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# On the Network Effects in the Hold-up Problem

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**Abstract**

I study the Hold-up problem in the case of a buyer-seller network, where sellers may simultaneously make specific investments, increasing expected qualities of joint projects with each buyer with whom there are connected. After the resolution of uncertainty, agents bargain over surpluses sharing and form buyer-seller pairs in order to undertake projects. I find that there always exists an ex-post bargaining rule, which provides sellers with the efficient ex-ante incentives to invest. According to this rule each seller gets the difference between the maximal social welfare from the trade in network with his presence and maximal social welfare from the trade in network without him. Then I focus on the case when each pair divides surplus according to the Nash Bargaining solution with endogenous outside options, consisting of opportunities to attract another partners over the network.

I find that in case of large uncertainty network structure may provide additional incentives to invest in comparison with the one-seller-one-buyer benchmark; however, this may also lead to overinvestment in comparison with the first-best solution. In the case of small uncertainty the graph of equilibrium investments turns out to be peculiar: each seller invests in no more than two links. The efficiency suffers since sellers make inefficient investments in order to increase their outside options. The natural attribute of small uncertainty case is the multiplicity of equilibria and, therefore, the coordination problem. I study this issue based on the simple examples and make the hypothesis that if agents may construct or break relationships (or credibly commit to not invest) prior to the investment game, this may help to solve the coordination problem.

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# 1 Introduction

Assume that two agents, a seller and a buyer, consider an opportunity to trade some good or make joint project. The Hold-up problem appears when agents may make sunk specific investments in their mutual relationships, increasing value or decreasing costs of joint projects (traded good), but these investments have non-observable nature, and in addition parties are uncertain about the conditions of these projects. Thus, parties may fail to write the complete contract in order to protect their ex-ante contributions, since they bargain ex-post. As a result, the optimal for agents levels of investments are lower than efficient ones.

I consider a Hold-up problem, appearing in case of multiple sellers and buyers, connected by some bipartite network. The key assumption is that before making the specific investment, a seller and a buyer have to know each other sufficiently good, otherwise they may fail to increase the value of their joint project. Thus, it is reasonable to restrict opportunities to invest and trade only on the existing network of economic relationships between agents. In the presence of multiple alternatives, outside options of an agent during the ex-post trade become endogenous: they depend on the investment decisions of this agent, on the investment decisions of other agents and on how the trade over network happens.

Let's imagine one example where Hold-up problem may appear in the network context. Consider some market of production equipment, where factories are buyers, and engineering firms, that design and make necessary equipment, are sellers. Relationship matters: if a given engineering firm has already worked with a particular company, specialists of both companies are familiar with each other and with peculiarities of existing equipment and production conditions. This forms a unique environment for the creation of additional value by designing new equipment. Thus, investments opportunities correspond to the network of relationships between factories and engineering firms. Ex-ante it is hard to understand what particular improvements can be made, and Research and Development outcomes may be not contractible, and thus engineering firms face a Hold-up problem, making their R&D decisions.

This paper is aimed to investigate how the network structure of buyer-seller relationships influence the resolution of the Hold-up problem. This is a multilateral issue, and a lot of questions could be asked. Firstly, since the resolution of the Hold-up problem in the classical one-buyer-one-seller case depends on how agents trade over ex-post surplus, the resolution of the Hold-up problem over network also depends on the ex-post bargaining rule. The question is, to what extend can bargaining rule with reasonable properties provide agents with incentives to invest efficiently? Secondly, natural bargaining rules generally lead to the inefficient levels of investment, but what is the role of network structure in these distortions in comparison with the one-seller-one-buyer benchmark? Could network help to provide additional incentives to invest, or could it damage social effectiveness in comparison with the situation when agents simply interact in pairs? Thirdly, mostly the existing studies of the Hold-up problem in the presence of competition assume a large number of agents who take the market environment exogenously when making investment decisions. But what if there are a few agents and there is a strategic interaction of their actions, how does it influence the resolution of the Hold-up problem? A few agents and simple networks is a good framework to study this question. Finally, if agents understand the nature of their future Hold-up problem, they may try to establish

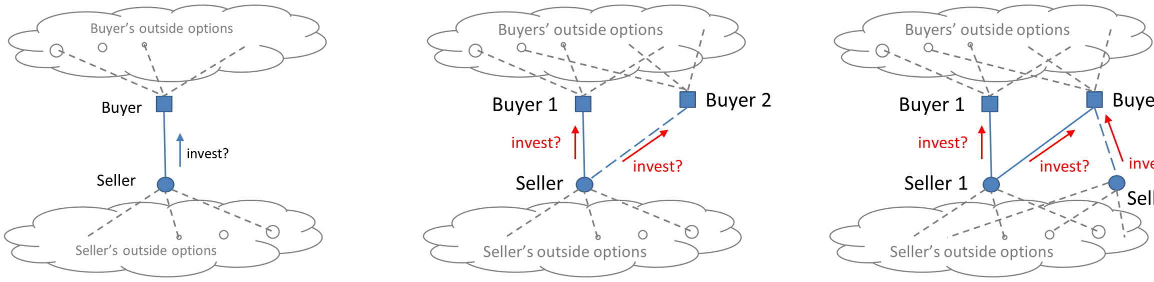


Figure 1: Illustration of the Hold-up problem in the case of 1 seller and 1 buyer (on the left), 1 seller and 1 buyers (in the center) and 2 seller, 2 buyers (on the right). The clouds stand for the exogenous outside options, and endogenous outside options are the opportunities two match with another agent in the network.

an optimal structure of economic relationships in order to increase their expected payoffs in the future investments' game. The question is, what kind of networks could emerge, if agents strategically build the network of their relationships, and how good are these networks in providing incentives to invest efficiently?

I start with the benchmark one-seller-one-buyer case, assuming that seller can make sunk investment $i$ in order to increase the quality of a joint project (good), and ex-post parties bargain over the surplus $k = i + \varepsilon$, consisting of an investment level and a noise term. Then I consider a network where sellers simultaneously make decisions about there buyer-specific investments. I study equilibriums of the corresponding game, assuming that sellers know ex-ante the way of future surplus division for different possible values of surpluses. Importantly, I do not allow buyers to make any investments and they do not make decisions in the considered game.

Since the game over network is computationally very complicated, I try to make the model as simple as I can (but still, non-trivial), saving the most important features which are needed in order to address the questions of the study. One important choice is characteristics of a traded good. I assume, that ex-post buyers and sellers bargain over single indivisible good; seller has a unit capacity for production, and buyer has a unit capacity for consumption. This choice has three reasons. Firstly, it simplifies the analysis of the ex-post bargaining, otherwise it may be too complicated issue for making any general conclusions. Secondly, this kind of trade over network is widely studied in the literature, hence I can compare the existing results with findings of this paper. Thirdly, this choice still represents some real situation: a marriage market or any market where agents may engage in joint projects by pairs. Other simplifications of the model are risk-neutrality of agents and quadratic costs of investments instead of some general cost function.

One of the questions of this study is the role of the ex-post bargaining rule in the resolution of the Hold-up problem. It is reasonable to think about possible natural properties of trade rules without modeling the process of the ex-post trade explicitly, since in at the end of the day the outcome of the trade depends on some way on the structure of a network and possible surpluses of trade. Thus, I design several axioms for the ex-post bargaining rules, and one set of results uses only the classification of rules through axioms, without appealing to the particular concept.

For the rest of results I use the concept, which seems to be a natural generalization of the Nash Bargaining solution on the case of trade over network with "endogenous" outside options, consisting of opportunities to attract alternative agents for the trade. Let's call in "Balanced Nash Bargaining (BNB) solution". I found this concept in the paper Kleinberg & Tardos (2008). Several works investigated properties of the BNB rule for the bipartite graph Kleinberg & Tardos (2008), Chakraborty & Kearns (2008), Azar, Birnbaum, Celis, Devanur & Peres (2009). However, a simple calculation with the full $2\times 2$ bipartite graph, presented in this paper, shows that BNB could have multiple solutions. In order to refine this multiplicity, I design a similar bargaining concept ("the BNB-delta solution"), where agents discount outside options by a factor $\delta < 1$ while trading over surplus . This bargaining concept gives a unique equilibrium outcome, and thus we may use it as an equilibrium refinement of the BNB rule, considering $\delta$ close to 1.

The relative scales of uncertainty and investments plays an important role in the determination of equilibrium investment levels. I focus on the two limit cases: the limit of the large noise and the limit of the small noise. This is reasonable because for both cases there are (distinct) promising approaches to the solution, and they both reveal some specific features, which are mixed up when the scale of uncertainty is comparable with the scale of investments. In particular, in the large noise limit players do not consider strategic interaction, and in contrary in the limit of small noise actions of other sellers are very important for the decision making process of each seller.

I consider the small noise limit, using the BNB-delta solution as a bargaining rule for the ex-post surplus division. I make the hypothesis (mostly proven at this moment, look at the Proposition (7)) that BNB-delta solution has some natural monotonicity properties. I assume that the BNB-delta solution is such that a slight increase of surplus of the link, corresponding to the seller's matching, is beneficial for him; and similar for the link, corresponding to the seller's outside option. However, it is not beneficial to increase the surplus of any link that corresponds only to the outside option of some other buyer, since it only may increase payoffs of buyers and decrease payoffs of sellers.

The existence and uniqueness of Nash equilibria in pure strategies of the investments game is one of the crucial questions. I do not consider mixed strategies equilibria, since they look a bit unnatural: they require a continuous mixture of actions. The situation seems to be different for different scales of noise. If uncertainty is large, then both questions of existence and uniqueness have positive answers. In case of the small noise, in contrary, no one of these results is guaranteed. The difference between existence and uniqueness of solutions for the considered two limit cases is based on the different convexities of the sellers' expected utilities as functions of investments. In case of large noise they are concave, but in case of small noise they may exhibit local convexities as well, giving birth to multiple equilibria or no equilibria. Importantly, at this moment I have no any general result about the existence of NE in case of small noise even for the BNB-delta trade rule, and I have no counterexample of its (non-existence) for this trade rule as well. Thus, all results for the small noise are subject to the existence of equilibria, i.e. if it exists, it has certain properties.

**Results.**

The core result of the paper is that there always exists an ex-post bargaining rule, which provides sellers with the efficient incentives to invest ex-ante. This rule satisfies all bargaining axioms (A1-7) and it is robust to change of various assumptions of the model. The idea behind this bargaining rule is to give each seller his contribution to the maximum social gain of trade over network in comparison with the maximum social gain of the trade over the rest of network without him. This allows to align sellers' incentives with social ones, since it turns out to be that ex-post alignment of payoffs from the trade leads to the ex-ante alignment of the total expected profits of society and each seller. One reservation should be made: this bargaining rule insures only that each first best profile of investments should be an equilibrium profile, but it could be that there exist also other Nash equilibria with non-efficient profiles of investments.

The other results are as follows. Firstly, I show, under mild assumptions on the bargaining rule in case of large uncertainty there always exists a unique Nash equilibrium of the investments game. I prove it, constructing the contraction mapping for which any NE should be fixed point. Secondly, it can be shown on the examples that network structure with some "natural" bargaining rule may help to make investment levels closer to the efficient levels, but it may also distort incentives in a bad way, decreasing the efficiency in comparison with one-seller-one-buyer benchmark case.

The rest of conclusions should be investigated more rigorously during further work, but the majority of proves are already made at this moment. I found that if ex-post bargaining concept is BNB-delta solution, then in case of the small noise at equilibrium each seller would invest in no more than two links. Hence, the graph of equilibrium investments in case of BNB solution and small noise limit has a peculiar structure, and if the initial network $G$ is dense, then a lot of links may have zero investments in this equilibrium, i.e. be useless. This allows to suggest the way of solving the problem of equilibria multiplicity. In particular, this problem mostly comes from the fact that there are several possible graphs of equilibrium investments. Let's fix one such graph. If agents may transform their network by removing all links which has zero investments, than they may drop out all other possible equilibria. Thus if agents may commit not to use some links (for example, by publicly breaking relationships or by some other way), they may solve the coordination problem; this may also include transfers to those agents who break their links, as a compensation for the decreasing of their bargaining power.

**Structure of the paper.**

In the next section I discuss the relevant literature and compare some of the existing models and findings with mine. In section 3 the general model is presented, then I solve several examples for the simple star networks and finish with the rigorous formulation of bargaining axioms. In section 4 I firstly investigate the first-best levels of investments for the sake of benchmark. Next, I present the key result about the existence of ex-ante efficient ex-post bargaining rule (Proposition 4). Finally, I prove the existence and uniqueness of the NE in case of large noise (Proposition 6). After that, I proceed to the small noise limit. Section 5 is a supporting section, where I describe the Balanced Nash Bargaining rule of trade as a natural generalization of the Nash Bargaining solution for the one-seller-one-buyer case. I discuss possible properties of BNB rule and suggest a way of refinement of the multiplicity of its solutions, the BNB-delta solution. I show that it satisfies bargaining axioms (A1-7) and additionally exhibits several important monotonicity properties (Proposition 7). Armed by the BNB-delta rule formalism and its properties, I study the small noise limit in Section 6, starting with the example of "$N$" network and example of the full $2\times2$ network. Then I formulate the theorem (Proposition 8) about properties of the graph of investments for the small noise limit (with ax-post trade happening according to the BNB-delta solution) and discuss further issues, such as multiplicity of equilibria of the investments game. Section 7 concludes, and bulky proves and calculations are given in Appendix.

## 2 Literature review

Paper, which are the most close to the investigated question papers, are Felli & Roberts (2002), Cole, Mailath & Postlewaite (2001) Georg Noldeke (2014) and Harold L. Cole (2001). Also Joseph Farrell and Paul Klemperer devoted the whole book Farrell & Klemperer (2007) to the problems of firms investments in the supplier chains and competition over the emerging networks.

In the Felli & Roberts (2002) the question "Does Competition Solve the Hold-up Problem?" is investigated. Authors consider the setup where "workers" and "firms" could make investments in their qualities prior to trade for the "wages". Authors investigate two possible types of inefficiency. The first one is the hold-up problem, arising from the fact that investment of parties in some cases may not fully realize in the appropriate increasing of their payoffs. The second investigated inefficiency is a possible coordination failure when the resulting matching may not be the best one. Considering different environments, authors come to the conclusion that the competition between players does help to solve the hold-up problem: the ex ante levels of investments are equal or tend to the efficient ones with the increasing of number of agents. However, two reservations about the results could be made. At first, in the considered models of the Bertrand competition all additional surpluses due to investments goes to sellers, thus it is not surprisingly that they have the correct incentives. The second reservation is that Leonardo Felli and Kevin W.S. Roberts consider a setup in which the investments are not specific, but general. Firms and workers invest in the increasing of their qualities, which may be used for the surplus generation with any other agents.

In the Cole et al. (2001)[1] authors study the hold-up problem with multiple sellers and buyers, where sellers and buyers may make ex-ante sunk investments in increasing of their scalar attributes $s$ and $b$ correspondingly. Then sellers and buyers matched and each pair perform a project, which gives the deterministic outcome $v(s, b)$. Authors find that under some reasonable assumptions, there always exists a bargaining rule, which gives ex-ante incentives to invest in attributes efficiently; however for the "natural" bargaining rule incentives are distorted and this may lead to underinvestment as well as to overinvestment. In comparison with my work, Harold Cole, George J. Mailath, Andrew Postlewaite considers more general cost functions, heterogeneous among agents; and the most important element of their study which is absent in my work is that they allow both sellers and buyers to make investments. However, my study is sufficiently different from their work in several other ways, which helps to study the efficiency of investments under other assumptions. In details, at first, Cole et al. (2001) consider the full bipartite graph: each seller may match with each buyer. I assume that this matching is constrained by the existing network of economic relationships. Secondly, Harold Cole, George J. Mailath, Andrew Postlewaite assume that each agent has only one-dimensional set of investment choices, and I assume that the dimension of this set for each seller is equal to the number of his adjacent links. Thus, in Cole et al. (2001) there is no room for the purely specific investments: each choice of attributes assumes both general and specific investments. Thirdly, Harold Cole, George J. Mailath, Andrew Postlewaite do not study the role of uncertainty: they assume that given the attributes (investment decisions) of matched buyer and seller, the surplus is deterministic. This corresponds to the "small noise limit" in my work. However, my investigations show that the picture of investments does depend on

[1] I found this paper recently, thus I can mistake about some of paper's results

the scale of the uncertainty, and it is meaningful to study non-zero noise as well. In sum, I study questions similar to those studied by Harold Cole, George J. Mailath, Andrew Postlewaite in Cole et al. (2001), but in a different framework. To my mind, together their study and my work give more complete picture of the Hold-up resolution in case of multiple sellers and buyers, bargaining ex-post and undertaking joint project in pairs.

The Generalization of Nash Bargaining over network (Balanced Nash Bargaining) were studied mostly by social and computer sciences, as there is some experimental evidence that people may trade over network according to this concept, and there is a computational question of finding these solutions. The studies include Kleinberg & Tardos (2008), Chakraborty & Kearns (2008) and Azar et al. (2009). One of the results of Kleinberg & Tardos (2008) paper is that for the bipartite graphs there always exists a balanced outcome (BNB solution in my notations); moreover, each balanced outcome is a stable outcome with respect to single and pairwise deviations. This result is in line with my result about the existence of BNB-delta solution. Also, I have to prove the point with the stability of the BNB-delta solution similar to those in Kleinberg & Tardos (2008).

In Chakraborty & Kearns (2008) authors consider the concept similar to the BNB solution, but for the general (non-linear) utility functions. They focus on the influence of network topology on the outcomes of the trade, and thus they do not consider in details weighted graphs, considering a situation, when all possible surpluses are equal to 1. Notwithstanding the fact, that authors get some results for this case (assuming general utility functions, i.e. considering risk averse agents), it is hard to use these results in order to guess what is going on in case of risk-averse agents when they play the investments' game over network, since various weighted graphs appear ex-post.

In Chakraborty & Kearns (2008) authors studied different dynamics which converge to the Balanced Nash Bargaining solution. In particular, they consider the dynamics, which I use in order to prove the existence and uniqueness of the BNB-delta solution (iteratively using the contraction mapping operator on the set of payoffs). However, for $\delta = 1$, when BNB-delta solutions becomes BNB-solution, this dynamics does not converge in case of certain cycles of outside options (the corresponding operator is no more the contraction mapping). Azar et al. single this out, but instead of more deep investigation of the consequences of this fact, they find out different dynamics, which indeed converge to the balanced outcomes (BNB solution). However these dynamics do not correspond to the iteratively using of some contraction mapping operator, and thus they may give multiple solutions, depending on the starting point of the process.

Interestingly, that in all papers Kleinberg & Tardos (2008), Chakraborty & Kearns (2008) and Azar et al. (2009) the question of uniqueness of the BNB solution is not investigated. In this work I show, that a multiplicity of solutions may be indeed a problem, and it is not easy to overcome this problem; however, the BNB-delta solution with $\delta \to 1$ seemed to be a nice candidate for the multiplicity refinement.

In this paper I do not study the nature of the Hold-up problem and ways of its contractual resolution. Instead, I focus on the role of network and bargaining concept. However, the properties of the traded good (joint project) determine the nature of the Hold-up problem as well as a way of how the ex-post trading happens and, to some extent, what buyer-seller networks emerge. What outside options are exogenous, and what are endogenous, when agents participate in the ex-post trade? If we assume, that buyers and sellers cannot write down a complete contract on the ex-post trade conditions, can we assume that they are able to contract on the structure of the buyer-seller network? There could be a lot of other interesting questions, involving the nature of the Hold-up problem in the network context. The following papers make a good overview of the potential issues, connected with the Hold-up problem: Hart & Moore (1999),Hart & Moore (1988), Tirole (1999) Maskin & Tirole (1999), Rogerson (1992), Che & Sákovics (2004), Siegel (2010).

# 3 Model

In this section I provide the formal model of the study. Firstly, I formulate the seller's game of investments over network, describing players, actions, payoffs, timing of the game, structure and scale of uncertainty. Secondly, I explore three examples of simple networks with a few agents and figure out what problems and peculiarities we can see based on these simple cases, trying to motivate further research direction. Thirdly, I present rigorous formulation of the bargaining rule (the concept of ex-post matching and surplus division), and suggest an axiomatic approach to the systematization of bargaining rules, which plays the role of fundament for main propositions of the paper.

## 3.1 General setup

There are $S \geq 1$ risk-neutral sellers and $B \geq 1$ risk-neutral buyers which have economic relationships, denoted by bipartite graph $G$ ($g_{sb} = 1$ if Seller $s$ and Buyer $b$ have a relationship, and $g_{sb} = 0$ otherwise). Each seller produces a single indivisible good, which may be sold to any connected with him buyer or to the outside agent in the market (let's call this "an exogenous outside option"). In his turn, each buyer requires one unit of a good, which he may buy from any connected seller or from the outside agent in the market (which is an exogenous outside option for the buyer). However, the cost of production of a good is exactly equal to the price in the outside market, and also it is equal to the valuation of a good for the buyer in case if he buys it from the market (let's call this a "general good"). That is, the exogenous outside option for each agent is zero. In what follows I normalize the cost of production of the general good as well as its price and valuation to zero.

If seller $s$ sells a good to the buyer $b$, then buyer gets some additional value $k_{sb}$, since the good is specific for him:

$$k_{sb} = i_{sb} + \varepsilon_{sb} \tag{1}$$

where $i_{sb}$ is the level of relationship-specific investment and $\varepsilon_s b$ is i.i.d. noise term, distributed according to some p.d.f. with the finite support, zero mean and median (the latter simplifies calculations for the examples):

$$E[\varepsilon_{sb}] = 0 \qquad \text{median}[\varepsilon_{sb}] = 0 \qquad \varepsilon_{sb} \text{ are i.i.d.} \tag{2}$$

For further calculations it is important to understand the scale of the noise term. One natural candidate for the measure of the noise scale is its standard deviation, however I introduce a different concept for the sake of computational simplicity. Namely, let's take some distribution with p.d.f. $f_0(\varepsilon)$, distributed on the $[-\frac{1}{2};\frac{1}{2}]$:

$$\text{support}(f_0(\varepsilon)) = [-\frac{1}{2};\frac{1}{2}] \tag{3}$$

and introduce the scale parameter $a$. Then distribution of noise with scale $a$ is given by the p.d.f.:

$$f(\varepsilon) = \frac{1}{a} f_0(\frac{\varepsilon}{a}). \tag{4}$$

I assume that $f_0(\varepsilon)$ is twice continuously differentiable function, and so does $f(\varepsilon)$.

The cost of production of a good for the seller is normalized to zero (since he always may sell it in the outside market for its cost of production). Each specific investment is costly with the convex cost:

$$C(i_{sb}) = \frac{i_{sb}^2}{2} \tag{5}$$

$C(i_{sb})$ are additive for sellers, i.e. the total specific investments expenditure of a seller $s$ is:

$$C_s = \sum_{b:sb\in G} C(i_{sb}) = \sum_{b:sb\in G} \frac{i_{sb}^2}{2} \tag{6}$$

This choice of investment costs is reasonable. We may think, that $\frac{i_{sb}^2}{2}$ are spent money, then the specific investments technology exhibits diminishing return to scale and the Inada conditions hold.

The timing is as follows:

1. Sellers simultaneously choose levels of relationship-specific investments in each of their adjacent links, i.e. seller $s$ chooses $i_{sb}:\ sb \in G$.

2. Uncertainty resolves.

3. Parties observes payoffs, bargain over the delivery's structure (matching) and payoffs. The discussion of the bargaining concept is given in the following subsections.

4. Payoffs realize.

Players are only sellers. It is a one-period game. The set of actions for each player is $[0;\infty)^{number\ of\ player's\ links}$; an action is a point from this set. The equilibrium concept is Bayes-Nash equilibrium. I assume the common knowledge of the game and rationality of players. The total expected payoff of a seller $i$, given levels of investments of other sellers, is:

$$EU_i = E[p_i(\mathbf{i_i} + \varepsilon_\mathbf{i}, \mathbf{i_{-i}} + \varepsilon_\mathbf{-i})] - \sum_{j:\ ij\in G} \frac{i_{ij}^2}{2} \tag{7}$$

where $p_i(\mathbf{i_i} + \varepsilon_\mathbf{i}, \mathbf{i_{-i}} + \varepsilon_\mathbf{-i})$ is a payoff of the $i$-th seller according to the bargaining rule; it depends on the levels of investment of all sellers and noise terms (here $\mathbf{i_i}$ is a vector of seller i's investments, $\mathbf{i_{-i}}$ is a vector of investments of all other sellers except of seller $i$; $\varepsilon_\mathbf{i}$ and $\varepsilon_\mathbf{-i}$ are the corresponding vectors of noise terms for the links, adjacent to seller $i$, and others).

**Discussion**

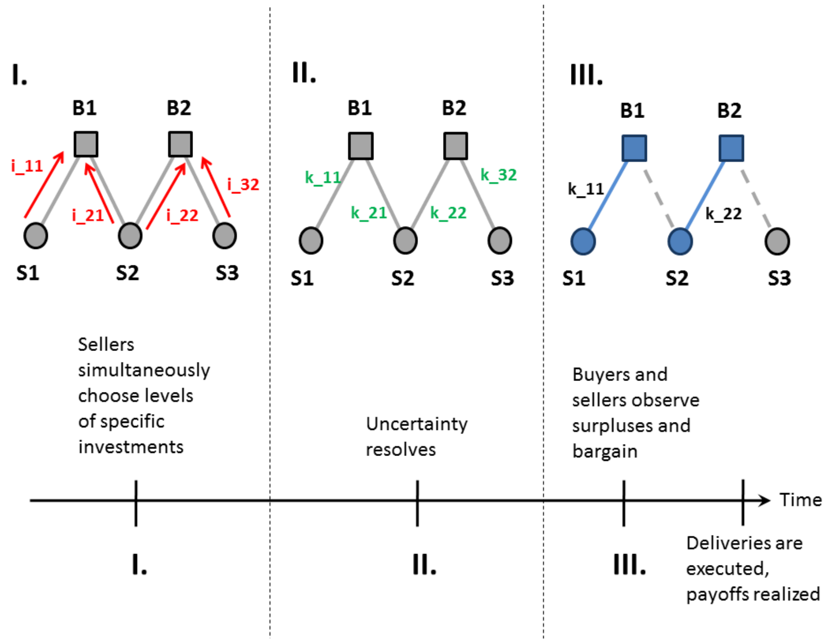


Figure 2: Timing of the game (An example of network with 3 sellers and 2 buyers).

An important specification of this model is that noise term is idiosyncratic and identically distributed for each link. Thus, I assume that shock $\varepsilon_{sb}$ is buyer-seller specific. Moreover, I assume that it is independent of any other shocks, including that for other links, adjacent to the seller $s$ and buyer $b$. This may seem a bit unnatural, since, for example, a seller may face some shock which influence possible surpluses of all projects in which he is involved in, and thus $\varepsilon_{sb}$ and $\varepsilon_{sb'}$ may be correlated. However, the assumption of independent shocks is in line with the assumption that sellers may make only specific investments, but not general ones, meaning that projects, corresponding to the different adjacent links, represent very different activities. A natural generalization of the study is consideration of general investments and correlated shocks for the adjacent links. However in this work I focus on the case where there are only specific investments and not general ones. There is also one more argument in favor of presence of at least some link-specific shocks (in comparison with the models like $k_{sb} = i_{sb} + \varepsilon_s + \mu_b$): in case of their absence, in certain situations there is no equilibria of the seller's game, since some combinations of surpluses become deterministic in this case (i.e. $k_{ij} - k_{ij'} - k_{i'j} + k_{i'j'} = i_{ij} - i_{ij'} - i_{i'j} + i_{i'j'}$), but even tiny uncertainty plays a crucial role in the existence of the equilibrium.

## 3.2 Simple examples: 1S-1B, 1S-2B and 2S-1B

**One seller, one buyer.** Here I consider 3 examples of simple networks. Since in this paper I consider mostly (but not always) limits of the large noise $a \to \infty$ and small noise $a \to 0$, I consider here these limits too for the sake of simplicity of calculations (even finding equilibria while operating with integrals over $R^2$ require bulky calculations). Thus in this section we consider two cases: noise is arbitrary small and noise is arbitrary large. Note, that I assume that the noise distribution is such that median$[\varepsilon] = 0$.

Let's consider firstly the classical benchmark case with one seller and one buyer. Assume that ex-post trade happens according to the Nash Bargaining solution: parties just divide the surplus equally, since their outside options are zero. Assume also that noise level is small ($a \to 0$). Then the seller's problem looks as:

$$EU_1 = E[\frac{1}{2}(i_{11} + \varepsilon_{11})\mathbb{1}(i_{11} + \varepsilon_{11} \geq 0)] - \frac{i_{11}^2}{2} \to underseti_{11}max \;\Rightarrow\; i_{11}^{eq} = \frac{1}{2} \tag{8}$$

Where I used the fact that in case of small noise, $Pr(i_{11} + \varepsilon_{11} \geq 0) = 1$ for $i_{11} > 0$ as $a \to 0$, and thus the indicator function always gives one for sufficiently large values of $i_{11}$. For the first best we have:

$$EW = E[(i_{11} + \varepsilon_{11})\mathbb{1}(i_{11} + \varepsilon_{11} \geq 0)] - \frac{i_{11}^2}{2} \to \max_{i_{11}} \;\Rightarrow\; i_{11}^{FB} = 1 \tag{9}$$

Social welfare at the equilibrium:

$$EW = E\sum(payoffs) - E\sum(investments) = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2}\right)^2 = \frac{3}{8} \tag{10}$$

Social welfare for the first best level of investments:

$$EW^{FB} = 1 - \frac{1}{2} \cdot 1^2 = \frac{1}{2} \tag{11}$$

And we can roughly say that the "efficiency coefficient" is

$$x = \frac{EW}{EW^{FB}} = 0.75 \tag{12}$$

If the noise is very large, then

$$\frac{\partial p_1}{\partial i_{11}} = \frac{\partial}{\partial i_{11}} \int \frac{1}{2}(i_{11} + \varepsilon_{11})\theta((i_{11} + \varepsilon_{11}))dF(\varepsilon_{11}) = \tag{13}$$

$$= \frac{1}{2}Pr(\varepsilon_{11} \geq 0) + O(a^{-1}) + \frac{1}{2}\int \frac{1}{2}(i_{11} + \varepsilon_{11})\delta((i_{11} + \varepsilon_{11}))dF(\varepsilon_{11}) \approx \frac{1}{4}$$

Similarly, the First Best is $i_{11}^{FB} = Pr(i_{11} + \varepsilon_{11} > 0) \approx \frac{1}{2}$. It is meaningful to compare here only investment levels, since both for competitive and for the first best case the expected social welfare is of order of $a$ (large noise creates large expected value by itself, since agents trade when positive shock occurs and do not trade in case of the negative shock) As we can see, again equilibrium investment level is half of the optimal one, as in case of the small noise. We will remind these results as a benchmark case.

**One seller, two buyers.** Now assume that there is one seller and two buyers. Assume that payoff is distributed according to the generalize Nash Bargaining solution. Let WLOG $k_{11} > k_{21}$. Then while trading with the first buyer, the seller considers opportunity to attract the second buyer buy giving him a small amount of money and taking $k_{21}$. By these reasonings, seller's outside option is $k_{12}$, and if both $k_{11}, k_{12} > 0$, then the seller's payoff is:

$$p_{(s)1} = OO_{(s)1} - \frac{1}{2}(k_{11} - OO_{(s)1}) = \frac{1}{2}k_{11} + OO_{(s)1} = \frac{1}{2}k_{11} + \frac{1}{2}k_{12} \tag{14}$$

If $a$ is small, then seller can be sure to get $k_{11}, k_{12} > 0$ if he invests something in both links. Clearly, his investment levels in this case would be:

$$i_{11} = i_{12} = \frac{1}{2} \tag{15}$$

We get two possible profiles for the First Best investments, using proposition (3):

$$i_{11} = 1,\ i_{21} = 0 \qquad \text{and} \ \ i_{11} = 0,\ i_{21} = 1 \tag{16}$$

and the First Best social welfare is again 0.5 for each of them. But the social welfare at equilibrium in this case is:

$$EW = \frac{1}{2} + O(a) - \frac{1}{2}\left(\frac{1}{2}\right)^2 - \frac{1}{2}\left(\frac{1}{2}\right)^2 = \frac{1}{4} + O(a) \tag{17}$$

and our "index of efficiency" becomes lower that in the benchmark scenario: $x = 0.5$. Thus we have learned that in this simple specific example the presence of network effects lowers the efficiency, since seller invests in his outside option in order to get total surplus from the trade with his counterpartner.
Now consider the large noise limit. Remind that if matching is $M = \{11\}$ and $k_{12} < 0$, then seller's outside option is zero. The payoff of the seller looks as:

$$p_1 = \begin{cases} \frac{1}{2}(i_{11} + \varepsilon_{11} + i_{12} + \varepsilon_{12}) & \text{if } i_{11} + \varepsilon_{11} > 0 \ \& \ i_{12} + \varepsilon_{12} > 0 \quad (A) \\ \frac{1}{2}(i_{11} + \varepsilon_{11}) & \text{if } i_{11} + \varepsilon_{11} > 0 \ \& \ i_{12} + \varepsilon_{12} < 0 \quad (B) \\ \frac{1}{2}(i_{12} + \varepsilon_{12}) & \text{if } i_{11} + \varepsilon_{11} < 0 \ \& \ i_{12} + \varepsilon_{12} > 0 \quad (C) \\ 0 & \text{if } i_{11} + \varepsilon_{11} < 0 \ \& \ i_{12} + \varepsilon_{12} < 0 \quad (D) \end{cases} \tag{18}$$

FOCs give us:

$$\begin{aligned} \frac{\partial Ep_{(s)1}}{\partial i_{11}} &= \frac{\partial}{\partial i_{11}} \frac{1}{2} \int\int (i_{11} + \varepsilon_{11} + i_{12} + \varepsilon_{12})\theta(i_{11} + \varepsilon_{11})\theta(i_{12} + \varepsilon_{12}) dF(\varepsilon) + \\ &+ \frac{\partial}{\partial i_{11}} \frac{1}{2} \int\int (i_{11} + \varepsilon_{11})\theta(i_{11} + \varepsilon_{11})(1 - \theta(i_{12} + \varepsilon_{12})) dF(\varepsilon) + \\ &+ \frac{\partial}{\partial i_{11}} \frac{1}{2} \int\int (i_{12} + \varepsilon_{12})(1 - \theta(i_{11} + \varepsilon_{11}))\theta(i_{12} + \varepsilon_{12}) dF(\varepsilon) = \\ = \frac{1}{2}(Pr(A) + Pr(B)) &+ \frac{1}{2} \int\int (i_{11} + \varepsilon_{11} + i_{12} + \varepsilon_{12})\delta(i_{11} + \varepsilon_{11})\theta(i_{12} + \varepsilon_{12}) dF(\varepsilon) + \\ &+ \frac{1}{2} \int\int (i_{11} + \varepsilon_{11})\delta(i_{11} + \varepsilon_{11})(1 - \theta(i_{12} + \varepsilon_{12})) dF(\varepsilon) + \\ &\frac{1}{2} \int\int (i_{12} + \varepsilon_{12})(1 - \delta(i_{11} + \varepsilon_{11}))\theta(i_{12} + \varepsilon_{12}) dF(\varepsilon) = \\ &= \frac{1}{2}(Pr(A) + Pr(B)) = i_{11} \end{aligned} \tag{19}$$

Similarly,

$$i_{12} = \frac{1}{2}(Pr(A) + Pr(C)) \tag{20}$$

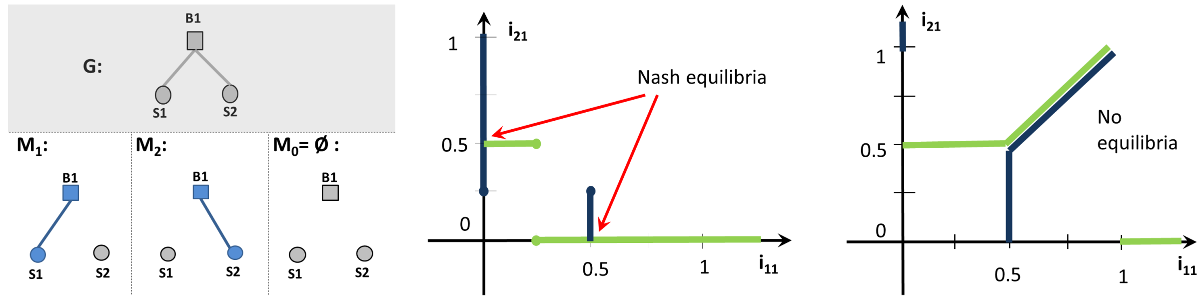


Figure 3: The game over star network with two sellers and one buyer. Graph $G$ and possible matchings $M_1, M_2, \varnothing$ on the left plot. Best responses of sellers for different bargaining rules middle figure for the $R_1$ rule and right figure for the $R_2$. Dark blue line - best response of first seller on the investment of the second seller; light green line - best response of the second seller on the investment of the first one.

Then we have:

$$i_{12} - i_{11} = \frac{1}{2}(Pr(C) - Pr(B)) \tag{21}$$

And since the difference between probabilities of events $B$ and $C$ should be next order of $a^{-1}$ in comparison with $i$ (because noise terms dominate over $i$ in the integrals), the only one opportunity is $Pr(C) = Pr(B)$ and $i_{11} = i_{12} = Pr(A) + Pr(B) = Pr(A) + Pr(B)$. Note, that up to the next order of $a^{-1}$, $Pr(A) = Pr(\varepsilon_{11} > 0 \ \& \ \varepsilon_{12} > 0) = \frac{1}{4}$. Similarly, $Pr(B) = Pr(C) = \frac{1}{4}$. Hence,

$$i_{11} = i_{12} = \frac{1}{2}(\frac{1}{4} + \frac{1}{4}) = \frac{1}{4} \tag{22}$$

What about first best levels of investments? Clearly, when $a \to \infty$, the probability that there will be no ex-post matchings, is $Pr(D) = \frac{1}{4}$. Then, if the there is ex-post matching, it is with equal probabilities $\frac{3}{8}$ either 11 or 12. Using proposition (1), we get: $i_{11}^{FB} = Pr(11 \in M) = \frac{3}{8}$ and similarly for the $i_{12}$. As we can see, the relative difference between investment levels is lower than that in the benchmark case (equilibrium investments $\frac{1}{4}$ is a two thirds of the first best level $\frac{3}{8}$). Thus, on this simple example we can see that in case of large noise network structure may give sellers additional incentives to invest.

**Two sellers, one buyer.** Now consider the situation when there are two sellers who compete for one buyer. Let's start with the same ex-post bargaining concept ($R_1$): the matching is effective, and for matched seller his payoff is equal to the half of the link's surplus minus half of the buyer's outside option (meaning that buyer considers an opportunity to trade with another seller as his outside option) That is:

$$p_1 = \begin{cases} \frac{1}{2}k_{11} - \frac{1}{2}k_{21} & \text{if } k_{11} > k_{21} > 0 \quad (X_1) \\ \frac{1}{2}k_{11} & \text{if } k_{11} > 0 > k_{21} \quad (X_2) \\ 0 & \text{if } 0 < k_{11} < k_{21} \quad (X_3) \\ 0 & \text{if } k_{11} < 0 < k_{21} \quad (X_4) \\ 0 & \text{if } k_{11} < 0 \ \& \ k_{21} < 0 \quad (X_5) \end{cases} \tag{23}$$

And similar for the $p_{(s)2}$. Here $X_1, ..., X_5$ are events that the corresponding conditions of $k$ holds. Consider the limit of the small noise. We want to find the NE of sellers' investments game. The best responses of players are represented in the middle of figures (3). For the first seller we get:

$$i_{11} = BR_1(i_{21}) = \begin{cases} \frac{1}{2} & \text{if } i_{21} < \frac{1}{4} \\ \frac{1}{2} \text{ or } 0 & \text{if } i_{21} = \frac{1}{4} \\ 0 & \text{if } i_{21} > \frac{1}{4} \end{cases} \tag{24}$$

And similar for the second seller. Thus, we have two Nash equilibria:

$$i_{11} = \frac{1}{2},\ i_{21} = 0 \qquad \text{and} \ \ i_{11} = 0,\ i_{21} = \frac{1}{2} \tag{25}$$

Again, there is two possible profiles for the first best investments: $i_{11} = 1,\ i_{21} = 0$ and $i_{11} = 0,\ i_{21} = 1$ (for for social welfare there is no difference between sellers' and buyers' investments, and thus we may use here the result for the one-seller-two-buyer case). As we can see, there are two equilibriums (and two first best levels of investments as well), with investments in each link equal to the half of the first best levels.

Now consider the large noise limit. We have:

$$Ep_1 = \frac{1}{2}\int\int [i_{11} + \varepsilon_{11} - i_{21} - \varepsilon_{21}]\theta(i_{11} + \varepsilon_{11} - i_{21} - \varepsilon_{21})\theta(i_{21} + \varepsilon_{21})dF(\varepsilon) + \tag{26}$$

$$+\frac{1}{2}\int\int [i_{11} + \varepsilon_{11}]\theta(-i_{21} - \varepsilon_{21})\theta(i_{11} + \varepsilon_{11})dF(\varepsilon)$$

Then,

$$\frac{\partial p_1}{\partial i_{11}} = \frac{1}{2}Pr(X_1) + \frac{1}{2}Pr(X_2) + \tag{27}$$

$$+\frac{1}{2}\int\int (i_{11} + \varepsilon_{11} - i_{21} - \varepsilon_{21})\delta(i_{11} + \varepsilon_{11} - i_{21} - \varepsilon_{21})\theta(i_{21} + \varepsilon_{21})dF(\varepsilon) +$$

$$+\frac{1}{2}\int\int (i_{11} + \varepsilon_{11})\delta(i_{11} + \varepsilon_{11})\theta(-i_{21} - \varepsilon_{21})dF(\varepsilon) = \frac{1}{2}Pr(X_1) + \frac{1}{2}Pr(X_2)$$

And we have:

$$i_{11} = \frac{1}{2}Pr(X_1) + \frac{1}{2}Pr(X_2) \tag{28}$$

analogously,

$$i_{12} = \frac{1}{2}Pr(X_3) + \frac{1}{2}Pr(X_4) \tag{29}$$

The fact that terms with delta-functions dropped out is not a general issue for the arbitrary network. Under large noise condition, the probabilities of events $X_1, .., X_5$ does not depend on the investment levels in the leading order of power expansion over $a^{-1}$. We have:

$$Pr(X_2) = Pr(X_4) = Pr(X_5) = \frac{1}{2} \qquad Pr(X_1) = Pr(X_3) = \frac{1}{4} \tag{30}$$

Thus, when $a \to \infty$, at NE we have:

$$i_{11} = i_{21} = \frac{3}{16} = \frac{1}{2} i_{11}^{FB} = \frac{1}{2} i_{21}^{FB} \tag{31}$$

Now suppose parties trade according to some "weird" bargaining rule ($R_2$):

$$p_1 = \begin{cases} \frac{1}{2}k_{11} & \text{if } k_{11} > k_{21} > 0 & (X_1) \\ \frac{1}{2}k_{11} & \text{if } k_{11} > 0 > k_{21} & (X_2) \\ 0 & \text{if } 0 < k_{11} < k_{21} & (X_3) \\ 0 & \text{if } k_{11} < 0 < k_{21} & (X_4) \\ 0 & \text{if } k_{11} < 0 \ \& \ k_{21} < 0 & (X_5) \end{cases} \tag{32}$$

Assume, that we are under the conditions of the small noise limit. The first-best responses of sellers are represented in right figure (3). In this case there are no interceptions of best responses, since up to a certain moment each seller is better off by investing a bit larger, than another seller up to the moment when it is no profitable to invest at all. Hence, we have no NE in pure strategies for the small noise limit here.

Now consider the large noise limit with the same "weird" bargaining rule. We have:

$$Ep_1 = \frac{1}{2} \int \int (i_{11} + \varepsilon_{11}) \theta(i_{11} + \varepsilon_{11} - i_{21} + \varepsilon_{21}) \theta(i_{11} + \varepsilon_{11}) dF(\varepsilon) \tag{33}$$

FOC gives us (neglecting terms $\sim O(a^{-1})$):

$$\begin{aligned} i_{11} &\approx \frac{1}{2}[Pr(X_1) + Pr(X_2)] + \frac{1}{2} \int (i_{21} + \varepsilon_{21}) \frac{1}{a} f_0\left(\frac{i_{21} - i_{11} + \varepsilon_{21}}{a}\right) \theta(i_{21} + \varepsilon_{21}) \frac{1}{a} f_0\left(\frac{\varepsilon_{21}}{a}\right) d\varepsilon_{21} \approx \\ &\approx \frac{1}{2}[Pr(X_1) + Pr(X_2)] + \frac{1}{2} \int \frac{\varepsilon_{21}}{a} f_0\left(\frac{\varepsilon_{21}}{a}\right) \theta\left(\frac{\varepsilon_{21}}{a}\right) f_0\left(\frac{\varepsilon_{21}}{a}\right) \frac{d\varepsilon_{21}}{a} \end{aligned} \tag{34}$$

and for the second seller we have the same FOC. Note, that the last term $\in (0;1)$ and it is of order of $O(1)$. As we can see, NE exists for the large noise even for this "weird" bargaining rule. We will see in the following that the existence of NE at the large noise limit is a general result for the wide class of bargaining rules.

**Discussion.**
We have studied a few simple examples, so what are the lessons that we learned from them? We have seen that even very simple networks may cause distortions of incentives (compared with the first-best incentives), different from that of one-seller-one-buyer case. Next thing to mention is that subject to the bargaining rule there could be or there could be no equilibrium in pure strategies; and when noise is small, then there could be multiple equilibria. These observations reveal an important role of bargaining concept and the scale of noise. Thus it is important to understand, what properties of the rules of trade lead to existence\non-existence of equilibria, multiplicity, etc. And it is useful to study large noise and small noise limits separately. I start the next section with the systematization

of bargaining rules, providing the set of possible properties (axioms). Then I continue with the discussion of conclusions which we can be made in case of the large noise limit. After that, I study the small noise limit under the assumption, that bargaining rule is Balanced Nash Bargaining (see the formal definition in the section (5).

## 3.3 Bargaining rule axioms

In this paper I do not model the ex-post trade explicitly as some game, instead I assume that sellers and buyers bargain over matching and surplus division according to some trade rule. This allows me to study the simple Bayes-Nash equilibria of the investments game. However, since the role of bargaining rule in the Hold-up resolution seems to be crucial, I do investigate this question. In order to address the issue, I think about natural properties of trade concepts and formalize them as axioms. The study benefits from it by at least three aspects. Firstly, I investigate the question of the existence of the bargaining rule, which provides sellers with incentives to invest efficiently ex-ante, and it is important to understand, in what class of rules it could exist (if it exists), i.e. how nice it could be. Secondly, it is important to know the sufficient conditions for equilibria existence and understand the connection between certain properties of bargaining rules and properties of equilibriums of the investments game. Thirdly, consideration of general bargaining rule allows me to avoid concerns with the structure of the ex-post information of sellers and buyers, since in at the end of the day the outcome of the trade depends on some way on the structure of a network and vector of possible surpluses. Of cause, further considerations of information asymmetry may shed light on what kind of bargaining rules could be reasonable.

Let $G$ be a bipartite graph of economic relationships between sellers and buyers. Denote $\mid G \mid$ to be a number of links of graph $G$.

**Definition 1.** *A surplus $k_{ij}$ : $ij \in G$ is a value of a good delivery from seller $i$ toward buyer $j$.*

**Definition 2.** *A matching $M \subseteq G$ is a subgraph of a relationships network, where each agent has no more than one link.*

Let $\mathcal{M}(G) = \{M \mid M \text{ is matching in } G\}$ to be the set of all possible matchings (including empty matching). Denote $\mid \mathcal{M} \mid$ to be a number of all possible matchings. If seller $i$ is matched with buyer $j$, that is, $ij \in M$, let's say that $j = M_{(s)}(i)$, and $i = M_{(b)}(j)$.

**Definition 3.** *A Bargaining rule over a buyer-sellers network is a correspondence*[2] *which for each graph of relationships and a vector of possible surpluses gives graph of matching*

[2] The reason why I allow bargaining rule to have multiple values is as follows. Firstly, there are concepts of trading where agents play some game (for example, take-it-or-leave-it offers), and there could be multiple equilibriums; in addition, there are concepts of trading which include random matching or order of turns of players, etc. Secondly, and more important, it could be that several matchings are efficient, and there could be multiple efficient solutions (we have a "coordination problem"). In this work I consider bargaining rules which lead to the efficient matching (axiom 4), and in addition I assume that if the matching is unique, the payoff vector is unique either (axiom 1). If the distribution of noise are such that no one value of noise appears with the positive probability, then the considered bargaining rules give unique outcomes with the probability one, and there are no equilibria selection problems.

*and vector of payoffs:*

$$R : (G, K) \rightrightarrows (M, P) \tag{35}$$

*where $G$ is a bipartite graph of relationships, $K = (k_{11}, ..., k_{sb}, ...)$ is a vector of possible surpluses, $M \subseteq G$ is the bipartite graph of matching between sellers and buyers and $P = (p, q) = (p_1, ..., p_S, q_1, ..., p_B)$ is the vector of payoffs of sellers $(p)$ and buyers $(q)$ correspondingly.*

Let's now think about the natural properties of bargaining rules over networks (keeping in mind examples). Firstly, if the bargaining rule defines matching, then for this matching the payoffs of players should be determined without ambiguity, otherwise we should think about the solution selection in each case.

**Axiom 1.** ***Payoff Uniqueness.*** *With fixed graph of relationships $G$, vector of surpluses $K$ and matching $M$, the vector of payoffs is unique.*

Next, if parties trade, then they should get non-negative payoffs, because they are rational and they may refuse to participate in trade (agents always have a zero outside options).

**Axiom 2.** ***Participation Rationality.*** *Each agent gets non-negative profit:*

$$P \geq 0 \tag{36}$$

Now, the bargaining rule is a rule of total surplus dividing. Thus the sum of payoffs should not exceed the overall surplus. It could be the case that parties may burn money, but this is out of scope of our work by now. Thus we consider rules for which the budget is balanced.

**Axiom 3.** ***Balanced Budget.*** *The total sum of gains is equal to the sum of surpluses over the matching:*

$$\sum_{j=1}^{S} p_j + \sum_{j=1}^{B} q_j = \sum_{ij \in M} k_{ij} \tag{37}$$

It is not always the case that bargaining rule leads to the ex-post efficient allocation of goods. However the aim of this paper is not to study ex-post efficiency, and we restrict our attention only to the efficient bargaining rules, although in might be interesting to study the Hold-up problem with some natural bargaining rules which do not necessarily lead to the efficient matching.

**Axiom 4.** ***Efficiency.*** *The matching is efficient:*

$$M \in \underset{M \in \mathcal{M}}{Argmax} \left[ \sum_{ij \in M} k_{ij} \right] \tag{38}$$

It is natural to assume that sellers (buyers) are initially homogenous and they may get different surpluses only because they are not in equal conditions in terms of economic relationships or because surpluses in different links are different. However, if all else is equal, sellers (buyers) should get equal payoffs. Many bargaining rules have this property, but some do not. However, it is an important property, since it may lead to the symmetry of the set of Nash equilibriums in the investment's game.

**Axiom 5.** ***Symmetry I.***[3] *The matching and payoffs depend only on the structure of network and surpluses, but not on the index number of agents. Let* $\pi = \pi_s \cdot \pi_b$ *be the permutation among sellers and among buyers (*$\pi_s$ *acts on the set of index number of sellers, and* $\pi_b$ *- the same for buyers). Then:*

$$R(\pi G, \pi K) = \pi R(G, K) \tag{39}$$

As we can see from simple examples, mentioned above, for common bargaining rules the payoffs of players for certain domains of surpluses' values tend to be linear in terms of surpluses; however, the payoff may jump up or down when vector of surpluses goes from one domain to another. We may think that subject to some global conditions (for example, $k_{sb} \geq 0$ for particular $sb$), agents have this or that "bargaining conditions" when they trade with each other, and they share surpluses differently subject to the global situation (for example, this or that link may be considered as a credible alternative in bargaining process or may not). I try to capture this property by the following axiom:

**Axiom 6.** ***Piecewise Linearity.*** *Let the space of surpluses* $R^{|G|}$ *be divided on the subsets* $X_1, ..., X_N$ *by the finite number of hyperplanes:* $\sum\limits_{ij \in G} \beta^l_{ij} k_{ij} + \beta_0 = 0$ *for* $l = 1, ..., L$*. Then vector of players' payoff is linear inside each subset* $k \in int(X_n)$*:*

$$k \in int(X_n) \ \Rightarrow P(k) = \sum_{ij \in G} \alpha^n_{ij} k_{ij} + \alpha^n_0 \tag{40}$$

*(here each of* $a^n_{ij}$ *and* $a^n_0$ *is a* $(B + S)$*-dimensional vector)*

Consider some matching $M$ and a link in this matching, $ij \in M$. Generally, we allow payoffs that of seller $i$ and buyer $j$ are not equal to $k_{ij}$. This means, that there is some monetary transfers (positive or negative) from this pair to other agents. However, it is more naturally to assume, that it is only buyer $j$ who makes monetary payment to seller $j$, that is: $q_j = k_{ij} - t$, $p_i = t$, and there is no any cross transfers between different buyer-seller pairs (but still monetary transfer $t$ inside each pair may depend on all surpluses over network G).

**Axiom 7.** ***No Cross-Transfers.*** *There is no cross transfers between different matched pairs of buyers and sellers:*

$$ij \in M \ \Rightarrow p_i + q_j = k_{ij} \tag{41}$$

Note, that it is a local variant of Budget Balanced axiom (A3), and it implies Budget Balance (proof - sum eq. (41) over all inks in matching). Note also, that some coalition bargaining rules, such as Shapley Value, do not satisfy this axiom, since they give payoffs even to the players who do not engage in the matching.

**Discussion.**

[3] There is also possible symmetry of bargaining concept with respect to the switching to dual graph with all buyers changed by all sellers and vice versa. Thus I use letter I to distinguish the symmetry of bargaining rule in the narrow sense (without considering transformation to the dual graph)

I design the following possible axioms for the bargaining rules: Payoff Uniqueness (after agents agreed about matching, payoffs are determined uniquely), Participation Rationality (agents get non-negative payoffs), Balanced Budget (agents redistribute the total gain of trade over network among themselves), No Cross Transfers (each pair of matched seller and buyer divides the surplus of their joint project without receiving any money from the outside), Efficiency, Symmetry (meaning that bargaining rule respects network symmetries), and Piecewise Linearity (payoff of each agent is a piecewise linear function of all surpluses). The two most questionable axioms are Efficiency and Piecewise Linearity. Indeed, we may easily imagine that agents trade ex-post inefficiently; however there are a lot of questions which may be studied under the assumption of efficient bargaining rules, and we may postpone the considerations of the ex-post inefficiency for further studies. Piecewise linearity of bargaining rule looks like a common issue: it is not so easy to imagine the local non-linearity in surpluses. Additionally, linearity greatly simplifies the analysis, allowing us to make otherwise intractable calculations.

One important property of a network bargaining rule (in case of when agents match in pairs) could be a pairwise stability: there is no an agent of a matched pair, who wants to break a relationship; and there is no a pair of agents who may break their matchings and trade with each other, getting better payoffs. I do not investigate pairwise stability here, but it necessarily should be studied during further work on this theme.

# 4 General results

In this section I discuss general conclusions which can be made for the wide class of bargaining rules. At the beginning, I study the first best levels of investments for arbitrary network, which are independent of the bargaining rule. Three statements are proven. Firstly, with the cost function $C = \sum \frac{i_{ij}^2}{2}$ the FB investment in each link is equal to its probability to participate it the efficient matching ex-post. Secondly, in case of large noise limit the first best profile of investments levels is unique, and this profile respects the network symmetry. Thirdly, in case of small noise the efficient profile of investments consists of units investments for all links, including in some maximal matching and zeros for the other links.

Next, I go to the core result of the study: the existence of the ex-post bargaining rule, providing ex-ante efficient incentives to invest. I show, that if sellers are paid for the positive externality which they bring when they are added to the network of other agents (similar to the Expected Externality mechanism), this provides the efficient incentives to invest ex-ante. Moreover, this bargaining rule is nice in the sense that it satisfies all axioms (A1-A7). Then I discuss the benefits and reservations of the result.

After brief consideration of the role of network's symmetry in characterization of the set of Nash equilibriums, I proceed to the study of the large noise limit case. Similarly to the first best calculations, I show that under mild assumptions on the bargaining rule (it should satisfy axioms Payoff Uniqueness (A1), Participation Rationality (A2), Balanced Budget (A3) and Piecewise Linearity (A6)), there always exists an unique Nash equilibrium in pure strategies, and if bargaining rule satisfies Symmetry axiom (A5), then equilibrium levels of investments respects the symmetry of the network.

## 4.1 First Best calculations

Let's describe the first best levels of investments. One interesting note is that for the society there is no difference between buyers and sellers: we may always say that it is buyers who invest in links, not sellers. Obviously, the first-best levels of investments do not depend on the bargaining rule.

**Efficient Matching**

The efficient matching is determined as follows:

$$M \in \underset{M \in \mathcal{M}(G)}{Argmax} \left[ \sum_{ij \in M} k_{ij} \right] \tag{42}$$

**Efficient Levels of Investments**

Central Planner solves:

$$EW = E[max_M \{ \sum_{sb \in M} k_{sb} \}] - \sum_{sb \in G} \frac{i_{sb}^2}{2} \quad \rightarrow \quad \underset{i_{sb} \geq 0}{max} \tag{43}$$

where $M \subseteq G$ is an ex-post matching. For further needs let's divide the social welfare on the benefits from trade $W^T$ and investment costs:

$$W = W^T - \text{Costs} = \sum_{sb \in M} k_{sb} - \sum_{sb \in G} \frac{i_{sb}^2}{2} \tag{44}$$

Gradient of welfare function with respect to investments in each link $ij$ look as:

$$\frac{\partial W}{\partial i_{ij}} = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{\partial}{\partial i_{ij}} max_M \{ \sum_{sb \in M} (i_{sb} + \varepsilon_{sb}) \} dF(\varepsilon) - i_{ij} = \tag{45}$$

$$= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \mathbb{1}(ij \in M^{efficient}) dF(\varepsilon) - i_{ij}$$

where I use the Envelope Theorem to get

$$\frac{\partial}{\partial i_{ij}} max_M \{ \sum_{sb \in M} (i_{sb} + \varepsilon_{sb}) \} = max \{ \frac{\partial}{\partial i_{ij}} \sum_{sb \in M} (i_{sb} + \varepsilon_{sb}) \} = \mathbb{1}(ij \in M^{efficient})$$

Let's show, that at $i^{FB}$ FOC always holds as $\nabla EW = 0$. The only one opportunity when FOC holds as inequality could be when we have a corner solution $i_{jk} = 0$ for some $jk \in G$. However, gradient of the expected social welfare could not have a negative projection on the $i_{jk}$ Axes, since the marginal cost of zero investment is zero, and investment does not decrease the expected surpluses. Since FOCs are necessary conditions for the maximum of welfare functional, we have proved the following proposition:

**Proposition 1.** *The first-best level of investment in link is equal to the probability of this link to be in the efficient matching:*

$$i_{ij} = Pr[ij \in M^{efficient}] \tag{46}$$

Note, that event $ij \in M^{efficient}$ means that $ij$ is included in at least one of the efficient matchings (in case when there are several of them); however the probability measure of this event is zero. Efficiency of matchings, in its turn, depends on the level of investments, thus $\mid G \mid$ equations (46) determine candidates for the FB levels of investment in the implicit way.

Obviously, $i_{sb}^{FB} \in [0;1]$ and in vector form: $i^{FB} \in [0;1]^{|G|}$. Consequently, the First Best levels of investments always exist, since social welfare is a continuous function of $i$, and $i$ lies in the compact set $[0;1]^{|G|}$. Now consider the mapping:

$$H : [0;1]^{|G|} \to [0;1]^{|G|} \qquad H_{ij}(i) = Pr[ij \in M^{efficient}(i)] \tag{47}$$

We may alternatively say that each first best level of investments is a fixed point of the mapping $H$. In general, there could be a number of fixed points and we should study which of them give us the largest expected social welfare in order to find the first best level of investments (SOC at point with largest social welfare will be satisfied automatically for this points, since global maximum always exists and it is one of the fixed points of $H$). However, in some cases we may provide more clear answer. Intuitively, when the noise is large in comparison with investments, the latter do not affect probabilities of matching to be efficient significantly, and when the scale of noise goes to the infinity, we have the only one candidate for the efficient level of investments. Let's formalize this intuition:

**Proposition 2.** *$\forall$ graph of economic relationships $G$ $\exists$ the scale of noise $\overline{a}$ such that $\forall a > \overline{a}$ there exists a unique vector of first best investments $i^{FB}$.*

*Proof.*

**Lemma 1.** *Under the conditions of proposition (2) $\exists \overline{a}$ such that $\forall a > \overline{a}$ H(i) is a contraction mapping with respect to the Euclidian metrics in $R^{|G|}$.*

*Proof.* See Appendix. □

Since lemma 1 holds, and $[0;1]^{|G|}$ a compact set, then by the Contraction Mapping Theorem $H(i)$ has a unique fixed point, which is the desired vector of first best investments. □

Now suppose that noise is small (that is, $a \to 0$). Then social planner could ex-ante determine future efficient matching, and marginal expected social benefits from the investment in a links of this matching would be one, while in other links it would be zero. We formalize this by the following proposition.

**Proposition 3. *First best investments in case of small noise.***
*$\exists \underline{a} > 0$ such that $\forall a < \underline{a}$ the first-best levels of investments is given by:*

$$i_{ij}^{FB} = \begin{cases} 1 & \textit{if } ij \in M^{max} \\ 0 & \textit{otherwise} \end{cases} \tag{48}$$

*where $M^{max} \subseteq G$ is one of the maximal matchings, i.e. matching with maximal possible number of links.*

*Proof.* In what follows, I use Proposition (1): $i_{ij} = Pr[ij \in M^{efficient}]$.
Firstly, as $a \to 0$, we always can choose some matching, which has ex-ante $\sim O(1)$ probability to be the efficient matching ex-post (the total number of matchings is finite

number independent of $a$ and the sum of all probabilities is $1 \sim O(1)$). Let's denote this matching as $M_1$, and the corresponding probability as $r_1$. Assume that $M_1$ is not maximal. Then there is a link $sb \in G$, $sb \notin M$ with investment $i_{sb} < \frac{a}{2}$, otherwise it always has positive surplus and hence ex-post efficient matching should be $M_1 + sb$. However, by investing $i_{sb} = r_1$ society may increase its expected profit by at least $\frac{1}{2}r_1^2$. Hence, our assumption is false and $M_1$ should be maximal, as well as any other matching which has $\sim O(1)$ probability to appear as ex-post efficient one. Assume there are at least two matchings, $M_1$ and $M_2$, which appear with probabilities $r_1, r_2 \sim O(1)$. The difference between payoffs of these two matchings in any state of the world could not exceed $n \cdot a \sim O(a)$, where n is the maximal number of links in matching, otherwise one of them would be for sure strictly worse than another. Next, since these matchings are different and maximal, there exists a pair of distinct links $sb \in M_1$ and $s'b' \in M_2$, $s'b' \neq sb$. Denote

$$c(M_1) = \frac{1}{2} \sum_{ij \in M_1} \frac{i_{ij}^2}{2} \tag{49}$$

and similar for $c(M_2)$. WLOG, $c(M_1) \leq c(M_2)$. Then, it is a profitable for the society not to invest in the link $s'b'$ at all. Indeed, the maximum loss in terms of ex-post trade would be $na \sim O(a)$, but costs of the society decrease by $\frac{r_2^2}{2} \sim O(1)$. Hence, there is only one matching (WLOG, $M_1$), which appears with probability $\sim O(1)$. Then $\forall ij \in M_1$ we have $i_{ij} - 1 \sim O(a)$, and if $ij \notin M_1$, then $i_{ij} \sim O(a)$. Assume $Pr(M = M_1) < 1$. Then $\exists M_2 \neq M_1 : \; Pr(M = M_2) > 0$. Since $M_2 \neq M_1$, there is some link $s'b' \in M_2 : \; s'b' \notin M$. Hence $W(M_1) - W(M_2) \geq 1 - O(a)$, and $M_1$ gives strictly better social gains in any state of the world. Thus, $Pr(M = M_1) = 1$, and $M_1$ is a maximal matching, which proves the proposition. $\square$

## 4.2 Bargaining rule, providing efficient incentives to invest

Here I investigate the existence of bargaining rule, which provides sellers with incentives to invest efficiently[4]. Let's call it "The ex-ante efficient bargaining rule". One idea how to design this kind of rule is to align individuals' (sellers') incentives with social ones. Let's try to find the bargaining rule, which gives each seller the expected utility, equal to

$$EU_i = Ep_i - \sum_{j:\; ij \in G} \frac{i_{ij}^2}{2} = EW - g(\mathbf{i_{-i}}) \tag{50}$$

Where $g(\mathbf{i_{-i}})$ is some function of investments of other players. Remind that total social welfare $W$ is a difference between total gains of trade and costs of investments: $W = W^T - Costs$. Then one natural candidate for this rule is to give each seller

$$p_i(k) = W^T(k) - h(\mathbf{k_{-i}}) \tag{51}$$

[4]Recently I found that my results are similar to that of Cole et al. (2001) for the hold-up problems in the finite economics. However, Harold and George Mailathy study the case of full graph, they allow only one-dimensional vector of investments for each player, and work under the assumption of zero noise. Thus my findings are generalize their results for the case of trade over network, more flexible investment choices and structure of uncertainty

where $h(\mathbf{k_{-i}})$ is a function of surpluses over links, not adjacent to seller $i$. The best response of seller $i$ is derived from the maximization of his expected utility:

$$EU_i = E[W^T(k)] - \sum_{j:\ ij \in G} \frac{i_{ij}^2}{2} - E[h(\mathbf{k_{-i}})] = EW + \sum_{i' \neq i, i'j \in G} \frac{i_{i'j}^2}{2} - E[h(\mathbf{k_{-i}})] = \tag{52}$$

$$= EW - g(\mathbf{i_{-i}}) \qquad \text{where } \ g(\mathbf{i_{-i}}) = E[h(\mathbf{k_{-i}})] + \sum_{i' \neq i, i'j \in G} \frac{i_{i'j}^2}{2}$$

What kind of function could be $h(\mathbf{k_{-i}})$ in order to get nice properties of the bargaining rule? In the spirit of Expected Externality mechanism, we can think about social gain from the trade, which other agents can gain together without the considered seller. More formally, the maximum social gain of trade is

$$W^T = \max_{M \in \mathcal{M}(G)} \sum_{ij \in M} k_{ij} \tag{53}$$

Next, let $L_{(s)}(i)$ be the set of links, adjacent in network $G$ to the seller $i$, and analogously $L_{(b)}(j)$ be the set of links, adjacent in network $G$ to the buyer $j$:

$$L_{(s)}(i) = \{ij : ij \in G\} \tag{54}$$

Denote $W_{-i}$ to be a maximum social welfare for the graph $G$ without seller $i$, that is:

$$W_{-i} = \max_{\hat{M} \in \mathcal{M}(G \setminus L_{(s)}(i))} \sum_{lm \in \hat{M}} k_{lm} \tag{55}$$

Then we simply set

$$h(\mathbf{k_{-i}}) = W_{-i} \tag{56}$$

We should also define buyers' payoffs. Let's set them such that No Cross Transfer axiom is satisfied automatically:

$$q_j = k_{M_{(s)}(j)j} - p_{M_{(s)}(j)} \tag{57}$$

The good think is that if we require matching to be efficient, then this bargaining rule satisfies all bargaining axioms.

**Proposition 4.** ***FB investment implementation through bargaining rule.***
*Let the bargaining rule $R$ be described as follows:*

$$R(G, K) = \Big( M \text{ is efficient }; \ p_i = W^T - W^T_{-i} \ , \ q_j = k_{M_{(b)}(j)j} - W^T + W^T_{-M_{(b)}(j)}, \tag{58}$$

$$q_j = 0 \text{ if buyer } j \text{ is unmatched} \Big)$$

*where $W^T$ is the maximum social gain from the trade over the whole network $G$ and $W^T_{-i}$ is the maximum social gain from the trade over the network $G$ without seller $i$. Then:*

1. *Bargaining rule $R$ satisfies axioms Payoff Uniqueness (A1), Participation Rationality (A2), Balanced Budget (A3), Efficiency (A4), Symmetry I (A5), Piecewise Linearity (A6) and No Cross-Transfers (A7).*

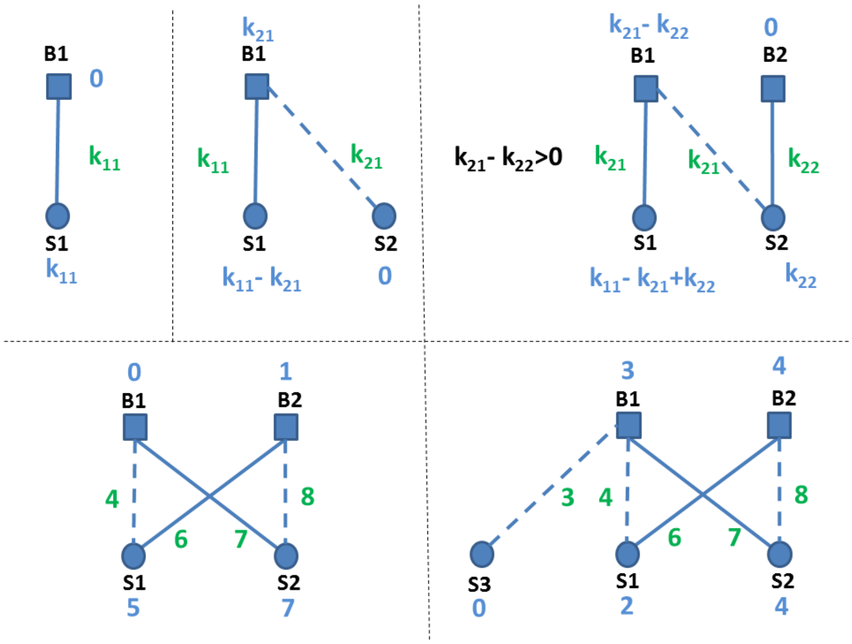


Figure 4: Examples of payoff distributions for the ex-ante effective bargaining rule. Green variables and numbers are values of links (surpluses), while blue variables and numbers are payoffs of the agents. Solid lines represent matching, while dashed lines represent non-realized opportunities to trade.

2. *Each efficient profile of investments $i^{FB}$ is a Nash equilibrium profile of sellers' investments in the game where payoffs are distributed according to the rule $R$.*

*Proof.* Let's show that any $i^{FB}$ is an equilibrium profile of investments. By construction, $W^T_{-i}$ does not depend on values of surpluses of links, adjacent to the seller $i$. Hence, seller $i$ solves the following problem (given that other sellers play $i^{FB}_{-i.}$):

$$EU_i = Ep_i - \sum_{j:\ ij\in G} \frac{i^2_{ij}}{2} = EW^T - EW^T_{-i} - \sum_{j:\ ij\in G} \frac{i^2_{ij}}{2} = \tag{59}$$

$$= EW^T - \sum_{i'j\in G} \frac{i^2_{i'j}}{2} + \sum_{i'\neq i,\ i'j\in G} \frac{i^2_{i'j}}{2} - EW^T_{-i} = EW + \sum_{i'\neq i,\ i'j\in G} \frac{i^2_{i'j}}{2} - EW^T_{-i} \ \rightarrow \ \max_{\mathbf{i_i}} \Leftrightarrow$$

$$\Leftrightarrow E[W(\mathbf{i_i} + \varepsilon_{\mathbf{i}}, \mathbf{i}^{FB}_{-\mathbf{i}} + \varepsilon_{-\mathbf{i}})] \ \rightarrow \ \max_{\mathbf{i_i}}$$

where $\mathbf{i_i}$ stands for the vector of investments of seller $i$, $\mathbf{i_{-i}}$ stands for the vector of investments for all other sellers, and similar for $\varepsilon_{\mathbf{i}}$ and $\varepsilon_{-\mathbf{i}}$. Clearly,

$$i^{FB} \in \underset{i}{\mathrm{Argmax}} E[W(i+\varepsilon)] \ \Rightarrow \ \mathbf{i}^{FB}_{\mathbf{i}} \in \underset{i_{i.}}{\mathrm{Argmax}} E[W(\mathbf{i_i} + \varepsilon_{\mathbf{i}}, \mathbf{i}^{FB}_{-\mathbf{i}} + \varepsilon_{-\mathbf{i}})] \tag{60}$$

Hence, there is an interception of players' best responses, i.e. NE, where sellers invest $i^{FB}$. Now let's check, that $R$ indeed satisfies Axioms (1-7):

1. (Payoff uniqueness). Sellers' payoffs are unique, since $W^T$ and $W^T_{-i}$ each have exactly one value for the given vector $K$ (it is a maximum over finite number of alternatives).

Buyers' payoffs are also unique by the same argument.

2. (Participation rationality). Clearly, $W^T \geq W^T_{-i}$, since we may always choose in $G$ the same matching as in $G \backslash L_{(s)}(i)$, and get at least $W^T_{-i}$. Hence,

$$p_i = W^T - W^T_{-i} \geq 0 \tag{61}$$

Next, consider $q_j$. We have:

$$p_i = k_{iM_{(s)}(i)} + \sum_{lm \in M \backslash L_{(s)}(i)} (k_{lm}) \; - \; W^T_{-i} \quad \leq \tag{62}$$

$$\leq k_{iM_{(s)}(i)} + \max_{\hat{M} \in \mathcal{M}(G \backslash L_{(s)}(i))} \sum_{lm \in \hat{M}} (k_{lm}) \; - \; W^T_{-i} = k_{iM_{(s)}(i)}$$

Hence,

$$q_j = \begin{cases} 0 & \text{if buyer } j \text{ is unmatched} \\ k_{M_{(b)}(j)j} - p_{M_{(b)}(j)} \geq k_{M_{(b)}(j)j} - k_{M_b(j)M_{(s)}(M_{(b)}(j))} = 0 & \text{otherwise} \end{cases} \tag{63}$$

where we have used $M_{(s)}(M_{(b)}(j)) = j$.

3. (Balanced budget). It follows from No Cross Transfers Axiom.

4. (Efficiency). Efficiency holds by the construction.

5. (Symmetry I). Firstly, the efficient matching does not depend on the index numbers of agents. Secondly, $\pi p_i = W^T - W^T_{\pi(i)} = p_{\pi(i)}$, and finally if buyer $j$ participates in matching M, then $\pi q_j = \pi k_{ij} - W^T_{\pi(i)} = q_{\pi j}$, and if not, then also $\pi q_j = 0 = q_{\pi(j)}$.

6. (Piecewise Linearity). Denote $M_{-i}$ to be the efficient matching for $G \backslash L_{(s)}(i)$. Each matching $M$ and $M_{-i}$ remains the same inside the region, where sum of surpluses over its links is maximal over all feasible matches. These regions are determined by the set of linear inequalities:

$$\forall \hat{M} \in \mathcal{M}(G), \hat{M} \neq M \;\; : \quad \sum_{ij \in M} k_{ij} - \sum_{ij \in \hat{M}} k_{ij} \geq 0 \tag{64}$$

for $M$, and similar for each $M_{-i}$. Clearly, it means that these regions are bounded by the corresponding hyperplanes (the inequalities become equations on the boundaries of regions). Consider now the set of regions $X_1, ..., X_L$, which appears when we consider all possible secant hyperplanes for all matchings in all graphs $G, G \backslash L_{(s)1}, ..., G \backslash L_{(s)S}$. Inside each region $X_l$ all efficient matchings for these graphs remains the same, and since $W^T$ and $W^T_{-i}$ are sums of surpluses over these matchings, there linear combinations are linear functions of $k$. Hence, $p_i$ and $q_j$ are linear functions of $k$, QED.

7. (No Cross-Transfers). Holds by construction:

$$ij \in M \;\Rightarrow\; p_i + q_j = W^T - W^T_i + k_{ij} - W^T + W^T_i = k_{ij} \tag{65}$$

$\square$

**Discussion**

One important reservation of this result is that the set of NE profiles of investments includes the set of first best profiles, but not vice versa. Thus, even with this bargaining rule there could be other Nash equilibria with inefficient levels of investments, and sellers may fail to coordinate on the FB equilibrium. This finding is in line with that of Cole et al. (2001).

Great advantage of this result is that it is robust to the changes of the model in several senses. Firstly, it does not matter, whether sellers make only specific investments, or they may make general investments as well: the core idea is that their incentives are in line with social ones. Secondly, the result is independent of our assumption on the noise terms: its joint distribution could have arbitrary structure. Indeed, all expectations are taken without any assumptions on the distribution of noise, except of existence of first moments. Finally, the result does not depend on the particular form of the cost function (which is otherwise limited by our quadratic in investments choice), since cost function of investments of the society is additive with respect to cost functions of investments of each seller.

There is also another important question, which needs further investigation: whether this bargaining rule is pairwise stable or not. I have a hypothesis that it is indeed pairwise stable; moreover, it seemed to be a coalitional stable concept (there is no one coalition of agents who may separate themselves from others and get from their trade larger payoff, than they get in sum according to our rule.

## 4.3 The role of network symmetries

Symmetry plays an important role in the structure of Nash equilibria of the game, and it is useful to clarify what conclusions can be made based on the symmetry considerations. I claim that if a bargaining rule satisfies Symmetry I axiom, then each Nash equilibrium in the investment game of sellers belongs to a family of Nash equilibriums, corresponding to the orbit of a symmetry group of graph $G$. Specifically, let $\pi = \pi_s \pi_b$ is a group of index permutations, for which graph $G$ is invariant. Let $i^{(1)}$ is an equilibrium vector of investments with components $i_{jk} = i^{(1)}_{jk}$. Then vector $\pi(i^{(1)})$ with components $i_{jk} = i^{(1)}_{\pi_s^{-1}(j)\pi_b^{-1}(k)}$ is also an equilibrium vector of investments. Let's formalize this:

**Proposition 5.** *Assume, bargaining rule satisfies Symmetry I axiom. Then each Nash equilibrium profile of investments $i^{NE}$ generates an orbit of equilibriums, corresponding to the symmetry group of a graph $G$.*

*Proof.* Sketch of the proof. The conclusion of the theorem follows from the following points:

- The bargaining rule is symmetric. Thus,

$$p_{\pi(i)}(k) = p_i(\pi k) \tag{66}$$

- The joint distribution of the noise is symmetric. This, $\forall$ function $g(\alpha; \varepsilon)$ we have:

$$E[g(\alpha; \pi\varepsilon) = E[g(\alpha; \varepsilon)]] \tag{67}$$

- Cost functions transforms one into another by the symmetry of graph G:

$$\pi \circ (\sum_{j:ij\in G} \frac{i_{ij}^2}{2}) = \sum_{j:\pi(i)j\in \pi G} \frac{i_{\pi(i)j}^2}{2} = \sum_{j:\pi(i)j\in G} \frac{i_{\pi(i)j}^2}{2} \tag{68}$$

Then, given the vector of other sellers' investments, being components of vector $\hat{i}_{lm} = i^{NE}_{\pi(l)\pi(m)}$, seller $i$ solves the same problem, as seller $\pi(i)$ with the vector of other seller's investments being components of vector $i^{NE}$, and since the solution of the latter problem yields NE outcome, the corresponding solution of the former problem give the NE outcome too. □

## 4.4 Large noise limit

Here I study the existence of NE under the assumption of large noise limit. I show that similar to the results of the proposition (2), if noise is large ($a \to \infty$), then under suitable conditions there exists a unique Nash equilibrium of the investment's game.

**Proposition 6.** ***Sufficient conditions for the unique NE in large noise limit.*** *Let the bargaining rule satisfies Payoff Uniqueness (A1), Participation Rationality (A2), Balanced Budget (A3) and Piecewise Linearity (A6) axioms. Then there exists $\overline{a}$ such that $\forall a > \overline{a}$ there is a unique Nash equilibrium of the investments' game. Additionally, if Symmetry I (A5) axiom holds, then at this equilibrium all links, who are translated to each other by the symmetry of graph G, have the same levels of investments.*

*Proof.*

**Lemma 2.** *Under the conditions of proposition (6) $\forall n > 0 \ \exists \overline{a} : \ \forall a > \overline{a}$ the following operator is a contraction mapping on the set $X : \ i \in [0; n(1+\sqrt{1+a})]^{|G|}$:*

$$\widehat{H}(i): \quad \widehat{H}_{jk}(i) = \begin{cases} \frac{\partial}{\partial i_{jk}} Ep_j(i+\varepsilon) & if \quad \frac{\partial}{\partial i_{jk}} Ep_j(i+\varepsilon) \geq 0 \\ 0 & otherwise \end{cases} \tag{69}$$

*Proof.* See Appendix. □

For each seller FOCs look as:

$$\begin{cases} \frac{\partial}{\partial i_{jk}} Ep_j(i+\varepsilon) - i_{jk} = 0 & \text{if } i_{jk} > 0 \\ \frac{\partial}{\partial i_{jk}} Ep_j(i+\varepsilon) - i_{jk} = \frac{\partial}{\partial i_{jk}} Ep_j(i+\varepsilon) \leq 0 & \text{if } i_{jk} = 0 \end{cases} \tag{70}$$

Hence, all Nash equilibria are fixed points of the mapping $\widehat{H}(i)$. Let $n$ be the number of links in maximal matching. Denote $i_m = \max(i_{ij})$. Since Participation Rationality (2) and Balanced Budget (3) implies non-negative payoffs of all sellers, we have:

$$0 \leq \sum_{sellers} (\text{payoffs}) < n(i_m + \frac{a}{2}) - \frac{i_m^2}{2} \ \Rightarrow \ i_m \leq n(1+\sqrt{1+a}) \tag{71}$$

Finally, for fixed $a > \overline{a}$ the set $[0; n(1+\sqrt{1+a})]^{|G|}$ is compact, and using lemma (2) and Contraction Mapping Theorem, we find that $\widehat{H}(i)$ has exactly one fixed point $\forall a > \overline{a}$.

Now let's show, that the unique fixed point of our operator is indeed a NE, i.e. the intersection of best responses of sellers. First order conditions are satisfied for all sellers in this point, so we should check the second order conditions. Consider arbitrary seller $i$. The matrix of second derivatives of his payoff is given by:

$$SOC_{jk} = \frac{\partial^2 EU_i}{\partial_{ik}\partial_{ij}} = -I_{jk} + \frac{\partial \widehat{H}_{ij}}{\partial i_{ik}} \tag{72}$$

where $I_{jk}$ is an identity matrix. In the appendix it is shown that

$$| \frac{\partial H_{jk}(i)}{\partial i_{sb}} | \leq \frac{1}{a}\left[c_1(\alpha,\beta)\text{sup}(f_0) + c_2(\alpha,\beta)\text{sup}(f_0)^2 + c_3(\alpha,\beta)\text{sup}(f_0) + c_4(\alpha,\beta)\text{sup}(f_0')\right] \tag{73}$$

where $c(\alpha,\beta)$ are some functions of parameters of the bargaining rule. Thus, choosing sufficiently small $a$, we always can make matrix $SOC_{jk}$ negatively defined, and our fixed point indeed gives maximum expected utility for each seller.
If Symmetry I axiom holds, than by proposition (5) Nash equilibrium generates an orbit of group of symmetry of graph $G$. Since the equilibrium is unique, all elements of symmetry group reflects vector of equilibrium investments in itself. □

**Discussion** Now let's discuss possible properties of the equilibrium levels of investments, if it satisfies axioms (A1,A2,A3,A6) in case of large noise. The optimal decisions of each seller then depend on the decisions of others only by the term of order of $O(a^{-1})$, as it can be seen from the proof of the contraction mapping lemma in the appendix, because for $i_{ij} > 0$ we have:

$$\frac{\partial \widehat{H}_{ij}}{\partial i_{sb}} = O(a^{-1}) \tag{74}$$

Let's consider the solution (which is determined by FOC) up to the terms of order $O(a^{-1})$ (assuming $i_{ij} > 0$). Remind, that payoff of seller $i$ could be rewritten as:

$$p_i = \sum_{n=1}^{N} \Theta_n(k) \cdot \left( \sum_{i'j'} \alpha^n_{i,i'j'} k_{i'j'} + \alpha^n_{i,0} \right) \tag{75}$$

where

$$\Theta_n = \begin{cases} 1 & if\ k \in X_n \\ 0 & \text{otherwise} \end{cases} \tag{76}$$

is an indicator function for the region $X_n$, corresponding to some bargaining conditions. Then we have:

$$i_{ij} = \widehat{H}_{ij} = \int \sum_{n=1}^{N} \alpha^n_{i,ij} \Theta_n dF(\varepsilon) + \int \sum_{n=1}^{N} \frac{\partial \Theta_n}{\partial i_{ij}} \left( \sum_{i'j'} \alpha^n_{i,i'j'} k_{i'j'} + \alpha^n_{i,0} \right) dF(\varepsilon) = \tag{77}$$

$$= \sum_{n=1}^{N} \alpha^n_{i,ij} Pr(X_n) + \int \sum_{n=1}^{N} \frac{\partial \Theta_n}{\partial i_{ij}} \left( \sum_{i'j'} \alpha^n_{i,i'j'} k_{i'j'} + \alpha^n_{i,0} \right) dF(\varepsilon)$$

The first term corresponds to the averaged share of surplus of the link $ij$, that seller $i$ gets, weighted on the probabilities of the different "bargaining situations" $X_n$. The second term consists of sum of integrals with delta-functions, which after the integration becomes integrals over the $\mid G \mid -1$ dimensional hyperplanes (the borders of bargaining regions $X_n$). This term takes into account the discontinuities of the payoff function on the borders of regions $X_n$: subject to the bargaining conditions $X_n$, the seller may get different limits of payoffs on the borders of these regions, and it is indeed the case for many bargaining rules (for example, it can be shown that it is the case for the "$N$" network with the Balanced Nash bargaining rule).

We can think about the two types of network effects which influence the resolution of the hold-up problem in the large noise limit. The first type correspond to the first term in eq. (77): locally (subject to the event $X_n$), bargaining rule determine coefficients $\alpha^n_{i,ij}$, which stand for the share of surpluses over the corresponding links $ij$, which are given to the seller $i$. Thus we should compare average $\alpha_{i,ij}$-s with the probabilities of the corresponding links to be in the efficient matching in order to understand, how far are these incentives from the social ones.

The second type of network effects corresponds to the second term in (77). As we can see, if seller gets better bargaining power and gets larger payoff when switching from $X_n$ to $X_m$, then he has an incentives to invest (or disinvest) in order to increase the probability of $X_m$ and decrease the probability of $X_n$. For natural bargaining rules we may intuitively expect that the role of the possible surplus of a particular link $k_{ij}$ in ex-post trade increases with the increasing of $k_{ij}$, thus we may expect that this term indeed provides sellers with the additional incentives to invest in links in order to get better bargaining position. This may be shown by the examples, and in particular, it may be shown that some bargaining rules provide incentives to overinvest in comparison with the first best profile of investments.

# 5 Nash Bargaining solution with endogenous outside options

In this section I discuss the generalization of Nash Bargaining solution for one-seller-one-buyer case on the case of agents, connected over network. Here I use a few results from the paper Kleinberg & Tardos (2008) of Jon Kleinberg and Éva Tardos (and also I use the notation "Balanced" Nash Bargaining solution over network - their "Balanced Outcomes in Social Exchange Networks"). The purpose of this section is firstly to find out a nice natural generalization of the Nash Bargaining solution, which is correctly defined and satisfies bargaining axioms, and secondly to investigate, how payoffs of sellers depend on the surpluses of their adjacent links in order to realize their incentives to invest.

The basic idea seems to be simple: let's search for some trade concept, which gives us Nash Bargaining solution for the case of 1 seller and 1 buyer; and additionally in case of several buyers and sellers let each matched pair bargain over the value of their joint project, taking into account outside options, consisting of opportunities to attract other agents. That is, payoffs of a seller and a buyer in each matched pair are determined as follows:

$$p = OO_s + \frac{1}{2}(k - OO_s - OO_b) = \frac{1}{2}k + \frac{1}{2}OO_s - \frac{1}{2}OO_b \tag{78}$$

$$q = OO_b + \frac{1}{2}(k - OO_s - OO_b) = \frac{1}{2}k - \frac{1}{2}OO_s + \frac{1}{2}OO_b \quad (79)$$

When we study the solution for one-seller-one-buyer case, we may consider outside options to be exogenously given. But in case of network, we have to determine them endogenously through the payoffs of other players. Assume seller 1 is matched with buyer 1. Suppose that he also is connected with buyers 2 and 3, and the possible value of their joint project are $k_{12}$ and $k_{13}$ correspondingly. Assume also that buyer 2 is matched with seller 2, and gets payoff $q_2$, and buyer 3 is unmatched. Seller 1 may consider to outside opportunities. Firstly, he may try to attract buyer 3 by giving him slightly larger than zero money and getting the rest of $k_{13}$. Another alternative is to try attract buyer 2 by giving him slightly more than $q_2$, and getting the rest $k_{12} - q_2$. The best of these options is seller 1's outside option (given that it is greater than zero). In our example,

$$OO_{(s)1} = max\{k_{13}, k_{12} - q_2, 0\} = max\{k_{13} - q_3, k_{12} - q_2, 0\} \quad (80)$$

Since the payoff of unmatched buyer 3 is zero. Analogously we may think about the outside options of the buyer 1 and about all other agents. Thus, we may define:

**Definition 8.** ***Balanced Nash Bargaining (BNB) solution*** *Let $M$ be the efficient matching. Balanced Nash Bargaining payoffs is a* <u>*non-negative*</u> *solution with respect to $(p, q)$ of the following system of equations:*

$$p = 0 \qquad \textit{if agent is unmatched, otherwise:} \quad (81)$$

$$p_i = \frac{1}{2}k_{iM^{(s)}(i)} + \frac{1}{2}OO_{(s)i} - \frac{1}{2}OO_{(b)M^{(s)}(i)}$$

$$q_j = \frac{1}{2}k_{iM^{(b)}(j)} + \frac{1}{2}OO_{(b)j} - \frac{1}{2}OO_{(s)M^{(b)}(j)}$$

$$OO_{(s)i} = \max_{ib \in G, b \neq M^{(s)}(i)} \{k_{ib} - q_b, 0\}$$

$$OO_{(b)i} = \max_{sj \in G, s \neq M^{(b)}(j)} \{k_{sj} - p_s, 0\}$$

*Here $M^{(s)}(i)$ is an index of buyer who is matched with seller $i$, and analogously $M^{(b)}(j)$ is a seller who is matched with buyer $j$.*

This concept gives reasonable solutions, however there are two subtle points. Firstly, it there is a possibility that there are several different profiles of outside options, giving different solutions. Another opportunity is that there are multiple solutions of these equations even with the same graph of outside options.

## 5.1 Multiplicity of Balanced Nash Bargaining solutions

**Example of multiplicity of BNB solutions:** Consider the full bipartite $2 \times 2$ graph: $G = \{11, 12, 21, 22\}$. Let the vector of surpluses is $k_{11} = k_{22} = 10$, $k_{12} = k_{21} = 9$. The unique efficient matching is $M = \{11, 22\}$. Then it is easy to see, that $\forall \lambda \in [0; 8]$ the

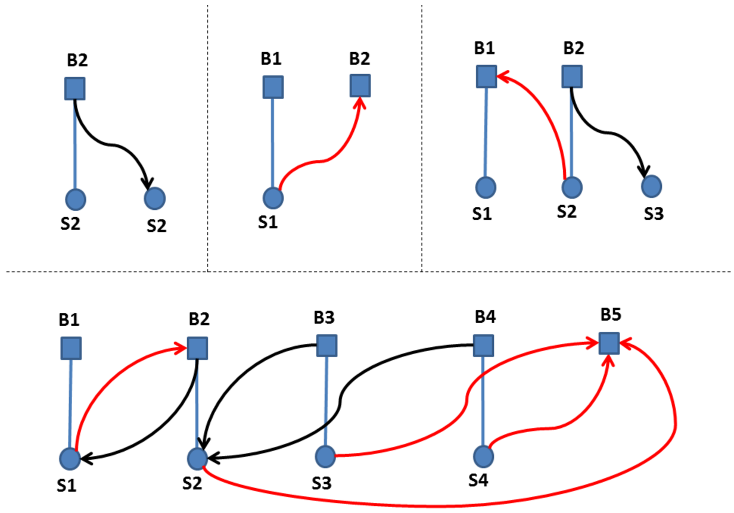


Figure 5: examples of BNB graphs $\mathcal{H} = \{M; O^{(s)}; O^{(b)}\}$ . Blue lines represent matching $M$, red arrows go from sellers to their outside options and represent the graph of sellers' outside options $O^{(s)}$, black arrows go from buyers to their outside options and represent the graph of buyers' outside options.

following is the BNB solution:

$$p_1 = p_2 = 1 + \lambda \tag{82}$$
$$q_1 = q_2 = 9 - \lambda$$

Indeed, consider the first seller. His outside option is $OO_{(s)1} = k_{12} - q_2 = 9 - 9 + \lambda = \lambda \geq 0$. The outside option of the first buyer is $OO_{(b)1} = k_{21} - q_2 = 9 - 1 - \lambda = 8 - \lambda \geq 0$. Hence, $p_1 = \frac{1}{2}k_{11} + \frac{1}{2}OO_{(s)1} - \frac{1}{2}OO_{(b)1} = 5 + \frac{\lambda}{2} - 4 + \frac{\lambda}{2} = 1 + \lambda$, which is correct. Analogous equations hold for the second seller and second buyer. Let's investigate, what is the source of multiplicity in the considered example. Assume that all outside options are positive. Let's express payoffs of buyers through sellers' payoffs:

$$q_1 = k_{11} - p_1 \qquad\qquad q_2 = k_{22} - p_2 \tag{83}$$

Then the system of equations looks as:

$$p_1 = \frac{1}{2}k_{11} + \frac{1}{2}(k_{12} - k_{22} + p_2) - \frac{1}{2}(k_{21} - p_2) = p_2 + \frac{1}{2}\left(k_{11} - k_{22} + k_{12} - k_{21}\right) \tag{84}$$
$$p_2 = \frac{1}{2}k_{22} + \frac{1}{2}(k_{21} - k_{11} + p_1) - \frac{1}{2}(k_{12} - p_1) = p_1 - \frac{1}{2}\left(k_{11} - k_{22} + k_{12} - k_{21}\right)$$

Clearly, these equations are linearly dependent, which gives us multiple solutions. Let's turn to the general case. Using the fact, that

$$q_{M^{(i)}(i)} = k_{iM^{(s)}(i)} - p_i \tag{85}$$

we can rewrite the BNB equations in terms of sellers' payoffs as:

$$p_i = 0 \qquad \text{if seller } i \text{ is unmatched, otherwise:} \tag{86}$$

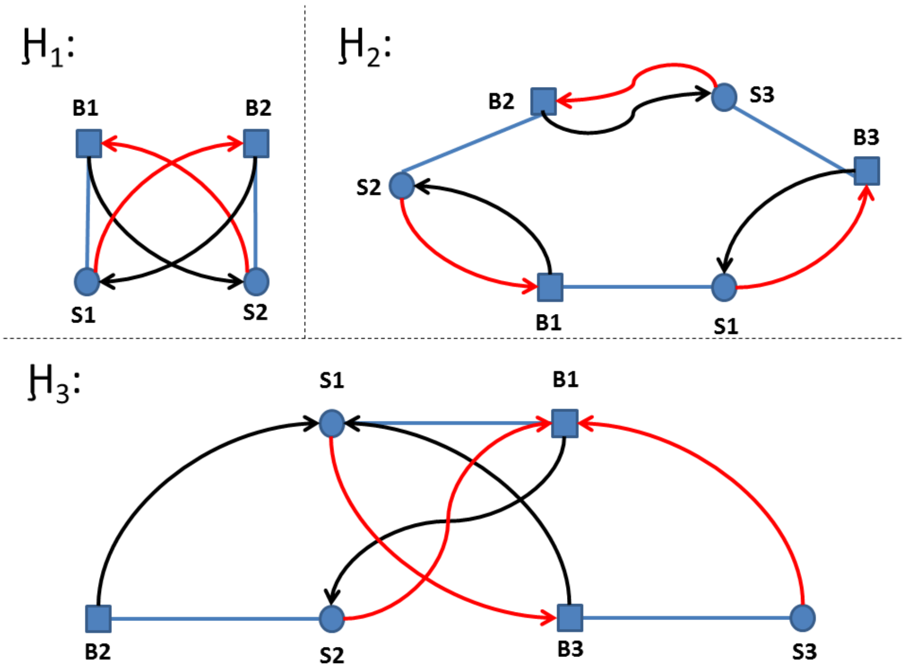


Figure 6: Candidates for BNB graphs $\mathcal{H} = \{M; O^{(s)}; O^{(b)}\}$ with the directed cycles, giving multiple solutions (top graphs $\mathcal{H}_1$, $\mathcal{H}_2$) and no solutions (bottom graph $\mathcal{H}_3$). Blue lines represents matching, red arrows - outside option of sellers, and black arrows - outside options of buyers.

$$p_i = \frac{1}{2}k_{iM^{(s)}(i)} + \frac{1}{2}\mathbb{1}(OO_{(s)i} \geq 0) \underset{ib\in G, b\neq M^{(s)}(i)}{max} \{k_{ib} - k_{M^{(b)}(b)b} + p_{M^{(b)}(b)}\} -$$

$$- \frac{1}{2}\mathbb{1}(OO_{(b)M^{(s)}(i)} \geq 0) \underset{sM^{(s)}(i)\in G, s\neq}{max} \{k_{sM^{(s)}(i)} - p_s\}$$

where $OO_{...}$ means the same as in the previous equations.

Consider some BNB solution and fix the outside options of all players. Then we have the triplet of graphs: $\mathcal{H} = \{M; O^{(s)}; O^{(b)}\}$ where $M$ represents matching, $O^{(s)}$ represents the directed graph of sellers' outside options and $O^{(b)}$ represents the directed graph of buyers' outside options. For the sake of simplicity, we will call the triplet of graphs $\mathcal{H}$ just a "BNB graph". If buyer $j'$ is an outside option for the seller $i$, let's denote $j' = O^{(s)}(i)$, and if seller $i'$ is an outside option for the buyer $j$, let's say that $i' = O^{(b)}(j)$. Let's say also that if seller $i$ has no match, than $M^{(s)}(i) = \diamond$, if he has no outside option, then $O^{(s)}(i) = \diamond$, and similar if some buyer has no outside option. Denote also $k_{i\diamond} = k_{\diamond j} = k_{\diamond\diamond} = 0$. Finally, let's introduce "fictive" seller $\diamond$ with payoff equal to $p_\diamond$ (we may think that it is an exogenous outside seller) and let's require $M^{(s)}(\diamond) = M^{(b)}(\diamond) = O^{(s)}(\diamond) = O^{(b)}(\diamond) = \diamond$, and also $p_\diamond = 0$. With these notations we can rewrite our equations as:

$$\forall i = 1, ..., S: \quad p_i = \frac{1}{2}k_{iM^{(s)}(i)} + \frac{1}{2}(k_{iO^{(s)}(i)} - k_{M^{(b)}(O^{(s)}(i))O^{(s)}(i)} + p_{M^{(b)}(O^{(s)}(i))}) - \tag{87}$$

$$- \frac{1}{2}(k_{O^{(b)}(M^{(s)}(i))M^{(s)}(i)} - p_{O^{(b)}(M^{(s)}(i))})$$

$$p_\diamond = 0$$

Finally, introducing $S+1$ dimensional vector $\hat{K}$ with components

$$\forall i = 1, ..., S: \quad \hat{K}_i = k_{iM^{(s)}(i)} + k_{iO^{(s)}(i)} - k_{M^{(b)}(O^{(s)}(i))O^{(s)}(i)} - k_{O^{(b)}(M^{(s)}(i))M^{(s)}(i)} \tag{88}$$

$$\hat{K}_\diamond = 0$$

we can rewrite these $S+1$ equations in matrix form:

$$(I - \frac{1}{2}M^{(b)}O^{(s)} - \frac{1}{2}O^{(b)}M^{(s)})p = \frac{1}{2}\hat{K} \tag{89}$$

where $I$ is $(S+1)\times(S+1)$ identity matrix;

$$\forall i = 1, ..., S: \quad M^{(s)}_{ij} = \begin{cases} 1 & \text{if } \ M^{(s)}(i) = j \\ 0 & \text{otherwise} \end{cases} \tag{90}$$

$$\forall j = 1, ..., B: \quad M^{(b)}_{ji} = \begin{cases} 1 & \text{if } \ M^{(s)}(j) = i \\ 0 & \text{otherwise} \end{cases}$$

$$\forall i = 1, ..., S: \quad O^{(s)}_{ij} = \begin{cases} 1 & \text{if } \ O^{(s)}(i) = j \\ 0 & \text{otherwise} \end{cases}$$

$$\forall j = 1, ..., B: \quad O^{(b)}_{ji} = \begin{cases} 1 & \text{if } \ O^{(b)}(j) = i \\ 0 & \text{otherwise} \end{cases}$$

and $M^{(s)}_{\diamond j} = M^{(b)}_{i\diamond} = O^{(s)}_{\diamond j} = O^{(b)}_{i\diamond} = 0$. Hence, we deal with the linear system of equations. Clearly, the matrix

$$A = (I - \frac{1}{2}M^{(b)}O^{(s)} - \frac{1}{2}O^{(b)}M^{(s)}) \tag{91}$$

plays a crucial role in the question of existence and multiplicity of solutions. If $rank(A) = full$, then the solution exists and it is unique; however if $rank(A) \neq full$, then it could have either a continuum of solutions (as in our case with $2\times 2$ full bipartite graph), or it could have no solutions.

## 5.2 Refinement of multiplicity of Balance Nash Bargaining solutions: BNB-delta solution.

Here I describe one possible approach of refinement of multiplicity of the BNB solution, which I call "BNB-delta solution" trade rule. Strictly speaking, it is a slightly different bargaining concept, but my definition coincide with the BNB solution in case when parameter $\delta = 1$. Thus, we may investigate the BNB-delta solution for $\delta$ close to 1, and intuitively our conclusions about the properties of investment game's equilibria would be the same as for the BNB solution with the advantage that our bargaining concept is not ill-defined and satisfies all bargaining axioms. I proceed as follows. Firstly, I define the bargaining rule, then investigate whether it satisfies bargaining axioms and show, that it exhibits continuity of payoffs given that matching remains the same. After that, I discuss local and global monotonicity properties of the BNB-delta solution and finish with the Proposition (7) that summarizes all findings.

**• Definition of the bargaining rule**

There could be several approaches of how to deal with multiplicity of solutions, and one seems to be straightforward: we may transform matrix $A$ in eq. (91) in such a way that it surely has full rank. Consider an alternative to Balanced Nash Bargaining concept, where agents discount outside option with some discount factor $\delta \in (0;1)$ (we may think that there that is, "$BNB^{\delta}$" equilibrium is defined as a solution of the set of equations:

$$p = 0 \;\; (\text{or } q = 0) \qquad \text{if agent is unmatched, otherwise:} \tag{92}$$

$$p_i = \frac{1}{2}k_{iM^{(s)}(i)} + \frac{1}{2}\delta OO_{(s)i} - \frac{1}{2}\delta OO_{(b)M^{(s)}(i)}$$
$$q_j = \frac{1}{2}k_{iM^{(b)}(j)} + \frac{1}{2}\delta OO_{(b)j} - \frac{1}{2}\delta OO_{(s)M^{(b)}(j)}$$
$$OO_{(s)i} = \max_{ib \in G, b \neq M^{(s)}(i)} \{k_{ib} - q_b, 0\}$$
$$OO_{(b)i} = \max_{sj \in G, s \neq M^{(b)}(j)} \{k_{sj} - p_s, 0\}$$

**Definition 9.** *BNB-delta solution is the following bargaining rule:*

1. *Matching $M$ is efficient.*
2. *Payoffs of agents are defined as solutions of eq. (92)*

**• Existence and Payoff Uniqueness**

Consider the region of surplus' vector $k$, where efficient matching remains constant. Using our trick with $\diamond$, we can rewrite the equations as:

$$\forall i = 1, ..., S: \tag{93}$$

$$p_i = \frac{1}{2}k_{iM^{(s)}(i)} + \frac{\delta}{2}\max_{j \neq M^{(s)}(i)}[k_{ij} - k_{M^{(b)}(j)j} + p_{M^{(b)}(j)}] - \frac{\delta}{2}\max_{l \neq i}[k_{lM^{(s)}(i)} - p_l]$$

Where first maximum goes over all buyers $j$ who are connected with seller $i$ except of seller $i$'s matching, but including exogenous outside option $\diamond$, which stands for the exogenous (external) outside option which gives seller $i$ zero payoff. Similarly, maximum over buyer's outside options include other sellers with whom he is connected and an exogenous outside option $\diamond$, giving him zero. Consider now the compact set $X = [0;x]^S \times \{0\}$. With sufficiently large $x$ (for example, $x = \sum_{ij \in M} k_{ij}$) we get: $(p_1, ..., p_S, p_\diamond) \in X$. Now consider an operator

$$G: \; X \to X \qquad G_\diamond(p_{(s)}) = 0 \qquad \forall i = 1, ..., S: \tag{94}$$

$$G_i(p_{(s)}) = \frac{1}{2}k_{iM^{(s)}(i)} + \frac{\delta}{2}\max_{j \neq M^{(s)}(i)}[k_{ij} - k_{M^{(b)}(j)j} + p_{M^{(b)}(j)}] - \frac{\delta}{2}\max_{l \neq i}[k_{lM^{(s)}(i)} - p_l]$$

Clearly, each solution of the equation 93 is a fixed point of operator $G$ and vice versa. The existence and payoff uniqueness of the BNB-delta bargaining rule follows from the fact, that as far as $\delta = [0;1)$, $G$ is a contraction mapping with respect to the metric in $X$, defined as the maximum of absolute value of differences between the corresponding coordinates of vector $p$:

**Lemma 3.** ***BNB-delta contraction mapping lemma.*** *Operator $G$, defined by equation (94), is a contraction mapping on the set $X = [0;x]^S \times \{0\}$ with respect to the metrics on $X$, defined as:*

$$d(y,z) = \max_{i \in \{1,\dots,S,\diamond\}} \mid y_i - z_i \mid \tag{95}$$

*In Particular,*

$$\forall p^{(1)}, p^{(2)} \in X : \quad d(G(p^{(2)}), G(p^{(1)})) \leq \delta d(p^{(2)}, p^{(1)}) \tag{96}$$

*Proof.* Proof is given in the appendix. □

Since the BNB-delta solution is unique, there is exactly one BNB-graph $\mathcal{H}$ of matchings and outside options, and payoffs of sellers are determined from:

$$(I - \frac{\delta}{2} M^{(b)} O^{(s)} - \frac{\delta}{2} O^{(b)} M^{(s)}) p = \frac{1}{2} \widehat{K}^{\delta} \tag{97}$$

This system of equations has a unique solution, and it is given by the following matrix expansion:

$$p = \frac{1}{2} \sum_{r=0}^{\infty} \delta^r \chi^r \widehat{K}^{\delta} \tag{98}$$

where

$$\chi = \frac{M^{(b)} O^{(s)} + O^{(b)} M^{(s)}}{2} \tag{99}$$

is a matrix, which represents the influence of outside options. Each row of this matrix looks as: $(0,\dots,0)$, or $(0,..,0,\frac{1}{2},0,..0$, or $(0,..,0,\frac{1}{2},0,..0,\frac{1}{2},0,..0)$, or $(0,..,0,1,0,..0)$, since each seller-buyer pair has at most two outside options together (one for the seller and one for the buyer).

**• Continuity of BNB-delta solution for the fixed efficient matching $M$.**
Lemma (3) guarantees that there always exists exactly one BNB-delta solution as long as $\delta \in (0;1)$. However, we still don't know anything about its properties, and what we are interested in is how the solutions for different vector $k$ are connected with each other. The first question is whether the solution is a continuous function of vector $k$. Intuitively, there is no guarantee for this in general case; however, under the condition of fixed effective matching $M$, the solution turns out to be indeed continuous in $k$. Note, that the continuity property does not depend on whether we use the standard Euclidian metrics on $X$, or our metrics $d(y,z) = \max_{i \in \{1,\dots,S,\diamond\}} \mid y_i - z_i \mid$, since they induce the same topology on $X$. With this remark we get the following:

**Lemma 4.** ***BNB-delta continuity lemma*** *BNB-delta solution $p(k)$ as a function of $k$ is continuous with respect to $d(y,z) = \max \mid y_i - z_i \mid$ metrics on each compact subset of $Y \subset R^{|G|} \times \{0\}$, where matching $M \in Argmax(\sum_{ij \in M} k_{ij})$ remains the same.*

*Proof.* Proof is given in the appendix. □

• **Monotonicity properties of the BNB-delta solution.**
Let's proceed to the core of this section: investigation of the monotonicity properties of the BNB-delta solution. I want to use them for the clarification of possible equilibria in sellers' investments game in the small noise limit. Let's think about local monotonicity properties firstly. When a seller $i$ think about the increasing of surplus of one of his adjacent links $ij$, there could be 5 different situations:

1. The link is his matching: $ij \in M$.
2. The link is his outside option, and also it is an outside option for the buyer j: $ij \in O^{(s)},\ ij \in O^{(b)}$
3. The link is his outside option, but it is not an outside option for the buyer j: $ij \in O^{(s)},\ ij \notin O^{(b)}$.
4. The link is not his outside option, but it is an outside option for the buyer j: $ij \notin O^{(s)},\ ij \in O^{(b)}$.
5. The link does not correspond to a matching or any outside option: $ij \notin M,\ ij \notin O^{(s)},\ ij \notin O^{(b)}$.

I claim that for our seller it is beneficial to increase $k_{ij}$ in the first three cases, but it is not beneficial for him to increase $k_{ij}$ in the last two cases:

**Lemma 5.** ***Local monotonicity properties of the BNB-delta solution.*** *Let the bargaining rule be BNB-delta solution. Consider the region $X_{ln}$ of the space of surpluses vectors, where the BNB graph $\mathcal{H}_{ln} = \{M_l; O^{(s)}_{ln}; O^{(b)}_{ln}\}$ remains the same. Consider some link $ij$. Then:*

1. *If link $ij$ corresponds to the matching or the outside option of seller $i$ (that is, $j = M^{(s)}(i)$ or $j = O^{(s)}(i)$), the payoff of seller $i$ in this surpluses' region is a linear function of surplus $k_{ij}$ with strictly positive coefficient:*

$$p_i(k) = \beta_{ij}k_{ij} + \sum_{i'j' \neq ij} \beta_{i'j'}k_{i'j'} \qquad \beta_{ij} > 0 \tag{100}$$

2. *Otherwise, and if link $ij$ corresponds to the outside option of buyer $j$, (that is, $j \neq M^{(s)}(i)$ and $j \neq O^{(s)}(i)$, but $i = O^{(b)}(j)$), the payoffs of all sellers in this surpluses' region $X_{ln}$ are a linear functions of surplus $k_{i'j'}$ with non-positive coefficient:*

$$\forall l = 1, ..., S: \quad p_l(k) = \beta_{ij}k_{ij} + \sum_{i'j' \neq ij} \beta_{i'j'}k_{i'j'} \qquad \beta_{ij} \leq 0 \tag{101}$$

3. *Finally, if link $ij$ does not correspond to any matching or outside option (that is, $j \neq M^{(s)}(i)$, and $j \neq O^{(s)}(i)$, and $i \neq O^{(b)}(j)$), then all payoffs are independent of $k_{ij}$ whenever $\mathcal{H}_{ln}$ remains fixed.*

*Proof.* The second and the third points are proven in the appendix, the first point remains as hypothesis □

Now let's turn to the global properties of the BNB-delta solution, i.e. properties which are not restricted for the region $X_{ln}$ with fixed BNB graph $\mathcal{H}$. For our purpose it is important to get two results. Firstly, that it is beneficial for the seller to increase the surplus over his matching even if the BNB-graph $\mathcal{H}$ changes. Secondly, it is important to understand, that if a seller to **decrease** the surplus of the link, which does not correspond to his matching or outside option, then it could not damage his payoff.

**Lemma 6.** ***Global positive responsiveness of investing in matching for the BNB-delta solution.***
*Let the bargaining rule be BNB-delta solution. Let $ij \in M_l$ be an arbitrary link in the matching with the initial surplus $k_{ij}^{(0)}$. Then the payoff of seller $i$ is strictly increasing piecewise linear continuous function of surplus $k_{ij}$ for $k_{ij} \geq k_{ij}^{(0)}$ under the condition that other surpluses $k_{-ij}$ remain the same.*

*Proof.* When we increase $k_{ij}$, the matching remains the same, and hence the payoff is a continuous function of $k_{ij}$. Inside each region $X_{ln}$ it is strictly positive linear function of $k_{ij}$ for $k_{ij} \geq k_{ij}^{(0)}$ by lemma (9). □

**Lemma 7.** ***Global non-positive responsiveness to the increasing of buyers' outside options for the BNB-delta solution.***
*Let the bargaining rule be BNB-delta solution. Consider some BNB graph $\mathcal{H}_{ln} = \{M_l; O_{ln}^{(s)}; O_{ln}^{(b)}\}$. Suppose that some link $i'j'$ stands for the outside option of buyer $j'$, but it does not stand for the outside option of seller $i$, that is: $i'j' \in O^{(b)} \backslash O^{(s)}$, and let $k_{i'j'}^{(0)}$ be the initial surplus for this link. Then the payoff of an arbitrary seller $i$ is a non-increasing piecewise linear continuous function of surplus $k_{i'j'}$ for $k_{i'j'} \leq k_{i'j'}^{(0)}$ under the condition that other surpluses $k_{-i'j'}$ remain the same.*

*Proof.* The idea of the proof is given in the appendix. Strict proof will be given during further studies. □

• **Summary of properties of the BNB-delta solution.**

**Proposition 7.** *The BNB-delta solution bargaining rule is correctly defined by (9) and has the following properties:*

1. *It satisfies axioms Payoff Uniqueness (A1), Participation Rationality (A2), Balanced Budget (A3), Efficiency (A4), Symmetry I (A5), Piecewise Linearity (A6) and No Cross-Transfers (A7).*

2. *The payoffs of agents are continuous functions of surpluses' vector $k$ in the region where matching remains the same.*

3. *It exhibits local monotonicity conditions (lemma 5): positive responsiveness of investing in matching (lemma 9) and seller's outside option (lemma 10), non-positive responsiveness to the increasing of buyers' outside options (lemma 11), local neutral responsiveness to the irrelevant options (lemma 12).*

4. *It exhibits global monotonicity conditions: positive responsiveness of investing in matching (lemma 6), non-positive responsiveness to the increasing of buyers' outside options (lemma 7).*

*Proof.* Proofs for the most points are given in the appendix. At this moment the Payoff Uniqueness (A1) is not strictly proven, as well as local positive responsiveness of investing in matching (lemma 9) and seller's outside option (lemma 10), and global non-positive responsiveness to the increasing of buyers' outside options (lemma 7). For the rest of points:
Balanced Budget (A3), Efficiency (A4) and No Cross-Transfers (A7) axioms hold by construction. Symmetry I (A5) follows from the fact that initial BNB-delta equations exhibit symmetry of a graph $G$; hence, each profile of payoffs for the BNB-delta solution forms an orbit of solutions with respect to symmetry group of graph $G$. However, the solution is unique, therefore it should be an invariant with respect to all elements of symmetry group of graph $G$. □

# 6 Small noise limit for the Balanced Nash Bargaining solution

In this section I explore the peculiarities of the small noise limit (meaning $a$ is small or $a \to 0$) under the assumption that agents trade ex-post according to the BNB-delta trade rule with $\delta$ arbitrary close to 1. In the examples I consider the limit $\delta \to 1$, and thus BNB-delta solution becomes just Balanced Nash Bargaining solution; however, at this moment I have no general result for the existence of such a limit, and thus for the general case I consider $\delta < 1$.

The small noise limit is important in understanding of strategic interactions of agents when they face the Hold-up problem over network, since investment levels of other sellers become good predictors of the ex-post surpluses, and each seller make his decisions taking into account actions of others. One more argument in favor of investigation of this case is the relatively simplicity of calculations in comparison with the general case, where it is very hard to calculate even simple examples.

When the uncertainty is small, sellers become more sure about the ex-post bargaining conditions. In particular, they better understand what matching will be, what outside options they will have, and what outside options buyers will have as well (that is, they become more sure in the ex-post $\mathcal{H} = \{M; O^{(s)}; O^{(b)}\}$). This allows sellers to concentrate their investments in a few links, because if they know with high probability the graph of matching and outside options, there is no sense for them to invest in any other links except of their own matchings and outside options.

I start with the consideration of two non-star networks of 2 sellers and 2 buyers, namely, "$N$" network and the full graph. While describing the $N$ network I show, that there is a unique equilibrium in the small noise limit, and in this limit only two graphs of matchings and outside options has ex-ante non-zero probabilities. I show, that in case of full network there are only one type of equilibria, which corresponds to the equilibrium for the "$N$" network. Since full graph has $Z_2 \times Z_2$ group of symmetry with respect to the permutations of the index numbers of sellers and buyers, and equilibrium investments levels for the "$N$" network has trivial group of symmetry, there are as many as 4 different

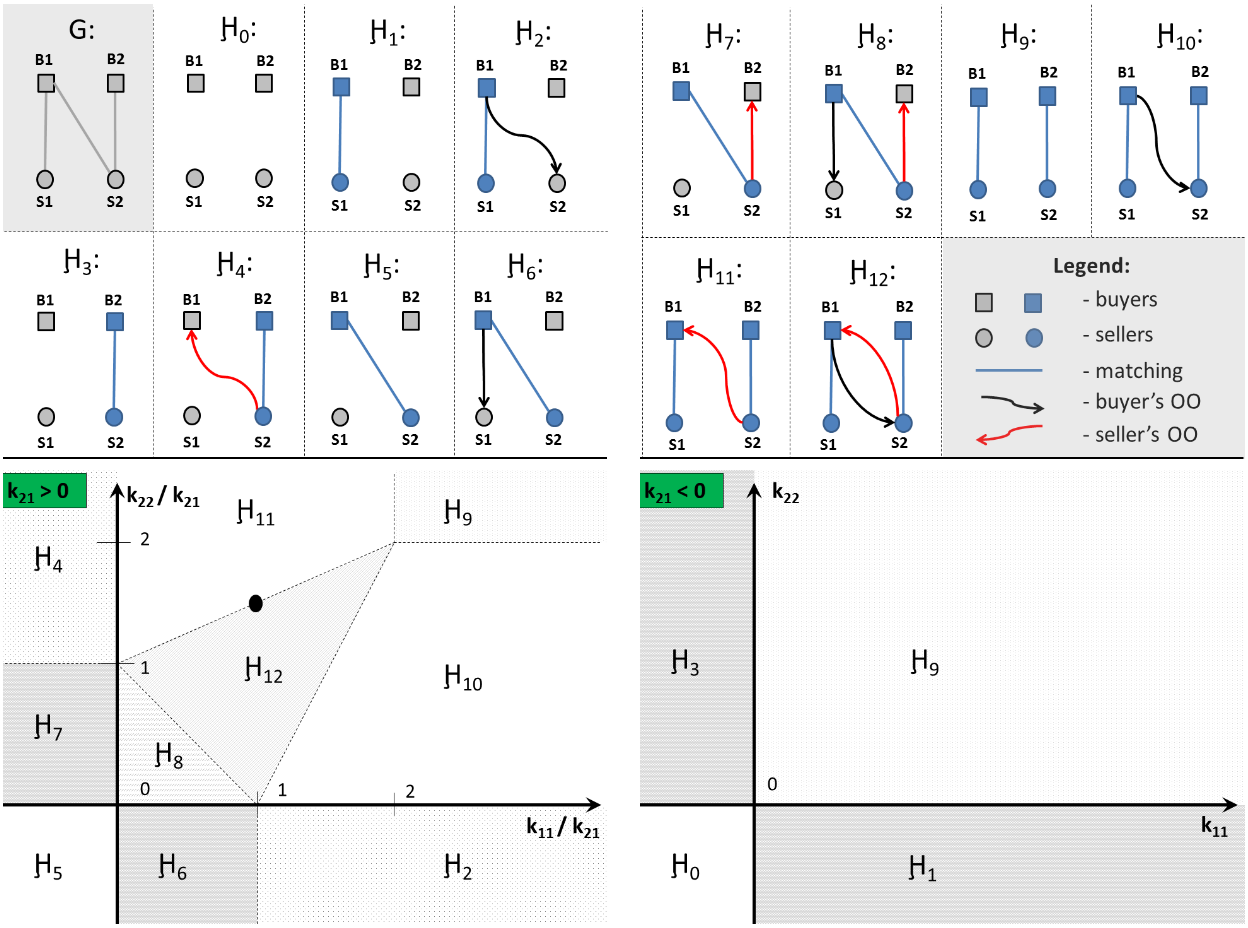


Figure 7: Top figures: different possible graphs of matchings and outside options $\mathcal{H} = (M, O^{(s)}, O^{(b)})$ for the "$N$" graph $G = \{11, 21, 22\}$ under Balanced Nash Bargaining solution. Bottom figures: regions of surpluses for different graphs of matchings and outside options $\mathcal{H}$. Left figure stands for the positive values of $k_{21}$. X Axes measures $\frac{k_{11}}{k_{21}}$, and Y Axes measures $\frac{k_{22}}{k_{21}}$. The black circle stands for the surpluses in Nash Equilibrium for the small value of noise. Right figure stands for the negative values of $k_{21}$. X Axes measures $k_{11}$, and Y Axes measures $k_{22}$.

equilibriums for the full graph, turning into each other by index permutations of the agents.

The result for the $2 \times 2$ network, that at equilibrium only two BNB graphs $\mathcal{H}$ have ex-ante non-zero probabilities, could be generalized for more complicated networks. In particular, I show that if BNB solution indeed satisfies a few natural properties (which is not proven by now in this work), then in any equilibrium each seller ex-ante considers only few opportunities of his own matching and outside option; moreover, he invests in only two links, and for sure one of two opportunities are realized: either the first link is seller's matching and the second is his outside option, or vice versa the first is his outside option, while the second is his matching.

## 6.1 Simple example: "$N$" network with 2 sellers and 2 buyers

Let's consider the simplest network which is distinct from the star. Partially, the details of calculations are presented in the appendix (section "Calculations for the "N" network"). Here I discuss the main points. Suppose we have two sellers and two buyers, and they form $N$ network:

$$G = \{11, 21, 22\} \tag{102}$$

The BNB solution operates with the triple $\mathcal{H} = \{M; O^{(s)}; O^{(b)}\}$, where $M$ is the undirected graph of the efficient matching, $A^{(s)}$ is a directed graph of the sellers' outside options, and $A^{(b)}$ is the directed graph of buyers' outside options. There are 13 different triples $\mathcal{H}$, describing possible BNB solutions. They are represented at the top figures (7). Conditions on $k$ for which these triples are possible, are represented at the bottom figures (7).

The most interesting graph is $\mathcal{H}_{12} = \{\{11, 22\}; \{21\}; \{21\}\}$:

$$k_{11} \le k_{21} + \tfrac{1}{2}k_{22} \qquad 0 \le k_{21} \le k_{11} + k_{22} \qquad k_{22} \le k_{21} + \tfrac{1}{2}k_{11} \tag{103}$$

$$p_1 = \frac{1}{3}k_{11} - \frac{1}{3}k_{21} + \frac{1}{3}k_{22} \qquad p_2 = \frac{2}{3}k_{22} + \frac{1}{3}k_{21} - \frac{1}{3}k_{11}$$

As we can see, the presence of link 21 allows second seller to increase his share of surplus from good delivery to the buyer 2. This appears, because when $k_{22}$ increases, the payoff of the second seller increases, and hence the outside option of buyer 1 when he trade with the seller 1, decreases. Then, payoff of the $1^{st}$ buyer decreases, which in turn increase the outside option of the $2^{nd}$ seller in his trade with the $2^{nd}$ buyer. That is why marginal return from increasing of $k_{22}$ by $\xi$ becomes $\frac{2}{3}\xi$ instead of usual $\frac{1}{2}\xi$ in other graphs $\mathcal{H}$ with $22 \in M$. Since our bargaining rule is neutral with respect to sellers and buyers, then the same arguments explain why the first buyer gets $\frac{2}{3}k_{11}$, and the first seller gets only $\frac{1}{3}k_{11}$, thus having less incentives to invest. If $k_{21}$ increases, it has two different effects on the payoffs of the second seller. Firstly, it directly increases his outside option, since $OO_{(s)2} = k_{21} - q_1$ in this case. However, it also increases the payoff of the first buyer, and thus decreases outside option of the second seller. That is why the marginal return of increasing $k_{21}$ by $\xi$ is $\frac{1}{3}\xi$ which is lower in comparison with $\frac{1}{2}\xi$ for the graphs with $22 \in M$ and $21 \in O^{(s)}$.

The careful investigation of NE for the "$N$" network proceed as follows. Firstly, I check FOCs, assuming that $Pr(X_n) = 1$ for some $n = 0, 1, ..., 12$. The only one candidate is $\mathcal{H}_9$. Next I examine the possibility that several $X_n$ have non-zero probabilities. This gives me the candidate ($\mathcal{H}_{11}$&$\mathcal{H}_{12}$). Then I check if these candidates for NE are stable with respect to global deviations of players. It turns out to be, that the NE candidate for $\mathcal{H}_9$: ($i_{11} = \frac{1}{2}, i_{21} = 0, i_{22} = \frac{1}{2}$) is not an NE, since the second seller wants to deviate by investing both in 21 and 22. The second candidate turns out to be the true NE, with equilibrium investment levels:

$$i_{11} = i_{21} = \frac{2}{5} - \frac{1}{6}\xi \qquad \frac{3}{5} - \frac{1}{6}\xi \tag{104}$$

$$where \;\; \xi \;\; solves \;\; Pr[\frac{1}{12}\xi + \varepsilon_{21} + \frac{1}{2}\varepsilon_{11} \ge \varepsilon_{22}] = \frac{3}{5} + \xi$$

$2 \times 2$ **full bipartite graph case.**
Let $G = \{11, 12, 21, 22\}$. Here I use the BNB-delta solution with $\delta \to 1$ as a refinement of BNB equilibrium. Let the graph $\mathcal{H} = \{\{11, 22\}; \{12, 21\}; \{12, 21\}\}$. Then:

$$p_1 = \frac{1}{2}k_{11} + \frac{1}{4}k_{12} - \frac{1}{4}k_{21} \tag{105}$$

$$p_2 = \frac{1}{2}k_{22} + \frac{1}{4}k_{21} - \frac{1}{4}k_{12}$$

Interestingly, that the solution for the case of "$N$" network still is a NE for the full graph (with $i_{12} = 0$), since it is not profitable for the first seller to invest differently from $i_{11} = \frac{2}{5} - \xi, i_{12} = 0$. There is also one more candidate for the NE, for which FOC holds: $(i_{11} = \frac{1}{2}; i_{12} = \frac{1}{4}; i_{21} = \frac{1}{4}; i_{22} = \frac{1}{2})$. But it turns out to be that this profile of investments gives sellers too low total profits (taking into account costs of investments) and thus for each seller it is better to invest in only one link as a best response for the investments of the other.

## 6.2 General result for small noise in case of BNB solution

Here I discuss the characterization of the investments' picture for the small noise limit. When we studied examples with 1-2 sellers and 1-2 buyers, we have seen, that sellers tend to invest in one or two links, but we did not study, what is going on when seller could invest in three or more links. However, we have seen one interesting feature: in the equilibrium for the full bipartite graph $G = \{11, 12, 21, 22\}$ one of the sellers did not invest in one of his links at all. It turns out to be the general case for the small noise limit. In particular, I claim that if the bargaining concept is the BNB-delta solution, then in small noise limit each seller should invest in no more than two links.

Let me illustrate this point in the example of 1 seller and 3 buyers, (assuming $\delta \to 1$, i.e. Balanced Nash Bargaining solution, since there is no problems with multiplicity of payoff profiles). Clearly, when $a \to 0$, the profile of equilibrium investment levels of the seller looks as:

$$i_{11} = \frac{1}{2} \qquad i_{12} = \frac{1}{2} \qquad i_{13} = 0 \tag{106}$$

or any other profile which may be made from this by indexes permutations; i.e. link 13 is redundant for the seller, and he is happy, using one of links $1, 2$ as an ex-post matching, and another - as an outside option, receiving almost all surplus from the ex-post trade. Why is it not profitable for the seller to invest in the third link? Let's clarify this. Suppose, all $i_{1j} \neq 0, \;\; j = 1, 2, 3$ at equilibrium. Then each link $1j$ is a matching or an outside option with a non-zero probability. Under the small noise limit it means that $i = i_{11} \approx i_{12} \approx i_{13}$ up to the terms of order of $O(a)$. However in this case seller cannot get ex-post surplus larger, than $i$ up to the terms of order of $O(a)$. Hence, if he wants to invest in all tree links, his best expected payoff is approximately:

$$EU^{(3)} = \max_i [i - 3 \cdot \frac{1}{2} i^2] + O(a) \tag{107}$$

But if he decides to invest in only two links, his best expected payoff is given by:

$$EU^{(2)} = \max_i [i - 2 \cdot \frac{1}{2} i^2] + O(a) \tag{108}$$

Since $i^{eq} \sim O(1)$, $EU^{(2)} > EU^{(3)}$, and it is not profitable for the seller to invest in more than two links: he would only increase his investment costs without increasing the expected ex-post payoff.

I claim, that similar picture holds for the general case as well:

**Proposition 8.** *Assume that the bargaining rule is BNB-delta solution. Then for any graph $G$ there exists $\underline{a} > 0$ such that $\forall a < \underline{a}$ in any equilibrium in pure strategies of the sellers' investments game, each seller invest in no more than two links. Moreover, ex-post one of these links is a matching for the seller, and another is his outside option.*

*Proof.* The proof is given in the appendix. The idea of the proof consists of the following steps:
**Step 1.** Assume that with probability 1 only one two adjacent to the seller $i$ links participate in the ex-post BNB-graph $\mathcal{H}$. Then seller would invest only in these two links, since by global monotonicity properties he do not lose, investing zero in other links, but he save investment costs.
**Step 2.** Assume that seller $i$ has more than two adjacent links, which participates in $\mathcal{H}$ with non-zero probability. Then all this possible graphs $\mathcal{H}$ should give seller $i$ payoffs which differs one from another by $O(a)$ as $a \to 0$, otherwise seller $i$ may invest a bit more in the link, which is the matching in the best for the $i$-s seller BNB-graph $\mathcal{H}$, and get this best payoff for sure.
**Step 3** Based on the result of step 2, if seller $i$ has more than two adjacent links, which participates in $\mathcal{H}$ with non-zero probability, than the worst $\mathcal{H}$ gives him approximately the same payoff, as others. Let he invest only in the matching and outside option of this BNB-graph. Using global monotonicity property of BNB-solution, we conclude, that he gets no less than in the worst case initially, but he saves costs of investments in other links, and thus it is a profitable deviation.
Thus, the only one opportunity is that seller $i$ invests in no more than two links (see step 1). □

## 6.3 Problem of multiplicity of NE and endogenous network formation as a possible way of its solution.

The basic problem seems to be serious: in case of the small noise limit we often get a multiplicity of equilibria. Moreover, even the profiles of first best levels of investments tend to be multiple. This fact allows us to make an observation, that the multiplicity of equilibria could be driven by the underlying properties of the network. In particular, if the bipartite graph $G$ of seller-buyer interactions exhibits some symmetry with respect to index permutations among sellers and buyers, then the first best levels of investments should necessarily exhibit multiplicity (see the Proposition 3), which might be reflected by the multiplicity of equilibria (see the Proposition 5). Thus, it turns out to be that the symmetry of the network plays an important role in the problem of coordination of agents when they play this or that Nash equilibrium. Therefore, thinking about the endogenous network formation, we may imagine that sellers do not like to build symmetric networks at all, since they are aware that they may fail to coordinate on the NE during the subsequent investments game, or because they could not get better payoffs, since in the emerging networks part of sellers may have less bargaining power.

Developing this idea, assume that there are several possible graphs of equilibrium investments $I$ for some graph $G$ (according to the Proposition (8), $I$ contains no more than two adjacent links for each seller, and for the dense enough graph $G$ we could have a lot of different graphs of equilibria investments). Let's fix one such graph $I \subset G$. If agents may transform their network by removing all links which has zero investments ($G \backslash I$), than they may drop out many other possible equilibria. Thus if agents may commit not to use some links (for example, by publicly breaking relationships or by some other way), they may solve the coordination problem at least partially; this may also include transfers to those agents who break their links, as a compensation for the decreasing of their bargaining power.

These basic ideas can be seen from the examples of "$N$" network and full $2 \times 2$ graph: at any NE equilibrium for the latter (and there are 4 of them) one link is not used for the investments, thus agents may break it (with the compensation for these pair of agents, who break it, say, seller 1 and buyer 2) and switch to the situation with "$N$" network and the unique NE.

# 7 Conclusion

In this paper I study, how the structure of relationships between buyers and sellers influence the resolution of the Hold-up problem, when each seller may make sunk specific investments in the quality of a joint project with a buyer, with whom he is connected, and these investments as well as the ex-post surplus have unverifiable nature. I consider the common used in literature framework where a seller may supply a buyer with a single indivisible good. In my specification only sellers could make investments.

I focus on several questions. Is there an ex-post bargaining rule over network, which provides sellers with the motivation to invest ex-ante efficient? What is role of network and bargaining concept in incentives' distortions, do they help to solve the hold-up problem, or do they provide "wrong incentives" to invest? What is the role of strategic interaction of players? If we allow for endogenous network formation, what kind of networks may emerge, do they better or worse in terms of efficiency of investments?

The key result of the paper is that there always exists the "ex-ante efficient" ex-post bargaining rule. According to this bargaining rule each seller gets an additional value which he adds to the maximum social gain from the trade over network by adding his links to the network of the rest agents. Each buyer gets non-negative payoff equal to the difference of the value of his match and payoff of the corresponding seller. The ex-ante efficiency of this (ex-post) bargaining rule is robust to the assumptions about the joined noise distribution, cost functions and presence of general investments. However non-efficient Nash equilibria may also appear together with the efficient ones.

The important finding is that the behavior of sellers crucially depends on the scale of noise. In particular, for the large scale of noise for a good enough bargaining rule there always exists a unique NE in pure strategies. Strategic interaction between sellers plays a diminishing role as uncertainty becomes large. In case of the small noise the picture changes sufficiently. Even the existence of the equilibrium is under the question, and multiplicity of NE becomes a natural property (as well as multiplicity of first-best levels of investments). This may cause the coordination problems. Also, in case of the small noise most bargaining rules provide sellers with incentives to invest in links which has otherwise zero first-best investment levels. This is because sellers wants to use the possible surpluses as outside options during the ex-post bargaining.

The rest of findings are partially in terms of hypothesis and some further research is needed in order to make rigorous conclusions. In order to investigate the small noise limit, I assume that each matched seller-buyer pair shares the value of their joint project according to the Nash Bargaining solution with outside options, endogenously determined through the opportunities to attract other agents, given their payoffs. This bargaining concept has some problems which should be resolved in order to use it for the general networks. I present the possible way of doing this, namely, the BNB-delta solution, which becomes the BNB solution as parameter $\delta$ goes to one. Using this bargaining concept, I conclude that in small noise limit sellers would invest in no more than two links. This allows to suggest an interesting method of equilibria refinement, based on endogenous network formation.

One interesting by-product of this paper is the uncovering of problems, connected with the Balanced Nash Bargaining solution, which is studied mostly in the experimental economics, sociologists, and computer science literature. To the best of my knowledge, the existing studies do not focus on the problem of multiplicity of solutions and their refinements. However, even the simple example with the $2 \times 2$ full bipartite graph shows that the problem does exists. The possible way of refinement of this multiplicity is to consider instead the BNB-delta solution bargaining concept, when we assume that agents discounts outside option by some factor $\delta$ close to 1. This approach may be useful in further studies of bargaining in social networks.

There are several issues which should be covered by the further research. Firstly, I have to prove several hypotheses about the properties of the BNB-delta bargaining rule (some of the monotonicity properties and Participation Rationality). Secondly, it is important to study, whether the ex-ante efficient bargaining rule and the BNB-delta solution exhibit pairwise stability property. More general, it could be that these bargaining rules are stable with respect to broader set of coalitions' deviations, or even they are coalitional stable. These are the important properties of a bargaining rule, which I do not consider in this paper. Indeed, if our ex-ante efficient bargaining rule turns out to be non-pairwise-stable, then it is hard to implement it. Thirdly, the ex-ante efficient bargaining rule seems to be similar to the Vickrey mechanism, and I should clarify this connection. Finally, it is interesting to assume the presence of information asymmetry during the ex-post trade, and understand, what constrains on the bargaining rules this assumption imposes, and what are the consequences of the ex-post information asymmetry on the resolution of the Hold-up problem.

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# A Appendix

## A.1 Contraction mapping lemma for the FB investments, large noise limit

**Proof of Lemma (1).**
Remind Lemma (1): Under the conditions of proposition (2) $\exists\overline{a}$ such that $\forall a > \overline{a}$ H(i) is a contraction mapping with respect to the Euclidian metrics in $R^{|G|}$.

*Proof.* Let's rewrite

$$H_{jk}(i) = \tag{A-1}$$

$$= \int_{-\infty}^{\infty} \cdot \int_{-\infty}^{\infty} \sum_{M:jk\in M} \left( \prod_{M'\neq M} \theta \left( \sum_{sb\in M} (i_{sb} + \varepsilon_{sb}) - \sum_{s'b'\in M'} (i_{s'b'} + \varepsilon_{s'b'}) \right) \right) \prod_{s''b''\in G} \frac{1}{a} f_0(\frac{\varepsilon_{s''b''}}{a}) d\varepsilon_{s''b''}$$

Where $\theta(x) = 1$ if $x \geq 0$ and $\theta(x) = 0$ if $x < 0$.
Let $i^{(1)}$ and $i^{(2)}$ be two points in $X = [0;1]^{|G|}$. Then:

$$H(i^{(2)})_{jk} - H(i^{(1)})_{jk} = \int_{i^{(1)}}^{i^{(2)}} (\nabla H_{jk}(i) \cdot di) \tag{A-2}$$

Hence, we have:

$$\mid H(i^{(2)})_{jk} - H(i^{(1)})_{jk} \mid \leq sup_{i\in X} \mid \nabla H(i) \mid \cdot \mid i^{(2)} - i^{(1)} \mid \leq \tag{A-3}$$

$$\leq \sqrt{\mid G \mid} \cdot max_{lm}[sup_{i\in X} \mid \frac{\partial H(i)}{\partial i_{lm}} \mid] \cdot \mid i^{(2)} - i^{(1)} \mid$$

Let's show, that $a \to \infty$ implies $\mid \frac{\partial H(i)}{\partial i_{lm}} \mid \to 0$. For the derivative of $\theta$-function we have: $\partial_x \theta(x) = \delta(x)$. From (A-1) we have:

$$\frac{\partial H_{jk}(i)}{\partial i_{lm}} = \int_{-\infty}^{\infty} \cdot \int_{-\infty}^{\infty} \sum_{M:jk\in M} \left( \sum_{\widetilde{M}\neq M} \left( \frac{\partial \theta \left( \sum_{sb\in M} (i_{sb} + \varepsilon_{sb}) - \sum_{\widetilde{s}\widetilde{b}\in \widetilde{M}} (i_{\widetilde{s}\widetilde{b}} + \varepsilon_{\widetilde{s}\widetilde{b}}) \right)}{\partial i_{lm}} \times \right.\right. \tag{A-4}$$

$$\left.\left.\times \prod_{M'\neq M,\widetilde{M}} \theta \left( \sum_{sb\in M} (i_{sb} + \varepsilon_{sb}) - \sum_{s'b'\in M'} (i_{j'b'} + \varepsilon_{j'b'}) \right) \right) \right) \prod_{s''b''\in G} \frac{1}{a} f_0(\frac{\varepsilon_{s''b''}}{a}) d\varepsilon_{s''b''}$$

Note, that $\theta(x) \in \{0, 1\}$. Each derivative in (A-4) is either zero or has a form of $\pm\delta(\varepsilon_{lm} + A(i, \varepsilon_{-lm}))$, where $A(i, \varepsilon_{-lm})$ is some linear function of investments and noise levels except of $\varepsilon_{lm}$. Then we can take an integral with respect to $d\varepsilon_{lm}$. As a result, we get a sum of no more then $\mid \mathcal{M} \mid^2$ terms, and absolute value of each term looks as:

$$\mid \int_{-\infty}^{\infty} \cdot \int_{-\infty}^{\infty} \prod \theta(...) \cdot \frac{1}{a} f_0(A(i, \varepsilon_{-lm})) \cdot \prod_{s''b''\in G-lm} \frac{1}{a} f_0(\varepsilon_{s''b''}) d\varepsilon_{s''b''} \mid \leq \frac{1}{a} \sup f_0(\varepsilon) \tag{A-5}$$

Hence, we have:

$$\mid \frac{\partial H_{jk}(i)}{\partial i_{lm}} \mid \leq \frac{1}{a} \mid \mathcal{M} \mid^2 \sup f_0(\varepsilon) \tag{A-6}$$

Finally, we get:

$$| H(i^{(2)}) - H(i^{(1)}) | \leq | G |^{\frac{1}{2}} \; max_{lm} \; | H(i^{(2)})_{lm} - H(i^{(1)})_{lm} | \leq \tag{A-7}$$

$$\leq | G | \cdot \frac{1}{a} \cdot | \mathcal{M} |^2 \sup f_0(\varepsilon) \cdot | i^{(2)} - i^{(1)} |$$

For some $q \in (0; 1)$ take

$$\overline{a} = q^{-1} \; | G || \mathcal{M} |^2 \sup f_0(\varepsilon) \tag{A-8}$$

Note, that since $f_0(\varepsilon)$ is a continuous p.d.f., defined on a compact set $[-\frac{1}{2}; \frac{1}{2}]$, then $\sup f_0(\varepsilon) < \infty$, and then for any $a > \overline{a}$ we get

$$| H(i^{(2)}) - H(i^{(1)}) | \leq q \; | i^{(2)} - i^{(1)} | \tag{A-9}$$

which is the condition of $H(i)$ to be the contraction mapping. Estimation (A-8) of $\overline{a}$ could be improved. Roughly, remark that essentially sums of products of $\theta$-functions in (A-4) represents different matchings, and integral with respect to $| G | -1$ dimensional noise over them is a $| G | -1$ dimensional probability measure of a surface, where efficient matching changes because of the increasing of $\varepsilon_{lm}$. This is much less, then $| \mathcal{M} |^2$ which we use for the upper bound of a sum. However, I have no aim to find the lowest $\overline{a}$, I only want to show that such $\overline{a}$ exists.

□

## A.2 Contraction mapping lemma for the equilibrium investments, large noise limit

**Proof of Lemma (2)**.
Remind Lemma (2): under the conditions of proposition (6) $\forall n > 0 \; \exists \overline{a} : \;\; \forall a > \overline{a}$ the following operator is a contraction mapping on the set $X : \; i \in [0; n(1 + \sqrt{1 + a})]^{|G|}$:

$$\widehat{H}(i) : \quad \widehat{H}_{jk}(i) = \begin{cases} \frac{\partial}{\partial i_{jk}} Ep_j(i + \varepsilon) & \text{if} \quad \frac{\partial}{\partial i_{jk}} Ep_j(i + \varepsilon) \geq 0 \\ 0 & otherwise \end{cases} \tag{A-10}$$

*Proof.* The proof is similar to that of Lemma (1). Since bargaining rule satisfies Piecewise Linearity Axiom, we can rewrite payoffs as:

$$p_i = \sum_{n=1}^{N} \Theta_n(k) \cdot \left( \sum_{i'j'} \alpha^n_{i'j'} k_{i'j'} + \alpha^n_0 \right) \tag{A-11}$$

where

$$\Theta_n = \begin{cases} 1 & if \;\; k \in X_n \\ 0 & \text{otherwise} \end{cases} \tag{A-12}$$

is an indicator function for the region $X_n$. For the sake of simplicity I also omit seller's index $i$ from $\alpha$s, since we consider here only payoff of this seller and other $\alpha$s does not play any role here (remind that initially $\alpha^n_{ij}$ is a $S + B$ dimensional vector, and I consider one component of it, corresponding to the seller $i$). Under the conditions of the lemma,

$$\Theta_n = \prod_{l \in B_n} \theta(\sum_{i'j'} \beta^l_{i'j'} k_{i'j'} + \beta^l_0) \tag{A-13}$$

Where we denote by $B_n$ the set of index numbers for all hyperplanes that bound region $X_n$ (i.e., the index number of active inequalities for $X_n$). Consider firstly the "unconstrained" operator

$$H_{jk}(i) = \frac{\partial}{\partial i_{jk}} Ep_j(i+\varepsilon) \tag{A-14}$$

and show, that is a contraction mapping for the sufficiently large values of $a$. Meanwhile, we have:

$$\widehat{H}_{jk}(i) = \begin{cases} H_{jk}(i) & \text{if } H_{jk}(i) \geq 0 \\ 0 & otherwise \end{cases} \tag{A-15}$$

In the spirit of the proof of Lemma (1), we consider:

$$\mid H(i^{(2)}) - H(i^{(1)}) \mid \leq \mid G \mid^{\frac{1}{2}} \underset{jk}{max} \mid H(i^{(2)})_{jk} - H(i^{(1)})_{jk} \mid \leq \tag{A-16}$$

$$\leq \mid G \mid \cdot \underset{jk}{max}[\underset{sb}{max}[\sup_X \mid \frac{\partial H_{jk}(i)}{\partial i_{sb}} \mid]] \cdot \mid i^{(2)} - i^{(1)} \mid$$

Hence, we want to show that for large $a$ we get $\mid \frac{\partial H_{jk}(i)}{\partial i_{sb}} \mid < \mid G \mid^{-1}$. We have:

$$H_{ij}(i) = \int \sum_{n=1}^{N} \Theta_n \alpha^n_{ij} dF(\varepsilon) + \int \sum_{n=1}^{N} \frac{\partial \Theta_n}{\partial i_{ij}} \left( \sum_{i'j'} \alpha^n_{i'j'} k_{i'j'} + \alpha^n_0 \right) dF(\varepsilon) \tag{A-17}$$

Denote the first integral by $J_1$ and the second by $J_2$. We have:

$$\frac{\partial \Theta_n}{\partial i_{sb}} = \sum_{l \in B_n} \left[ \beta^l_{sb} \delta(\sum_{i'j'} \beta^l_{i'j'} k_{i'j'} + \beta_0) \prod_{l' \in B_n,\ l' \neq l} \theta(\sum_{i'j'}^{l'} k_{i'j'} + \beta_0) \right] \tag{A-18}$$

Hence,

$$\frac{\partial}{\partial i_{sb}} J_1 = \sum_{n=1}^{N} \alpha^n_{sb} \sum_{l \in B_n} \mathbb{1}(\beta^l_{sb} \neq 0) \int \prod_{l' \in B_n,\ l' \neq l} \theta \left( \sum_{i'j' \neq sb} (\beta^{l'}_{i'j'} - \frac{\beta^{l'}_{sb}}{\beta^l_{sb}} \beta^l_{i'j'}) k_{i'j'} + \beta^{l'}_0 - \frac{\beta^{l'}_{sb}}{\beta^l_{sb}} \beta^l_0 \right) \times$$
$$\times \frac{1}{a} f_0 \left( \frac{1}{a} \left( -i_{sb} - \frac{1}{\beta^l_{sb}} ( \sum_{i'j' \neq sb} \beta^l_{i'j'} k_{i'j'} + \beta^l_0) \right) \right) dF(\varepsilon_{-sb}) \tag{A-19}$$

Where we have integrated over $\varepsilon_{sb}$, using delta-functions. The product of a number of theta functions is either 1 or 0, and hence the absolute value of integral is bounded by $a^{-1}\sup(f_0)$. Then,

$$\mid \frac{\partial}{\partial i_{sb}} J_1 \mid \leq \frac{1}{a} c_1(\alpha, \beta) \sup(f_0) \tag{A-20}$$

where $c_1(\alpha, \beta) = \mid \sum_{n=1}^{N} \alpha^n_{sb} \sum_{l \in B_n} \mathbb{1}(\beta^l_{sb} \neq 0) \mid$ is a bounded function of parameters $\alpha, \beta$.

Let's evaluate $J_2$ analogously:

$$J_2 = \sum_{n=1}^{N} \sum_{l \in B_n} \mathbb{1}(\beta^l_{ij} \neq 0) \int \prod_{l' \in B_n,\ l' \neq l} \theta \left( \sum_{i'j' \neq ij} (\beta^{l'}_{i'j'} - \frac{\beta^{l'}_{ij}}{\beta^l_{ij}} \beta^l_{i'j'}) k_{i'j'} + \beta^{l'}_0 - \frac{\beta^{l'}_{ij}}{\beta^l_{ij}} \beta^l_0 \right) \times$$
$$\times \left[ \sum_{i'j' \neq ij} (\alpha^n_{i'j'} - \frac{\beta^l_{i'j'}}{\beta^l_{ij}} \alpha^n_{ij}) k_{i'j'} + \alpha^n_0 - \frac{\beta^l_0}{\beta^l_{ij}} \alpha^n_{ij} \right] \frac{1}{a} f_0 \left( \frac{1}{a} \left( -i_{ij} - \frac{1}{\beta^l_{ij}} ( \sum_{i'j' \neq ij} \beta^l_{i'j'} k_{i'j'} + \beta^l_0 ) \right) \right) dF(\varepsilon_{-ij}) \tag{A-21}$$

Now consider what kind of terms could be in $\frac{\partial J_2}{\partial_{sb}}$. We should be careful, since now there is a term $\sim k \sim i + \varepsilon$ in the integral. Derivatives of theta functions gives us delta functions and after integration over $\varepsilon_{sb}$ (in case if $sb \neq ij$), and they gives terms $\sim a^{-2} f_0^2 \cdot (i + \varepsilon) \sim a^{-1} f_0^2 + O(a^{-\frac{3}{2}})$, since $0 \leq i_{lm} < n(1 + \sqrt{1+a})$ and $\mid \varepsilon_{lm} \mid \leq a$. If $sb = ij$, this gives us zero (theta functions does not depend on $i_{ij}$ after integration over $\varepsilon_{ij}$, since in all theta functions $i_{ij}$ comes together with $\varepsilon_{ij}$ in a combination $k_{ij} = i_{ij} + \varepsilon_{ij}$, and $k_{ij}$ becomes a linear combination of other $k_{i'j'}$ after the integration).

Next, the derivatives of middle brackets (expression with $\alpha$-s) gives us terms $\sim a^{-1} f_0$. Finally, derivative of $f_0(...)$ gives us one more $\frac{1}{a}$ multiple, and this term is of order of $\sim a^{-2} f_0' \cdot (i + \varepsilon) \sim a^{-1} f_0' + O(a^{-\frac{3}{2}})$. Overall,

$$\mid \frac{\partial}{\partial i_{sb}} J_2 \mid \leq \frac{1}{a} c_2(\alpha, \beta) \sup(f_0)^2 + \frac{1}{a} c_3(\alpha, \beta) \sup(f_0) + \frac{1}{a} c_4(\alpha, \beta) \sup(f_0') \tag{A-22}$$

Since $f_0$ is twice continuously differential function, defined on the compact set, then $\sup(f_0') < \infty$. Hence,

$$\mid \frac{\partial H_{jk}(i)}{\partial i_{sb}} \mid \leq \frac{1}{a} \left[ c_1(\alpha, \beta) \sup(f_0) + c_2(\alpha, \beta) \sup(f_0)^2 + c_3(\alpha, \beta) \sup(f_0) + c_4(\alpha, \beta) \sup(f_0') \right] \tag{A-23}$$

As we can see, by choosing $a$ sufficiently large, we always can get

$$\mid \frac{\partial H_{jk}(i)}{\partial i_{sb}} \mid \leq q \cdot \mid G \mid^{-1} \tag{A-24}$$

for some $q \in (0; 1)$. Hence, we have proven, that $H(i)$ is indeed a contraction mapping. Finally, let's consider $\widehat{H}$. For each $ij \in G$ we have:

$$\mid \widehat{H}_{ij}(i^{(2)}) - \widehat{H}_{ij}(i^{(1)}) \mid = \tag{A-25}$$

$$= \begin{cases} \mid \widehat{H}_{ij}(i^{(2)}) - \widehat{H}_{ij}(i^{(1)}) \mid = \mid H_{ij}(i^{(2)}) - H_{ij}(i^{(1)}) \mid & \text{if } H_{ij}(i^{(2)}) \geq 0 \ \&\ H_{ij}(i^{(1)}) \geq 0 \\ \widehat{H}_{ij}(i^{(2)}) < \mid H_{ij}(i^{(2)}) - H_{ij}(i^{(1)}) \mid & \text{if } \widehat{H}_{ij}(i^{(2)}) \geq 0 \ \&\ \widehat{H}_{ij}(i^{(1)}) < 0 \\ \widehat{H}_{ij}(i^{(1)}) < \mid H_{ij}(i^{(2)}) - H_{ij}(i^{(1)}) \mid & \text{if } \widehat{H}_{ij}(i^{(2)}) < 0 \ \&\ \widehat{H}_{ij}(i^{(1)}) \geq 0 \\ 0 \leq \mid H_{ij}(i^{(2)}) - H_{ij}(i^{(1)}) \mid & \text{if } \widehat{H}_{ij}(i^{(2)}) < 0 \ \&\ \widehat{H}_{ij}(i^{(1)}) < 0 \end{cases}$$

Hence, in any way

$$\mid \widehat{H}_{ij}(i^{(2)}) - \widehat{H}_{ij}(i^{(1)}) \mid \leq \mid H_{ij}(i^{(2)}) - H_{ij}(i^{(1)}) \mid \tag{A-26}$$

And for large $a > \overline{a}$ we get:

$$\mid \widehat{H}(i^{(2)}) - \widehat{H}(i^{(1)}) \mid \leq \mid H(i^{(2)}) - H(i^{(1)}) \mid \leq q \mid i^{(2)} - i^{(1)} \mid \tag{A-27}$$

And $\widehat{H}$ is a contraction mapping too. $\square$

## A.3 Properties of the BNB-delta solution

### A.3.1 Existence and Uniqueness: contraction mapping lemma.

**Proof of lemma (3)**. Remind lemma (3): Consider a compact set $X = [0; x]^S \times \{0\}$. Define operator $G$ as:

$$G: \; X \to X \qquad G_\diamond(p) = 0, \qquad \forall i = 1, ..., S: \tag{A-28}$$

$$G_i(p) = \frac{1}{2} k_{iM^{(s)}(i)} \; + \; \frac{\delta}{2} \cdot \underset{j \in (\{j:\; ij \in G\} \backslash M^{(s)}(i)) \cup \{\diamond\}}{max} [k_{ij} - k_{M^{(b)}(j)j} + p_{M^{(b)}(j)}] -$$
$$- \; \frac{\delta}{2} \cdot \underset{l \in (\{l:\; lM^{(s)}(i) \in G\} \backslash i) \cup \{\diamond\}}{max} [k_{lM^{(s)}(i)} - p_l]$$

Then $G$ is a contraction mapping with respect to the metrics on $X$, defined as:

$$d(y, z) = \underset{i \in \{1, ..., S, \diamond\}}{max} \mid y_i - z_i \mid \tag{A-29}$$

In details,

$$\forall p^{(1)}, p^{(2)} \in X: \quad d(G(p^{(2)}), G(p^{(1)})) \leq \delta d(p^{(2)}, p^{(1)}) \tag{A-30}$$

*Proof.* In what follows, for simplicity I will use $j \neq M^{(s)}(i)$ instead of $j \in (\{j:\; ij \in G\} \backslash M^{(s)}(i)) \cup \{\diamond\}$, and similarly $l \neq M^{(s)}(i)$ instead of $j \in (\{j:\; ij \in G\} \backslash M^{(s)}(i)) \cup \{\diamond\}$. Consider two arbitrary points $p^{(1)}, p^{(2)} \in X$. We are interesting in $d(G(p^{(2)}), G(p^{(1)}))$. We have:

$$G_i(p^{(2)}) - G_i(p^{(1)}) = \frac{\delta}{2}\rho_i - \frac{\delta}{2}\sigma_i \tag{A-31}$$

where

$$\rho_i = \underset{j \neq M^{(s)}(i)}{max} [k_{ij} - k_{M^{(b)}(j)j} + p^{(2)}_{M^{(b)}(j)}] \; - \; \underset{j \neq M^{(s)}(i)}{max} [k_{ij} - k_{M^{(b)}(j)j} + p^{(1)}_{M^{(b)}(j)}] \tag{A-32}$$

and

$$\sigma_i = \underset{l \neq M^{(s)}(i)}{max} [k_{lM^{(s)}(i)} - p^{(2)}_l] \; - \; \underset{l \neq M^{(s)}(i)}{max} [k_{lM^{(s)}(i)} - p^{(1)}_l] \tag{A-33}$$

Consider $\rho_i$. Let

$$m \in \underset{j \neq M^{(s)}(i)}{\text{Argmax}} [k_{ij} - k_{M^{(b)}(j)j} + p^{(1)}_{M^{(b)}(j)}] \tag{A-34}$$

Then

$$\rho_i = \underset{j \neq M^{(s)}(i)}{max} [k_{ij} - k_{M^{(b)}(j)j} + p^{(2)}_{M^{(b)}(j)}] \; - [k_{im} - k_{M^{(b)}(m)m} + p^{(1)}_{M^{(b)}(m)}] \geq \tag{A-35}$$

$$\geq [k_{im} - k_{M^{(b)}(m)m} + p^{(2)}_{M^{(b)}(m)}] - [k_{im} - k_{M^{(b)}(m)m} + p^{(1)}_{M^{(b)}(m)}] =$$

$$= p^{(2)}_{M^{(b)}(m)} - p^{(1)}_{M^{(b)}(m)} \geq -d(p^{(2)}, p^{(1)})$$

Now let

$$r \in \underset{j \neq M^{(s)}(i)}{\text{Argmax}}[k_{ij} - k_{M^{(b)}(j)j} + p^{(2)}_{M^{(b)}(j)}] \tag{A-36}$$

Then

$$\rho_i \leq [k_{ir} - k_{M^{(b)}(r)r} + p^{(2)}_{M^{(b)}(r)}] - [k_{ir} - k_{M^{(b)}(r)r} + p^{(1)}_{M^{(b)}(r)}] = \tag{A-37}$$

$$= p^{(2)}_{M^{(b)}(r)} - p^{(1)}_{M^{(b)}(r)} \leq d(p^{(2)}, p^{(1)})$$

Hence,

$$-\, d(p^{(2)}, p^{(1)}) \leq \rho_i \leq d(p^{(2)}, p^{(1)}) \quad \Leftrightarrow \quad \mid \rho_i \mid \leq d(p^{(2)}, p^{(1)}) \tag{A-38}$$

Analogously, considering $\underset{l \neq M^{(s)}(i)}{\text{Argmax}}[k_{lM^{(s)}(i)} - p^{(1)}_l]$ and $\underset{l \neq M^{(s)}(i)}{\text{Argmax}}[k_{lM^{(s)}(i)} - p^{(2)}_l]$, we get:

$$\mid \sigma_i \mid \leq d(p^{(2)}, p^{(1)}) \tag{A-39}$$

Finally,

$$d(G(p^{(2)}), G(p^{(1)})) = \underset{i=1,\dots,S,\diamond}{max} \mid G_i(p^{(2)}) - G_i(p^{(1)}) \mid \leq \frac{\delta}{2} \mid \rho_i \mid + \frac{\delta}{2} \mid \sigma_i \mid \leq \delta d(p^{(2)}, p^{(1)}) \tag{A-40}$$

Since $\delta \in (0; 1)$, this proves the lemma. □

### A.3.2 Continuity, conditional on matching

**Proof of lemma (4)**. Remind lemma (4): BNB-delta solution $p_{(s)}(k)$ as a function of $k$ is continuous with respect to $d(y, z) = max \mid y_i - z_i \mid$ metrics on each compact subset of $Y \subset R^{|G|} \times \{0\}$, where matching $M \in Argmax(\sum_{ij \in M} k_{ij})$ remains the same.

*Proof.* Let $p^{(0)}$ be the BNB-delta solution for vector $k$. Consider vector $\hat{k} = k + \triangle\, k \in Y$, and let's try to find the solution for this vector. Since $G$ is a contraction mapping, we can start from arbitrary point $p \in X$, and $\underset{n \to \infty}{lim} G^n p$ gives us a solution[5] $p(k + \triangle\, k)$. In particular, $p(k + \triangle\, k) = \underset{n \to \infty}{lim} G^n p^{(0)}$. Let

$$|\triangle\, k\, | \equiv d(\triangle\, k, 0) = d(k + \triangle\, k, k) \tag{A-41}$$

Under the conditions of lemma (fixed efficient matching M), we have:

$$\mid G_i p^{(0)} - p^{(0)} \mid = \tag{A-42}$$

$$= \mid \frac{1}{2} \triangle\, k_{iM^{(s)}(i)} + \frac{\delta}{2} \max_{j \neq M(i)} [k_{ij} - k_{M^{(b)}(j)j} + p^{(0)} + \triangle\, k_{ij} - \triangle\, k_{M^{(b)}(j)j}] - \frac{\delta}{2} \max_{j \neq M(i)} [k_{ij} - k_{M^{(b)}(j)j} + p^{(0)}] -$$

$$- \frac{\delta}{2} \max_{l \neq i} [k_{lM^{(s)}(i)} - p_l + \triangle\, k_{lM^{(s)}(i)}] + \frac{\delta}{2} \max_{l \neq i} [k_{lM^{(s)}(i)} - p_l] \mid \leq$$

$$\leq \frac{1}{2} \mid \triangle\, k \mid + \frac{\delta}{2} \mid \triangle\, k \mid + \frac{\delta}{2} \mid \triangle\, k \mid + \frac{\delta}{2} \mid \triangle\, k \mid = 2 \mid \triangle\, k \mid$$

[5] I use $lim$ with respect to metrics $d(y, z) = max \mid y_i - z_i \mid$

Hence,

$$d(Gp^{(0)}, p^{(0)}) = \underset{i\in\{1,\ldots,S,\diamond\}}{max} \mid G_i p^{(0)} - p_i^{(0)} \mid \leq 2 \mid \triangle\ k \mid \tag{A-43}$$

Overall, using lemma (3), we get:

$$d(p(k+ \triangle\ k), p^{(0)}) = d(\underset{n\to\infty}{lim} G^n p^{(0)}, p^{(0)}) \leq \tag{A-44}$$

$$\leq \sum_{n=1}^{\infty} d(G^n p^{(0)}, G^{n-1} p^{(0)}) \leq \frac{1}{1-\delta} 2 \mid \triangle\ k \mid$$

Hence,

$$d(p(k+ \triangle\ k), p(k)) \leq \frac{2}{1-\delta} d(k+ \triangle\ k, k) \tag{A-45}$$

and since $\delta \in (0;1)$, this means that $p$ is a continuous function of surpluses' vector $k$ with respect to metrics $d$. □

### A.3.3 Piecewise linearity.

**Lemma 8.** ***Piecewise linearity lemma for the BNB-delta solution.***
*BNB-delta rule satisfies Piecewise Linearity axiom (A6).*

*Proof.* The fact that BNB-delta solution is continuous with respect to vector $k$ allows us better understand its other properties. In particular, each term under maximums in eq. (92) stands for the possible outside option of a seller or a buyer. There is a finite set of possible graphs $\mathcal{H} = \{M; O^{(s)}; O^{(b)}\}$, and for the fixed $\mathcal{H}$ we get the following matrix equation for BNB-delta solution:

$$A^{\delta} p = \frac{1}{2} \hat{K}^{\delta} \tag{A-46}$$

where

$$A^{\delta} = (I - \frac{\delta}{2} M^{(b)} O^{(s)} - \frac{\delta}{2} O^{(b)} M^{(s)}) \tag{A-47}$$

and $\hat{K}^{\delta}$ have the following components:

$$\hat{K}_i^{\delta} = k_{iM^{(s)}(i)} + \delta k_{iO^{(s)}(i)} - \delta k_{M^{(b)}(O^{(s)}(i))O^{(s)}(i)} - \delta k_{O^{(b)}(M^{(s)}(i))M^{(s)}(i)} \tag{A-48}$$

This equation has a unique solution (as we have proven previously), and hence it is given by

$$p(k) = \frac{1}{2} (A^{\delta})^{-1} \hat{K}^{\delta} \tag{A-49}$$

Let's divide the space of vectors $k$ on the regions $\hat{X}_l$, corresponding to different matchings $M$. As we know, the borders of these regions are a finite number of hyperplanes. Consider some region $\hat{X}_l$, and let's divide it regions $X_{ln}$, corresponding to different graphs $\mathcal{H} = \{M; O^{(s)}; O^{(b)}\}$ (matching remains the same, but graphs of outside options may be different). Assume that for some vector $k$, all maximums in eq. (92) are unique, that is, there is no agent who is indifferent between two outside options. Then from the continuity of BNB-delta solution, in sufficiently small neighborhood of $k$ payoffs of the sellers are close to their values at point $k$, and outside options remains the same (since the corresponding expressions for the outside options are strictly larger than other alternatives). Hence, all these points are internal for the corresponding region $X_{ln}$. The only one candidate for the border between two regions $X_{ln_1}, X_{ln_2}$ is the set of vectors $k$ for which there is at least one agent who is indifferent between two outside options. Since the payoff is unique, and it is defined inside each region by eq. (A-49), it is a linear function of vector $k$ inside each region $X_{ln}$ (because $A^\delta$ consists of constants inside each region $X_{ln}$, and $\hat{K}^\delta$ is linear in $k$); it is one of the conditions for the bargaining rule to be piecewise linear. Moreover, each outside option is the linear function of surplus vector $k$ and payoff vector $p_{(s)}$, and thus it is a linear function of vector $k$ after the substitution of $p_{(s)}(k)$. It means that the condition for a point to be in the border between two regions looks as:

$$OO^1(k) = \alpha^1 + \sum_{ij\in G} \beta^1_{ij} k_{ij} = \alpha^2 + \sum_{ij\in G} \beta^2_{ij} k_{ij} = OO^2(k) \qquad \text{(A-50)}$$

This equation defines a hyperplane in the space of surpluses' vectors $k$. Finally, $q_j = k_{M^{(b)}(j)} - p_{M^{(b)}(j)}$ and thus it is also linear inside each region $X_{ln}$, since sellers' payoffs are linear. □

### A.3.4 Monotonicity properties of the BNB-delta solution

Remind lemma (5) "Local monotonicity properties of the BNB-delta solution": Let the bargaining rule be BNB-delta solution. Consider the region $X_{ln}$ of the space of surpluses vectors, where the BNB graph $\mathcal{H}_{ln} = \{M_l; O^{(s)}_{ln}; O^{(b)}_{ln}\}$ remains the same. Consider some link $ij$. Then:

1. If link $ij$ corresponds to the matching or the outside option of seller $i$ (that is, $j = M^{(s)}(i)$ or $j = O^{(s)}(i)$), the payoff of seller $i$ in this surpluses' region is a linear function of surplus $k_{ij}$ with strictly positive coefficient:

$$p_i(k) = \beta_{ij} k_{ij} + \sum_{i'j'\neq ij} \beta_{i'j'} k_{i'j'} \qquad \beta_{ij} > 0 \qquad \text{(A-51)}$$

2. Otherwise, and if link $ij$ corresponds to the outside option of buyer $j$, (that is, $j \neq M^{(s)}(i)$ and $j \neq O^{(s)}(i)$, but $i = O^{(b)}(j)$), the payoffs of all sellers in this surpluses' region $X_{ln}$ are a linear functions of surplus $k_{i'j'}$ with non-positive coefficient:

$$\forall l = 1, ..., S: \quad p_l(k) = \beta_{ij} k_{ij} + \sum_{i'j'\neq ij} \beta_{i'j'} k_{i'j'} \qquad \beta_{ij} \leq 0 \qquad \text{(A-52)}$$

3. Finally, if link $ij$ does not correspond to any matching or outside option (that is, $j \neq M^{(s)}(i)$, and $j \neq O^{(s)}(i)$, and $i \neq O^{(b)}(j)$), then all payoffs are independent of $k_{ij}$ whenever $\mathcal{H}_{ln}$ remains fixed.

*Proof.* Let's prove it, considering different situations separately.

**Lemma 9.** ***Local positive responsiveness of investing in matching for the BNB-delta solution.***
*Let the bargaining rule be BNB-delta solution. Consider the region $X_{ln}$ of the space of surpluses vectors, where the BNB graph $\mathcal{H}_{ln} = \{M_l; O^{(s)}_{ln}; O^{(b)}_{ln}\}$ remains the same. Let $ij \in M_l$ be an arbitrary link in the matching. Then the payoff of seller $i$ in this region is a linear function of surplus $k_{ij}$ with strictly positive coefficient:*

$$p_i(k) = \beta_{ij} k_{ij} + \sum_{i''j'' \neq ij} \beta_{i''j''} k_{i''j''} \qquad \beta_{ij} > 0 \tag{A-53}$$

*Proof.* The idea of the proof. Inside region $X_{ln}$ the vector of sellers' payoffs is given by the eq. (A-49). The idea is to solve it by steps, expressing payoffs of each seller $i' \neq i$ through the payoffs of other sellers, and finally get an expression like $a \cdot p_i = \beta \cdot k$. Since $\delta < 1$, $a > 1 - \delta$. Then, careful investigation of cases, when $k_{ij}$ enters with negative sign (corresponding only to the buyers' outside options), gives us a coefficient[6] $b_{ij} \geq \frac{1}{4}(1 - \delta)$. Strict proof should be done during further studies. □

**Lemma 10.** ***Local positive responsiveness of investing in seller's outside option for the BNB-delta solution.***
*Let the bargaining rule be BNB-delta solution. Consider the region $X_{ln}$ of the space of surpluses vectors, where the BNB graph $\mathcal{H}_{ln} = \{M_l; O^{(s)}_{ln}; O^{(b)}_{ln}\}$ remains the same. Let seller $i$ is such that he participates in matching $M$, and he has an endogenous[7] outside option $O^{(s)}(i) = j' \in \{1, ..., B\}$. Then the payoff of seller $i$ in this region is a linear function of surplus $k_{ij'}$ with strictly positive coefficient:*

$$p_i(k) = \beta_{ij'} k_{ij} + \sum_{i''j'' \neq ij'} \beta_{i''j''} k_{i''j''} \qquad \beta_{ij'} > 0 \tag{A-54}$$

*Proof.* The idea is the same as in the proof of lemma (9). Strict proof should be done during further studies. □

**Lemma 11.** ***Local non-positive responsiveness to the increasing of buyers' outside options for the BNB-delta solution.***
*Let the bargaining rule be BNB-delta solution. Consider the region $X_{ln}$ of the space of surpluses vectors, where the BNB graph $\mathcal{H}_{ln} = \{M_l; O^{(s)}_{ln}; O^{(b)}_{ln}\}$ remains the same. Suppose that some link $i'j'$ stands for the outside option of buyer $j'$, but it does not stand for the outside option of seller $i'$, that is: $i'j' \in O^{(b)} \backslash O^{(s)}$. Then the payoff of an arbitrary seller $i$ in this surpluses' region is a linear function of surplus $k_{i'j'}$ with non-positive coefficient:*

$$\forall i: \quad p_i(k) = \beta_{i'j'} k_{i'j'} + \sum_{i''j'' \neq i'j'} \beta_{i''j''} k_{i''j''} \qquad \beta_{i'j'} \leq 0 \tag{A-55}$$

*Proof.* Remind that locally

$$p = \frac{1}{2} \sum_{r=0}^{\infty} \delta^r \chi^r \cdot \widehat{K}^{\delta} \tag{A-56}$$

[6] this is not a strict calculation up to this moment

[7] meaning that seller $i$'s outside option is not external (exogenous), but there is some buyer j' such that $k_{ij'} \geq p_{(b)j'}$

Consider the change of surplus vector $k$ such that $\widehat{K}^{\delta} \to \widehat{K}^{\delta} + \triangle\, \widehat{K}^{\delta}$. Then for the change of payoff vector $p$, $\triangle\, p$, we get:

$$\triangle\, p = \frac{1}{2} \sum_{r=0}^{\infty} \delta^r \chi^r \cdot \triangle\, \widehat{K}^{\delta} \tag{A-57}$$

Note, that matrix $\chi$ has only non-negative entries. In its turn, the only one change in the surpluses' vector $k$ is $k_{i'j'} \to k_{i'j'} + \triangle\, k_{i'j'}$, Hence under the conditions of lemma we have:

$$\triangle\, \widehat{K}_i^{\delta} = \begin{cases} -\triangle\, k_{i'j'} & if\ i = M^{(b)}(j') \\ 0 & otherwise \end{cases} \tag{A-58}$$

Thus,

$$\triangle\, p_i = \sum [(\text{non-negative terms}) \cdot (\text{non-positive terms})] \leq 0 \tag{A-59}$$

Since locally vector of payoffs is a linear function of surpluses, this proves the lemma. □

Intuitively, lemma (11) holds because increasing of outside option of buyer $j'$ increases his payoff, which in turn decreases the payoff of his matched seller $O^{(b)}(j')$, increasing values of outside options of buyers for which this seller is an outside option, etc. And there is no opposite effect, because the surplus of this link does not influence the outside option of seller $i'$ directly, but only through the network effects, which turn out to be negative or zero.

**Lemma 12.** ***Local neutral responsiveness to the irrelevant options for the BNB-delta solution.***
*Let the bargaining rule be BNB-delta solution. Consider the region $X_{ln}$ of the space of surpluses vectors, where the BNB graph $\mathcal{H}_{ln} = \{M_l; O_{ln}^{(s)}; O_{ln}^{(b)}\}$ remains the same. Suppose that some link $i'j' \notin M \cup O^{(s)} \cup O^{(b)}$ does not participate in $\mathcal{H}_{ln}$ at all. Then all payoffs are independent of $k_{i'j'}$ whenever $\mathcal{H}_{ln}$ remains fixed.*

*Proof.* It follows from eq. (A-49), since $\hat{K}^{\delta}$ does not depend on $k_{i'j'}$ in this case. □

At this moment, there are proves for the last two lemmas, and the first two lemmas should be proved during further studies. □

**Global monotonicity properties of the BNB-delta solution.**

Remind lemma (7):

Let the bargaining rule be a BNB-delta solution. Consider some BNB graph $\mathcal{H}_{ln} = \{M_l; O_{ln}^{(s)}; O_{ln}^{(b)}\}$. Suppose that some link $i'j'$ stands for the outside option of buyer $j'$, but it does not stand for the outside option of seller $i$, that is: $i'j' \in O^{(b)} \backslash O^{(s)}$, and let $k_{i'j'}^{(0)}$ be the initial surplus for this link. Then the payoff of an arbitrary seller $i$ is a non-increasing piecewise linear continuous function of surplus $k_{i'j'}$ for $k_{i'j'} \leq k_{i'j'}^{(0)}$ under the condition that other surpluses $k_{-i'j'}$ remain the same.

*Proof.* Sketch of the proof. When we decrease $k_{i'j'}$, the matching remains the same, and hence the payoff is a continuous function of $k_{i'j'}$. The only one peculiar moment is that it might be that during the decreasing of $k_{i'j'}$ after change of $\mathcal{H}$ the link $i'j'$ becomes outside option of some seller, and we cannot use lemma (11) further. Intuitively, it might not be the case, since we decrease the surplus over this link; however we should investigate this moment more strictly. Then, inside each region $X_{ln}$ it is a non-positive linear function of $k_{ij}$ for $k_{ij} \geq k_{ij}^{(0)}$ by lemmas (11) and (12). Strict proof of this lemma should be given during further studies. □

### A.3.5 Other properties of the BNB-delta solution

- **Participation Rationality**

  Let's investigate, whether BNB-delta bargaining rule satisfies Participation Rationality (A2) axiom; that is, $P \geq 0$. Consider the following definition of the pairwise stable bargaining rule in case when alternative matchings are discounted by the agents:

**Definition 10.** ***Delta-Pairwise Stability.*** *The bargaining rule $R$ is pairwise stable if for any seller-buyer pair $lm \notin M$ who are not matched with each other, both $p_l \geq \delta(k_{lm} - q_m)$ and $q_m \geq \delta(k_{lm} - p_l)$.*

**Lemma 13.** ***BNB-delta solution is a delta-pairwise stable bargaining rule.***

*Proof.* Idea of the proof. This result is a consequence of efficiency of the bargaining rule. When $\delta = 1$, delta-pairwise stability turns out to be pairwise stability: $lm \notin M \;\Rightarrow\; k_{lm} \leq p_l + q_m$. Jon Kleinberg and Éva Tardos in Kleinberg & Tardos (2008) have shown that for the bipartite graph the BNB solution (which is our BNB-delta solution with $\delta = 1$) satisfies pairwise stability condition. Intuitively, this result could be extrapolated for $\delta < 1$ as well by the same methods. I postpone proof of this point to the MT defence. □

Now let's show that delta-pairwise stability implies Participation Rationality.

**Lemma 14.** *If lemma (13) for the BNB-delta solution holds, then this bargaining rule also satisfies Participation Rationality axiom.*

*Proof.* $P \geq 0$ for BNB-delta solution as far as

$$\forall ij \in M \;:\quad k_{ij} - \delta OO^{(s)}(i) - \delta OO^{(b)}(j) \geq 0 \tag{A-60}$$

WLOG (using our trick with $\diamond$), let $O^{(s)}(i) = j'$ and $O^{(b)}(j) = i'$. Consider

$$k_{ij} - \delta OO^{(s)}(i) - \delta OO^{(b)}(j) = k_{ij} - \delta(k_{ij'} - q_{j'}) - \delta(k_{i'j} - p_{i'}) = \tag{A-61}$$

$$= [p_i - \delta(k_{ij'} - q_{j'})] + [q_j - \delta(k_{i'j} - p_{i'})] \geq 0$$

where I use lemma (13) in the last inequality. □

## A.4 Theorem about the investments' graph in case of small noise limit for the BNB-delta solution

Remind Proposition (8): Assume that the bargaining rule is BNB-delta solution. Then for any graph $G$ there exists $\underline{a} > 0$ such that $\forall a < \underline{a}$ in any equilibrium in pure strategies of the sellers' investments game, each seller invest in no more than two links. Moreover, ex-post one of these links is a matching for the seller, and another is his outside option.

*Proof.* Here I extensively use the Proposition (7) for the BNB-delta solution. Consider arbitrary seller $i$. If the number of his adjacent links is less than 3, then he trivially can't invest in more than two links. Suppose, he has more than two adjacent links. Then:

1. Assume that with probability 1 the ex-post BNB-graph $\mathcal{H}$ is such that there are only two links, WLOG $i1$ and $i2$, which may seller $i$'s matching and outside option, that is: $Pr(M^{(s)}(i) = j) = 0 \ for \ j \neq 1, 2 : ij \in G$ and $Pr(O^{(s)}(i) = j) = 0 \ for \ j \neq 1, 2 : ij \in G$. Then his investments profile in this equilibrium is such that $i_{ij} = 0 \ for \ j \neq 1, 2 : ij \in G$.

   *Proof.* Assume by contradiction that $i_{ij}^{eq} > 0$ for some $j \neq 1, 2 : ij \in G$. Assume at first that $i_{ij}^{eq} \sim O(1)$ when $a \to 1$. Consider a deviation when seller $i$ invest less in this link. Then link $ij$ does not participate in the ex-post matching $M$ for sure, because it was not in $M$ with probability 1 initially, and we decrease $k_{ij}$ for each realization of uncertainty. Moreover, by the same reason this deviation does not influence the probabilities of different matchings $M_l$ to be the efficient ones. Then the expected utility of seller $i$ is given by the following expression, which is valid as long as $i_{ij} \leq i_{ij}^{eq}$ and $i_{-ij} = i_{-ij}^{eq}$:

   $$EU_i(i) = \sum_{l=1}^{L} \Pr(X_l) g_l(i_{ij}) + O(a) - \frac{i_{ij}^2}{2} + h(i_{-ij}) \quad \text{(A-62)}$$

   where $g_l$ for each $l$ is some piecewise linear continuous non-increasing function of $k_{ij}$ according to the Proposition (7). Consider the deviation $i_{ij} = 0, i_{-ij} = i_{-ij}^{eq}$. Then:

   $$EU_i(i^{dev}) - EU_i(i^{eq}) = \sum_{l=1}^{L} \Pr(X_l)[g_l(0) - g_l(i_{ij}^{eq})] + \frac{(i_{ij}^{eq})^2}{2} + O(a) > 0 \quad \text{(A-63)}$$

   Hence, this deviation is profitable and we get the contradiction. Let's postpone the case when $i_{ij}^{eq} \sim O(a)$. Thus we have proved, that $i_{ij} > 0 : \ i_{ij} \sim O(1)$ for $j \neq 1, 2 : ij \in G$ is impossible in the equilibrium. □

2. Now assume that there are more than two links, WLOG, $i1, i2, ..., it$ with $t > 2$ such that they appear ex-post as $M^{(s)}(i)$ or $O^{(s)}(i)$ with non-zero probability of order $O(1)$ when $a \to 0$. Let's firstly show that at the equilibrium the ex-post payoff of seller $i$ could be distinct subject to the uncertainty realization only by the value of order of $O(a)$.

   *Proof.* Assume by contradiction that there are two distinct possible realizations $p_i(k^1)$ and $p_i(k^2)$ such that, WLOG, $p_i(k^2) > p_i(k^1)$ and $p_i(k^2) - p_i(k^1) = y = O(1)$.

We may consider[8] $p_i(k^1) = \underset{\varepsilon}{max} p_i(i^{eq}+\varepsilon)$ and $p_i(k^2) = \underset{\varepsilon}{min} p_i(i^{eq}+\varepsilon)$, because if there are two realizations with payoffs' difference $\sim O(a)$, then the maximum possible payoff and the minimum possible payoff realizations satisfies this as well. These realizations should correspond to the different matchings $M_1 \neq M_2$, otherwise by the continuity property of the BNB-delta solution it should be $p_i(k^2) - p_i(k^1) = O(a)$. Let at the equilibrium $Pr(M = M_2) = r_2 > 0$, $r_2 = O(1)$. Then, seller $i$ could make a profitable deviation. Indeed, let WLOG $i_{i1} \in M_1$ and $i_{i2} \in M_2$. Consider a deviation:

$$i_{i1}^{dev} = i_{i1}^{eq} + \frac{1}{2}v_+, \quad i_{-i1}^{dev} = i_{-i1}^{eq}. \tag{A-64}$$

$$\text{where } v_+ \sim O(1) \text{ is a positive root of } \quad r_2 y - i_{i1}^{eq} v - \frac{v^2}{2} = 0$$

When $a \to 0$, it should be that

$$\sum_{ij \in M_1} k_{ij} - \sum_{ij \in M_l} k_{ij} = O(a) \tag{A-65}$$

for any matching $M_l$, which appears as the ex-post efficient matching with non-zero probability. Then, for sufficiently small $a$, after our deviation it should be $Pr(M = M_1) = 1$, and the expected utility of seller $i$ increases:

$$EU_i(i^{dev}) - EU_i(i^{eq}) \geq r_2 y - i_{i1} v_+ - \frac{1}{4}v_+^2 + O(a) > 0 \tag{A-66}$$

Hence, we get a contradiction. □

3. Next, let's consider the case when at the equilibrium the ex-post payoff of seller $i$ could be distinct subject to the uncertainty realization only by the value of order of $O(a)$, and seller $i$ makes investments of order of $O(1)$ in more than two links: $i1, i2, i3, ....$ Consider some ex-post BNB graph $\mathcal{H}_1$ which realizes with probability $\sim O(1)$. Let $i1 \in M_1$ and $i2 \in O_1^{(s)}$ be the matching and the outside option[9] of seller $i$. Then he has a profitable deviation to invest $v_1 \sim O(1)$ and $v_2 \sim O(1)$ in links $i1$ and $i2$ such that BNB graph $\mathcal{H}_1$ realizes with probability 1, and do not invest in any other links. Indeed, by the Proposition (7) and in particular, by the lemma (7), the ex-post payoff of seller $i$ could not decrease larger than on the value of order of $O(a)$. However, as $a \to 0$, we also may choose $v_1, v_2 \sim O(1)$ such that seller $i$ wins on investment costs quantity of order of $O(1)$:

$$C_i(i^{dev}) - C_i(i^{eq}) = -i_{i1}^{eq} v_1 - \frac{1}{2}v_1^2 - i_{i2}^{eq} v_2 - \frac{1}{2}v_2^2 + \sum_{j \neq 1,2} \frac{i_{ij}^2}{2} > 0 \tag{A-67}$$

Hence, there is always a profitable deviation in case when seller $i$ makes investments of order of $O(1)$ in more than two links: $i1, i2, i3, ....$

[8]Since $\varepsilon$ is defined on the compact set $[-\frac{a}{2}; \frac{a}{2}]^{|G|}$, and payoff function is continuous at finite set of compacts $X_l$, then after introducing the correct tie-breaking rule on the borders of $X_n$ (where payoff vector could be non-unique since there are multiple efficient matchings), we way consider $max$ instead of $sup$, and similar for the $min$ instead of $inf$

[9]case when seller $i$ has no endogenous outside option in $\mathcal{H}_1$ could be considered analogously

4. Finally, we should consider technical cases when probabilities of some BNB-graphs $\mathcal{H}$ are of order of $O(a)$ and similar for the equilibrium levels of investments $i^{eq} \sim O(a)$. We may do it with the help of Proposition (7), and in particular by the local monotonicity lemmas for the BNB-delta solution. I postpone this technical part for further research.

$\square$

## A.5 Calculations for the "N" network

Here is the part of calculations for the "N" network in case of small noise.
Let's explicitly write down the conditions on $k$ (we further refer to them as $k \in X_n$ for the corresponding region $X$ of the space of possible surpluses $R^3$) and payoffs of sellers for different $\mathcal{H}$ (the payoff of buyers are $q_j = k_{M^{(b)}(j)j} - p_{M^{(b)}j}$ if they are matched and zero otherwise):

- $\mathcal{H}_0 = \{\varnothing; \varnothing; \varnothing\}$:

$$k_{11} \leq 0 \qquad k_{21} \leq 0 \qquad k_{22} \leq 0 \tag{A-68}$$

$$p_1 = 0 \qquad p_2 = 0$$

- $\mathcal{H}_1 = \{\{11\}; \varnothing; \varnothing\}$:

$$k_{11} \geq 0 \qquad k_{21} \leq 0 \qquad k_{22} \leq 0 \tag{A-69}$$

$$p_1 = \frac{1}{2}k_{11} \qquad p_2 = 0$$

- $\mathcal{H}_2 = \{\{11\}; \varnothing; \{21\}\}$:

$$k_{11} \geq k_{21} \geq 0 \qquad k_{22} \leq 0 \tag{A-70}$$

$$p_1 = \frac{1}{2}k_{11} - \frac{1}{2}k_{21} \qquad p_2 = 0$$

- $\mathcal{H}_3 = \{\{22\}; \varnothing; \varnothing\}$:

$$k_{11} \leq 0 \qquad k_{21} \leq 0 \qquad k_{22} \geq 0 \tag{A-71}$$

$$p_1 = 0 \qquad p_2 = \frac{1}{2}k_{22}$$

- $\mathcal{H}_4 = \{\{22\}; \{21\}; \varnothing\}$:

$$k_{22} \geq k_{21} \geq 0 \qquad k_{11} \leq 0 \tag{A-72}$$

$$p_1 = 0 \qquad p_2 = \frac{1}{2}k_{22} - \frac{1}{2}k_{21}$$

- $\mathcal{H}_5 = \{\{21\}; \varnothing; \varnothing\}$:

$$k_{11} \leq 0 \qquad k_{21} \geq 0 \qquad k_{22} \leq 0 \tag{A-73}$$

$$p_1 = 0 \qquad p_2 = \frac{1}{2}k_{21}$$

- $\mathcal{H}_6 = \{\{21\}; \varnothing; \{11\}\}$:

$$k_{21} \geq k_{11} \geq 0 \qquad k_{22} \leq 0 \tag{A-74}$$

$$p_1 = 0 \qquad p_2 = \frac{1}{2}k_{21} - \frac{1}{2}k_{11}$$

- $\mathcal{H}_7 = \{\{21\}; \{22\}; \varnothing\}$:

$$k_{21} \geq k_{22} \geq 0 \qquad k_{11} \leq 0 \tag{A-75}$$

$$p_1 = 0 \qquad p_2 = \frac{1}{2}k_{21} + \frac{1}{2}k_{22}$$

- $\mathcal{H}_8 = \{\{21\}; \{22\}; \{11\}\}$:

$$k_{21} \geq k_{22} + k_{11} \qquad k_{11} \geq 0 \qquad k_{22} \geq 0 \tag{A-76}$$

$$p_1 = 0 \qquad p_2 = \frac{1}{2}k_{21} + \frac{1}{2}k_{22} - \frac{1}{2}k_{11}$$

- $\mathcal{H}_9 = \{\{11, 22\}; \varnothing; \varnothing\}$:

$$k_{11} \geq 2k_{21} \qquad k_{11} \geq 0 \qquad k_{22} \geq 2k_{21} \qquad k_{22} \geq 0 \tag{A-77}$$

$$p_1 = \frac{1}{2}k_{11} \qquad p_2 = \frac{1}{2}k_{22}$$

- $\mathcal{H}_{10} = \{\{11, 22\}; \varnothing; \{21\}\}$:

$$k_{11} \geq k_{21} + \tfrac{1}{2}k_{22} \qquad k_{21} \geq 0 \qquad 0 \leq k_{22} \leq 2k_{21} \tag{A-78}$$

$$p_1 = \frac{1}{2}k_{11} - \frac{1}{2}k_{21} + \frac{1}{4}k_{22} \qquad p_2 = \frac{1}{2}k_{22}$$

- $\mathcal{H}_{11} = \{\{11, 22\}; \{21\}; \varnothing\}$:

$$0 \leq k_{11} \leq 2k_{21} \qquad k_{21} \geq 0 \qquad k_{22} \geq k_{21} + \tfrac{1}{2}k_{11} \tag{A-79}$$

$$p_1 = \frac{1}{2}k_{11} \qquad p_2 = \frac{1}{2}k_{22} + \frac{1}{2}k_{21} - \frac{1}{4}k_{11}$$

- $\mathcal{H}_{12} = \{\{11, 22\}; \{21\}; \{21\}\}$:

$$k_{11} \leq k_{21} + \tfrac{1}{2}k_{22} \qquad 0 \leq k_{21} \leq k_{11} + k_{22} \qquad k_{22} \leq k_{21} + \tfrac{1}{2}k_{11} \tag{A-80}$$

$$p_1 = \frac{1}{3}k_{11} - \frac{1}{3}k_{21} + \frac{1}{3}k_{22} \qquad p_2 = \frac{2}{3}k_{22} + \frac{1}{3}k_{21} - \frac{1}{3}k_{11}$$

Let's think now about the NE of the investments game on the network "$N$" in case when noise is small. In what follows we assume parameter to be arbitrary small, but positive $a > 0$. Since all expected payoffs are integrals of piecewise linear functions of $k = i + \varepsilon$ with respect to noise $\varepsilon$, these payoffs are differentiable functions of investments $i$. Hence, at equilibrium FOCs should hold.

One possibility is that the vector of investments $i$ lies inside one of the region $X_n$ for $k$ with constant graph $\mathcal{H}_n$, than the probability that this graph will appear in the ex-post NBN solution, is 1. Indeed, possible values of $k$ lies within a sphere with radius $a$ and center $i$. Hence:

$$i \in int(X_n) \quad \Rightarrow \quad \exists \underline{a} > 0 : \quad \forall a < \underline{a} \quad Pr(\mathcal{H} = \mathcal{H}_n) = Pr(X = X_n) = 1 \tag{A-81}$$

In other words, sellers in this case are sure about the ex-post bargaining conditions. Let's check if we have this kind of equilibria. FOCs for each seller should be satisfied. Clearly, in our case FOCs give us that $i_{11}, i_{21}, i_{22}$ should be equal to the corresponding coefficient of $k_{11}$ in the payoff of the first seller, and coefficients of $k_{21}$ and $k_{22}$ in the payoff of the second seller. Before we start to check FOCs, we can say, that $\mathcal{H}_0$, $\mathcal{H}_1$ and $\mathcal{H}_3$ could not support the equilibrium solely(meaning that it could not be that they have probability 1 to be the ex-post BNB graphs), since one of the sellers always may invest a bit and get positive expected return. Let's consider others:

$$\mathcal{H}_2 : \quad i_{21} = 0 \quad \Rightarrow \quad Pr(k_{21} \geq 0) < 1 \;\Rightarrow\; Pr(\mathcal{H} = \mathcal{H}_2) < 1 \;\Rightarrow\; \text{contradiction}$$

$$\mathcal{H}_4 : \quad i_{11} = 0 \quad \Rightarrow \quad Pr(k_{11} \geq 0) < 1 \;\Rightarrow\; Pr(\mathcal{H} = \mathcal{H}_4) < 1 \;\Rightarrow\; \text{contradiction}$$

$$\mathcal{H}_5 : \; Pr(k_{22} \geq 0) > 0 \;\Rightarrow\; Pr(\mathcal{H} = \mathcal{H}_5) < 1 \;\Rightarrow\; \text{contradiction}$$

$$\mathcal{H}_6 : \quad i_{11} = 0 \quad \Rightarrow \quad Pr(k_{11} \geq 0) < 1 \;\Rightarrow\; Pr(\mathcal{H} = \mathcal{H}_6) < 1 \;\Rightarrow\; \text{contradiction}$$

$$\mathcal{H}_7 : \quad i_{11} = 0 \quad i_{21} = i_{22} = \frac{1}{2} \;\Rightarrow\; Pr(\mathcal{H} = \mathcal{H}_4) > 0 \;\Rightarrow\; \text{contradiction}$$

$$\mathcal{H}_8 : \quad i_{11} = 0 \;\; Pr(\mathcal{H} = \mathcal{H}_8) < 1 \;\Rightarrow\; \text{contradiction}$$

$$\mathcal{H}_9 : \quad i_{11} = i_{22} = \frac{1}{2} \;\; i_{21} = 0 \quad \Rightarrow \quad Pr(\mathcal{H} = \mathcal{H}_9) = 1 \;\Rightarrow\; \text{FOC} \checkmark$$

$$\mathcal{H}_{10} : \quad i_{21} = 0 \quad \Rightarrow \quad Pr(\mathcal{H} = \mathcal{H}_{10}) < 1 \;\Rightarrow\; \text{contradiction}$$

$$\mathcal{H}_{11} : \quad i_{11} = i_{21} = i_{22} = \frac{1}{2} \quad \Rightarrow \quad Pr(\mathcal{H} = \mathcal{H}_{11}) < 1 \;\Rightarrow\; \text{contradiction}$$

$$\mathcal{H}_{12} : \quad i_{11} = i_{21} = \frac{1}{3} \quad = i_{22} = \frac{2}{3} \quad \Rightarrow \quad Pr(\mathcal{H} = \mathcal{H}_{12}) < 1 \;\Rightarrow\; \text{contradiction}$$