) \\ &= \eta\, \mathcal{E}_q \max(0, \tilde{\varphi} - \hat{\varphi})\, D^s(1-k). \end{aligned}$$

If A were equal to zero, $\mathcal{E}_q \max(0, \tilde{\varphi} - \hat{\varphi})$ would also be zero, and $\beta(\tilde{\varphi}, \hat{\varphi}))$ would be equal to one with probability one. Expected liquidity benefits from the bank's deposits would then be zero. The bank's deposits would be like shares under pure equity funding, earning returns in proportion to the depositors' share of the bank's debt. However, pure equity funding itself is strictly dominated by some mix of equity and deposit funding, with expected benefits from liquidity that are positive. Because pure equity funding is strictly dominated by such a mix, so must be pure deposit funding.[40]

In the constant-returns-to-scale case, deposit provision does not yield any producer's surplus. In equilibrium, the portion of the liquidity benefits from deposits that the bank can appropriate are eaten up by the costs. Without equity funding, therefore, the bank has no buffer against return shocks, and the potential liquidity benefits from deposits are destroyed by default.

In the general case of arbitrary convex deposit cost functions, producer's surplus can provide a substitute for equity funding as a buffer against return shocks. If the uncertainty about the bank's returns is sufficiently small, this buffer may even be large enough so that no equity funding is needed at all. For example, if consumer savings are large enough so that $\eta\ \mathcal{E}_q\tilde{\varphi} = 1$, and if $D^s(b) = D^*(\mathcal{E}_q\tilde{\varphi})$ for all b, the requirement that $\hat{\varphi}(b) \leq \varphi_1$ for all b translates into the inequality

$$\begin{aligned}\eta\,\varphi_1\,(A + B^s + D^*(\mathcal{E}_q\tilde{\varphi}) - K(D^*(\mathcal{E}_q\tilde{\varphi}))) \quad &\geq \quad B^s + D^*(\mathcal{E}_q\tilde{\varphi}) - \eta\theta'(D^*(\mathcal{E}_q\tilde{\varphi}))\, D^*(\mathcal{E}_q\tilde{\varphi}) \\ &= \quad \eta\,\mathcal{E}_q\tilde{\varphi}\,[B^s + D^*(\mathcal{E}_q\tilde{\varphi}) - K'(D^*(\mathcal{E}_q\tilde{\varphi}))D^*(\mathcal{E}_q\tilde{\varphi})],\end{aligned}$$

which is satisfied regardless of A, if

$$\frac{\varphi_1}{\mathcal{E}_q\tilde{\varphi}} \geq \frac{B^s + D^*(\mathcal{E}_q\tilde{\varphi}) - K'(D^*(\mathcal{E}_q\tilde{\varphi}))D^*(\mathcal{E}_q\tilde{\varphi})}{B^s + D^*(\mathcal{E}_q\tilde{\varphi}) - K(D^*(\mathcal{E}_q\tilde{\varphi}))}$$

or, equivalently, if

$$\frac{K'(D^*(q))D^*(q) - K(D^*(q))}{B^s + D^*(q) - K(D^*(q))} \geq 1 - \frac{\varphi_1}{\mathcal{E}_q\tilde{\varphi}}. \tag{3.47}$$

The left-hand side of (3.47) is a measure of producer's surplus relative to the bank's investments, the right-hand side a measure of the bank's return uncertainty. If the former exceeds the latter, the bank does not need equity to avoid default with probability one. If $K'(D^*(q)) > 0$, this is necessarily the case if B^s is small and, moreover, φ_1 is close to $\mathcal{E}_q\tilde{\varphi}$ so that the uncertainty about the bank's returns is small.

In contrast, since $K'(D^*(q)) < 1$, the inequality (3.47) cannot be satisfied if B^s is large or if φ_1 is close to or equal to zero.[41] In this case, producer's surplus is too small to fully substitute for equity funding.

[40] If $k > 0$, nonoptimality of pure deposit funding already follows because, in this case, (3.45) and $\beta(\tilde{\varphi}, \hat{\varphi})) = 1$ with probability one imply $\eta\ \mathcal{E}_q\tilde{\varphi} > 1$, which is incompatible with the existence of an optimal level of A. If $k = 0$ and $\eta\ \mathcal{E}_q\tilde{\varphi} = 1$, any combination of $A = B^s = 0, D^s > 0$, and $\hat{\varphi} = \varphi_2$ satisfies the first-order conditions for maximizing (3.43) subject to (3.44), but is dominated by one involving positive equity funding. Note the underlying non-convexity in the bank's optimization problem.

[41] If $\varphi_1 = 0$, the bank cannot avoid having a positive probability of default. In this case, in fact, $\hat{e}_0 = \infty$, $B^s = 0$ regardless of e_0, and $\alpha^s E > 0$ if e_0 is large.

Efficiency

To study the efficiency properties of equilibrium allocations, I need a concept of constrained efficiency. Otherwise, there is no hope for an efficiency results because any allocation with a positive default probability is trivially dominated by some feasible allocation which involves pure deposit funding with a deposit rate equal to zero.

Proposition 3.6 *Under the stated assumptions, any equilibrium of the model with full commitment and communication of banks' choices involves an allocation that is Pareto efficient in the set of all allocations that are feasible and that are compatible with the banks' budget constraints at the equilibrium price system* $(\eta, \lambda(\cdot), q(\cdot))$.

In contrast to the efficiency result in Proposition 2.1, Proposition 3.6 does not follow from the first welfare theorem. In fact, unlike the proof of the first welfare theorem, the proof of Proposition 3.6 makes essential use of the concavity of the functions $u(\cdot)$, $v(\cdot)$, and $\theta(\cdot)$.

By keeping the price system fixed, the constrained-efficiency concept in Proposition 3.6 controls for all externalities that might be induced by the effects of participants' decisions of the price system. External effects that are due to the effects of agents' actions on prices are usually irrelevant for efficiency. Here, however, these "pecuniary" externalities are relevant for efficiency because default probabilities and expected liquidity benefits from deposits depend on interest rates and therefore on the equilibrium price system. This consideration suggests that equilibrium allocations might actually be improved upon by government interventions that change the equilibrium price system, for example a specific tax on consumption in period 0, in combination with a lump sum subsidy that neutralizes the revenue effects of the tax. By increasing consumer saving incentives, such a tax might raise the equilibrium value of η and thereby reduce interest rates and the banks' default points.

3.3 Lack of Equity Funding under Interest Rate Taking

In this final part of the analysis, I abandon the assumption that banks are able to commit their funding choices and to communicate them credibly to consumers. If banks are unable to commit and credibly communicate their choices to investors, banks will be unable to take account of the fact that, in equilibrium, the interest rates they must pay will depend on the funding mix they choose. In the following, I will therefore assume that banks take these interest rates as given.

Equilibrium with Interest Rate Taking

A price system now consists of a discount factor η, a system q of contingent-claims prices at $t = \frac{1}{2}$, and a list $\{r_B(b), r_D(b)\}_{b\in[0,1]}$ of interest rates that the different banks must offer if they want to get funding through bonds and deposits. An *equilibrium in the model with price taking* is given by a price system $(\eta, q, \{r_B(b), r_D(b)\}_{b\in[0,1]})$ and measurable mappings $b \to (\alpha^s(b), B^s(b), D^s(b), L(b))$ and $a \to (c_0(a), c_1(\cdot, a), \alpha^d(\cdot, a), B^d(\cdot, a), D^d(\cdot, a))$, indicating the plans of the different banks and consumers such that

(E.1*) given the price system, the plan $(\alpha^s(b), B^s(b), D^s(b), L(b), E(b))$ of (almost) every bank b maximizes the value $(1 - \alpha^s)E$ of the initial shareholders' stock in bank b subject to nonnegativity, the period 0 budget constraint

$$L \leq \alpha^s E + B^s + D^s - K(D^s), \tag{3.48}$$

and the condition

$$E \leq \eta \mathcal{E}_q \max[0, \varphi L - (r_B(b)B^s + r_D(b)D^s] \tag{3.49}$$

for the valuation of the bank's equity;

(E.2*) given the price system and the banks' plans, for (almost) every $a \in [0, 1]$, the plan $(c_0(a), c_1(\cdot, a), \alpha^d(\cdot, a), B^d(\cdot, a), D^d(\cdot, a))$ of (almost) every consumer a maximizes the consumer's, expected utility (3.17) subject to the constraints (3.18), (3.19), (3.20), and nonnegativity of $c_0, c_1(\cdot), 1-\alpha^s(\cdot)+\alpha^d(\cdot), B^d(\cdot)$, and $D^d(\cdot)$, where, for any φ, $R(\varphi)$ is given by (3.22) and, for any b, $\hat{\varphi}(b) = [r_B(b)B^s(b) + r_D(b)D^s(b)]/L(b)$;

(E.3*) the banks' and the consumers' plans satisfy the market clearing conditions

$$\int_0^1 L(b)db = e_0 - \int_0^1 c_0(a)da \tag{3.50}$$

and

$$\alpha^s(b) = \int_0^1 \alpha^d(a, b)da, \quad B^s(b) = \int_0^1 B^d(a, b)da, \quad D^s(b) = \int_0^1 D^d(a, b)da, \tag{3.51}$$

for the goods market and the markets for the securities of bank b, for (almost) every $b \in [0, 1]$ in $t = 0$; the banks' and consumers' plans also satisfy the market-clearing conditions

$$\int_0^1 c_1(\varphi, a)da = \int_0^1 \varphi L(b)db \tag{3.52}$$

for the contingent-claims markets in $t = \frac{1}{2}$, for (almost) all φ;

In this definition of an equilibrium in the model with interest rate taking, conditions (E.2*) and (E.3*) are identical with conditions (E.2) and (E.3) in the definition of an equilibrium in the model with full commitment and communication. Condition (E.1*) replaces the previous condition (E.1). The previous condition (E.4) has become superfluous because now there is no need to specify the liquidity premium in the deposit rate for funding mixes and default points that do not correspond to the banks' equilibrium plans.

Proposition 3.7 *a: Regardless of the value of* e_0*, there always exists an equilibrium of the model with interest rate taking in which there is no equity funding at all, interest rates are equal to* φ_2*, and all banks default almost surely on their debt.*

b: If the marginal cost of deposit provision is constant and the initial endowment e_0 *is too small to admit an equilibrium with satiation in deposits and zero default risk, the equilibrium with zero equity funding and certainty of default is the only equilibrium with interest rate taking. In this equilibrium, the liquidity benefits from deposits are zero.*

c:If the marginal cost of deposit provision is strictly increasing, equity funding is still zero in any equilibrium in which there is no satiation in deposits and the default probability is positive. In any such equilibrium, the equilibrium indebtedness and the default probability of any bank are inefficiently high.

c: If $\varphi_1 > 0$ *and* $D^*(\varphi_1) < \infty$ *and if the initial endowment* e_0 *is sufficiently large for the model with full commitment and communication to have an equilibrium with satiation in deposits, then this equilibrium, which satisfies* $L(b) > 0$*,* $D^s(b) = D^*(q)$*,* $\alpha^s(b)E(b) + B^s(b) > 0$*, and* $\hat{\varphi}(b) \leq \varphi_1$ *for almost all* b*, is also an equilibrium of the model with interest rate taking.*

Discussion

To understand the logic behind this result, observe that the reformulation

$$\eta\ \mathcal{E}_q\tilde{\varphi}\ L - \eta\ \mathcal{E}_q \min(1, \frac{\tilde{\varphi}}{\hat{\varphi}})\ (r_B B^s + r_D D^s) - A$$
$$= \eta\ \mathcal{E}_q\tilde{\varphi}\ (A + B^s + D^s - K(D^s) - \eta\ \mathcal{E}_q \min(1, \frac{\tilde{\varphi}}{\hat{\varphi}})\ (r_B B^s + r_D D^s) - A \quad (3.53)$$

of the banks' objective function applies in the present setting as in the setting with interest rate taking as well as the setting with commitment and communication. Now, however, the interest rates r_B and r_D are taken as given.

What is the effect of an increase in A on the objective (3.53)? The direct effect is the same as before; it hinges on the difference $\eta\ \mathcal{E}_q\tilde{\varphi} - 1$ between the present value of the additional returns and the marginal cost of the additional equity. In addition, there is an indirect effect because, under the constraint

$$\hat{\varphi}L = r_B B^s + r_D D^s, \quad (3.54)$$

the increase in L that is made possible by the additional equity lowers the default point $\hat{\varphi}$. However, whereas previously, the increase in the default point was considered to lower the expected value of the bank's debt service, now the opposite is true. As the interest rates r_B and r_D are taken as given, the decrease in the default point *raises* the expected value of the bank's debt service. As the additional equity raises the bank's investment, it raises the amount that debtholders receive in default, as well as reducing the default probability. This effect is akin to the debt overhang effect identified by Myers (1977), namely equity funding is discouraged if some of the benefits accrue to debtholders.[42] In the present setting, this debt overhang effect implies that banks never issue equity.

If the interest rates r_B and r_D are very high (above φ_2), then trivially, one obtains an equilibrium in which the bank defaults for sure. With constant returns to scale, there is no other equilibrium unless the value of the initial endowment e_0 is sufficiently high so that an equilibrium with satiation in deposits and zero default exists. With decreasing returns to scale, producer's surplus can substitute for the missing equity in order to reduce default points. However, even then, equilibrium allocations are not constrained efficient. Bank borrowing is inefficiently high because they neglect the impact of their borrowing on the interest rates they must pay.

Proposition 3.7 implies that, in a world with interest rate taking, even in the absence of systemic considerations, it may be desirable to have some statutory regulation of bank funding. Formally, one obtains:

Proposition 3.8 *Except for the equilibria with satiation in deposits that can be obtained if the consumers' initial endowment e_0 is large enough, the allocations that are generated by equilibria in the model with interest rate taking can be improved upon by requiring banks to satisfy a minimum equity requirement. In the case of constant returns to scale in deposit provision,* ***any*** *equity requirement below 100% will make for a strict improvement.*

As discussed above, the assumption of interest rate taking reflects an inability of banks to commit and communicate their funding choices ex ante. Commitment to a funding mix may not be feasible, and a price-taking assumption appropriate, if banks can repeatedly issue additional debt. For suppose that period 0 is split into a large number of sub-periods so that in each subperiod, banks can issue new claims. In this case, interest rates on deposits and bonds issues in the first subperiod are afterwards fixed, i.e., in later periods, banks deciding on additional issues of deposits and bonds do not have to worry that their choices might raise the rates they have to pay on their previously issued debt. The situation is similar to that of Coase's durable-good monopolist, who is unable to realize the monopoly profit because everybody believes that, after he has sold the monopoly quantity, he will return to the market and make additional sales.[43] In durable-good monopoly, lack of commitment implies that, if the number of potential trading dates is unbounded and the discount

[42] See also, Admati et al. (2012/2014).

[43] Coase (1972), Gul, Sonnenschein and Wilson (1986).

rates between trading dates is positive, the monopolist is reduced to behaving approximately like a price taker at the competitive market clearing price.

In practice, equity and different classes of debt are issued at different frequencies. Equity and bonds are issued infrequently, short-term borrowing may be decided on a daily basis, and contracting for the latest overnight repo loan is hardly committed by covenants in earlier contracts. Lack of commitment seems like a realistic assumption. Under this assumption, there is a market failure because banks fail to internalize the effects of restraining leverage on the interest rates they must pay. They appreciate the effects at the margin, but the average interest cost is driven by bondholders and depositors expecting further increases in the bank's borrowing.

4 Concluding Remarks

The paper has several simple messages:

- To understand the relation of liquidity provision and equity funding of banks, one must look at market equilibria and not just the optimizing behaviour of banks. Looking at what is optimal for banks is also insufficient for any serious welfare analysis.
- In a world of certainty, funding by deposits and funding by equity tend to be substitutes. Any funding by equity detracts from deposits and vice versa.
- In a world of uncertainty, in which liquidity benefits from deposits depend on the bank's not going into default, funding by deposits and funding by equity tend to be complements because the equity reduces the bank's default probability and raises the **expected** liquidity benefits from deposits.
- If banks can commit to their funding policies and communicate these policies to investors *ex ante*, they will take account of the complementarity between liquidity provision and equity funding so that competitive equilibrium allocations are constrained efficient. In this setting, banks appreciate that an increase in equity funding will lower the interest they must pay on deposits.
- If banks cannot commit to their funding policies and communicate these policies to investors *ex ante* and if therefore they take interest rates as given, they will not take account of the complementarity between liquidity provision and equity funding. Because they do not expect an increase in equity funding to affect their deposit rates, competitive equilibrium allocations in this setting involve zero equity issuance. The absence of equity issuance may prevent any of the potential liquidity benefits of deposits from being obtained.

- Minimum equity requirements may serve to reduce or even to eliminate the distortions that are caused by the inability of banks to commit to their funding policies *ex ante* and to communicate these policies to investors.

A Appendix: Proofs

A.1 Proof of Proposition 2.1

Proof of Statements (a) and (b). The market system is complete, there are no external effects, and preferences satisfy local non-satiation. By the First Welfare Theorem, therefore, any equilibrium allocation is Pareto efficient. Any symmetric equilibrium allocation thus maximizes

$$u(e_0 - L - K(D)) + v(\varphi L + \theta(D)) \tag{A.1}$$

subject to

$$L = \alpha E + B + D - K(D) \tag{A.2}$$

and nonnegativity. Such an allocation must therefore satisfy the first-order conditions

$$-u'(e_0 - L - K(D)) + v'(\varphi L + \theta(D))\,\varphi - \lambda \leq 0, \tag{A.3}$$

with a strict inequality implying that $L = 0$;

$$-u'(e_0 - L - K(D))\,K'(D) + v'(\varphi L + \theta(D))\,\theta'(D) + \lambda(1 - K'(D)) \leq 0, \tag{A.4}$$

with a strict inequality implying that $D = 0$;

and

$$\lambda \leq 0, \tag{A.5}$$

with a strict inequality implying that $\alpha E + B = 0$,

where λ is the Lagrange multiplier of the constraint (A.4).

The possibility of a corner solution for L can be neglected, i.e., (A.3) must hold as an equation.. By (A.2) and nonnegativity, $L = 0$ would imply $D = 0$, and therefore, $v'(\varphi L + \theta(D)) = \infty$, which is incompatible with (A.3).

Next consider the Lagrange multiplier λ. If $\lambda < 0$, $L = D - K(D) > 0$, so $D > 0$, and (A.4) holds with equality. In this case, (A.3) and (A.4) jointly imply

$$u'(e_0 - D) = v'(\varphi(D - K(D) + \theta(D) + \theta(D))\,[\varphi + m(D, \varphi)], \tag{A.6}$$

as well as

$$v'(\varphi(D - K(D)) + \theta(D)\,m(D, \varphi) = -\lambda > 0, \tag{A.7}$$

where, as in the text,

$$m(D,\varphi) := [\theta'(D) - \varphi K'(D)], \tag{A.8}$$

By contrast, if $\lambda = 0$, (A.3) and (A.4) yield:

$$u'(e_0 - L - K(D)) = \ v'(\varphi L + \theta(D))\, \varphi \tag{A.9}$$

and

$$v'(\varphi L + \theta(D))\, m(D,\varphi) \leq 0, \tag{A.10}$$
$$\text{with a strict inequality implying } D = 0$$

Under the given assumptions about the functions u, v, K, and θ, the left-hand side of (A.6) is increasing and the right-hand side is decreasing in D. Moreover, the left-hand side goes out of bounds as D converges to e_0, and the right-hand side goes out of bounds as D goes to zero. For any e_0, therefore, there exists a unique $D(e_0)$ such that $D = D(e_0)$ satisfies (A.6). Under the given assumptions about the functions u, v, K, and θ, it is easy to see that the mapping $D(e_0)$ is increasing in e_0, with $\lim_{e_0 \to 0} D(e_0) = 0$ and $\lim_{e_0 \to \infty} D(e_0) = \infty$. By (2.15), it follows that there exists a critical $\hat{e}_0$ such that $D(\hat{e}_0) = D^*$ and, moreover, $m(D(e_0), \varphi) \gtreqless 0$ as $e_0 \lesseqgtr \hat{e}_0$.

Thus, (A.10) cannot hold for $e_0 \in (0, \hat{e}_0)$. In this case, $\lambda < 0$, and therefore $\alpha E + B = 0$ and $L = D - K(D)$, which proves (2.17) and (2.18); (2.16) is just (A.6). For $e_0 > \hat{e}_0$, $m(D(e_0), \varphi) < 0$, so that (A.7) cannot hold. It follows that, in this case, $\lambda = 0$, and therefore, by (A.10) and (2.15), that $D = D^*$, which proves (2.19); (2.20) follows from (A.9) and (2.21) from the constraint (A.2). Statements (a) and (b) of the proposition are thereby proved. ■

Proof of Statement (c). To prove statement (c), I write down the first-order conditions of the typical bank and the typical consumer. The bank chooses (α^s, B^s, D^s, L) to maximize

$$(1 - \alpha^s)\eta\pi \tag{A.11}$$

subject to

$$L \leq \alpha^s \eta\, \pi + B^s + D^s - K(D^s), \tag{A.12}$$

and

$$\pi = \varphi L - r_B B^s - r_D D^s. \tag{A.13}$$

and nonnegativity. Using (A.13), one can rewrite (A.11). and (A.12) as

$$\eta(\varphi L - r_B B^s - r_D D^s) - A \tag{A.14}$$

and

$$L \leq A + B^s + D^s - K(D^s), \tag{A.15}$$

where

$$A := \alpha^s \eta(\varphi L - r_B B^s - r_D D^s) \tag{A.16}$$

First-order conditions for maximizing (A.14) subject to (A.15) are given as

$$-1 + \mu \leq 0, \tag{A.17}$$

$$-\eta r_B + \mu \leq 0, \tag{A.18}$$

$$-\eta r_D + \mu(1 - K'(D^s) \leq 0, \tag{A.19}$$

and

$$\eta\varphi - \mu \leq 0, \tag{A.20}$$

where μ is the multiplier of the constraint (A.15), and, in each case, a strict inequality means that the corresponding maximization variable takes the value zero. Because the objective function is concave and the constraint set is convex, these first-order conditions are sufficient as well as necessary for the vector (A, B^s, D^s, L) to maximize (A.14) subject to (A.15) and nonnegativity. Given that, by parts (a) and (b) of the proposition, any equilibrium satisfies $D^s = D > 0$ and $L > 0$, any price system (η, r_B, r_D) supporting a plan (α^s, B^s, D^s, L) for a bank that is part of an equilibrium must satisfy (A.19) and (A.20) with equality, implying

$$r_D = \varphi(1 - K'(D^s), \tag{A.21}$$

as claimed in (2.24). From (A.17) and (A.18), in combination with (A.20), one also obtains

$$\eta \leq \frac{1}{\varphi} \text{ and } r_B \geq \varphi. \tag{A.22}$$

Turning to the consumer, maximization of

$$u(e_0 - \alpha^d E - B^d - D^d) + v((1 - \alpha^s + \alpha^d)\pi + r_B B^d + r_D D^d + \theta(D^d)) \tag{A.23}$$

requires that the following first-order conditions be satisfied:

$$-u'(e_0 - \alpha^d E - B^d - D^d)\; E + v'((1 - \alpha^s + \alpha^d)\pi + r_B B^d + r_D D^d + \theta(D^d))\; \pi \leq 0, \tag{A.24}$$

$$-u'(e_0 - \alpha^d E - B^d - D^d) + v'((1 - \alpha^s + \alpha^d)\pi + r_B B^d + r_D D^d + \theta(D^d))\; r_B \leq 0, \tag{A.25}$$

and

$$-u'(e_0 - \alpha^d E - B^d - D^d) + v'((1 - \alpha^s + \alpha^d)\pi + r_B B^d + r_D D^d + \theta(D^d))\; [r_D + \theta'(D^d)] \leq 0, \tag{A.26}$$

where in each case a strict inequality means that the corresponding nonnegativity condition is binding. As before, these first-order conditions are sufficient as well as necessary for a vector (α^d, B^d, D^d) to maximize (A.22).

By the equilibrium condition $\alpha^s = \alpha^d$, $1 - \alpha^s + \alpha^d > 0$. By statements (a) and (b) of the proposition, a symmetric equilibrium also satisfies $D^d = D > 0$. For any price system (η, r_B, r_D) that supports a symmetric equilibrium allocation, therefore, conditions (A.24) and (A.26) must hold with equality. Thus,

$$E = \frac{v'(\varphi L + \theta(D))}{u'(e_0 - \alpha E - B - D)}\, \pi \tag{A.27}$$

and

$$r_D + \theta'(D) = \frac{u'(e_0 - \alpha E - B - D)}{v'(\varphi L + \theta(D))}, \tag{A.28}$$

where the right-hand sides have been simplified by use of the market-clearing conditions and the definition of π.

Upon combining (A.27) and (A.28) with the equation $E = \eta\pi$, one obtains

$$\eta\,[r_D + \theta'(D)] = 1. \ \ .$$

By (A.21), it follows that

$$\eta = \frac{1}{\varphi + m(D,\varphi)}, \tag{A.29}$$

as claimed in (2.22). From (A.25) and (A.28), one also obtains

$$r_B \leq r_D + \theta'(D),$$

so (A.21) implies

$$r_B \leq \varphi + m(D,\varphi).. \tag{A.30}$$

Now (2.23) follow from (A.22) and (A.30). ■

Proof of Statement (d). By (A.12) and (A.13), bank profits are given as

$$\pi = \varphi(\alpha^s \eta\pi + B + D - K(D)) - r_B B - r_D D. \tag{A.31}$$

Using (A.21) and rearranging terms, one can rewrite this equation as

$$\pi - \varphi\alpha^s\eta\pi = (\varphi - r_B)B + \varphi(K'(D)D - K(D). \tag{A.32}$$

If $e_0 \leq \hat{e}_0$, $\alpha^s = B^s = 0$, so (A.32) becomes

$$\pi = \varphi(K'(D)D - K(D), \tag{A.33}$$

and (2.25) follows because $\alpha^s = 0$. If $e_0 > \hat{e}_0$, statement (b) of the proposition implies $D = D^*$ and $m(D,\varphi) = 0$, so (2.22) and (2.23) yield $\eta\varphi = 1$ and $r_B = \varphi$; in this case, (A.32) becomes

$$(1 - \alpha^s)\pi = \varphi(K'(D)D - K(D), , \tag{A.34}$$

and (2.25) again follows. This completes the proof of Proposition 2.1. ■

A.2 Proofs for Section 3.1

Proof of Lemma 3.1. If $(c_0, c_1(\cdot), \alpha^d(\cdot), B^d(\cdot), D^d(\cdot))$ is an optimal plan for a

consumer, there exist Lagrange multipliers $\nu > 0$ and $\mu > 0$ for the constraints (3.19) and (3.20), such that the following first-order conditions hold:

$$u'(c_0) = \nu; \tag{A.35}$$

for almost all φ,

$$v'(c_1(\varphi) + \theta(\bar{D}(\varphi))) = \mu\, q(\varphi); \tag{A.36}$$

for almost all b,

$$-\nu E(b) + \mu \mathcal{E}_q \pi(\tilde{\varphi}, b) \leq 0, \tag{A.37}$$

with a strict inequality implying that $1 - \alpha^s(b) + \alpha^d(b) = 0$;

for almost all b,

$$-\nu + \mu r_B(b) \mathcal{E}_q \min(1, \frac{\tilde{\varphi}}{\hat{\varphi}(b)}) \leq 0, \tag{A.38}$$

with a strict inequality implying that $B^d(b) = 0$;

and

$$\begin{aligned} &-\nu + \mu r_D(b)\ \mathcal{E}_q \min(1, \frac{\tilde{\varphi}}{\hat{\varphi}(b)}) \\ &+ \int_{\varphi_1}^{\varphi_2} v'(c_1(\varphi) + \theta((\bar{D}(\varphi)))(1 - \beta(\varphi, \hat{\varphi}(b)))\theta'(\bar{D}(\varphi)) f(\varphi) d\varphi \leq 0, \end{aligned} \tag{A.39}$$

with a strict inequality implying that $D^d(b) = 0$.

Let $\chi := \frac{\nu}{\mu}$. Then (3.29) follows from (A.35) and (A.36), (3.30) follows from (A.37), (3.31) follows from (A.38), and (3.32) follows from (A.39) and (A.36). ■

Proof of Lemma 3.2. I first prove (3.34). If the consumers' plans are all the same, there is nothing to prove. Suppose therefore that the consumers' plans are not all the same. Let $(c_0, c_1(\cdot), \alpha^d(\cdot), B^d(\cdot), D^d(\cdot))$ and $(\hat{c}_0, \hat{c}_1(\cdot), \hat{\alpha}^d(\cdot), \hat{B}^d(\cdot), \hat{D}^d(\cdot))$ be two distinct plans chosen by consumers a and $\hat{a}$. Because the two consumers have the same budget set and the same utility functional, it must be the case that consumers a and $\hat{a}$ are actually indifferent between the two plans. Because their budget set is convex and the utility functions u and v are concave, they are also indifferent between these two plans and any convex combination of the form

$$\lambda(c_0, c_1(\cdot), \alpha^d(\cdot), B^d(\cdot), D^d(\cdot)) + (1 - \lambda)(\hat{c}_0, \hat{c}_1(\cdot), \hat{\alpha}^d(\cdot), \hat{B}^d(\cdot), \hat{D}^d(\cdot))$$

where $\lambda \in [0, 1]$. Because u and v are strictly concave, it follows that

$$c_0 = \hat{c}_0 \tag{A.40}$$

and that, for all $\lambda \in [0, 1]$

$$\lambda c_1(\tilde{\varphi}) + (1 - \lambda) c_2(\tilde{\varphi}) + \theta(\lambda \bar{D}^d(\tilde{\varphi}) + (1 - \lambda)\hat{D}^d(\tilde{\varphi})) = c_1(\tilde{\varphi}) + \theta(\bar{D}^d(\tilde{\varphi})), \tag{A.41}$$

almost surely.The latter is possible only if $\theta'(\lambda\bar{D}(\tilde{\varphi}) + (1-\lambda)\hat{D}(\tilde{\varphi}))$ is almost surely independent of λ, i.e., if $\theta'(\cdot)$ is almost surely constant on the interval between $\bar{D}(\tilde{\varphi})$ and $\hat{D}(\tilde{\varphi})$.

Now let $d_1(\varphi), d_2(\varphi)$ be the infimum and the supremum over $a \in [0,1]$ of $\theta'(\bar{D}^d(\varphi, a))$. The argument just given implies that $\theta'(\cdot)$ is almost surely constant on the interval between $d_1(\varphi)$ and $d_2(\varphi)$. Moreover, by definition

$$\bar{D}^d(\varphi, a) \in [d_1(\varphi), d_2(\varphi)] \text{ for all } a \tag{A.42}$$

and, therefore,

$$\int_0^1 \bar{D}^d(\varphi, \bar{a})d\bar{a} \in [d_1(\varphi), d_2(\varphi)]. \tag{A.43}$$

Hence,

$$\theta'\left(\int_0^1 \bar{D}^d(\varphi, \bar{a})d\bar{a}\right) = \theta'(\bar{D}^d(\varphi, a)) \tag{A.44}$$

almost surely, for all a.

I next prove that, for almost all banks, in equilibrium, the constraints (3.23) and (3.10), (3.13), or (3.14) are satisfied with equality. For the date 0 budget constraint (3.23), this is trivial because a strict inequality would imply that either the investment $L(b)$ can be increased or one of the funding terms $\alpha^s(b)E(b)$, $B^s(b), D^s(b)$ can be decreased, which is incompatible with the optimality of the bank's plan.

For the pricing constraints, (3.10), (3.13), and (3.14), the argument is more complicated. A strict inequality in one of these constraints implies that $E(b)$ can be increased or $r_B(b)$ or $r_D(b)$ can be decreased, which would raise the value of the bank's objective function unless the corresponding funding variable, $\alpha^s(b)$, $B^s(b)$,or $D^s(b)$, is zero.

So suppose that, for a nonnull set of banks, one of the pricing constraints, (3.10), (3.13), or (3.14), is strict. For example, suppose that the set $\mathcal{B}$ of banks for which

$$r_D(b) > \frac{\hat{\varphi}(b)}{\eta\, \mathcal{E}_q \min(\hat{\varphi}(b), \tilde{\varphi})} - \lambda(\hat{\varphi}(b)) \tag{A.45}$$

has positive measure. By the equilibrium conditions (E.4) and (E.3), it follows that

$$\begin{aligned} r_D(b)\mathcal{E}_q \min(1, \frac{\tilde{\varphi}}{\hat{\varphi}(b)}) &> \frac{1}{\eta} - \mathcal{E}_q(1 - \beta(\tilde{\varphi}, \hat{\varphi}(b)))\theta'(\bar{D}^s(\tilde{\varphi})) \\ &= \frac{1}{\eta} - \mathcal{E}_q(1 - \beta(\tilde{\varphi}, \hat{\varphi}(b)))\theta'(\bar{D}^d(\tilde{\varphi})) \end{aligned}$$

for every $b \in \mathcal{B}$.

For every consumer $a \in [0,1]$, therefore, condition (3.32) in Lemma 3.1 therefore implies that

$$\begin{aligned} \chi(a) \quad > \quad & \frac{1}{\eta} - \mathcal{E}_q(1 - \beta(\tilde{\varphi}, \hat{\varphi}(b)))\theta'(\bar{D}^d(\tilde{\varphi})) \\ & + \mathcal{E}_q(1 - \beta(\tilde{\varphi}, \hat{\varphi}(b))\theta'(\bar{D}^d(\tilde{\varphi}, a)) \end{aligned} \tag{A.46}$$

for almost every $b \in \mathcal{B}$. By (A.44), the last two terms cancel on the right-hand side of (A.46) cancel out. For every $a \in [0, 1]$, therefore, the parameter $\chi(a)$ in Lemma 3.1 satisfies

$$\chi(a) > \frac{1}{\eta}. \tag{A.47}$$

By Lemma 3.1 then,

$$\begin{aligned} 1 - \alpha^s(b, a) + \alpha^d(b, a) &= 0 \qquad \text{(A.48)} \\ \text{for almost every bank } b \text{ such that } E(b) &= \eta\, \mathcal{E}_q \pi(\tilde{\varphi}), \end{aligned}$$

$$B^d(b, a) = 0 \tag{A.49}$$

for almost every bank b such that

$$r_B(b) = \mathcal{E}_q \min(1, \frac{\tilde{\varphi}}{\hat{\varphi}(b)}) = \frac{1}{\eta},$$

and

$$D^d(b, a) = 0 \tag{A.50}$$

for almost every bank b such that

$$\begin{aligned} r_D(b)\mathcal{E}_q \min(1, \frac{\tilde{\varphi}}{\hat{\varphi}(b)}) &= \frac{1}{\eta} - \mathcal{E}_q(1 - \beta(\tilde{\varphi}, \hat{\varphi}(b))\theta'(\bar{D}^d(\tilde{\varphi}, a)) \\ &= \frac{1}{\eta} - \mathcal{E}_q(1 - \beta(\tilde{\varphi}, \hat{\varphi}(b))\theta'(\bar{D}^s(\tilde{\varphi})). \end{aligned}$$

By market clearing and the banks' budget constraints for $t = 0$, it follows that $L(b) = B^s(b) = D^s(b) = 0$ for almost every bank b. By market clearing at $t = \frac{1}{2}$, $L(b) = 0$ for almost all b implies $c_1(\tilde{\varphi}, a) = 0$ almost surely for all a. Also by market clearing, $D^s(b) = 0$ for almost all bank b implies $\theta(\bar{D}^d(\tilde{\varphi})) = 0$ almost surely and therefore $\theta(\bar{D}^d(\tilde{\varphi}, a) = 0$ almost surely for all a. But then, $v'(c_1(\tilde{\varphi}, a) + \theta(\bar{D}(\tilde{\varphi}, a))) = \infty$ almost surely for all a. But then the equation for $\chi(a)$, together with the optimization condition (A.36) in the proof of Lemma 3.1, implies that $\chi(a) = 0$ for almost all a, which is incompatible with (A.47). The assumption that the set of banks for whom the constraint (3.14) holds with a strict inequality has positive measure thus led to a contradiction and must be false. The proofs that the constraints (3.6) and (3.13) must also hold with equality for almost all banks are similar and are left to the reader.

Given that almost all banks' plans satisfy the constraints (3.10), (3.13), and (3.14) with equality, the parameter $\chi(a)$ in Lemma 3.1 satisfies $\chi(a) \geq \frac{1}{\eta}$ for all a. If the inequality were strict, then, along the lines of the argument just given, consumer a's plan would have $1 - \alpha^s(b, a) + \alpha^d(b, a) = B^d(b, a) = D^d(b, a)$ for almost all b and hence $c_1(\tilde{\varphi}, a) + \theta(\bar{D}(\tilde{\varphi}, a)) = 0$ and $v'(c_1(\tilde{\varphi}, a) + \theta(\bar{D}(\tilde{\varphi}, a))) = \infty$ almost surely. But then, he equation for $\chi(a)$, together with the optimization condition (A.36) in the proof of Lemma 3.1, implies that $\chi(a) = 0$ for almost all a, contrary to the assumption that $\chi(a) > \frac{1}{\eta}$. Hence $\chi(a) = \frac{1}{\eta}$. This in turn implies that the consumers' first-order conditions hold as equations. Equation

(??) follows upon combining (3.32) holding as an equation, (3.34), and (3.14) holding as an equation. ■

A.3 Proof of Proposition 3.4

As far as I know, Proposition 3.4 is not subsumed by known existence results, nor does it follow from standard arguments. The proof must overcome several difficulties: First, the dependence of consumers' liquidity benefits on the banks' funding choices implies that there is an external effect in the model. Second, the banks' optimization problems are not convex. Third, the assignment of plans to banks must be measurable. Fourth, there is no natural bound on the banks' plans.

To overcome these difficulties, I will exploit the special structure of the model. First, along the lines of Lemma 3.2, the special structure of the consumer specification will be used to deal with the pricing effects of the externalities in liquidity provision. Second, the continuum of banks will be used to smooth over any discontinuities that might arise from the nonconvexity of the banks' optimization problem. Third, along the lines of Hart, Hildenbrand and Kohlberg (1974), the technical difficulties associated with measurability will be avoided by working with cross-section distributions of banks' plans rather than functions assigning plans to banks. Finally, boundedness of banks' plans will be obtained from an interplay of aggregate feasibility and individual optimization. The argument here is more involved than usual because, with a continuum of banks, the boundedness of aggregates that follows from feasibility considerations does not automatically yield boundedness of individual banks' plans.

I first use Lemmas 3.1 and 3.2 to simplify the conditions for an equilibrium, in particular, the conditions for consumer behaviour.

Lemma A.1 *Let $\eta > 0$ and $q(\cdot)$ be a discount factor and a price system for $t = \frac{1}{2}$. Consider two mappings $b \to (L(b), A(b), B^s(b), D^s(b), \hat{\varphi}(b))$ from $[0,1]$ to $\mathbb{R}^5_+$ and $\varphi \to \bar{D}(\varphi)$ from $[\varphi_1, \varphi_2]$ to $\mathbb{R}_+$ such that the following conditions hold:*

(a) for any $\varphi \in [\varphi_1, \varphi_2]$,

$$\bar{D}(\varphi) = \int_0^1 (1 - \beta(\varphi, \hat{\varphi}(b))D^s(b)db; \tag{A.51}$$

(b)

$$\eta = \frac{\int_{\varphi_1}^{\varphi_2} v'\left(\varphi \int_0^1 L(b)db + \theta\left(\bar{D}(\varphi)\right)\right) f(\varphi)d\varphi}{u'\left(e_0 - \int_0^1 (A(b) + B^s(b) + D^s(b))db\right)}; \tag{A.52}$$

(c) for any $\varphi \in [\varphi_1, \varphi_2]$,

$$q(\varphi) = \frac{v'\left(\varphi \int_0^1 L(b)db + \theta\left(\bar{D}(\varphi)\right)\right)}{\int_{\varphi_1}^{\varphi_2} v'\left(\bar{\varphi} \int_0^1 L(b)db + \theta\left(\bar{D}(\bar{\varphi})\right)\right) f(\bar{\varphi})d\bar{\varphi}}; \tag{A.53}$$

(d) for (almost) all $b \in [0,1]$, the vector $(L(b), A(b), B^s(b), D^s(b), \hat{\varphi}(b))$ is a solution to the problem of maximizing the objective

$$\eta\ \mathcal{E}_q \tilde{\varphi} L - (A+B+D) + \eta D\ \mathcal{E}_q (1-\beta(\tilde{\varphi},\hat{\varphi}))\theta'\left(\bar{D}(\tilde{\varphi})\right) \tag{A.54}$$

under the constraints

$$L \leq A+B+D-K(D), \tag{A.55}$$

$$\eta\ \mathcal{E}_q \min(\tilde{\varphi},\hat{\varphi}) L = B+D-\eta D\ \mathcal{E}_q (1-\beta(\tilde{\varphi},\hat{\varphi}))\theta'\left(\bar{D}(\tilde{\varphi})\right), \tag{A.56}$$

and nonnegativity.

Then an equilibrium of the model with full commitment and communication of banks' choices is obtained by setting

$$c_0(a) = e_0 - \int_0^1 (A(b)+B(b)+D(b))db, \tag{A.57}$$

$$c_1(\varphi, a) = \varphi \int_0^1 L(b)db \tag{A.58}$$

$$\alpha^d(b,a) = \alpha^s(b) = \frac{A(b)}{E(b)}, \tag{A.59}$$

$$B^d(b,a) = B^s(b) = B(b), \tag{A.60}$$

$$D^d(b,a) = D^s(b) = D(b), \tag{A.61}$$

$$E(b) = \eta\ \mathcal{E}_q \max[0, \tilde{\varphi}L(b) - r_B(b)B(b) - r_D(b)D(b)] \tag{A.62}$$

$$r_B(b) = \frac{1}{\eta\ \mathcal{E}_q \min(1, \frac{\tilde{\varphi}}{\hat{\varphi}(b)})}, \tag{A.63}$$

$$r_D(b) = \frac{1}{\eta\ \mathcal{E}_q \min(1, \frac{\tilde{\varphi}}{\hat{\varphi}(b)})} - \lambda(\hat{\varphi}(b), \tag{A.64}$$

for all a and b, and, for any $\hat{\varphi}$,

$$\lambda(\hat{\varphi}) = \frac{\mathcal{E}_q (1-\beta(\tilde{\varphi},\hat{\varphi}))\theta'\left(\int_{\{\bar{b}|\hat{\varphi}(\bar{b})<\tilde{\varphi}} D(\bar{b})d\bar{b}\right)}{\mathcal{E}_q \min(1, \frac{\tilde{\varphi}}{\hat{\varphi}(b)})}. \tag{A.65}$$

Proof. The equilibrium conditions (E.3) and (E.4) hold by construction. By construction, the consumer's plan $(c_0(a), c_1(\cdot, a), \alpha^d(\cdot, a), B^d(\cdot, a), D^d(\cdot, a))$ also satisfies the first-order conditions (A.35) - (A.39), with $\nu = u'(c_0(a))$ and $\mu = \eta\nu$, as well as the constraints for a solution to the consumer's problem in condition (E.2). Because the objective function is concave and the constraint set convex, it follows that condition (E.2) is also satisfied. To show that condition (E.1) also holds, I note that, for any plan $(\alpha^s, B^s, D^s, L, E, r_B, r_D)$ that satisfies the constraints for the bank's optimization problem in (E.2), the value of the bank's objective function is given as

$$\begin{aligned}
(1-\alpha^s)E &\leq \eta\,\mathcal{E}_q \max[\tilde{\varphi}L - r_B B^s - r_D D^s, 0] - \alpha^s E \\
&= \eta\,\mathcal{E}_q \tilde{\varphi}L - \eta\,\mathcal{E}_q \min[\tilde{\varphi}L, r_B B^s + r_D D^s] - \alpha^s E \\
&= \eta\,\mathcal{E}_q \tilde{\varphi}L - \eta\,\mathcal{E}_q \min(\frac{\tilde{\varphi}}{\hat{\varphi}}, 1)\,(r_B B^s + r_D D^s) - \alpha^s E \\
&\leq \eta\,\mathcal{E}_q \tilde{\varphi}L - B^s - D^s + \eta\,\mathcal{E}_q \min(\frac{\tilde{\varphi}}{\hat{\varphi}}, 1)\,\lambda(\hat{\varphi})D^s - \alpha^s E \\
&= \eta\,\mathcal{E}_q \tilde{\varphi}L - B^s - D^s + \eta\,\mathcal{E}_q (1-\beta(\tilde{\varphi},\hat{\varphi}))\theta' \left(\int_{\{\bar{b}|\hat{\varphi}(\bar{b})<\tilde{\varphi}\}} D(\bar{b})d\bar{b}\right) D^s - \alpha^s E,
\end{aligned} \tag{A.66}$$

where the last equation follows from (A.65). Moreover, the inequalities hold as equations if the constraints for E, r_B, r_D are satisfied with equality.

By construction, for any b, the plan $(\alpha^s(b), B^s(b), D^s(b), L(b), E(b), r_B(b), r_D(b))$ satisfies the constraints for E, r_B, r_D with equality. Therefore,

$$\begin{aligned}
&(1-\alpha^s(b))E(b) \qquad\qquad (A.67) \\
&= \eta\,\mathcal{E}_q \tilde{\varphi}L(b) - B^s(b) - D^s(b) + \eta\,\mathcal{E}_q (1-\beta(\tilde{\varphi},\hat{\varphi}(b)))\theta' \left(\int_{\{\bar{b}|\hat{\varphi}(\bar{b})<\tilde{\varphi}\}} D(\bar{b})d\bar{b}\right) D^s(b) - A(b),
\end{aligned}$$

where $A(b) := \alpha^s(b)E(b)$, by (A.59). For any b for which the vector $(L(b), A(b), B(b), D(b), \hat{\varphi}(b))$ solves the problem of maximizing (A.54) under the constraints (A.55), (A.56), and nonnegativity, it follows that

$$(1-\alpha^s(b))E(b) \geq \eta\,\mathcal{E}_q \tilde{\varphi}L - B - D + \eta\,\mathcal{E}_q (1-\beta(\tilde{\varphi},\hat{\varphi}))\theta' \left(\int_{\{\bar{b}|\hat{\varphi}(\bar{b})<\tilde{\varphi}\}} D(\bar{b})d\bar{b}\right) D - A$$

for any vector $(L, A, B, D, \hat{\varphi}$ that also satisfies (A.55), (A.56), and nonnegativity, and hence, by (A.66) that

$$(1-\alpha^s(b))E(b) \geq (1-\alpha^s)E$$

for any plan $(\alpha^s, B^s, D^s, L, E, r_B, r_D)$ that satisfies the constraints for the bank's optimization problem in (E.2). Validity of (E.2) follows immediately. This completes the proof of Lemma A.1 ■

I next reformulate the conditions in Lemma A.1 in terms of the cross-section distribution H of the vectors $(L(b), A(b), B^s(b), D^s(b), \hat{\varphi}(b))$ that induced by the mapping $b \to (L(b), A(b), B^s(b), D^s(b), \hat{\varphi}(b))$. For this purpose, I note that, with an abuse of notation, equations (A.51), (A.52), (A.53) can be rewritten in the form

$$\bar{D}(\varphi) = \bar{D}(\varphi|H), \quad \eta = \eta(H), \quad q(\varphi) = q(\varphi|H), \tag{A.68}$$

where

$$\bar{D}(\varphi|H) := \int_{\{L,A,B,D,\hat{\varphi})\in\mathbb{R}^5_+|\hat{\varphi}<\varphi\}} D\; dH(L,A,B,D,\hat{\varphi}), \tag{A.69}$$

$$\eta(H) :== \frac{\int v'\left(\bar{\varphi}\int_{\mathbb{R}^5_+} L\; dH(L,A,B,D,\hat{\varphi}) + \theta\left(\bar{D}(\bar{\varphi})\right)\right) f(\bar{\varphi})d\bar{\varphi}}{u'\left(e_0 - \int_{\mathbb{R}^5_+}(A+B+D)\; dH(L,A,B,D,\hat{\varphi})\right)}, \tag{A.70}$$

and

$$q(\varphi|H) = \frac{v'\left(\varphi\int_{\mathbb{R}^5_+} L\; dH(L,A,B,D,\hat{\varphi}) + \theta\left(\bar{D}(\varphi)\right)\right)}{\int v'\left(\bar{\varphi}\int_{\mathbb{R}^5_+} L\; dH(L,A,B,D,\hat{\varphi}) + \theta\left(\bar{D}(\bar{\varphi})\right)\right) f(\bar{\varphi})d\bar{\varphi}}. \tag{A.71}$$

Lemma A.2 *Let $H \in \mathcal{M}(\mathbb{R}^5_+)$ be a measure that assigns all probability mass to set of vectors $(L, A, B^s, D^s, \hat{\varphi})$ that maximize the objective*

$$\eta(H)\; \mathcal{E}_{q(H)}\tilde{\varphi}L - (A+B^s+D^s) + \eta(H)\; D^s\; \mathcal{E}_{q(H)}(1-\beta(\tilde{\varphi},\hat{\varphi}))\theta'\left(\bar{D}(\tilde{\varphi}|H)\right) \tag{A.72}$$

under the constraints

$$L \leq A + B^s + D^s - K(D^s), \tag{A.73}$$

$$\eta(H)\; \mathcal{E}_{q(H)}\min(\tilde{\varphi},\hat{\varphi})L = B^s + D^s - \eta(H)\; D^s\; \mathcal{E}_{q(H)}(1-\beta(\tilde{\varphi},\hat{\varphi}))\theta'\left(\bar{D}(\tilde{\varphi}|H)\right). \tag{A.74}$$

and nonnegativity. Further let $b \to (L(b), A(b), B^s(b), D^s(b), \hat{\varphi}(b))$ be a mapping from $[0,1]$ to $\mathbb{R}^5_+$ such that the distribution of the vector $(L(b), A(b), B^s(b), D^s(b), \hat{\varphi}(b))$ that is induced by the uniform distribution on $[0,1]$ is equal to H. Then the discount factor $\eta(H)$ and price system $q(\cdot|H)$ for $t = \frac{1}{2}$ and the mappings $b \to (L(b), A(b), B^s(b), D^s(b), \hat{\varphi}(b))$ and $\varphi \to \bar{D}(\varphi|H)$ satisfy conditions (a) - (d) of Lemma A.1.

Proof. Statements (a) - (c) follow because (A.69) - (A.71) imply (A.51) - (A.53) if H is the distribution of $(L(b), A(b), B^s(b), D^s(b), \hat{\varphi}(b))$ that is induced by Lebesgue measure. Statement (d) follows from the fact that H assigns full measure to the set of solutions to the problem of maximizing (A.72) subject to (A.73), (A.74), and nonnegativity. ■

For any $H \in \mathcal{M}(\mathbb{R}^5_+)$, the existence of a mapping $b \to (L(b), A(b), B^s(b), D^s(b), \hat{\varphi}(b))$ from $[0,1]$ to $\mathbb{R}^5_+$ such that the distribution of the vector $(L(b), A(b), B^s(b), D^s(b), \hat{\varphi}(b))$ that is induced by the uniform distribution on $[0,1]$ is equal to H follows by standard arguments; see, e.g., Hart, Hildenbrand, and Kohlberg (1974), pp. 164 ff. The problem of proving the existence of an equilibrium with full commitment and communication of banks' choices has thus been reduced to proving the existence of a measure H on $\mathbb{R}^5_+$ that satisfies the conditions of Lemma A.2.

To deal with this latter problem, I will use a standard fixed-point argument. Let $\mathcal{M}(\mathbb{R}^5_+)$ be endowed with the topology of weak convergence, i.e., the weak* topology. For any $y \in (e_0, \infty)$, let $\mathcal{H}^y$ be the subspace of $\mathcal{M}(\mathbb{R}^5_+)$ such that, for any $H \in \mathcal{H}^y$,

$$H([0, y]|^4 \times [0, \varphi_2]\} = 1, \tag{A.75}$$

and

$$0 < \int (L + K(D^s))dH \leq \int (A + B^s + D^s)dH < e_0. \tag{A.76}$$

Lemma A.3 *For any $y \in (e_0, \infty)$, let $\{H^n\}$ be any sequence of measures in $\mathcal{H}^y$ that converges to a measure $H \in \mathcal{H}^y$. Then, as n goes out of bounds, the following are true:*

(a) The integrals $\int LdH^n$, $\int AdH^n$, $\int B^s dH^n$, $\int D^s dH^n$ converge to $\int LdH$, $\int AdH$, $\int BdH$, and $\int D^s dH$, respectively.

(b) For almost every $\varphi \in [\varphi_1, \varphi_2]$, $\bar{D}(\varphi|H^n)$ converges to $\bar{D}(\varphi|H)$.

(c) For almost every $\varphi \in [\varphi_1, \varphi_2]$, $v'(\varphi \int LdH^n + \theta(\bar{D}(\varphi|H^n)))$ converges to $v'(\varphi \int LdH + \theta(\bar{D}(\varphi|H)))$.

(d) For any uniformly bounded sequence $\{g^n\}$ of measurable function $g^n : [\varphi_1, \varphi_2] \to \mathbb{R}$, if $g^n(\varphi)$ converges to $g(\varphi)$ for almost all φ, then $\int g^n(\varphi)v'(\varphi \int LdH^n + \theta(\bar{D}(\varphi|H^n)))f(\varphi)d\varphi$ converges to $\int g(\varphi)v'(\varphi \int LdH + \theta(\bar{D}(\varphi|H)))f(\varphi)d\varphi$.

(e) $\eta(H^n)$ converges to $\eta(H)$.

(f) For any uniformly bounded sequence $\{g^n\}$ of measurable function $g^n : [\varphi_1, \varphi_2] \to \mathbb{R}$, if $g^n(\varphi)$ converges to $g(\varphi)$ for almost all φ, then $\mathcal{E}_{q(H^n)} g^n(\tilde{\varphi})$ converges to $\mathcal{E}_{q(H)} g(\tilde{\varphi})$.

Proof. (a) This is immediate from the definition of weak convergence and the fact that, for $H^n \in \mathcal{H}^y$, L, A, B^s, and D^s are bounded by y, H^n-almost surely.

(b) Fix φ and assume that $H(\mathbb{R}^4_+ \times \{\varphi\}) = 0$, i.e., under the marginal distribution on $[0, \varphi_2]$ that is induced by H, φ is not an atom. Observe that the map $(D^s, \hat{\varphi}) \to (1 - \beta(\varphi, \hat{\varphi}))D^s$ is continuous *except* at points $(D^s, \hat{\varphi})$ with $\hat{\varphi} = \varphi$. By Theorem 5.1, p. 30, in Billingsley (1968), it follows that the distributions of $(1 - \beta(\varphi, \hat{\varphi}))D^s$ that are induced by $H^n, n = 1, 2, ...$, converge to the distribution that is induced by H. Because, for $H^n \in \mathcal{H}^y$, $(1 - \beta(\varphi, \hat{\varphi}))D^s$ is bounded by y, it follows that $\bar{D}(\varphi|H^n) = \int (1 - \beta(\varphi, \hat{\varphi}))D^s dH^n$ converges to $\bar{D}(\varphi|H) = \int (1 - \beta(\varphi, \hat{\varphi}))D^s dH$. The statement now follows from the observation that the marginal distribution on $[0, \varphi_2]$ that is induced by H has at most countably many atoms, and therefore, that $H(\mathbb{R}^4_+ \times \{\varphi\}) = 0$ for almost every $\varphi \in [0, \varphi_2]$.

(c) Because the function $v'(\cdot)$ is continuous, this statement follows immediately from statements (a) and (b).

(d) Since $H^n \in \mathcal{H}$ and $H \in \mathcal{H}$ imply $\int LdH^n > 0$ and $\int LdH > 0$, statement (a) implies that there exists $\hat{L} > 0$ such that $\int LdH^n \geq \hat{L}$ for all n. Because the function $v'(\cdot)$ is decreasing, it follows that, for any $\varphi \in [\varphi_1, \varphi_2]$, $v'(\varphi \int LdH^n + \theta(\bar{D}(\varphi|H^n))) \leq v'(\varphi \int LdH^n) \leq v'(\varphi\hat{L})$ for all n. For any uniformly bounded sequence $\{g^n\}$ of measurable function $g^n : [\varphi_1, \varphi_2] \to \mathbb{R}$, therefore, $|g^n(\varphi)|\, v'(\varphi \int LdH^n + \theta(\bar{D}(\varphi|H^n))) \leq \bar{g}v'(\varphi\hat{L})$ for all n, where $\bar{g}$ is the uniform bound for $|g^n(\varphi)|$, $\varphi \in [\varphi_1, \varphi_2]$. Moreover,

$$\int_{\varphi_1}^{\varphi_2} \bar{g} v'(\varphi \hat{L}) f(\varphi) d\varphi \leq \frac{\bar{f}\bar{g}}{\hat{L}} \left[v(\varphi_2 \hat{L}) - v(\varphi_1 \hat{L}) \right]$$

where f is the maximum of $f(\varphi)$ on the interval $[\varphi_1, \varphi_2]$. Statement (d) now follows from Lebesgue's Dominated-Convergence Theorem (see, e.g., Hildenbrand 1974, p. 46).

(e) By statement (d), the integrals $\int_{\varphi_1}^{\varphi_2} v'(\varphi \int L dH^n + \theta(\bar{D}(\varphi|H^n))) f(\varphi) d\varphi$ converge to $\int_{\varphi_1}^{\varphi_2} v'(\varphi \int L dH + \theta(\bar{D}(\varphi|H))) f(\varphi) d\varphi$. By the continuity of $u'(\cdot)$ and statement (a), the expressions $u'(e_0 - \int (A + B^s + D^s) dH^n)$ converge to $u'(e_0 - \int (A + B^s + D^s) dH)$. Convergence of $\eta(H^n)$ to $\eta(H)$ follows immediately.

(f) For any bounded measurable function $g^n : [\varphi_1, \varphi_2] \to \mathbb{R}$ and any n,

$$\mathcal{E}_{q(H^n)} g(\tilde{\varphi}) = \frac{\int_{\varphi_1}^{\varphi_2} g(\varphi) v'(\varphi \int L dH^n + \theta(\bar{D}(\varphi|H^n))) f(\varphi) d\varphi}{\int_{\varphi_1}^{\varphi_2} v'(\varphi \int L dH^n + \theta(\bar{D}(\varphi|H^n))) f(\varphi) d\varphi}; \tag{A.77}$$

similarly, for any $g : [\varphi_1, \varphi_2] \to \mathbb{R}$,

$$\mathcal{E}_{q(H)} g(\tilde{\varphi}) = \frac{\int_{\varphi_1}^{\varphi_2} g(\varphi) v'(\varphi \int L dH + \theta(\bar{D}(\varphi|H))) f(\varphi) d\varphi}{\int_{\varphi_1}^{\varphi_2} v'(\varphi \int L dH + \theta(\bar{D}(\varphi|H))) f(\varphi) d\varphi}. \tag{A.78}$$

If the sequence $\{g^n\}$ is uniformly bounded and, moreover, $g^n(\varphi)$ converges to $g(\varphi)$ for almost all φ, statement (d) implies that the numerators in (A.77) converge to the numerator in (A.78) and the denominators in (A.77) converge to the denominator in (A.78). Convergence of $\mathcal{E}_{q(H^n)} g(\tilde{\varphi})$ to $\mathcal{E}_{q(H)} g(\tilde{\varphi})$ follows immediately. ■

Lemma A.4 *Let $y \in (e_0, \infty)$ be given. For any $H \in \mathcal{H}^y$ and any $X \in \mathbb{R}_+ \cup \{\infty\}$, let $\Psi(H, X)$ be the set of vectors $(L, A, B^s, D^s, \hat{\varphi}) \in \mathbb{R}_+^5$ that solve the problem of maximizing the objective (A.72) under the constraints (A.73), (A.74), and*

$$A + B^s + D^s \leq X. \tag{A.79}$$

Then the graph of the relation Ψ from $\mathcal{H}^y \times (\mathbb{R}_+ \cup \{\infty\})$ is closed. Moreover, the restriction of Ψ to $\mathcal{H}^y \times \mathbb{R}_+$ is upper hemi-continuous and compact-valued.

Proof. The proof proceeds in three steps.

Step 1: The objective (A.72) is jointly continuous in $(H, X) \in \mathcal{H}^y \times (\mathbb{R}_+ \cup \{\infty\})$ and $(L, A, B^s, D^s, \hat{\varphi}) \in \mathbb{R}_+^5$.

To prove this statement, let $\{(H^n, X^n)\}$ and $\{(L^n, A^n, (B^s)^n, (D^s)^n, \hat{\varphi}^n)\}$ be any two sequences in $\mathcal{H}^y \times (\mathbb{R}_+ \cup \{\infty\})$ and $\mathbb{R}_+^5$, respectively, and suppose that (H^n, X^n) converges to $(H, X) \in \mathcal{H}^y \times (\mathbb{R}_+ \cup \{\infty\})$ and $(L^n, A^n, (B^s)^n, (D^s)^n, \hat{\varphi}^n)$ converges to $(L, A, B^s, D^s, \hat{\varphi}) \in \mathbb{R}_+^5$ as n goes out of bounds. By statements (e) and (f) of Lemma A.3, $\eta(H^n)\, \mathcal{E}_{q(H^n)} \tilde{\varphi} L^n$ converges to $\eta(H)\, \mathcal{E}_{q(H)} \tilde{\varphi} L$. Trivially also, $-(A^n + (B^s)^n + (D^s)^n)$ converges to $-(A + B^s + D^s)$. To deal with the terms $\eta(H^n)\ (D^s)^n\ \mathcal{E}_{q(H^n)}(1-\beta(\tilde{\varphi},\hat{\varphi}^n))\theta'\left(\bar{D}(\tilde{\varphi}|H^n)\right)$, introduce functions $g^n : [\varphi_1, \varphi_2] \to \mathbb{R}_+$, $n = 1, 2, ...,$ and $g : [\varphi_1, \varphi_2] \to \mathbb{R}_+$ such that, for any φ,

$$g^n(\varphi) = (1 - \beta(\varphi, \hat{\varphi}^n))\theta'(\bar{D}(\varphi|H^n))$$

and

$$g(\varphi) = (1 - \beta(\varphi, \hat{\varphi}))\theta'(\bar{D}(\varphi|H)).$$

The functions $g^n, n = 1, 2, ...,$ and g are obviously uniformly bounded. Moreover, as n becomes large, $g^n(\varphi)$ converges to $g(\varphi)$ *unless* $\bar{D}(\varphi|H^n)$ fails to converge to $\bar{D}(\varphi|H)$ or *unless* $\varphi = \hat{\varphi}$, so φ is a discontinuity point of $\beta(\cdot, \hat{\varphi})$. By statement (b) of Lemma A.3,it follows that $g^n(\varphi)$ converges to $g(\varphi)$ for almost all φ. By statements (e) and (f) of Lemma A.3, therefore, $\eta(H^n)\ (D^s)^n\ \mathcal{E}_{q(H^n)}(1 - \beta(\tilde{\varphi},\hat{\varphi}^n))\theta'\left(\bar{D}(\tilde{\varphi}|H^n)\right)$ converges to $\eta(H)\ D^s\ \mathcal{E}_{q(H)}(1-\beta(\tilde{\varphi},\hat{\varphi}))\theta'\left(\bar{D}(\tilde{\varphi}|H)\right)$ as n becomes large. Continuity of the objective (A.72) in $(H, X) \in \mathcal{H}^y \times (\mathbb{R}_+ \cup \{\infty\})$ and $(L, A, B^s, D^s, \hat{\varphi}) \in \mathbb{R}^5_+$ follows immediately.

Step 2: For any $(H, X) \in \mathcal{H}^y \times (\mathbb{R}_+ \cup \{\infty\})$, let $\Gamma(H, X)$ be the set of vectors $(L, A, B^s, D^s, \hat{\varphi}) \in \mathbb{R}^5_+$ that satisfy the constraints (A.73), (A.74), and (A.79). Then Γ is a continuous and closed-valued correspondence from $\mathcal{H}^y \times (\mathbb{R}_+ \cup \{\infty\})$ to $\mathbb{R}^5_+$.If $X \in \mathbb{R}_+$, $\Gamma(H, X)$ is compact.

Trivially, $(0, 0, 0, 0, 0) \in \Gamma(H, X)$ for all (H, X) so $\Gamma(H, X)$ is nonempty. To prove that Γ is upper hemi-continuous and closed-valued, let $\{(H^n, X^n)\}$ and $\{(L^n, A^n, (B^s)^n, (D^s)^n, \hat{\varphi}^n)\}$ be any two sequences in $\mathcal{H}^y \times (\mathbb{R}_+ \cup \{\infty\})$ and $\mathbb{R}^5_+$, respectively, and suppose that (H^n, X^n) converges to $(H, X) \in \mathcal{H}^y \times (\mathbb{R}_+ \cup \{\infty\})$ and $(L^n, A^n, (B^s)^n, (D^s)^n, \hat{\varphi}^n)$ converges to $(L, A, B^s, D^s, \hat{\varphi})$ as n goes out of bounds. Suppose that $(L^n, A^n, (B^s)^n, (D^s)^n, \hat{\varphi}^n) \in \Gamma(H^n, X^n)$ for all n. Then, trivially, $(L, A, B^s, D^s, \hat{\varphi}) \in \mathbb{R}^5_+$ and, moreover, $(L, A, B^s, D^s, \hat{\varphi})$ satisfies (A.73) and (A.79). Moreover, the same continuity arguments as in Step 1 show that $\eta(H^n)\ \mathcal{E}_{q(H^n)} \min(\tilde{\varphi}, \hat{\varphi}^n)L^n$ converges to $\eta(H)\ \mathcal{E}_{q(H)} \min(\tilde{\varphi}, \hat{\varphi})L$ and that $\eta(H^n)\ (D^s)^n\ \mathcal{E}_{q(H^n)}(1-\beta(\tilde{\varphi},\hat{\varphi}^n))\theta'\left(\bar{D}(\tilde{\varphi}|H^n)\right)$ converges to $\eta(H)\ D^s\ \mathcal{E}_{q(H)}(1-\beta(\tilde{\varphi},\hat{\varphi}))\theta'\left(\bar{D}(\tilde{\varphi}|H)\right)$. The vector $(L, A, B^s, D^s, \hat{\varphi})$ thus also satisfies (A.74) and must be an element of $\Gamma(H, X)$.

To prove lower hemi-continuity, I first show that, for any $X \in \mathbb{R}_+ \cup \{\infty\}$,the section $\Gamma(\cdot, X)$ of the correspondence Γ that is determined by X is lower hemi-continuous. For this purpose, I fix $(H, X) \in \mathcal{H}^y \times (\mathbb{R}_+ \cup \{\infty\})$ and consider the behaviour of $\Gamma(\hat{H}, X)$ for $\hat{H}$ close to H. Observe that $\Gamma(\hat{H}, X)$ depends on $\hat{H}$ only through the constraint (A.74). If $(L, A, B^s, D^s, \hat{\varphi}) \in \Gamma(H, X)$ satisfies $B^s + D^s = 0$, then, trivially, (A.74) implies $\hat{\varphi} \cdot L = 0$ and $(L, A, B^s, D^s, \hat{\varphi}) \in \Gamma(\hat{H}, X)$ for all $\hat{H}$. Alternatively, if $(L, A, B^s, D^s, \hat{\varphi}) \in \Gamma(H, X)$ satisfies $B^s > 0$, the effects of a small deviation of $\hat{H}$ from H on (A.74) can be neutralized by a small change in B^s, combined with an equal change of opposite sign in A while keeping L, D^s, and $\hat{\varphi}$ constant. If $(L, A, B^s, D^s, \hat{\varphi}) \in \Gamma(H, X)$ satisfies $D^s > 0$ and $\hat{\varphi} > 0$, the effects of a small deviation of $\hat{H}$ from H on (A.74) can be neutralized by small changes in L, A, and D^s, while keeping B^s and $\hat{\varphi}$ constant. If $(L, A, B^s, D^s, \hat{\varphi}) \in \Gamma(H, X)$ satisfies $D^s > 0$ and $\hat{\varphi} = 0$, the effects of a small deviation of $\hat{H}$ from H on (A.74) can be neutralized by a small change in D^s

and $\hat{\varphi}$, possibly in combination with small increases in L and A. In all cases, $\Gamma(\hat{H}, X)$ contains a vector that is close to $(L, A, B^s, D^s, \hat{\varphi}) \in \Gamma(\hat{H}, X)$ if $\hat{H}$ is close to H.

I next show that, for any $H \in \mathcal{H}^y$, the section $\Gamma(H, \cdot)$ of Γ that is determined by H is also lower hemi-continuous. If $X = 0$, then, trivially $(L, A, B^s, D^s, \hat{\varphi}) \in \Gamma(H, X)$ implies $(L, A, B^s, D^s, \hat{\varphi}) = (0, 0, 0, 0, 0)$ and hence $(L, A, B^s, D^s, \hat{\varphi}) \in \Gamma(H, \hat{X})$ for all $\hat{X} \in \mathbb{R}_+$. If $X > 0$, then, by inspection of (A.73), (A.74), and (A.79), $(L, A, B^s, D^s, \hat{\varphi}) \in \Gamma(H, X)$ implies that, for any $\hat{X} \in \mathbb{R}_+$, $(\frac{\hat{X}}{X}L, \frac{\hat{X}}{X}A, \frac{\hat{X}}{X}B^s, \frac{\hat{X}}{X}D^s, \hat{\varphi}) \in \Gamma(H, \hat{X})$. In either case, it follows that, if $\hat{X}$ is close to X, then $\Gamma(H, \hat{X})$ contains a vector that is close to $(L, A, B^s, D^s, \hat{\varphi}) \in \Gamma(H, X)$.

From these two findings, I conclude that, if $(\hat{H}, \hat{X})$ is close to (H, X), then, for any $(L, A, B^s, D^s, \hat{\varphi}) \in \Gamma(H, X)$, $\Gamma(\hat{H}, X)$ contains a vector that is close to $(L, A, B^s, D^s, \hat{\varphi})$ and $\Gamma(\hat{H}, \hat{X})$ contains a vector that is close to the latter element of $\Gamma(\hat{H}, X)$. Hence, Γ is lower hemi-continuous.

Step 3: Given Steps 1 and 2, the lemma follows from the maximum theorem (see, e.g., Hildenbrand, 1974, pp. 29 f.) ■

The following lemma contains the basic fixed-point argument. The argument is more intricate than usual because it involves two different spaces $\mathcal{H}^X \subset \mathcal{M}([0, X]^4 \times [0, \varphi_2])$ and $\mathcal{H}^y \subset \mathcal{M}([0, y]^4 \times [0, \varphi_2])$, where $X > y > e_0$. Of these two spaces, $\mathcal{H}^X$ is the one to which the actual fixed-point argument will be applied (with a suitable compactification). Heuristically, $\mathcal{H}^X$ should be thought of as a space of measures that can result from the banks' optimizing against the price system they expect to prevail. The parameter X corresponds to the bound in Lemma A.4. In the equilibrium condition (E.1) of course, there is no such bound. Subsequently, I will therefore eliminate this bound by letting X become large and showing that whatever limits I obtain will be among the solutions to the banks' optimization problem in the limit.

As X becomes large, however, there may be a discontinuity in the banks' optimization problem. This problem itself depends on the distribution H as this distribution affects expectations and the price system. As indicated by equations (A.68) - (A.71), the effects of H on the price system depend on the expected values $\int L dH$, $\int A dH$, etc. Unless the vectors $(L, A, B^s, D^s, \hat{\varphi})$ in the supports of the distributions under consideration are uniformly bounded, these expected values may exhibit a discontinuity in the limit as X goes out of bounds. To avoid this discontinuity, I introduce a second bound, y and a second space $\mathcal{H}^y$ of measures so that, for any vector $(L, A, B^s, D^s, \hat{\varphi})$ in the support of a measure in $\mathcal{H}^y$, $L \in [0, y]$, $A \in [0, y]$, etc. Heuristically, $\mathcal{H}^y$ should be thought of as a space of measures that induce the price systems against which banks optimize. As X goes out of bounds, y is kept fixed, so that the continuity and closed-graph properties in Lemmas A.3 and A.4 can be used.

As I introduce the distinction between the space $\mathcal{H}^y$ of measures that induce the price systems against which banks optimize and the space $\mathcal{H}^X$ of measures that can result from the banks' optimization, I must make sure that the elements of $\mathcal{H}^y$ that I focus on are compatible with the banks' optimization. This is the point of the following lemma.

Lemma A.5 *Fix $y \in (e_0, \infty)$ and $X \in (y, \infty)$. There exists a measure $H^*_X \in \mathcal{H}^y$ such that $H^*_X(\Psi(H^*_X, X)) = 1$, where, as in Lemma A.4, for any $(H, X) \in \mathcal{H}^y \times (\mathbb{R}_+ \cup \{\infty\})$, $\Psi(H, X)$ is the set of vectors $(L, A, B^s, D^s, \hat{\varphi}) \in \mathbb{R}^5_+$ that maximize the objective (A.72) under the constraints (A.73), (A.74), and (A.79).*

Proof. Fix $X \in \mathbb{R}_+$, and let $\bar{\mathcal{H}}^X$ be the closure of the set $\mathcal{H}^X \subset \mathcal{M}([0, X]^4 \times [0, \varphi_2])$. The definition of Ψ implies that, for any $H \in \mathcal{H}^y$, the set $\Psi(H, X)$ is a subset of $[0, X]^4 \times [0, \varphi_2]$ and, therefore, $\mathcal{M}(\Psi(H, X)) \subset \bar{\mathcal{H}}^X$. By Lemma A.4, it follows that, for any continuous function $\chi : \bar{\mathcal{H}}^X \to \mathcal{H}^y$, the mapping $H \to \mathcal{M}(\Psi(\chi(H), X))$ is an upper-hemi-continuous, compact- and convex-valued correspondence from $\bar{\mathcal{H}}^X$ into itself. Since, obviously, $\bar{\mathcal{H}}^X$ is a compact convex subset of $\mathcal{M}[0, X]^4 \times [0, \varphi_2])$, the fixed-point theorem of Glicksberg (1952) and Ky Fan (1952) implies the existence of a measure $H^X_\chi \in \mathcal{H}^X$ such that $H^X_\chi \in \mathcal{M}(\Psi(\chi(H), X))$.

The function mapping $\bar{\mathcal{H}}^X$ into $\mathcal{H}^y$ will be the composition of three distinct functions, $r(\cdot|\rho)$, $s(\cdot)$, and $t(\cdot|\tau)$, where $\rho \in (0, 1)$ and $\tau \in (0, e_0)$ are two parameters that I introduce in order to deal with difficulties arising from the non-compactness of $\mathcal{H}^y$ and from the fact that $\mathcal{H}^y$ is larger than the domain of the functions defined by (A.68) - (A.71). .

For any $\rho \in [0, 1)$ and any $H \in \bar{\mathcal{H}}^X$, I define

$$r(H|\rho) = \rho\delta_{(e_0 - K(e_0), 0, 0, e_0, \varphi_1)} + (1 - \rho)H. \tag{A.80}$$

For $\rho \in (0, 1)$, the map $r(\cdot|\rho)$ so defined takes $\bar{\mathcal{H}}^X$ into the set of measures that assign positive mass to the interior of the set $[0, y]^4 \times [0, \varphi_2]$.

Next, using the fact that $y > e_0$ let g be a continuous function from $\mathbb{R}^5_+$ into $[0, 1]$ that takes the value one if $(L, A, B^s, D^s, \hat{\varphi}) \in [0, e_0]^4 \times [0, \varphi_1]$ and the value zero if $(L, A, B^s, D^s, \hat{\varphi}) \notin [0, y]^4 \times [0, \varphi_2]$. For any measure $H \in \mathcal{M}(\mathbb{R}^5_+)$ that assigns positive mass to the interior of the set $[0, y]^4 \times [0, \varphi_2]$, define a new measure $s(H)$ by setting

$$s(B|H) = \frac{\int_B g dH}{\int_{\mathbb{R}^5_+} g dH}. \tag{A.81}$$

Notice that, for any H, the support of $s(H)$ is a subset of the support of H. The mapping s takes values in the set of measures satisfying (A.75) as well as

$$0 < \int (L + K(D^s))dH \leq \int (A + B^s + D^s)dH < y. \tag{A.82}$$

Finally, for $\tau \in (0, e_0)$, I define a function $t(\cdot|\tau)$ from the set of measures satisfying (A.75) and (A.82) to $\mathcal{H}^y$ by setting

$$\gamma(H|\tau) := \min\left(1, \frac{e_0 - \tau}{\int (A + B^s + D^s)dH}\right), \tag{A.83}$$

$$\hat{g}(L, A, B^s, D^s, \hat{\varphi}|H, \tau) = (\gamma(H|\tau)(L, A, B^s, D^s), \hat{\varphi}), \tag{A.84}$$

and

$$t(H|\tau) = H \circ \hat{g}^{-1}(\cdot|H, \tau). \tag{A.85}$$

I claim that, for any H and τ, $t(H|\tau) \in \mathcal{H}^y$. To prove this claim, I note that, for any H and τ,

$$\int L dt(H|\tau) = \gamma(H|\tau) \int L dH, \tag{A.86}$$

$$\int K(D^s) dt(H|\tau) = \int K(\gamma(H|\tau) D^s) dH|, \tag{A.87}$$

and

$$\int (A + B^s + D^s) dt(H|\tau) = \gamma(H|\tau) \int (A + B^s + D^s) dH. \tag{A.88}$$

Moreover, since $\gamma(H|\tau) \leq 1$ and the function K is convex,

$$\int K(\gamma(H|\tau) D^s) dH \leq \gamma(H|\tau) \int K(D^s) dH, \tag{A.89}$$

Finally, $\int L dH > 0$ implies $\int L dt(H|\tau) > 0$, and $\int K(D^s) dH > 0$ implies $\int D^s dH > 0$ and $\int \gamma(H|\tau)\ D^s dH > 0$, hence $\int K(\gamma(H|\tau) D^s) dH > 0$. Upon collecting these results and using (A.83), one finds that, if the measure H satisfies (A.82), then the measure $t(H|\tau)$ also satisfies (A.82). The conclusion that $t(H|\tau)$ belongs to $\mathcal{H}^y$ follows immediately.

The functions $(H, \rho) \to r(H|\rho)$, $H \to s(H)$, and $(H, \tau) \to t(H|\tau)$ are obviously continuous. For any $\rho \in (0, 1)$ and $\tau \in (0, e_0)$ therefore, the composition

$$\chi_{\rho\tau} := t(\cdot|\tau) \circ s \circ r(\cdot|\rho) \tag{A.90}$$

is a continuous function from $\bar{\mathcal{H}}^X$ to $\mathcal{H}^y$. As discussed above, it follows that, for any $\rho \in (0, 1)$, $\tau \in (0, e_0)$ and $X > e_0$, the correspondence $H \to \mathcal{M}(\Psi(t(s(r(H|\rho))|\tau), X))$ has a fixed point, i.e., there exists a measure $H^X_{\rho\tau} \in \bar{\mathcal{H}}^X$ such that

$$H^X_{\rho\tau}(\Psi(t(s(r(H^X_{\rho\tau}|\rho))|\tau), X)) = 1. \tag{A.91}$$

Equivalently,

$$H^X_{\rho\tau}(\Psi(H^{*X}_{\rho\tau}, X)) = 1, \tag{A.92}$$

where $H^{*X}_{\rho\tau} \in \mathcal{H}^y$ is given as

$$H^{*X}_{\rho\tau} := t(s(r(H^X_{\rho\tau}|\rho))|\tau). \tag{A.93}$$

I will show that, as ρ and τ converge to zero, the pairs $(H^X_{\rho\tau}, H^{*X}_{\rho\tau})$ have a limit $(H^X_{\rho\tau}, H^{*X}_{\rho\tau}) \in \bar{\mathcal{H}}^X \times \mathcal{H}^y$, and that this limit satisfies

$$H^X(\Psi(H^{*X}, X)) = 1 \quad \text{and} \quad H^{*X} := s(H^X). \tag{A.94}$$

In the process, I have to deal with the difficulty that, because of the strict inequalities in (A.76), the set $\mathcal{H}^y$ is not compact. To deal with this difficulty, for any $\varepsilon > 0$, I introduce the set $\mathcal{H}^y_\varepsilon$of measures $H \in \mathcal{H}^y$ such that

$$\varepsilon \leq \int (L + K(D^s)) dH \leq \int (A + B^s + D^s) dH \leq e_0 - \varepsilon. \tag{A.95}$$

Claim: There exists $\bar{\varepsilon} > 0$ such that, for any $\rho \in (0,1)$, $\tau \in (0,\bar{\varepsilon})$, the measure $H^{*X}_{\rho\tau}$ that is defined by (A.93) and the measure $H^X_{\rho\tau}$ is an element of the subset $\mathcal{H}^y_{\bar{\varepsilon}}$ of $\mathcal{H}^y$.

To proves this claim, I proceed in several steps.

Step 1: For any $\rho \in (0,1)$, $\tau \in (0,e_0)$ and $X > y$, the measure $H^{*X}_{\rho\tau}$ that is defined by $H^X_{\rho\tau}$ and (A.93) satisfies

$$\int L dH^{*X}_{\rho\tau} \geq \Delta \int (A + B^s + D^s) dH^{*X}_{\rho\tau}, \tag{A.96}$$

where $\Delta > 0$ is the parameter given by the assumption that $K'(D^s) \leq 1 - \Delta$ for all D^s.

To prove this claim, I note, that, for any vector $(L, A, B^s, D^s, \hat{\varphi}) \in \Psi(H^{*X}_{\rho\tau}, X)$,

$$L + K(D^s) = A + B^s + D^s. \tag{A.97}$$

By (A.91), therefore, (A.97) holds $H^X_{\rho\tau}$-almost surely. By (A.80) and (A.81), it follows that, for any ρ, (A.97) also holds $r(H^X_{\rho\tau}|\rho)$-almost surely and $s(r(H^X_{\rho\tau}|\rho))$-almost surely. Thus,

$$\int (L + K(D^s)) ds(r(H^X_{\rho\tau}|\rho)) = \int (A + B^s + D^s) ds(r(H^X_{\rho\tau}|\rho)).$$

Because $K(\cdot)$ is convex and $K(0) = 0$, it follows that

$$\int L ds(r(H^X_{\rho\tau}|\rho)) \geq \int (A + B^s + D^s) ds(r(H^X_{\rho\tau}|\rho)) - \int D^s K'(D^s) ds(r(H^X_{\rho\tau}|\rho)).$$

By (A.81) - (A.83) and (A.93) therefore,

$$\int L dH^{*X}_{\rho\tau} \geq \int (A + B^s + D^s) dH^{*X}_{\rho\tau} - \int D^s K'(D^s) dH^{*X}_{\rho\tau}.$$

Now (A.96) follows because $0 \leq K'(D^s) \leq 1 - \Delta$ for all D^s.

Step 2: Let $\varepsilon_1 > 0$ be such that

$$u'(\varepsilon_1) > \frac{v(\varphi_2 \Delta(e_0 - \varepsilon_1))}{\Delta(e_0 - \varepsilon_1)} \ \bar{f} \ \left[\int_{\varphi_1}^{\varphi_2} \varphi f(\varphi) d\varphi + \theta'(0) \right], \tag{A.98}$$

where $\bar{f}$ is again the maximum of $f(\varphi)$ on the interval $[\varphi_1, \varphi_2]$. Then, for any $\rho \in (0, 1 - \varepsilon_1/e_0)$ and any $\tau \in (0, e_0)$,

$$\int (A + B^s + D^s) dH \leq e_0 - \varepsilon_1. \tag{A.99}$$

If this claim is false, there exist $\rho \in (0, 1 - \varepsilon_1/e_0)$ and $\tau \in (0, e_0)$ such that

$$\int (A + B^s + D^s) dH^{*X}_{\rho\tau} > e_0 - \varepsilon_1. \tag{A.100}$$

By (A.83) - (A.85), (A.81), and (A.80), it follows that

$$\begin{aligned}\int (A+B^s+D^s)dH^{*X}_{\rho\tau} &\leq \int (A+B^s+D^s)ds(r(H^X_{\rho\tau}|\rho)) \\ &\leq \int (A+B^s+D^s)dr(H^X_{\rho\tau}|\rho) \\ &= \rho e_0 + (1-\rho)\int (A+B^s+D^s)dH^X_{\rho\tau},\end{aligned}$$

so (A.100) and $\rho \in (0, 1-\varepsilon_1/e_0)$ imply

$$\int (A+B^s+D^s)dH^X_{\rho\tau} > 0. \tag{A.101}$$

I will show that (A.101) is incompatible with the requirement that $H^X_{\rho\tau}(\Psi(H^{*X}_{\rho\tau}, X)) = 1$. The idea is that, if (A.100) holds, date 0 consumption is so small that the returns banks must offer to consumers are so high that profit maximization requires banks to be inactive.

Consider the problem of maximizing the objective (A.72) under the constraints (A.73), (A.74), and (A.79). For $H = H^{*X}_{\rho\tau}$, the objective (A.72) takes the form

$$\eta(H^{*X}_{\rho\tau})\ \mathcal{E}_{q(H^{*X}_{\rho\tau})}\tilde{\varphi}L - (A+B^s+D^s) + \eta(H^{*X}_{\rho\tau})\ D^s\ \mathcal{E}_{q(H^{*X}_{\rho\tau})}(1-\beta(\tilde{\varphi},\hat{\varphi}))\theta'\left(\bar{D}(\tilde{\varphi}|H^{*X}_{\rho\tau})\right). \tag{A.102}$$

The definition of $\eta(H^{*X}_{\rho\tau})$ and $\mathcal{E}_{q(H^{*X}_{\rho\tau})}\tilde{\varphi}$ yields

$$\eta(H^{*X}_{\rho\tau}) = \frac{\int_{\varphi_1}^{\varphi_2} v'(\varphi \int L dH^{*X}_{\rho\tau} + \theta(\bar{D}(\varphi|H^{*X}_{\rho\tau}))\ f(\varphi)d\varphi}{u'(e_0 - \int (A+B^s+D^s)dH^{*X}_{\rho\tau})}$$

and

$$\eta(H^{*X}_{\rho\tau})\mathcal{E}_{q(H^{*X}_{\rho\tau})}\tilde{\varphi} = \frac{\int_{\varphi_1}^{\varphi_2} \varphi\ v'(\varphi \int L dH^{*X}_{\rho\tau} + \theta(\bar{D}(\varphi|H^{*X}_{\rho\tau}))\ f(\varphi)d\varphi}{u'(e_0 - \int (A+B^s+D^s)dH^{*X}_{\rho\tau})}.$$

By (A.96), (A.100) and the concavity of $v(\cdot)$ and $u(\cdot)$, it follows that

$$\begin{aligned}\eta(H^{*X}_{\rho\tau}) &\leq \frac{\int_{\varphi_1}^{\varphi_2} v'(\varphi\Delta(e_0-\varepsilon_1))\ f(\varphi)d\varphi}{u'(\in_1)} \\ &\leq \frac{v(\varphi_2\Delta(e_0-\varepsilon_1))}{\Delta(e_0-\varepsilon_1)u'(\in_1)}\ \bar{f}\end{aligned}$$

and

$$\begin{aligned}\eta(H^{*X}_{\rho\tau})\mathcal{E}_{q(H^{*X}_{\rho\tau})}\tilde{\varphi} &\leq \frac{\int_{\varphi_1}^{\varphi_2} \varphi\ v'(\varphi\Delta(e_0-\varepsilon_1))\ f(\varphi)d\varphi}{u'(\varepsilon_1)} \\ &\leq \frac{\int_{\varphi_1}^{\varphi_2} v'(\varphi\Delta(e_0-\varepsilon_1))\ f(\varphi)d\varphi}{u'(\varepsilon_1)} \int_{\varphi_1}^{\varphi_2} \varphi f(\varphi)d\varphi \\ &\leq \frac{v(\varphi_2\Delta(e_0-\varepsilon_1))}{\Delta(e_0-\varepsilon_1)u'(\in_1)}\ \bar{f} \int_{\varphi_1}^{\varphi_2} \varphi f(\varphi)d\varphi.\end{aligned}$$

The objective (A.102) is thus bounded above by

$$
\begin{aligned}
& (\eta(H^{*X}_{\rho\tau})\ [\mathcal{E}_{q(H^{*X}_{\rho\tau})}\tilde{\varphi} + \theta'(0)] - 1)(A + B^s + D^s) \\
\leq \quad & \cdot \left(\frac{v(\varphi_2 \Delta(e_0 - \varepsilon_1))}{\Delta(e_0 - \varepsilon_1) u'(\in_1)} \left[\int_{\varphi_1}^{\varphi_2} \varphi f(\varphi) d\varphi + \theta'(0) \right] - 1 \right) (A + B^s + D^s).
\end{aligned}
\tag{A.103}
$$

By the definition of ε_1, the first factor in the product on the right-hand side of (A.103) is strictly negative. The right-hand side of (A.103), as well as the objective (A.102) is therefore maximized by setting $A + B^s + D^s = 0$. More precisely, for $H^{*X}_{\rho\tau}$ satisfying (A.98), $\Psi(H^{*X}_{\rho\tau}, X)$ consists of the set of vectors $(L, A, B^s, D^s, \hat{\varphi})$ that satisfy $L = A = B^s = D^s = 0$. Since $H^X_{\rho\tau}(\Psi(H^{*X}_{\rho\tau}, X)) = 1$, it follows that $\int (A + B^s + D^s) dH^X_{\rho\tau} = 0$, which contradicts (A.101). Step 2 is thereby completed.

Step 3: Let $\varepsilon_2 > 0$ be such that

$$
u'(e_0 - \varepsilon_2/\Delta) < v'(\varphi_2(\varepsilon_2 + \theta(\varepsilon_2/\Delta)) \cdot \int_{\varphi_1}^{\varphi_2} \varphi f(\varphi) d\varphi. \tag{A.104}
$$

Then, for any $\rho \in (0, 1)$, $\tau \in (0, e_0 - \varepsilon_2/\Delta)$,

$$
\varepsilon_2 \leq \int L dH^{*X}_{\rho\tau}. \tag{A.105}
$$

If this claim is false, there exist $\rho \in (0, 1)$ and $\tau \in (0, e_0 - \varepsilon_2/\Delta)$ such that

$$
\int L dH^{*X}_{\rho\tau} < \varepsilon_2. \tag{A.106}
$$

By (A.96), it follows that

$$
\int (A + B^s + D^s) dH^{*X}_{\rho\tau} < \frac{\varepsilon_2}{\Delta} \tag{A.107}
$$

and therefore,

$$
\bar{D}(\varphi | H^{*X}_{\rho\tau}) \leq \int D^s dH^{*X}_{\rho\tau} \leq \frac{\varepsilon_2}{\Delta} \Delta. \tag{A.108}
$$

Using a similar logic as in Step 2, I will show that (A.106), (A.107), and (A.108) are incompatible with the requirement that $H^X_{\rho\tau}(\Psi(H^{*X}_{\rho\tau}, X)) = 1$. The idea is that, if (A.106), (A.107), and (A.108) hold, then the returns that banks must offer to consumer are so low that profit maximization requires banks to raise lots of funds, so that the bound X is binding.

By (A.96), (A.106), (A.107) and the concavity of $v(\cdot)$ and $u(\cdot)$,

$$\begin{aligned}\eta(H^{*X}_{\rho\tau})\mathcal{E}_{q(H^{*X}_{\rho\tau})}\tilde{\varphi} &= \frac{\int_{\varphi_1}^{\varphi_2}\varphi v'(\varphi\int LdH^{*X}_{\rho\tau}+\theta(\bar{D}(\varphi|H^{*X}_{\rho\tau}))\ f(\varphi)d\varphi}{u'(e_0-\int(A+B^s+D^s)dH^{*X}_{\rho\tau})} \\ &> \frac{\int_{\varphi_1}^{\varphi_2}\varphi v'(\varphi\varepsilon_2+\theta(\varepsilon_2/\Delta))\ f(\varphi)d\varphi}{u'(e_0-\varepsilon_2/\Delta)} \\ &> \frac{v'(\varphi_2\varepsilon_2+\theta(\varepsilon_2/\Delta))\cdot\int_{\varphi_1}^{\varphi_2}\varphi\ f(\varphi)d\varphi}{u'(e_0-\varepsilon_2/\Delta)}.\end{aligned}$$

By (A.104) therefore, $\eta(H^{*X}_{\rho\tau})\mathcal{E}_{q(H^{*X}_{\rho\tau})}\tilde{\varphi} > 1$. By inspection of the objective (A.72) and the constraints (A.73) and (A.74), it follows that any element of $\Psi(H^{*X}_{\rho\tau}, X)$ must satisfy $L = X$ or $A = X$. Since $X > y$, therefore, $\Psi(H^{*X}_{\rho\tau}, X)\cap[0, y]^4 \times [0, \hat{\varphi}] = \emptyset$. By (A.92), (A.80) and (A.81), it follows that

$$s(r(H^X_{\rho\tau}|\rho)) = \delta_{(e_0-K(e_0),0,0,e_0)}$$

and, hence, that

$$\int(A+B^s+D^s)ds(r(H^X_{\rho\tau}|\rho)) = e_0.$$

By (A.83) - (A.85), therefore,

$$\int(A+B^s+D^s)dH^{*X}_{\rho\tau} = e_0-\tau,$$

which is incompatible with (A.107) if $e_0 - \tau > \varepsilon_2/\Delta$. The assumption that $\int LdH^{*X}_{\rho\tau} < \varepsilon_2$ for some $\rho \in (0, 1)$ and $\tau \in (0, e_0 - \varepsilon_2/\Delta)$ has thus led to a contradiction and must be false. This completes step 3.

Upon setting $\bar{\varepsilon} = \min(\varepsilon_1, \varepsilon_2)$, one finds that, for any $\rho \in (0, \bar{\varepsilon})$, $\tau \in (0, e_0-\bar{\varepsilon})$, the measure $H^{*X}_{\rho\tau}$ that is defined by (A.93) and $H^X_{\rho\tau}$ satisfying (A.91) is an element of the subset $\mathcal{H}^y_{\bar{\varepsilon}}$ of $\mathcal{H}^y$.

To complete the proof of Lemma A.5, let $\{\rho_k\}_{k=1}^{\infty}$ and $\{\tau_k\}_{k=1}^{\infty}$ be two null sequences and consider the associated sequence $\{(H^X_{\rho_k\tau_k}, H^{*X}_{\rho_k\tau_k})\}$ in $\bar{\mathcal{H}}^X \times \mathcal{H}^y$. By the preceding claim, the sequence $\{(H^X_{\rho_k\tau_k}, H^{*X}_{\rho_k\tau_k})\}$ may be taken to lie in the set $\bar{\mathcal{H}}^X \times \mathcal{H}^y_{\bar{\varepsilon}}$, which is compact. Therefore this sequence has a limit point $(H_X, H^*_X) \in \bar{\mathcal{H}}^X \times \mathcal{H}^y_{\bar{\varepsilon}}$. By Lemma A.4 and (A.91),

$$H_X(\Psi(H^*_X, X)) = 1,$$

i.e. the support of H_X is a subset of $\Psi(H^*_X, X)$.

Moreover, by the continuity of the functions $(H, \rho) \to r(H|\rho)$, $H \to s(H)$, and $(H, \tau) \to t(H|\tau)$, the measures H_X and H^*_X satisfy the equation

$$H^*_X = s(H_X).$$

By (A.81),therefore, every point in the support of H^*_X is also contained in the support of H_X and is thus contained in the set $\Psi(H^*_X, X)$. Thus,

$$H^*_X(\Psi(H^*_X, X)) = 1. \tag{A.109}$$

The proof of Lemma A.5 is complete. ■

Proof of Proposition 3.4. Fix some $y > e_0$ and let $\{X_k\}_{k=1}^{\infty}$ be any sequence in $\mathbb{R}_+$ that goes out of bounds. Consider the associated sequence $\{H^*_{X_k}\}_{k=1}^{\infty}$ of measures in $\mathcal{H}^y$ that is given by Lemma A.5. By the argument given in the proof of Lemma A.5, the measures $H^*_{X_k}$ all belong to the compact subset $\mathcal{H}^y_{\bar{\varepsilon}}$ of $\mathcal{H}^y$ that satisfy (A.95) for $\varepsilon = \bar{\varepsilon}$: with $\bar{\varepsilon} = \min(\varepsilon_1, \varepsilon_2)$ given by (A.98) and (A.104), which is independent of X. Therefore the sequence $\{H^*_{X_k}\}_{k=1}^{\infty}$ has a convergent subsequence. By Lemmas A.5 and A.4, any limit H^* of the sequence $\{H^*_{X_k}\}_{k=1}^{\infty}$ satisfies

$$H^*(\Psi(H^*, \infty)) = 1.$$

The measure H^* thus satisfies the premises of Lemma A.2.

As mentioned above, standard arguments, as for example in Hart, Hildenbrand, and Kohlberg (1974), pp. 164 ff., also imply the existence of a mapping $b \to (L(b), A(b), B^s(b), D^s(b), \hat{\varphi}(b))$ from $[0,1]$ to $\mathbb{R}^5_+$ such that the distribution of the vector $(L(b), A(b), B^s(b), D^s(b), \hat{\varphi}(b))$ that is induced by the uniform distribution on $[0,1]$ is equal to H^*. The proposition therefore follows from Lemmas A.1 and A.2. ■

A.4 Proof of Propositions 3.5 and 3.6

As discussed in the text, the banks' maximization problem in condition (E.1) is equivalent to the problem of $A = \alpha^s E, B^s, D^s, \hat{\varphi}$ so as to maximize the objective

$$\eta\, \mathcal{E}_q \tilde{\varphi}\, (A + B^s + D^s - K(D^s)) - B^s - D^s + \eta\, \mathcal{E}_q (1 - \beta(\tilde{\varphi}, \hat{\varphi}))\theta'(\bar{D}^s(\tilde{\varphi}))\, D^s - A \tag{A.110}$$

under the constraint

$$\eta\, \mathcal{E}_q \min(\tilde{\varphi}, \hat{\varphi})\, (A + B^s + D^s - K(D^s)) = B^s + D^s - \eta\, \mathcal{E}_q (1 - \beta(\tilde{\varphi}, \hat{\varphi}))\theta'(\bar{D}^s(\tilde{\varphi}))\, D^s. \tag{A.111}$$

Proof of Proposition 3.5. The proof proceeds through a sequence of claims. The ordering of these claims is not the same as the ordering of the corresponding statements in the proposition but instead follows the internal logic of the argument.

Claim 1: $\eta\, \mathcal{E}_q \tilde{\varphi} \leq 1$.

To see this, it suffices to note that, if $\eta\, \mathcal{E}_q \tilde{\varphi} > 1$, then, by setting $B^s = D^s = \hat{\varphi} = 0$ and making A arbitrarily large, one can make the value of the objective function arbitrarily large. In this case, the bank's maximization problem has no solution.

Claim 2: If $\eta\, \mathcal{E}_q \tilde{\varphi} = 1$, then $\hat{\varphi}(b) \leq \varphi_1$ for almost all b with $D^s(b) > 0$; moreover, $\bar{D}(\varphi) = \int_0^1 D^s(b)db = D^*(\mathcal{E}_q \tilde{\varphi})$, regardless of φ.

If $\eta\, \mathcal{E}_q \tilde{\varphi} = 1$. the objective (A.110) takes the form

$$-\eta\, \mathcal{E}_q \tilde{\varphi}\, K(D^s) + \eta\, \mathcal{E}_q (1 - \beta(\tilde{\varphi}, \hat{\varphi}))\theta'(\bar{D}^s(\tilde{\varphi}))\, D^s, \tag{A.112}$$

which depends only on D^s and $\hat{\varphi}$. Suppose that the bank chooses a plan with $D^s > 0$. If $\hat{\varphi} > \varphi_1$, the value of the term

$$\eta\, \mathcal{E}_q(1-\beta(\tilde{\varphi},\hat{\varphi}))\theta'(\bar{D}^s(\tilde{\varphi}))\; D^s = \eta \int_{\hat{\varphi}}^{\varphi_2} \theta'(\bar{D}^s(\varphi))q(\varphi)f(\varphi)d\varphi \cdot D^s$$

in the objective function (A.112) can be increased by lowering $\hat{\varphi}$. Such a decrease is compatible with the constraint (A.111) if at the same time A is increased by the requisite amount.

Thus if $\eta\, \mathcal{E}_q\tilde{\varphi} = 1$, then for any bank b, $D^s(b) > 0$ implies $\hat{\varphi}(b) \leq \varphi_1$.hence $\beta(\tilde{\varphi},\hat{\varphi}(b)) = 0$ and

$$(1.\beta(\tilde{\varphi},\hat{\varphi}(b)))D^s(b) = D^s(b) \tag{A.113}$$

almost surely. Trivially, equation (A.113) also holds if $D^s(b) = 0$. Therefore $\eta\, \mathcal{E}_q\tilde{\varphi} = 1$ implies that

$$\bar{D}^s(\tilde{\varphi}) = \bar{D}^s := \int_0^1 D^s(b)db \tag{A.114}$$

almost surely. (A.112) thus takes the form

$$-\eta\, \mathcal{E}_q\tilde{\varphi}\; K(D^s) + \eta\, \theta'(\bar{D}^s)\; D^s. \tag{A.115}$$

Maximization of (A.115) with respect to D^s, using suitable adjustments in A to keep the default point below φ_1, requires that

$$\mathcal{E}_q\tilde{\varphi}\; K'(D^s) = \theta'(\bar{D}^s). \tag{A.116}$$

This first-order condition must be satisfied for $D^s = D^s(b)$, for almost all banks b. Because the cost function K is convex, it follows that (A.116) must also hold for $\bar{D}^s = \int_0^1 D^s(b)db$. Thus, $\bar{D}^s = D^*(\mathcal{E}_q\tilde{\varphi})$.

If $K'(\bar{D}^s) > K'(0)$, the maximum value of the objective (A.112) is strictly positive. In this case, $D^s(b) > 0$ and hence, by the argument given before, $\hat{\varphi}(b) \leq \varphi_1$ for almost all b. If $K'(\bar{D}^s) = K'(0)$, the maximum value of (A.112) is zero, and I cannot rule out the possibility that a bank with zero deposit supply chooses a mix of equity and bonds that induces a positive probability of default.

Claim 3: If $\eta\, \mathcal{E}_q\tilde{\varphi} = 1$, any plan with a deposit level D^s that satisfies (A.116) and a default point $\hat{\varphi} \leq \varphi_1$ is optimal for the bank. In particular, as long as the default probability is zero, the bank is indifferent about the mix of shares and bonds that it issues.

This claim follows immediately from the observation that, if $\eta\, \mathcal{E}_q\tilde{\varphi} = 1$, the value of the bank's objective function is independent of the amounts of shares and bonds that it issues.

Claim 4: If e_0 is sufficiently small or if $\varphi_1 = 0$, then $\eta\, \mathcal{E}_q\tilde{\varphi} < 1$.

If $e_0 < D^*(\bar{\varphi})$, the aggregate demand for deposits is less than $D^*(\bar{\varphi})$. In equilibrium, therefore, the aggregate supply of deposits is also less than $D^*(\bar{\varphi})$. As discussed in the text, the concavity of $v(\cdot)$ implies that $\mathcal{E}_q\tilde{\varphi} \leq \bar{\varphi}$. By the definition of the function $D^*(\cdot)$, therefore $D^*(\bar{\varphi}) \leq D^*(\mathcal{E}_q\tilde{\varphi})$. For $e_0 < D^*(\bar{\varphi})$, therefore, the aggregate supply of deposits is less than $D^*(\mathcal{E}_q\tilde{\varphi})$. Because, by Step 2, $\eta\ \mathcal{E}_q\tilde{\varphi} = 1$ implies $\int_0^1 D^s(b)db = D^*(\mathcal{E}_q\tilde{\varphi})$, it follows that $e_0 < D^*(\bar{\varphi})$ implies $\eta\ \mathcal{E}_q\tilde{\varphi} < 1$.

Alternatively, if $\varphi_1 = 0$, then, by Claim 2, $\eta\ \mathcal{E}_q\tilde{\varphi} = 1$ implies $\hat{\varphi}(b) = 0$ and, by $(A.111)$, $D^s(b) = 0$ for almost all b. However, by Claim 2, $\eta\ \mathcal{E}_q\tilde{\varphi} = 1$ also implies $\int_0^1 D^s(b)db = D^*(\mathcal{E}_q\tilde{\varphi})$. The assumption that there is an equilibrium with $\eta\ \mathcal{E}_q\tilde{\varphi} = 1$ when $\varphi_1 = 0$ thus leads to a contradiction and must be false.

Claim 5: Let $\varphi_1 > 0$ and $D^*(\varphi_1) < \infty$. If e_0 is sufficiently large, there exists an equilibrium with $\eta\ \mathcal{E}_q\tilde{\varphi} = 1$.

Using the fact that $\varphi_1 > 0$ and $D^*(\varphi_1) < \infty$, let e_0 be large enough so that

$$\int_{\varphi_1}^{\varphi_2} v'(\varphi(D\frac{1}{\varphi_1} + D - K(D)) + \theta(D))\varphi f(\varphi)d\varphi < u'(e_0 - D\frac{1}{\varphi_1} - D) \quad \text{(A.117)}$$

for all $D \in [0, D^*(\varphi_1)]$. Using the strict concavity of v and u and the boundary conditions on v' and u', one can define a continuous function $D \to A(D)$ such that, for any $D \in [0, D^*(\varphi_1)]$,

$$\int_{\varphi_1}^{\varphi_2} v'(\varphi(A(D) + D - K(D)) + \theta(D))\varphi f(\varphi)d\varphi = u'(e_0 - A(D) - D). \quad \text{(A.118)}$$

I next define a mapping $\rho \to \hat{\rho}(\rho)$ on $[\varphi_1, \varphi_2]$, by setting

$$\hat{\rho}(\rho) := \frac{\int_{\varphi_1}^{\varphi_2} v'(\varphi(A(D^*(\rho)) + D^*(\rho) - K(D^*(\rho))) + \theta(D^*(\rho)))\varphi f(\varphi)d\varphi}{\int_{\varphi_1}^{\varphi_2} v'(\varphi(A(D^*(\rho)) + D^*(\rho) - K(D^*(\rho))) + \theta(D^*(\rho)))f(\varphi)d\varphi}. \quad \text{(A.119)}$$

The function $\rho \to \hat{\rho}(\rho)$ maps the compact interval $[\varphi_1, \varphi_2]$ continuously into itself and has a fixed point ρ^*.

Given this fixed point, set

$$\eta = \frac{\int_{\varphi_1}^{\varphi_2} v'(\varphi(A(D^*(\rho^*)) + D^*(\rho^*) - K(D^*(\rho^*))) + \theta(D^*(\rho^*)))f(\varphi)d\varphi}{u'(e_0 - A(D^*(\rho^*)) - D^*(\rho^*))} \quad \text{(A.120)}$$

and

$$q(\varphi') = \frac{v'(\varphi'(A(D^*(\rho^*)) + D^*(\rho^*) - K(D^*(\rho^*))) + \theta(D^*(\rho^*)))}{\int_{\varphi_1}^{\varphi_2} v'(\varphi(A(D^*(\rho^*)) + D^*(\rho^*) - K(D^*(\rho^*))) + \theta(D^*(\rho^*)))f(\varphi)d\varphi}. \quad \text{(A.121)}$$

For any b let $A(b) = A(D^*(\rho^*))$, $B^s(b) = 0$, $D^s(b) = D^*(\rho^*)$. Then for $L(b) = A(D^*(\rho^*)) + D^*(\rho^*) - K(D^*(\rho^*))$, the bank's constraint for $t = 0$ is satisfied. Moreover, by (A.117) and (A.118),

$$A(D^*(\rho^*)) > D^*(\rho^*)\frac{1}{\varphi_1} \quad \text{(A.122)}$$

and therefore

$$\begin{aligned} \varphi_1(A(D^*(\rho^*)) + D^*(\rho^*) - K(D^*(\rho^*))) &> D^*(\rho^*) \quad \text{(A.123)} \\ &> D^*(\rho^*) - \eta D^*(\rho^*)\mathcal{E}_q\theta'(\bar{D}(\tilde{\varphi})), \end{aligned}$$

implying that the bank's constraint for $t = 1$ is satisfied with $\hat{\varphi}(b) < \varphi_1$. Upon setting

$$\bar{D}(\varphi) = D^*(\rho^*) \tag{A.124}$$

for all φ, one finds that, for the given η and q, the specified maps $b \to (L(b), A(b), B^s(b), D^s(b), \hat{\varphi}(b))$ and $\varphi \to \bar{D}(\varphi)$ satisfy all the conditions of Lemma A.1 so an equilibrium is obtained by the construction given in that lemma.

Claim 6: If $\eta\ \mathcal{E}_q\tilde{\varphi} < 1$, then $B^s(b) = 0$ for almost all b.

If $\eta\ \mathcal{E}_q\tilde{\varphi} < 1$ and $B^s(b) > 0$, a reduction in the bank's bond issue would raise the value of the objective both directly, through the impact of B^s on (A.110), and indirectly, through the impact of B^s on the constraint (A.111) and the implies value of the default point $\hat{\varphi}$, a lowering of which raises (A.110).

Claim 7: If $\eta\ \mathcal{E}_q\tilde{\varphi} < 1$, then $\int D^s(b)db > 0$.

For any bank b, $\eta\ \mathcal{E}_q\tilde{\varphi} < 1$ and $D^s(b) = 0$ implies that the objective (A.110) is maximized by setting $A(b) = B^s(b) = 0$. Hence $\int D^s(b)db = 0$ would imply $\int A(b)db = \int B^s(b)db = 0$. Upon using market clearing conditions as in the proof of Proposition 2.1, one then obtains $c_1(\varphi) = 0$ and $\theta(\bar{D}^d(\varphi))$ for all φ. By Lemmas 3.1 and 3.2, it follows that $\eta = \infty$, which is impossible.

Claim 8: If $\eta\ \mathcal{E}_q\tilde{\varphi} < 1$ and $D^s(b) > 0$, then

$$\mathcal{E}_q\tilde{\varphi}\ K'(D^s(b)) < \mathcal{E}_q(1 - \beta(\tilde{\varphi}, \hat{\varphi}(b)))\theta'(\bar{D}^s(\tilde{\varphi})) \tag{A.125}$$

and

$$\varphi_2 > \hat{\varphi}(b) \geq \varphi_1 \tag{A.126}$$

for almost all b; moreover, the latter inequality is strict if $\varphi_1 = 0$.

The necessary first-order conditions for the bank's maximization with respect to D^s and $\hat{\varphi}$ are given as:

$$\begin{gathered}\eta\ \mathcal{E}_q\tilde{\varphi}(1 - K'(D^s)) - 1 + \eta\ \mathcal{E}_q(1 - \beta(\tilde{\varphi}, \hat{\varphi}))\theta'(\bar{D}^s(\tilde{\varphi})) \\ +\lambda\eta\ \mathcal{E}_q \min(\tilde{\varphi}, \hat{\varphi})(1 - K'(D^s)) - \lambda \\ +\lambda\eta\ \mathcal{E}_q(1 - \beta(\tilde{\varphi}, \hat{\varphi}))\theta'(\bar{D}^s(\tilde{\varphi})) \leq 0, \\ \text{with a strict inequality implying that } D^s = 0,\end{gathered} \tag{A.127}$$

and

$$\begin{gathered}-\eta q(\hat{\varphi})f(\hat{\varphi})\theta'(\bar{D}^s(\hat{\varphi}))\ D^s(1 + \lambda) \\ +\lambda\eta\ \mathcal{E}_q(1 - \beta(\tilde{\varphi}, \hat{\varphi}))\ (A + B^s + D^s - K(D^s)) \leq 0, \\ \text{with a strict inequality implying that } \hat{\varphi} = 0.^{44}\end{gathered} \tag{A.128}$$

In these conditions, λ is the Lagrange multiplier for the constraint (A.111). I first claim that $D^s > 0$ implies $\hat{\varphi} < \varphi_2$. For suppose that $\hat{\varphi} \geq \varphi_2$. Then the term $\eta\ \mathcal{E}_q(1 - \beta(\tilde{\varphi}, \hat{\varphi}))\theta'(\bar{D}^s(\tilde{\varphi}))D^s$ in the objective function (A.110) and the constraint (A.111) is equal to zero. Since $\eta\ \mathcal{E}_q \min(\tilde{\varphi}, \hat{\varphi})(D^s - K(D^s)) < \eta\ \mathcal{E}_q\tilde{\varphi}D^s < D^s$, the constraint (A.111) can only be satisfied if $A > 0$. But then the value of (A.110) is strictly negative and can be increased by reducing both

A and D^s to zero. The assumption that $\hat{\varphi} \geq \varphi_2$ when $D^s > 0$ thus leads to a contradiction and must be false.

I next claim that $D^s > 0$ implies $1 + \lambda > 0$. For this purpose, I rewrite condition (A.127) in the form

$$\begin{gathered} \eta\, \mathcal{E}_q \max(0, \tilde{\varphi} - \hat{\varphi})(1 - K'(D^s)) \\ +(1+\lambda)(\eta\, \mathcal{E}_q \min(\tilde{\varphi}, \hat{\varphi})(1 - K'(D^s) - 1 + \eta\, \mathcal{E}_q(1 - \beta(\tilde{\varphi}, \hat{\varphi}))\theta'(\bar{D}^s(\tilde{\varphi})))) \leq 0. \end{gathered}$$

Because $\hat{\varphi} < \varphi_2$, the first term in (A.129) is positive. Therefore the second term must be negative. To show that $1 + \lambda > 0$, it suffices to show that

$$\eta\, \mathcal{E}_q \min(\tilde{\varphi}, \hat{\varphi})(1 - K'(D^s) - 1 + \eta\, \mathcal{E}_q(1 - \beta(\tilde{\varphi}, \hat{\varphi}))\theta'(\bar{D}^s(\tilde{\varphi}))) \leq 0. \quad \text{(A.129)}$$

For this purpose, I note that, by the concavity of K and the fact that $A(b) \geq 0$ and $B^s(b) = 0$,

$$\begin{aligned} & (\eta\, \mathcal{E}_q \min(\tilde{\varphi}, \hat{\varphi})(1 - K'(D^s)) - 1 + \eta\, \mathcal{E}_q(1 - \beta(\tilde{\varphi}, \hat{\varphi}))\theta'(\bar{D}^s(\tilde{\varphi}))))D^s \\ \leq\ & \eta\, \mathcal{E}_q \min(\tilde{\varphi}, \hat{\varphi})(D^s - K(D^s) - D^s + \eta\, \mathcal{E}_q(1 - \beta(\tilde{\varphi}, \hat{\varphi}))\theta'(\bar{D}^s(\tilde{\varphi}))))D^s \\ \leq\ & \eta\, \mathcal{E}_q \min(\tilde{\varphi}, \hat{\varphi})(A + B^s + D^s - K(D^s) - B^s - D^s + \eta\, \mathcal{E}_q(1 - \beta(\tilde{\varphi}, \hat{\varphi}))\theta'(\bar{D}^s(\tilde{\varphi}))))D^s, \end{aligned}$$

so that, with $D^s > 0$, the inequality (A.129) follows from the constraint (A.111).

I next show that $D^s > 0$ implies $\hat{\varphi} > 0$. For suppose that $\hat{\varphi} = 0$. Then $\min(\tilde{\varphi}, \hat{\varphi}) = \beta(\tilde{\varphi}, \hat{\varphi}) = 0$ almost surely, so the constraint (A.111) yields,

$$\eta\, \mathcal{E}_q(1 - \beta(\tilde{\varphi}, \hat{\varphi}))\theta'(\bar{D}^s(\tilde{\varphi}))D^s - D^s = 0,$$

and the left-hand side of (A.127) is equal to $\eta\, \mathcal{E}_q\tilde{\varphi}(1 - K'(D^s)) > 0$.

Given that $\hat{\varphi} > 0$, condition (A.128) must hold as an equation. Because $1 + \lambda > 0$, the first term on the left-hand side is as negative. Therefore the second term must be positive. Therefore, the Lagrange multiplier λ must be positive.

To complete the proof of Claim 8, I note that, with $\lambda > 0$ and $\hat{\varphi} < \varphi_2$, the left-hand side of (A.127) is strictly less than

$$(1+\lambda)(\eta\, \mathcal{E}_q\tilde{\varphi}(1 - K'(D^s)) - 1 + \eta\, \mathcal{E}_q(1 - \beta(\tilde{\varphi}, \hat{\varphi}))\theta'(\bar{D}^s(\tilde{\varphi}))).$$

Since $(1 + \lambda) > 0$ and $\eta\, \mathcal{E}_q\tilde{\varphi} < 1$, the inequality (A.125) follows.

Claim 9: If the deposit provision cost function exhibits constant returns to scale, i.e., if $K'(D)$ is a constant, $D^s(b) > 0$ implies $A(b) > 0$ for almost all b.

Suppose first that $\eta\, \mathcal{E}_q\tilde{\varphi} < 1$. If the claim is false, the banks' maximization problem has a solution with $D^s(b) > 0$ and $A(b) = 0$. If $\eta\, \mathcal{E}_q\tilde{\varphi} < 1$, then, by Claim 6, $B^s(b)$ is also zero. The constraint (A.111) then takes the form

$$\eta\, \mathcal{E}_q \min(\tilde{\varphi}, \hat{\varphi})\ D^s(1 - k) = D^s(1 - \eta\, \mathcal{E}_q(1 - \beta(\tilde{\varphi}, \hat{\varphi}))\theta'(\bar{D}^s(\tilde{\varphi}))), \quad \text{(A.130)}$$

where k is the constant value of the marginal cost function K'. Notice that the validity of (A.130) is independent of D^s. This conditions determines $\hat{\varphi}$, regardless of D^s. For the given $\hat{\varphi}$, maximization of (A.110) with respect to D^s then yields the first-order condition

$$\eta\,\mathcal{E}_q\tilde{\varphi}(1-k) - 1 + \eta\,\mathcal{E}_q(1-\beta(\tilde{\varphi},\hat{\varphi}))\theta'(\bar{D}^s(\tilde{\varphi}))) = 0. \tag{A.131}$$

By (A.130), this equation can be rewritten as

$$\eta\,\mathcal{E}_q \max(0, \tilde{\varphi}-\hat{\varphi})(1-k) = 0. \tag{A.132}$$

Since $k < 1$, it follows that $\mathcal{E}_q \max(0, \tilde{\varphi}-\hat{\varphi}) = 0$, or $\hat{\varphi} \geq \varphi_2$, contrary to Claim 8. The assumption that $D^s(b) > 0$ and $A(b) = 0$ has thus led to a contradiction and must be false.

Suppose next that $\eta\,\mathcal{E}_q\tilde{\varphi} = 1$. If the claim is false, the banks' maximization problem has a solution with $D^s(b) > 0$ and $A(b) = 0$. By Claim 2, the constraint (A.111) then takes the form

$$\eta\,\hat{\varphi}\,[B^s + D^s(1-k)] = B^s + D^s(1-\eta\,\theta'(D^*(\mathcal{E}_q\tilde{\varphi}))). \tag{A.133}$$

Since $\eta\hat{\varphi} < \eta\,\mathcal{E}_q\tilde{\varphi} = 1$, the same argument as was used to establish Claim 6 implies that $B^s(b)$ must be zero. The validity of the constraint (A.133) is thus independent of D^s. Maximization of (A.110) with respect to D^s again yields (A.131). By substitution from (A.133), one again obtains (A.132), hence $\hat{\varphi} \geq \varphi_2$, contrary to Claim 2.

Claim 10: If $\eta\,\mathcal{E}_q\tilde{\varphi} = 1$ and

$$\frac{\varphi_1}{\mathcal{E}_q\tilde{\varphi}} < \frac{D^*(\mathcal{E}_q\tilde{\varphi}) - K'(D^*(\mathcal{E}_q\tilde{\varphi}))D^*(\mathcal{E}_q\tilde{\varphi})}{D^*(\mathcal{E}_q\tilde{\varphi}) - K(D^*(\mathcal{E}_q\tilde{\varphi}))}, \tag{A.134}$$

then $\int \alpha^s(b)E(b)db > 0$.

By Claim 2, $\eta\,\mathcal{E}_q\tilde{\varphi} = 1$ implies $\int D^s(b)db = D^*(\mathcal{E}_q\tilde{\varphi}) > 0$. Moreover, in this case, $D^s(b) > 0$ imply $\hat{\varphi}(b) \leq \varphi_1$ and $\int D^s(b')db' = D^*(\mathcal{E}_q\tilde{\varphi}) > 0$. If $\int A(b)db = 0$, then $A(b) = 0$ for almost all b. For any b such that $D^s(b) > 0$ and $A(b) = 0$, the constraint (A.111) implies

$$\eta\varphi_1(D^s(b) - K(D^s(b))) \geq D^s(b)(1-\eta\theta'(D^*(\mathcal{E}_q\tilde{\varphi})). \tag{A.135}$$

Upon integrating over all banks, one obtains

$$\eta\varphi_1(D^*(\mathcal{E}_q\tilde{\varphi}) - \int K(D^s(b))db \geq D^*(\mathcal{E}_q\tilde{\varphi})(1-\eta\theta'(D^*(\mathcal{E}_q\tilde{\varphi})). \tag{A.136}$$

The first-order condition for $D^s(b)$ implies

$$\eta\mathcal{E}_q\tilde{\varphi}K'(D^s(b)) = \eta\theta'(D^*(\mathcal{E}_q\tilde{\varphi})), \tag{A.137}$$

i.e. K' is constant over the set $\{D^s(b)|b \in [0,1]\}$. In particular, $K'(D^s(b)) = K'(D^*(\mathcal{E}_q\tilde{\varphi}))$ for all b. Therefore,

$$\begin{aligned}\int K(D^s(b))db &= K(D^*(\mathcal{E}_q\tilde{\varphi})) + K'(D^*(\mathcal{E}_q\tilde{\varphi}))\int(D^s(b) - D^*(\mathcal{E}_q\tilde{\varphi}))db \\ &= K(D^*(\mathcal{E}_q\tilde{\varphi})).\end{aligned} \tag{A.138}$$

Using (A.137) and (A.138), one can rewrite (A.136) as

$$\eta\varphi_1(D^*(\mathcal{E}_q\tilde{\varphi}) - K(D^*(\mathcal{E}_q\tilde{\varphi})) \geq D^*(\mathcal{E}_q\tilde{\varphi})(1 - K'(D^*(\mathcal{E}_q\tilde{\varphi}))).$$

Since $\eta\ \mathcal{E}_q\tilde{\varphi} = 1$, one finds that this inequality is incompatible with (A.134). The assumption that $\eta\ \mathcal{E}_q\tilde{\varphi} = 1$ and $A(b) = 0$ when (A.134) holds has thus led to a contradiciton and must be false.

Claim 11: If the deposit provision cost function does *not* exhibit constant returns to scale and e_0 is close to zero, then, for almost all b, $D^s(b) > 0$ implies $A(b) > 0$.

If e_0 is close to zero, then for $D^s < e_0$, the difference $\delta(D^s) := K'(D^s) - \frac{K(D^s)}{D^s}$ between the marginal and the average cost of deposit provision at D^s is close to zero. ■

Proof of Proposition 3.6. Consider an equilibrium of the model with full commitment and communication of banks' choices, with price system $(\eta, \lambda(\cdot), q(\cdot))$, and plans $(\alpha^s(b), B^s(b), D^s(b), L(b), E(b), r_B(b), r_D(b))$, $b \in [0, 1]$, and $(c_0(a), c_1(\cdot, a), \alpha^d(\cdot, a), B^d(\cdot, a), D^d(\cdot, a))$, $a \in [0, 1]$, of banks and consumers. Suppose that the equilibrium allocation is Pareto-dominated by some other feasible allocation with banks' plans $(\bar{\alpha}^s(b), \bar{B}^s(b), \bar{D}^s(b), \bar{L}(b), \bar{E}(b), \bar{r}_B(b), \bar{r}_D(b))$, $b \in [0, 1]$, and consumers' plans $(\bar{c}_0(a), \bar{c}_1(\cdot, a), \bar{\alpha}^d(\cdot, a), \bar{B}^d(\cdot, a), \bar{D}^d(\cdot, a))$, $a \in [0, 1]$ satisfying the respective budget constraints as implied by the price system $(\eta, \lambda(\cdot), q(\cdot))$. Without loss of generality, one may assume that, for any bank b,

$$\bar{L}(b) \leq \bar{\alpha}^s(b)\bar{E}(b) + \bar{B}^s(b) + \bar{D}^s(b) - K(D^s(b). \tag{A.139}$$

Moreover, one may define

$$\bar{\varphi}(b) := \frac{1}{\bar{L}(b)}(\bar{r}_B(b)\bar{B}^s(b) + \bar{r}_D(b)\bar{D}^s(b)) \tag{A.140}$$

as the associated bankruptcy point of bank b. Pareto dominance implies that

$$\begin{aligned} & u(c_0(a)) + \int_{\varphi_1}^{\varphi_2} v(c_1(\varphi, a) + \theta\left(\int_0^1 (1 - \beta(\varphi, \hat{\varphi}(b))D^d(b, a)db\right))f(\varphi)d\varphi \\ \leq \quad & u(\bar{c}_0(a)) + \int_{\varphi_1}^{\varphi_2} v(\bar{c}_1(\varphi, a) + \theta\left(\int_0^1 (1 - \beta(\varphi, \bar{\varphi}(b))\bar{D}^d(b, a)db\right))f(\varphi)d\varphi \end{aligned} \tag{A.141}$$

for almost all $a \in [0, 1]$, and the inequality is strict for a non-null set of consumers. By the concavity of $u(\cdot), v(\cdot)$, and $\theta(\cdot)$ and by Lemmas 3.1 and 3.2, it follows that, for almost all $a \in [0, 1]$,

$$\begin{aligned} 0 \quad \leq \quad & u'(c_0(a))(\bar{c}_0(a) - c_0(a)) + u'(c_0(a)\eta\mathcal{E}_q(\bar{c}_1(\varphi, a) - c_1(\varphi, a)) \\ & u'(c_0(a)\eta\mathcal{E}_q\theta'\left[\int_0^1 (1 - \beta(\varphi, \bar{\varphi}(b))\bar{D}^d(b, a)db - \int_0^1 (1 - \beta(\varphi, \hat{\varphi}(b))D^d(b, a)db\right], \end{aligned} \tag{A.142}$$

where θ' is evaluated at $\int\int(1-\beta(\varphi,\hat{\varphi}(b))D^d(b,a')dbda'$. Upon dividing by $u'(c_0(a))$ and integrating over the set of consumers, taking account of the fact that, for a nun-null set of consumers, the inequality is strict, one obtains

$$\begin{aligned} 0 &< \int_0^1 [\bar{c}_0(a) - c_0(a) + \eta\mathcal{E}_q(\bar{c}_1(\varphi,a) - c_1(\varphi,a))]da \\ &+ \int_0^1 \eta\mathcal{E}_q\theta' \left[\int_0^1 (1-\beta(\varphi,\bar{\varphi}(b))\bar{D}^d(b,a)db - \int_0^1 (1-\beta(\varphi,\hat{\varphi}(b))D^d(b,a)db\right] da \end{aligned} \tag{A.143}$$

and hence

$$0 < \int_0^1 [\bar{c}_0(a) - c_0(a) + \eta\mathcal{E}_q(\bar{c}_1(\varphi,a) - c_1(\varphi,a))]da$$

By feasibility and (A.139),,

$$\begin{aligned} \int_0^1 \bar{c}_0(a)da &\leq e_0 - \int_0^1 \bar{L}(b)db - \int_0^1 K(\bar{D}^s(b))db \\ &= \int_0^1 (\bar{\alpha}^s(b)\bar{E}(b) + \bar{B}^s(b) + \bar{D}^s(b))db. \end{aligned}$$

$$\int_0^1 \bar{c}_1(\varphi,a)da \leq \varphi\int_0^1 \bar{L}(b)db,$$

almost surely, and

$$\int_0^1 \bar{D}^d(b,a)da = \bar{D}^s(b),$$

Similarly, by the consumers' budget constraints, and market clearing, and Lemma 3.2,

$$\begin{aligned} \int_0^1 c_0(a)da &= e_0 - \int_0^1\int_0^1 [\alpha^d(b,a)E(b) + B^d(b,a) + D^d(b,a)]dbda \\ &= e_0 - \int_0^1\int_0^1 [\alpha^d(b,a)E(b) + B^d(b,a) + D^d(b,a)]daddb \\ &= e_0 - \int_0^1 [\alpha^s(b)E(b) + B^s(b) + D^s(b)]db \end{aligned}$$

$$
\begin{aligned}
\int_0^1 c_1(\varphi,a)da &= \int_0^1\int_0^1 [(1-\alpha^s(b)+\alpha^d(b,a))\max(0,\varphi L(b)-r_B B^s(b)-r_D D^s(b)) \\
&\quad +(r_B B^d(b,a)+r_D D^s(b,a))\min(1,\frac{\varphi L(b)}{r_B B^s(b)+r_D D^s(b})]dbda \\
&= \int_0^1 [(1-\alpha^s(b)+\int_0^1 \alpha^d(b,a)da)\max(0,\varphi L(b)-r_B B^s(b)-r_D D^s(b)) \\
&\quad +(r_B\int_0^1 B^d(b,a)da + r_D\int_0^1 D^s(b,a)da)\min(1,\frac{\varphi L(b)}{r_B B^s(b)+r_D D^s(b)})]db \\
&= \int_0^1 [\max(0,\varphi L(b)-r_B B^s(b)-r_D D^s(b)) \\
&\quad +\min(r_B B^s(b)+r_D D^s(b),\varphi L(b))]db \\
&= \int_0^1 \varphi L(b)db,
\end{aligned}
$$

almost surely, and

$$\int_0^1 D^d(b,a)da = D^s(b),$$

Thus (A.143) implies

$$
\begin{aligned}
0 &< -\int_0^1 \bar{L}(b)db - \int_0^1 K(\bar{D}^s(b))db + \int_0^1 L(b)db + \int_0^1 K(D^s(b))db \\
&\quad +\eta\mathcal{E}_q\varphi\int_0^1 \bar{L}(b)db - \eta\mathcal{E}_q\varphi\int_0^1 L(b)db \\
&\quad +\eta\mathcal{E}_q\theta'\left[\int_0^1 (1-\beta(\varphi,\bar{\varphi}(b))\bar{D}^s(b)db - \int_0^1 (1-\beta(\varphi,\hat{\varphi}(b))D^s(b)db\right].
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
&\int_0^1 \left[\eta\mathcal{E}_q\varphi L(b) - (\alpha^s(b)E(b)+B^s(b)+D^s(b)) + \eta\mathcal{E}_q\theta'(1-\beta(\varphi,\hat{\varphi}(b))D^s(b)\right] db \\
<\ &\int_0^1 \left[\eta\mathcal{E}_q\varphi\bar{L}(b) - (\bar{\alpha}^s(b)\bar{E}(b)+\bar{B}^s(b)+\bar{D}^s(b)) + \eta\mathcal{E}_q\theta'(1-\beta(\varphi,\hat{\varphi}(b))D^s(b)\right] db, \qquad \text{(A.144)}
\end{aligned}
$$

where, throughout, the derivative θ' in the last term is evaluated at the point $\int\int(1-\beta(\varphi,\hat{\varphi}(b))D^d(b,a')dbda'$.

From (A.144), one infers that

$$
\begin{aligned}
&\eta\mathcal{E}_q\varphi L(b) - (\alpha^s(b)E(b)+B^s(b)+D^s(b) + \eta\mathcal{E}_q\theta'(1-\beta(\varphi,\hat{\varphi}(b))D^s(b) \qquad \text{(A.145)} \\
<\ &\eta\mathcal{E}_q\varphi\bar{L}(b) - (\bar{\alpha}^s(b)\bar{E}(b)+\bar{B}^s(b)+\bar{D}^s(b)) + \eta\mathcal{E}_q\theta'(1-\beta(\varphi,\hat{\varphi}(b))D^s(b)
\end{aligned}
$$

for a non-null set of banks $b \in [0,1]$. The left-hand side of (A.145) can be rewritten as

$$\begin{aligned}
&\eta\mathcal{E}_q \max[\varphi L(b) - r_B B^s(b) - r_D D^s(b), 0] + \eta\mathcal{E}_q \min[\varphi L(b), r_B B^s(b) + r_D D^s(b)] \\
&\quad -(\alpha^s(b)E(b) + B^s(b) + D^s(b)) + \eta\lambda(\hat{\varphi}(b))\eta\mathcal{E}_q \min(1, \frac{\varphi}{\hat{\varphi}(b)})D^s(b) \\
&= \eta\mathcal{E}_q\tilde{\pi}(b) + [(r_B B^s(b) + r_D D^s(b) + \lambda(\hat{\varphi}(b))D^s(b)]\mathcal{E}_q \min(1, \frac{\varphi}{\hat{\varphi}(b)}) \\
&\quad -(\alpha^s(b)E(b) + B^s(b) + D^s(b) \\
&= (1 - \alpha^s(b))E(b).
\end{aligned}$$

Similarly, the right-hand side of (A.145) can be rewritten as $(1 - \bar{\alpha}^s(b))\bar{E}(b)$, so (A.145) implies $(1 - \alpha^s(b))E(b) < (1 - \bar{\alpha}^s(b))\bar{E}(b)$, which is incompatible with the equilibrium condition (E.1).

The assumption that the equilibrium allocation is Pareto dominated has thus led to a contradiction and must be false. ■