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Abstract

The paper considers a simple battle for a throne game. A winner may either execute or spare a contender. Each player has either opportunistic or bloody type; the latter can only execute contenders. Each execution lowers both incentives of other players to pretend for a throne and the dictator's chances to survive next lost fight. Depending on the parameters, in equilibrium opportunistic players either execute all contenders or almost always save their lives due to different belief of their types. Comparative statics shows that the “killing” equilibrium is more probable in countries with strong and bloody dictators, while arrival of a contender to a throne is less probable in countries with the same characteristics and, in addition, with larger dictator’s benefits and smaller death penalty.

Content

1 Introduction

Dictatorial rulers often use inhumane methods such as execution fighting their political opponents. In some dictatorial countries the winner of a battle for a throne execute those who were just removed from power and/or even those who might pretend for a dictator's place meanwhile in other non-democratic countries contenders are usually spared. The objective of this paper is to provide a simple game-theoretical model illustrating incentives lying behind such behavior.

The same subject was investigated in Egorov, Sonin (2005), where a similar model was analyzed in the context of Nash equilibrium: in that set-up dictators are concerned of their reputation presented in a strategy profile they choose and, thus, commit different behavior based on parameters of the model. Debs (2010) is another recent paper investigating the winner's decision on the fate of the loser in non-democratic succession. Dynamic models of leadership contest are widely presented in literature on the political economics of dictatorial regimes: Acemoglu, Robinson (2001), Bueno de Mesquita et al. (2003), Acemoglu (2003), Acemoglu, Robinson, Verdier (2004), Galliego, Pitchik (2004), Egorov, Sonin (2010), Egorov, Sonin (2011). In this work the notion of sequential equilibrium introduced in Kreps, Wilson (1982) is employed in order to present reputation concerns in more intuitive way: reputation is a belief of other players about a particular dictator's type. The paper follows classical literature on reputation effects pioneered by Kreps, Wilson (1982), Milgrom, Roberts (1982), and generalized by Fudenberg, Levine (1989). The former two papers consider typical sequential entrant-incumbent game with commitment-type players in the context of sequential equilibrium. In both models a long-run player being opposed to a sequence of short-lived players benefits from reputational effects. The later one considers abstract infinitely repeated game, where a long-run player can benefit from showing commitment behavior – its results were extended and considerably strengthened by several other papers of the same and other authors: Fudenberg, Levine (1992), Schmidt (1993), Cripps, Schmidt, Thomas (1996), Cripps, Dekel, Pesendorfer (2005).

In particular, the model presented in this work considers an infinitely-repeated game with possibly an infinite number of long-run players, which fight for the dictator's place. Players are of two types: opportunistic and bloody. Each turn defeated and spared later or new contender may appear and start fighting for a throne, and the opportunistic winner can choose whether to spare or to execute the loser, while the bloody winner always executes losers. Since player’s type is private information, other players have to assess the probability of a particular player being bloody, and, subsequently, each player can affect her reputation by her actions. There are two major incentives to enhance or diminish reputation of a bloody dictator, these are: on the one hand, other players can become threaten of a possibility to be executed and avoid starting fighting against a dictator who is known to bloody, on the other hand, a dictator who usually executes contenders is unlikely to survive next defeat in a battle for the throne.

The findings of the paper is that there can be one of three equilibria: first, the one in which no player ever spares another, second, the one in which player spares only certainly opportunistic players – the first and the second equilibria differ insignificantly in a sense that they cannot be visibly distinguished over the equilibrium paths – and in the last one players execute only those players who behave as though they are of a bloody type. Each equilibrium exists only under certain conditions on parameters. Moreover, under some refinements and under some conditions these are the unique equilibria of the game. Comparative statics shows that dictator is more likely to face a contender if either probability of dictator’s defeat is larger or benefits of being a dictator is greater, or execution is less threatening, that is, death penalty is lower, or reservation utility of players is larger. In the case of “sparing” equilibrium, opportunistic players also benefit from higher discount factor. That is, it is more likely that a dictator faces a contender in countries where (keeping all equal): dictatorship is “unstable” – high probability of dictator’s defeat; the dictator possess more power or the country itself richer – benefits of being a dictator are larger; possible contenders are not so afraid of death, e. g. religious zealots – death penalty is lower; inequality is less and citizens are relatively rich – reservation utility of players is greater. Also the area in the space of all possible values of parameters, in which “killing” equilibrium exists, tends to shrink with an increase in probability of dictator’s defeat or a decrease in prior probability of a player being bloody. In other words, our model predicts that “killing” equilibrium is more likely to occur in countries with "stable" dictatorships and high prior probability of a bloody dictator – the latter may take place due to, for instance, cultural or religious reasons.

History of dictatorial countries provides an enormous number of examples of equilibria mentioned above. A good overview of the historical examples is presented in Egorov, Sonin (2005). I refer to this paper here. Latin America provides an abundance of examples of dictators who lost the power and took it back up to five times without being executed or execution of opponents. Russian empire might be considered as another illustration of sparing equilibrium. Meanwhile in Ottoman Empire dozens of contenders and ex-dictators were executed over six centuries. Other examples of executed dictators include Afghanistan, Bangladesh, Iraq, Nigeria, Comoros, and Liberia, and monarchies in European countries.

The rest of the paper structured as follows: section two provides formal set-up of the game; section three presents the results: equilibrium and its properties: uniqueness and comparative statics; and section three concludes. Proofs of propositions and claims are provided in the appendix.

2 Formal Set-up

In this section formal set-up of the game is presented.

Time is discrete: $t = 1, 2, \dots$; each player is marked with a number: $i = 1, 2, \dots$ At the beginning of each period *t* there is a dictator D_t and (possibly) a leader of an opposition LO_t: D_t = 1, 2, …; LO_t = 0, 1, 2, … If LO_t = 0, this means there is no leader of an opposition. In each period there are determined, first, whether there is a contender, $C_t = 0, 1, 2, \dots$ (if $C_t = 0$, this means there is no contender), and, second, winner and loser, W_t and L_t, respectively. Let N_t be the identity of player with the least number who has not joined the active part of the game yet.

Each player has a type $T(i) \in \{o, b\} \times \Theta, \Theta = [0, 1]$, where o and b refer to 'opportunistic' and 'bloody', respectively, the components of a type are independent of each other and the probabilities of being a particular type are determined the following way:

$$Pr(T(i)_1 = s) = \varepsilon, \qquad Pr(T(i)_1 = n) = 1 - \varepsilon, \qquad 0 < \varepsilon < 1, \qquad T(i)_2 \sim \mathrm{U}[\Theta]$$

Here, $\mathrm{U}[\Theta]$ denotes uniform distribution over Θ. A type of a player is her private information. The first component determines which actions can be performed by a particular player, and the second one – reservation utility which might be received by a player.

Each player maximizes her utility which is equal to a sum of discounted payoffs in each period $U(i, T(i)) = E_t \sum_{\tau=1}^{\infty} \delta^{\tau} U_{t+\tau}(i, T(i))$

In each period *t*, the sequence of actions and events is as follows:

First, particular players decide whether there will be a contender in this period or not:

- If $LO_t = 0$, then player N_t decides whether to become a contender or not:
  - If N_t decides to stay out, then $U(N_t, T(N_t)) = u(T(N_t)_2)$, and $C_t = 0,\ N_{t+1} = N_t + 1$, where function $u(\cdot): \Theta \rightarrow [-D, \alpha Y],\ 0 < \alpha < 1$
  - If N_t decides to fight against the dictator, then $C_t = N_t,\ N_{t+1} = N_t + 1$
- If $LO_t \neq 0$, then player LO_t decides whether to continue fighting against the dictator or not:
  - If LO_t decides to quit, then $U(LO_t, T(LO_t)) = \min\{u(T(LO_t)_2), 0\}$, and, $C_t = 0, N_{t+1} = N_t$
  - If LO_t decides to fight against the dictator, then $C_t = LO_t,\ N_{t+1} = N_t$

Second, the winner and the loser are determined.

1. If $C_t = 0$, then $W_t = D_t,\ L_t = 0$, and there is no leader of an opposition in the next period, $LO_{t+1} = 0$, and steps 4-6 are skipped, o/w, proceed to step 4
2. The fight breaks out, and the contender wins with probability $0 < \mathrm{p} < 1$.
3. If $\mathrm{T}(\mathrm{W}_t)_1 = \mathrm{o}$, then W_t decides on its action A_t whether to spare ($\mathrm{A}_t = \mathrm{S}$) or to execute ($\mathrm{A}_t = \mathrm{E}$) the loser L_t, o/w, $\mathrm{A}_t = \mathrm{E}$
4. If $\mathrm{A}_t = \mathrm{E}$, then $\mathrm{U}_t(L_t, \mathrm{T}(L_t)) = -\mathrm{D},\ \mathrm{D} > 0$, and there is no leader of an opposition in the next period, $\mathrm{LO}_{t+1} = 0$. If $\mathrm{A}_t = \mathrm{S}$, then $\mathrm{U}_t(\mathrm{L}_t, \mathrm{T}(\mathrm{L}_t)) = 0$, the loser become a leader of an opposition in the next period ($\mathrm{LO}_{t+1} = \mathrm{L}_t$)
5. The winner receives a payoff $\mathrm{Y} > 0$ and becomes a dictator in the next period: $\mathrm{U}(\mathrm{W}_t, \mathrm{T}(\mathrm{W}_t)) = \mathrm{Y}, \mathrm{D}_{t+1} = \mathrm{W}_t$

3 Equilibrium and its properties

3.1 Equilibrium

In this section the results are presented.

As we wish to employ a notion of sequential equilibrium in this work, we have to move from the game with incomplete information to a game with complete but imperfect information, which can be performed by introducing a new player, “Nature”, which moves first and “distributes” types between the players according to the prior distributions, that is, “Nature” follows mixed strategy, and its move determines types of all other players, and the other players do not know which move is made by the nature.

Let H_t be a full history of actions and decisions made by all players before the start of period t. Let $p_i(H_t)$ be a belief about the first component of the i’th player’s type possessed by the other players, that is, it is the probability of player i being a bloody type assessed by the other players. Obviously, in any equilibrium with beliefs satisfying Bayes’ rule $p_i(H_t) = 0$ if player i spared someone before period t. Later we will see that players are totally indifferent what the second component of any other player’s type is. These definitions are employed in the definition of sequential equilibrium for the game analyzed here, which is defined as follows. An equilibrium includes a strategy for each player and, for each period $t = 1, 2, \dots$ a set of functions $\{p_i(\cdot), i = 1, 2, \dots\}$, taking histories of moves up to stage t into numbers in $[0, 1]$ such that: (a) starting from any point in the game, a player’s strategy (contingent on a history H_t, of prior play) is a best response to the others’ strategies given that a player j is bloody with probability $p_j(H_t)$; (b) Each $p_j(H_t)$, is computed from H_t using Bayes’ rule whenever possible.

Lemma 1. In any sequential equilibrium player i always spares player j from some period t, then player j always become a contender after being spared by player i.

Proof. Consider the following strategy of player LO_t: she always continues to fight once she is defeated in the previous turn and always spares a contender, then in this case neither LO_t nor D_t is ever executed (D_t always spares LO_t). Thus, in this case $U_{continue\ \ fighting}\,(W_{t-1}, T(W_{t-1})) > 0$, since p > 0, while $U_{quit}\,(W_{t-1}, T(W_{t-1})) = \min\{u(T(W_{t-1})_2), 0\} \leq 0$. So, there is a strategy which gives strictly greater payoff than quitting. Subsequently, continuing fighting dominates quitting. ■

Now, the key proposition of existence of an equilibrium in pure strategies can be formulated. For further purposes a linear function determining reserved utility is employed $u(\theta) = \theta(\alpha Y + D) - D$. Notice that $u(1) = \alpha Y$, and $u(0) = -D$.

Proposition 1. There is a sequential equilibrium of the game.

Proof: Full proof is provided in the appendix. Here the exact form of an equilibrium is presented. In particular, the equilibrium might take one of the following forms:

Case 1: if $\frac{Y((1-\delta)(1-\bar{\theta})-\delta p\bar{\theta})}{(1-\delta)(1-\delta+2p\delta)} > \bar{\theta}D$, then every player i follow the strategy: a) start fighting against a dictator (become a contender) if $T(i)_2 < \bar{\theta}$; b) always continue fighting (become a contender) after being spared; c) if opportunistic type, execute any player no matter what her reputation is; d) beliefs are $p_j\,(H_t) = \varepsilon\ \forall j, H_t$ such that player j spares no player during the history H_t, otherwise, $p_j\,(H_t) = 0$;

Case 2: if $\frac{Y((1-\delta)(1-\bar{\theta})-\delta p\bar{\theta})}{(1-\delta)(1-\delta+2p\delta)} < \bar{\theta}D$ and $\{\bar{\theta}(1-\delta+\delta p) - \varepsilon(1-\delta+\delta p\bar{\theta})\}D < \Big\{(1-\bar{\theta}) -$ $-1-\delta+\delta p\theta p\delta 1-\varepsilon 1-\delta 1-\delta+2p\delta Y$, then every player i follow the strategy: a) start fighting against a dictator (become a contender) if either $T(i)_2 < \bar{\theta}, p_{D_t}(H_t) > 0$ or $p_{D_t}(H_t) = 0$; b) always continue fighting (become a contender) after being spared; c) if opportunistic type, execute player j at period t if $p_j\,(H_t) > 0$, spare, otherwise; d) beliefs are the following: $p_j\,(H_t) = 1$, if player j executes certainly opportunistic player during the history H_t, $p_j\,(H_t) = 0$, player j spares any player during the history H_t $p_j\,(H_t) = \varepsilon$, otherwise;

Case 3: if $\{\bar{\theta}(1-\delta+\delta p) - \varepsilon(1-\delta+\delta p\bar{\theta})\}D > \left\{(1-\bar{\theta}) - \frac{(1-\delta+\delta p\bar{\theta})p\delta(1-\varepsilon)}{(1-\delta)(1-\delta+2p\delta)}\right\}Y$, then every player i follow the strategy: a) start fighting (become a contender) against a dictator if either

$\begin{cases} T(i)_2 < \hat{\theta}_o, T(i)_1 = o \\ T(i)_2 < \hat{\theta}_b, T(i)_1 = b \end{cases}, p_{D_t}(H_t) = 1,$ or $\begin{cases} T(i)_2 < \tilde{\theta}_o, T(i)_1 = o \\ T(i)_2 < \tilde{\theta}_b, T(i)_1 = b \end{cases}, 1 > p_{D_t}(H_t) > 0,$ or $p_{D_t}(H_t) = 0$; b) always continue fighting (become a contender) after being spared; c) if opportunistic type, spare player j at period t if $p_j(H_t) < 1$, execute, otherwise; d) beliefs are the following: $p_1(H_1) = \varepsilon$, $p_j(H_t) = \hat{\varepsilon}\ \forall j >, H_t$ such that player j "entered" against certainly bloody dictator and either performed no action (sparing or execution) or executed only certainly bloody players during the history H_t, $p_j(H_t) = \tilde{\varepsilon}\ \forall j, H_t$ such that player j "entered" against dictator of uncertain type and performed no action (sparing or execution) during the history H_t, $p_j(H_t) = 0$ if player j performed sparing, $p_j(H_t) = 1$ if player j performed execution of not certainly bloody players.

where:

$$\bar{\theta} = \frac{\sqrt{(1-\delta)^2 + 4p^2\delta \frac{Y + (1-\delta)D}{\alpha Y + D}} - (1-\delta)}{2p\delta}$$

And $\hat{\theta}_o$, $\hat{\theta}_b$, $\tilde{\theta}_o$, $\tilde{\theta}_b$, $\hat{\varepsilon}$, and $\tilde{\varepsilon}$ solve a system of the following equations:

$$\hat{\theta}_o(\alpha Y + D) = p\left(D + \frac{1}{1-\delta+\tilde{\theta}\delta}\left(Y + \frac{\delta\tilde{\theta}}{1-\delta+p\delta}\left\{(1-p)Y + \frac{(1-\tilde{\varepsilon})p^2\delta Y}{(1-\delta)(1-\delta+2p\delta)} - \tilde{\varepsilon}pD\right\}\right)\right)$$

$$\hat{\theta}_b(\alpha Y + D) - pD = \frac{p}{1-\delta+\delta\tilde{\theta}}\left\{Y - \delta\varepsilon\tilde{\theta}_b pD + \delta\tilde{\theta}\frac{(1-\delta-p+2p\delta-\tilde{\varepsilon}p^2\delta)(Y-\delta p\hat{\theta}D)}{(1-\delta+\delta p)(1-\delta+\delta p\hat{\theta})}\right\}$$

$$\tilde{\theta}_o(\alpha Y + D) - D = \frac{pY}{1-\delta+p\delta} + \frac{1-\delta-p+2p\delta}{1-\delta+p\delta}\left(\hat{\varepsilon}(-D) + (1-\hat{\varepsilon})\frac{pY}{(1-\delta)(1-\delta+2p\delta)}\right)$$

$$\tilde{\theta}_b(\alpha Y + D) - (1-\hat{\varepsilon}(1-p))D = \frac{(1-\hat{\varepsilon}(1-p)\delta)p}{(1-\delta+\delta p)\left(1-\delta+\delta p\hat{\theta}\right)}(Y - \delta p\hat{\theta}D)$$

$$\hat{\theta} = \varepsilon\hat{\theta}_b + (1-\varepsilon)\hat{\theta}_o$$

$$\hat{\varepsilon} = \frac{\varepsilon\hat{\theta}_b}{\varepsilon\hat{\theta}_b + (1-\varepsilon)\hat{\theta}_o} = \frac{\varepsilon\hat{\theta}_b}{\hat{\theta}}$$

$$\tilde{\theta} = \varepsilon\tilde{\theta}_b + (1-\varepsilon)\tilde{\theta}_o$$

$$\tilde{\varepsilon} = \frac{\varepsilon\tilde{\theta}_b}{\varepsilon\tilde{\theta}_b + (1-\varepsilon)\tilde{\theta}_o} = \frac{\varepsilon\tilde{\theta}_b}{\tilde{\theta}}$$

More precisely, $\hat{\theta}_o$, $\hat{\theta}_b$, $\tilde{\theta}_o$, $\tilde{\theta}_b$, $\hat{\varepsilon}$, and $\tilde{\varepsilon}$ solve this system of equations, but with each unknown replaced on the right-hand side with minimum of it and one, since each of it cannot exceed one, and thus each marginal agent solving his "enter"/"stay out" problem should not expect that these parameters exceed one.■

It is important to notice that the conditions mentioned in the sketch proof are cumulatively exhaustive and mutually exclusive. The former property is obvious, while the latter one is not so trivial - it is shown in the following claim.

Claim 1. Cases 1 – 3 of Proposition 1 are mutually exclusive.

Proof of the claim is provided in the appendix.

Notice that the first two cases do not differ in a sense that the equilibrium paths of both Case 1 and Case 2 are the same, but they are different conceptually: in the former case players just do not have enough incentives to spare any player regardless of their types, while in the later one players would like to spare opportunistic opponents, but they are too threaten by the possibility of being executed a by bloody dictator in a case of defeat.

Moreover, given the conditions of Case 2, there might be more than one equilibrium – it is discussed in details in the next subsection.

3.2 Properties of equilibrium

Before stating formulating of a proposition of uniqueness of the equilibrium a refinement on the set of possible equilibria is conducted. Consider the following definitions

Definition. Equilibrium is called to be *monotone in beliefs* if $p_i(H_{t+1}) \geq p_i(H_t)$ once player i performed execution at period t, and if $p_i(H_{t+1}) \leq p_i(H_t)$ once player i performed sparing at period t.

It is easy to see that the latter condition always holds in any equilibrium with beliefs satisfying Bays' rule, since in this case $p_i(H_{t+1}) = 0$ and $0 \leq p_i(\cdot) \leq 1\ \forall i$

Definition. Equilibrium is called to be *monotone in actions* if on the equilibrium path at any period t the probability of a loser, L_t, being executed is not decreasing with $p_{W_t}(H_t)$ and $p_{L_t}(H_t)$.

Put simply, the last definition states that in monotone in actions equilibrium for each history H_t there are thresholds $\bar{p}_1(H_t)$ and $\bar{p}_2(H_t)$ such that if $p_j(H_t) \leq \bar{p}_1(H_t)$ and $p_i(H_t) \leq \bar{p}_2(H_t)$, player *i* spares player *j* at period *t*, otherwise, – executes.

The intuition behind these definitions (strictly speaking, restrictions) is straightforward: first, players should not have any kind of “strange” beliefs assessing killing players with non-bloody reputation, second, players’ strategies have to be consistent in a sense that: if player once spared someone with a particular reputation, sparing players with “more peaceful” reputation should be a best response, similarly, if players with “more peaceful” reputation should behave “more peaceful” than those with “more bloody” reputation.

Trivially, the equilibrium from Proposition 1 is monotone in beliefs and actions. The following lemma is employed in the proof of the proposition of uniqueness.

Lemma 2. In any equilibrium which is monotone in beliefs and actions, if player *i* spares player *j* at period *t*, then player *i* always spares player *j* thereafter if she has to decide on whether to execute or to spare player *j*.

Proof. Assume player *i* spares player *j* at period *t*. Consider period $t + k$ of the game at which player *i* has to decide on whether to execute or to spare player *j*. This can occur in one of two cases: first, player *j* wins battle (or several battles) for a throne and spares player *i* between periods *t* and $t + k$, or player *j* never wins player *i* between periods *t* and $t + k$. Moreover, in both cases player *i* can only spare player *j* and cannot execute another player between periods *t* and $t + k$. Thus, by monotonicity in beliefs, in both cases $p_j(H_{t+k}) \leq p_j(H_t)$, $p_i(H_{t+k}) \leq p_i(H_t)$. Subsequently, by monotonicity in actions player *i* always spares player *j* at period $t + k$. ■

There is one extremely useful implication of the lemma, which is formulated in the subsequent corollary.

Corollary. In any equilibrium which is monotone in beliefs and actions, if $LO_t \neq 0$ at some period *t*, then player LO_t always decides to continue fighting against the dictator, that is, $C_t = LO_t$.

Proof. If $LO_t \neq 0$ at some period t, then D_t spared LO_t at period $t-1$, thus, in any equilibrium which is monotone in beliefs and actions, D_t always spares LO_t thereafter, hence, by Lemma 1, LO_t always become a contender against player D_t. ■

Further obvious implication can be formulated as follows. If in a sequential equilibrium $p_{W_t}(H_t) = 0$, then L_t is spared.

Finally, everything is ready for the proposition of uniqueness to be formulated. As it mentioned earlier the conditions of Case 2 from Proposition 1 allow for more than one equilibrium. This holds true for a class of monotone in beliefs and actions equilibria: there are multiple equilibria, which actually can be as well as the equilibrium stated in Proposition 1 classified as either "killing" or "sparing". These equilibria are formulated after the Proposition 2.

It is also worth to notice that even under conditions of Case 1 and 3 one cannot state that the equilibrium is completely unique as well, since actually there is a certain leeway in definition of equilibrium out of the equilibrium path: for instance, in Case 3 player could execute loser *j* at period *t* with $p_j(H_t) > \varepsilon$, and this would be still an equilibrium, but it would change completely nothing over the equilibrium path, since the only players who have $p(H_t) > \varepsilon$ are those for whom it is equal to 1. Hence, the following proposition does not state complete uniqueness. Instead, it says that the equilibrium is unique only over the equilibrium path, which means that if there is another equilibrium of the game under the stated conditions, then this equilibrium is absolutely the same over the equilibrium path as those stated in Proposition 1.

Proposition 2. If $\frac{Y((1-\delta)(1-\bar{\theta})-\delta p\bar{\theta})}{(1-\delta)(1-\delta+2p\delta)} > \bar{\theta}D$ or $\left\{\frac{(1-\delta+\delta p\bar{\theta})p\delta(1-\varepsilon)}{(1-\delta)(1-\delta+2p\delta)} - (1-\bar{\theta})\right\}Y > \{\varepsilon(1-\delta+$ $\delta p\theta - \theta 1 - \delta + \delta pD$, then the equilibrium stated in Proposition 1 is unique over the equilibrium path in the class of monotone in beliefs and actions sequential equilibria.

Proof of the proposition is provided in the appendix.

In Proposition 2 only two out of three cases stated in Proposition 1 are considered for there are other equilibria from the class of monotone in beliefs and actions sequential equilibria, which can take place under the condition of Case 2 of Proposition 1. The following is informal speculation about other possible under the condition equilibria – unfortunately, formal proof of the statements made below cannot be derived analytically.

First, in the proof of Proposition 1 it is shown that at the boundary of Case 2 and 3 in the “killing” equilibrium players are indifferent between killing and sparing players who are assessed to be bloody with probability ε, but in the “sparing” equilibrium, as shown in the same proof, present value of executing and sparing any player with uncertain reputation (player is not assessed by the others to be certainly opportunistic or bloody) are lower and greater, respectively. Thus, in the “sparing” equilibrium it is suboptimal to execute any player with uncertain reputation at the boundary of Case 2 and 3, that is, the boundary does not actually binds the area of existence of the equilibrium. In other words, the “sparing” equilibrium can take place under the condition of Case 2 of Proposition 1.

Second, monotonicity in actions requires that at any period t for any belief about winner’s type, $p_{W_t}(H_t)$, there is a threshold, $\bar{p}(H_t)$, such that the loser is spared, if $p_{L_t}(H_t) < \bar{p}(H_t)$, and executed, if $p_{L_t}(H_t) > \bar{p}(H_t)$. A simple corollary of Propositions 1 and 2 is that under the condition of Case 3 of Proposition 1 the threshold has to be greater than ε, which is definitely does not need to be true under the condition of Case 2, where there is a set of other equilibria in which some players after joining the active part of the game have to improve their reputation via sparing some opponent in order to be consequently spared by other players, while some players do not have to. In other words, under conditions of Case 2 of Proposition 1, there is a (possibly infinite) sequence of other equilibria with “killing” threshold for player’s reputation (some critical value for player’s reputation, under which players are spared, and above - executed) changing from ε to 1.

Now, we proceed with comparative statics of the equilibrium. In particular, we provide comparative statics of the “entering” thresholds, players’ beliefs about players with uncertain reputation, and, the most interesting, the conditions of existence of “killing” and “sparing” equilibria. Some of the comparative statics is provided in analytical form, meanwhile some – in quantitative due to impossibility of employing of analytical approach.

We start with comparative statics of the thresholds, beliefs and existence conditions the "killing" equilibria, that is, equilibria in which every winner executes the loser along the equilibrium path. Form Proposition 1 we have an expression for the threshold:

$$\bar{\theta} = \frac{\sqrt{(1-\delta)^2 + 4p^2\delta \frac{Y + (1-\delta)D}{\alpha Y + D}} - (1-\delta)}{2p\delta}$$

Trivially, it is decreasing with α, since, keeping all equal, higher α leads to higher reservation utility. Take the derivatives of the expression with respect to p, Y, and D, respectively.

$$\bar{\theta}'_p = \frac{2\delta}{(2p\delta)^2}\left((1-\delta) - \frac{(1-\delta)^2}{\sqrt{(1-\delta)^2 + 4p^2\delta \frac{Y + (1-\delta)D}{\alpha Y + D}}} \right) > 0$$

$$\bar{\theta}'_Y = \frac{pD\big(1 - \alpha(1-\delta)\big)}{(\alpha Y + D)^2 \sqrt{(1-\delta)^2 + 4p^2\delta \frac{Y + (1-\delta)D}{\alpha Y + D}}} > 0$$

$$\bar{\theta}'_D = \frac{-pDY\big(1 - \alpha(1-\delta)\big)}{(\alpha Y + D)^2 \sqrt{(1-\delta)^2 + 4p^2\delta \frac{Y + (1-\delta)D}{\alpha Y + D}}} < 0$$

We see that willingness of a player to get involved in the battle for a throne increases with an increase in probability of a dictator's defeat and winner's payoff, while decreases with an increase in death penalty. The latter two properties are quite intuitive: an increase in winner's payoff or a decrease in death penalty lead to higher expected payoff of being a contender, and thus, increase incentives to become a contender. However, the former property, the reaction of the threshold on an increase in probability of a dictator's defeat is not so trivial, since on the one hand, higher probability of a dictator's defeat leads to higher probability of the player becoming the winner, but on the other hand, the winner becomes a dictator in the next period and higher probability of a dictator's defeat decreases her expected payoff as a dictator, since we consider "killing" equilibrium, where defeated dictator is killed for sure. The former property implies that the first period benefits from an increase in probability of a dictator's defeat exceed consequents losses from it.

Consider the derivative of the expression with respect to δ:

$$\bar{\theta}_{\delta}' = \frac{2p}{(2p\delta)^2}\left(\frac{\delta - 1 - 2p^2\delta\dfrac{Y+D}{\alpha Y+D}}{\sqrt{(1-\delta)^2 + 4p^2\delta\dfrac{Y+(1-\delta)D}{\alpha Y+D}}} + 1\right)$$

The threshold is increasing $\bar{\theta}_{\delta}' > 0$ if and only if $p^2 < \frac{Y(\alpha Y+D)}{(Y+D)^2}$, that is, if probability of a dictator's defeat is relatively small. Indeed, players are more likely to "positively react" to an increase in discount factor if payoffs of future periods are sufficiently large, which is only the case if probability of keeping the dictators place is sufficiently large as well (p is small).

It is important to notice that the threshold does not depend on the probability of a player being bloody, which occurs due to the fact that on the equilibrium path both types behave similarly and, subsequently, their types do not affect players' intention to enter. By the same reason, as shown in Proposition 1, in the "killing" equilibrium beliefs do not depend on any parameters except for the threshold does not depend on the probability of a player being bloody, and hence, there is no need to provide comparative statics for beliefs about players' types.

Consider the conditions of "killing" equilibrium existence. The first condition allows for an equilibrium in which even players known to be certainly opportunistic are executed; the condition is provided below

$$F = \bar{\theta}D + \frac{Y(\delta p\bar{\theta} - (1-\delta)(1-\bar{\theta}))}{(1-\delta)(1-\delta+2p\delta)} < 0$$

We investigate how the left hand side of the inequality reacts to changes in parameters, which allows us to infer whether the area of existence of the equilibrium shrinks or expands with an increase in certain parameters.

First, consider the derivative of F with respect to p.

$$F_p' = \bar{\theta}_p' D + \frac{Y}{(1-\delta)} \frac{\bar{\theta}_p'\big(\delta p + (1-\delta)\big)(1-\delta+2p\delta) + (2-\bar{\theta})\delta(1-\delta)}{(1-\delta+2p\delta)^2} > 0$$

This implies that the area of existence of the equilibrium shrinks with an increase in p, which might be interpreted in the following sense: "killing" equilibrium takes place in countries where the probability of dictator's defeat is sufficiently small, while in countries with sufficiently small probability of dictator's defeat "killing" equilibrium cannot exist.

Now, consider the derivatives of F with respect to Y and D.

$$F_Y' = \bar{\theta} \frac{1-\delta+\delta p}{(1-\delta)(1-\delta+2p\delta)} + \bar{\theta}_Y' \left(D + \frac{Y\delta p}{(1-\delta)(1-\delta+2p\delta)}\right) - \frac{1}{(1-\delta+2p\delta)}$$

$$F_D' = \bar{\theta} + \bar{\theta}_D' \left(D + \frac{Y(1-\delta+\delta p)}{(1-\delta)(1-\delta+2p\delta)}\right)$$

Both of them can have either positive or negative sign due to the following reasons: on the one hand, if dictator's payoff is higher, then each player has more incentives to threaten possible contentedness via executing losers, but on the other hand, higher dictator's payoff makes it harder to threaten other players, since they have more incentives to "enter"; similarly, higher death penalty makes it easier to threaten possible contenders, since those have less incentives to "enter", nevertheless, it reduces dictator's incentives to build bloody reputation.

Apparently, reaction of the condition to changes in discount factor is ambiguous, and also changes in prior probability of a player being bloody has no effect on the condition because of the nature of the equilibrium expressed in Case 1 of Proposition 1: each player executes losers regardless to their types. Another trivial fact is that the area expands with α, since higher α decreases player's incentives to "enter", and thus, makes it easier to threaten possible contenders through imitating bloody type behavior.

The second condition to be considered here is the one which bounds an area in the set of all possible values parameters, in which an equilibrium expressed in Case 2 of Proposition 2 takes place: the one where only players known to be certainly opportunistic are spared. The condition is provided below.

$$G = \left\{ \frac{(1-\delta+\delta p\bar{\theta})p\delta(1-\varepsilon)}{(1-\delta)(1-\delta+2p\delta)} - (1-\bar{\theta}) \right\} Y + \{\bar{\theta}(1-\delta+\delta p) - \varepsilon(1-\delta+\delta p\bar{\theta})\} D < 0$$

Consider the derivatives of G with respect to ε and p.

$$G_{\varepsilon}^{'} = -\frac{(1-\delta+\delta p\bar{\theta})p\delta}{(1-\delta)(1-\delta+2p\delta)} Y - (1-\delta+\delta p\bar{\theta}) D < 0$$

$$G_{p}^{'} = \frac{\delta(1-\varepsilon)(1-\delta)}{(1-\delta+2p\delta)^2} Y + \bar{\theta}_p^{'} Y \left(1 + \frac{\delta^2 p^2 (1-\varepsilon)}{(1-\delta)(1-\delta+2p\delta)} \right) + \bar{\theta}_p^{'} D \big(1-\delta+\delta p(1-\varepsilon) \big)$$
$$+ \bar{\theta}\delta(1-\varepsilon) \left(D + 2\delta p Y \frac{(1-\delta+p\delta)}{(1-\delta)(1-\delta+2p\delta)^2} \right) > 0$$

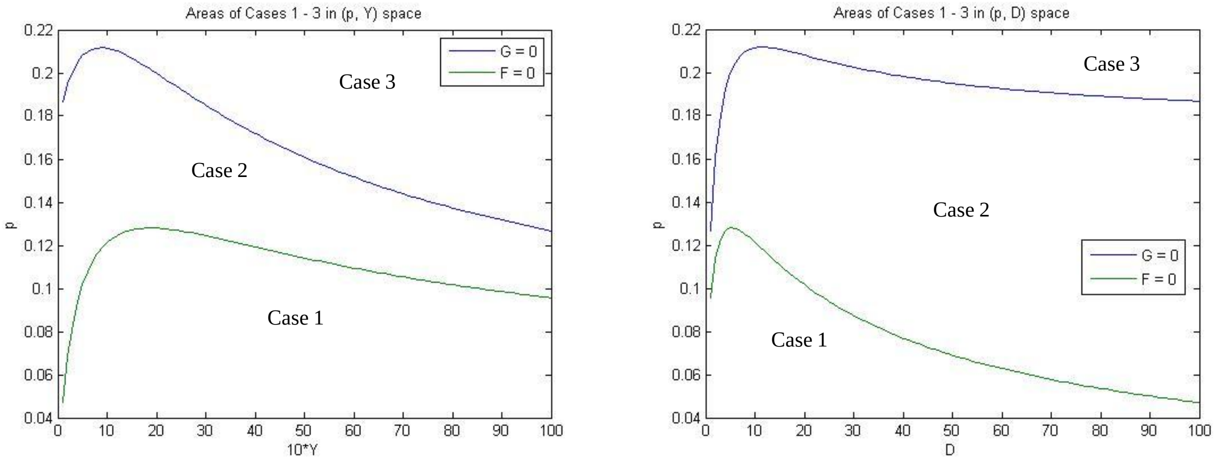
From the obtained expressions for the derivatives we may infer that the area shrinks with an increase in probability of a dictator's defeat and expands with an increase in prior probability of a player being bloody. Intuition behind these results is trivial: first, higher probability of dictator's defeat reduces dictator's incentives to imitate bloody type, since once such a dictator loses she gets executed, thus, "killing" equilibrium is "less possible"; second, an increase in prior probability of a dictator being bloody makes players more threaten of possibility of facing a bloody opponent and, thus, decreases incentives to spare losers. And again, an increase in α leads to an increase in the area bounded by the condition.

Due to the same reasons as provided for the previously considered condition the derivatives of G with respect to Y and D can have either positive or negative sign, which can be seen from the expressions for the derivatives provided below.

$$G_Y^{'} = \left\{\frac{(1-\delta+\delta p\bar{\theta})p\delta(1-\varepsilon)}{(1-\delta)(1-\delta+2p\delta)} - (1-\bar{\theta})\right\} + \bar{\theta}_Y^{'}\left\{\frac{p^2\delta^2(1-\varepsilon)}{(1-\delta)(1-\delta+2p\delta)} + 1\right\}Y$$
$$+ \bar{\theta}_Y^{'}\big(1-\delta+\delta p(1-\varepsilon)\big)D$$

$$G_D^{'} = \bar{\theta}_D^{'}\left\{\frac{p^2\delta^2(1-\varepsilon)}{(1-\delta)(1-\delta+2p\delta)} + 1\right\}Y + \bar{\theta}_D^{'}\big(1-\delta+\delta p(1-\varepsilon)\big)D$$
$$+ \{\bar{\theta}(1-\delta+\delta p) - \varepsilon(1-\delta+\delta p\bar{\theta})\}$$

Also, one can observe it on the graph below, where the areas of Cases 1 – 3 are presented on (p, Y) and (p, D) spaces, respectively. For the left-hand part other parameters are the following: $Y = 1, \delta = 0.9, \varepsilon = 0.05, \alpha = 0.6$; for the right-hand part – $D = 10, \delta = 0.9, \varepsilon = 0.3, \alpha = 0.6$.

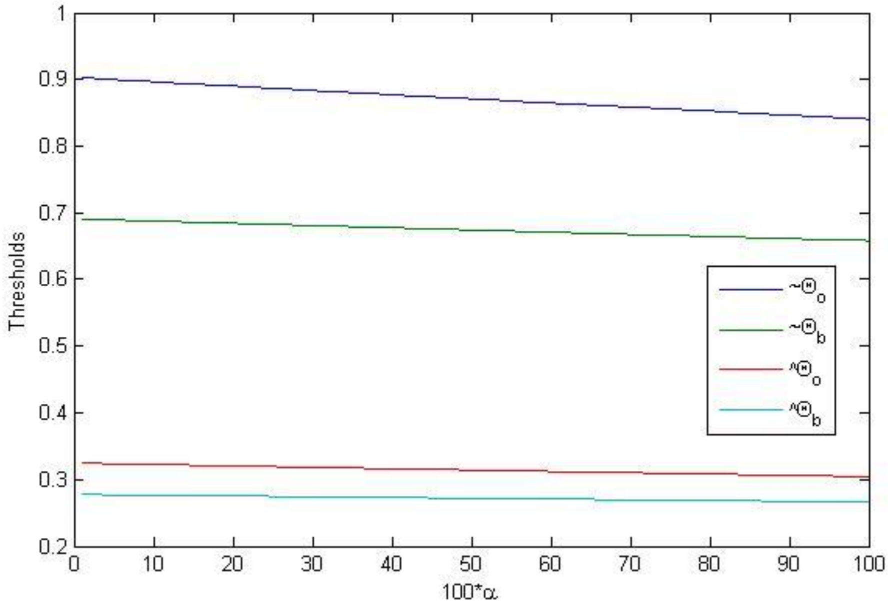


Graph 1. Areas of Cases 1 – 3 in (p, Y) and (p, D) spaces. Other parameters are: for the left-hand part: $Y = 1, \delta = 0.9, \varepsilon = 0.05, \alpha = 0.6$; for the right-hand part: $D = 10, \delta = 0.9, \varepsilon = 0.3, \alpha = 0.6$

Now, proceed with comparative statics of the "sparing equilibrium". We need to analyze how "enter" thresholds and beliefs react to changes in parameters. Since both the thresholds and beliefs cannot be found analytically, their comparative statics is provided in computational form: the results stated here are obtained via a number of repeated estimations of on a grid over possible values of parameters and supported with representative graphs with one changing parameter and others fixed.

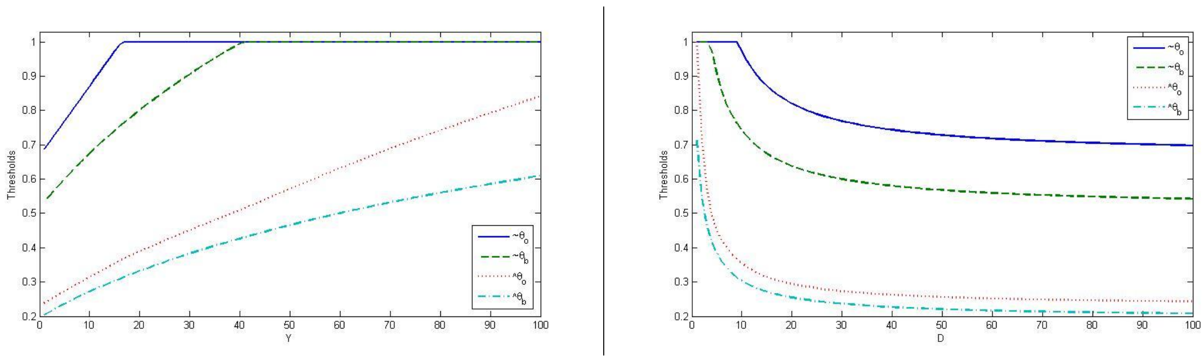
Comparative statics of beliefs is not as rich as of the thresholds, since they almost ambiguously react to increase in other parameters (in some cases they rise, in some – fall), and thus, no certain conclusion can be made based on the analysis.

Consider the “enter” thresholds. Recall that these determine whether a player of a certain type “enters” (becomes a contender). As well as in the “killing” equilibrium, they are decreasing with α, since larger α means larger reservation utility. The following graph illustrates the dynamics.



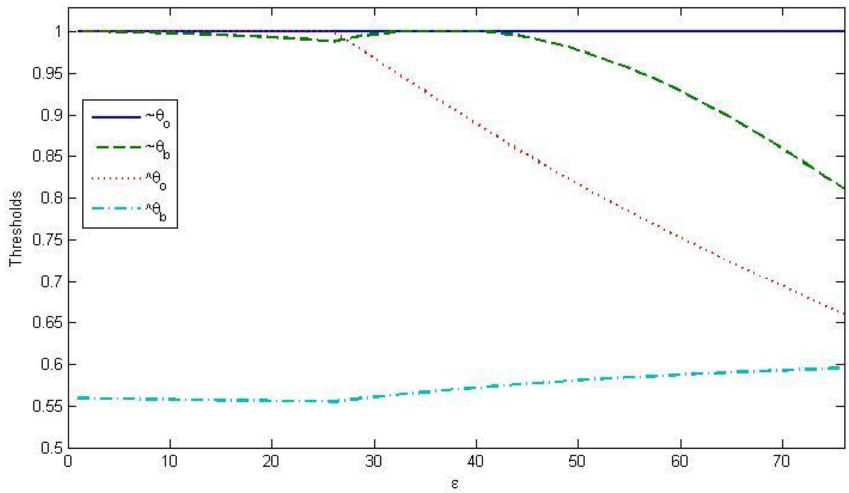
Graph 2. Illustration of negative dependence of the thresholds on α. Other parameters are the following: $Y = 10, D = 150, \delta = 0.9, p = 0.3, \varepsilon = 0.3, \alpha = 0.5$

The thresholds are increasing with dictator’s payoff, Y, and decreasing with death penalty, $\boldsymbol{D}$: both an increase in Y and a decrease in D make an option of becoming a contender more tempting and, thus, shift the thresholds up. An illustration is provided below.



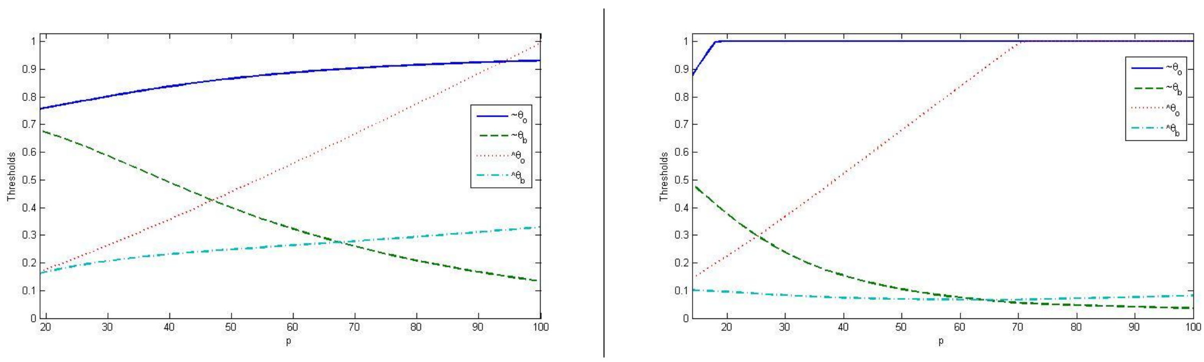
Graph 3. Illustration of positive and negative dependence of the thresholds on Y and D, respectively. Other parameters are the following: $Y = 1, D = 150, \delta = 0.9, p = 0.3, \varepsilon = 0.4, \alpha = 0.5$

Reaction of the thresholds to an increase in ε is positive for opportunistic players, since those do not try to threaten other players, and, thus do not benefit from existence of bloody type – they lose from it, since higher ε leads to higher expected probability of execution by a player with uncertain reputation; and ambiguous for bloody players, since they both benefit and lose from an increase in ε, and cumulative effect depends on which of the two dominates. The following graph provides an illustration.



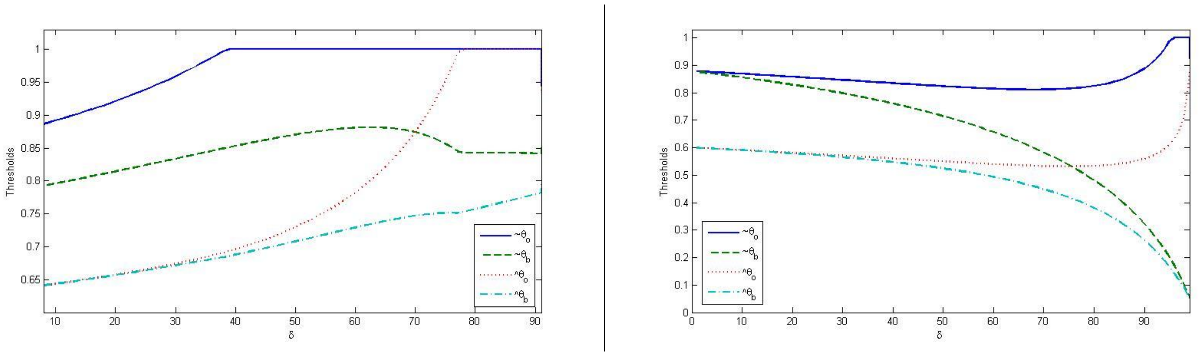
Graph 4. Illustration of dependence of the thresholds on ε. Other parameters are the following: $Y = 10, D = 16, \delta = 0.9, p = 0.4, \alpha = 0.8$

Dependence of the thresholds on p is not so trivial as well: for opportunistic player it is positive due to similar reasons as in the "killing" equilibrium; for bloody players it can be either positive or negative, if the dictator is known to be bloody, due to tradeoff between ability to win a battle against current dictator and consequent contenders, but if the dictator is not known to be bloody, then the second incentive dominates the first one, and the dependence is negative. The effect is illustrated below.



Graph 5. Illustration of dependence of the thresholds on p. Other parameters are the following: $Y = 1, D = 100, \delta = 0.9, \varepsilon = 0.3, \alpha = 0.9$, and $Y = 1, D = 100, \delta = 0.99, \varepsilon = 0.5, \alpha = 0.9$, respectively.

And finally, the thresholds increase with discount factor for opportunistic players, since they spare other players in order to be spared in future and benefit from possibility of comebacks, thus a higher discount factor increases present value of these future benefits and, subsequently, incentives to “enter”, meanwhile for bloody players the dependence is uncertain, since, for instance, if death penalty is relatively large to the winner’s payoff, then bloody players “would prefer” to value future less than in case of relatively small death penalty. An illustration is provided on the following graph.



Graph 6. Illustration to dependence of the thresholds on δ. Other parameters are the following: $Y = 10, D = 10, p = 0.6, \varepsilon = 0.3, \alpha = 0.9$, and $Y = 1, D = 100, p = 0.6, \varepsilon = 0.3, \alpha = 0.9$, respectively.

Comparative statics provided here can be outlined in the following way. Generally, “enter” thresholds are higher which means higher probability of arrival of a contender, if: dictator’s benefits, Y, which might be either amount of power the dictator possesses or prosperity of the country she rules, are larger; death penalty, D, which might be considered as players’ perception of death caused by religious and cultural views accepted in the country, is smaller; probability of a dictator’s defeat, p, caused by amount of military power, loyal aristocrats, citizens’ support and other factors, is lower; reservation utility, in particular, α, which represents level of inequality among citizens, prosperity of elites, etc. The area in the space of all possible values of parameters, in which “killing” equilibrium (Cases 1, 2 of Proposition 1) exists, “expands” if probability of dictator’s defeat, p, decreases, and/or prior probability of a player being bloody, ε, increases, and/or reservation utility, in particular, α, increases. In other words, if either the first parameter is sufficiently small, or at least one of the other two is sufficiently large, then our model predicts “killing” equilibrium, on the contrary if we have reversed, that is, inequality in the country is high, elites are weak, and dictators are likely to lose and are not expected to be bloody, then would expect “sparing” equilibrium.

4 Conclusion

The objective of this paper is to provide a simple game-theoretical model illustrating incentives lying behind different behavior of dictatorial rulers: in some dictatorial countries the winner of a battle for a throne execute those who were just removed from power and/or even those who might pretend for a dictator’s place meanwhile in other non-democratic countries contenders and ex-dictators are usually spared.

The paper considers a simple battle for a throne game with incomplete (imperfect) information with possibly an infinite number of long-run players. Each player has either opportunistic or bloody type (the latter can only execute opponents), and is able to make at maximum two kind of decisions: whether to enter or not the active part of the game – fighting for a throne – and whether to kill unlucky opponents or not.

The game is analyzed in the context of sequential equilibrium. Since player’s type is private information, each player has to develop her beliefs about other players’ types and to be concerned of her reputation – beliefs of other players about her type – and response to others’ actions accordingly: each execution lowers both incentives of other players to pretend for a throne and the dictator's chances to survive next lost fight.

There are two major incentives driving the choice of a winner of a battle for a throne: on the one hand, execution of opponents may threaten other possible contenders and, thus, secure her position as a dictator, on the other hand, a dictator that used to execute opponents is likely to be executed by other players as well, since those are afraid of possible comebacks of a bloody dictator. Due to the latter reason, a player who committed herself to executions has even less incentives to spare opponents, since spared opponent will likely execute her in future, as a consequence, the player cannot escape this death circle and has to execute new opponents more and more.

The resulting equilibria are the following: depending on parameters, in equilibrium opportunistic players either execute all opponents or almost always save their lives due to different beliefs about their types. Moreover, under some refinements and some conditions these are unique equilibria of the game; if the conditions are not satisfied, then there is a set of equilibria in which some players are spared and some – executed based on players' reputation. Comparative statics shows that, first, arrival of a contender is more probable if: dictator's benefits are larger, death penalty is smaller, probability of a dictator's defeat is lower, and reservation utility is higher; second, the area in the space of all possible values of parameters, in which "killing" equilibrium exists, "shrinks" if probability of dictator's defeat increases, prior probability of a player being bloody decreases, and reservation utility decreases.

The equilibria and their properties seem to provide well explanation for the "real equilibria" observed in the history of non-democratic regimes. Indeed, in countries with relatively often change of rulers (high probability of dictator's defeat), weak elites and/or high inequality (low reservation utility), and little history of executed dictators (little prior probability of a dictator being bloody) we observe "sparing" equilibria, e. g. Venezuela, 1830 – 1970. By contrast, in countries with opposite characteristics, that is, rulers changing usually by natural cause (low probability of dictator's defeat), strong elites and/or low inequality (high reservation utility), and long cultural tradition of executing of dictators and contenders frequently supported by country's legislation (high prior probability of a dictator being bloody) "killing" equilibrium takes place, e. g. Ottoman Empire, 1230 – 1932. Moreover, contenders to a throne appear more often in countries, where dictator possess more power (dictator's benefit is larger), contenders are less afraid of death (death penalty is lower) due to, for instance, religious beliefs, inequality is higher and/or elites are weaker (reservation utility is smaller), the dictator is less able to keep his place in case of political turmoil (probability of dictator's defeat is higher), and there is not much history of executed dictators (lower prior probability of a dictator being bloody). For instance, our model predicts little probability of arrival of a contender to a throne in countries with hereditary monarchies, high income per capita and low inequality, and where power of the ruler is somehow limited, e. g. United Arab Emirates.

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Appendix

Proof of Proposition 1

Case 1: Check that this is indeed an equilibrium. Assume that every player follows the equilibrium strategy, and show that no player has incentives to deviate. First, consider conditions under which a player, who has not joined the active part of the game, starts fighting. An observation can be made that in this equilibrium on the equilibrium path no player spares another, thus, both opportunistic and bloody types behave the same way. So, the conditions will be the same for the both types: $u(T(i)_2) \leq U(enter)$, where $U(enter)$ is the present value of all future payoffs to player i if she starts fighting. In particular, we need to find $\bar{\theta}$ such that $T(i)_2(\alpha Y + D) - D \leq U(enter)\ \forall T(i)_2 \leq \bar{\theta}$, that is, to find a marginal player who is indifferent between "entering" and "staying out": $\bar{\theta}(\alpha Y + D) - D = U(enter)$. If a player enters and loses, she gets executed, if she wins, she executes ex-dictator, thus, $U(enter) = p(Y + \delta PV_1) + (1-p)(-D)$, where PV_1 is the present value of all future payoffs of a dictator at the beginning of a period: $PV_1 = (1-\bar{\theta})(Y + \delta PV_1) + \bar{\theta}[(1-p)(Y + \delta PV_1) + p(-D)]$, where the former term refers to a case when no players enter in the next period, which happens with probability $1 - \bar{\theta}$ (only players with $T(i)_2 \leq \bar{\theta}$ enter, while $T(i)_2$is distributed uniformly over [0, 1]), the latter term – to the opposite case. Solving for PV_1 we get:

$$PV_1 = \frac{(1 - p\bar{\theta})Y - pD\bar{\theta}}{1 - \delta + \delta p\bar{\theta}}$$

Then, the equation for $\bar{\theta}$ takes the form:

$$\bar{\theta}(\alpha Y + D) - D = p\left(Y + \delta\frac{(1 - p\bar{\theta})Y - pD\bar{\theta}}{1 - \delta + \delta p\bar{\theta}}\right) + (1-p)(-D)$$

$$\bar{\theta}^2(\alpha Y + D)p\delta + \bar{\theta}(1-\delta)(\alpha Y + D) - pY - p(1-\delta)D = 0$$

Solving for $\bar{\theta}$, we get:

$$\bar{\theta} = \frac{\sqrt{(1-\delta)^2 + 4p^2\delta\frac{Y + (1-\delta)D}{(\alpha Y + D)}} - (1-\delta)}{2p\delta}$$

This is the exact threshold stated in the theorem.

By the way, obviously, $\bar{\theta}$ is always positive, while it is not always smaller than one: $\bar{\theta} < 1$ if $p[Y + (1-\delta)D] < (\alpha Y + D)p\delta + [(1-\delta)(\alpha Y + D)]$, which holds if and only if $(p - \alpha(1-\delta + p\delta)Y<1-\delta 1-p+p\delta D$.

Rewrite the condition of Case 1 in the following form:

$$\frac{Y(1-\delta)}{(1-\delta)(1-\delta+2p\delta)} > \bar{\theta}\left(\frac{Y\big((1-\delta)+\delta p\big)}{(1-\delta)(1-\delta+2p\delta)} + D\right)$$

One can see that if $\bar{\theta} = 1$, then it transforms to

$$0 > \frac{Y\delta p}{(1-\delta)(1-\delta+2p\delta)} + D > 0$$

which cannot take place, that is, if $\bar{\theta} = 1$, then the condition is not satisfied. Similar implication can be made for $\bar{\theta} > 1$. Thus, we may conclude that under the condition of Case 1 the threshold, $\bar{\theta}$, is always smaller than 1.

Now, show that executing player with certainly opportunistic reputation ($p_j(H_t) = 0, j = L_t$) is more profitable than sparing. Notice, that actually, on the equilibrium path there will be no player with $p_j(H_t) = 0$, since no player ever spares another player. But assume that player *j* spares another player *i*, that is, player *j* finds it more profitable to spare than to execute player *i*. Apparently, in this case player *j* spares player *i* all consequent periods, since the present value of execution of player *i* is the same as it is when player *j* spares player *i* first time (other players still execute every opponent), and the present value of sparing of player *i* is at least this much as it is when player *j* spared player *i* first time (it is so, since belief about player *i*'s type could only remain unchanged or change to zero - certainly opportunistic type). Subsequently, $W_t = i$, that is, current "decision-maker" is exactly that player spared by player *j*. Assume by contrary that sparing player *j* is profitable than executing, that is, the present value of one action is greater than of the other. Derive these present values, $U(ex)$ and $U(sp)$.

$$U(sp) = Y + \delta PV_2$$

$$PV_2 = (1-p)(Y + \delta PV_2) + p\delta PV_3$$

$$PV_3 = p(Y + \delta PV_2) + (1-p)\delta PV_3$$

PV_2 and $PV_3$are the present values of being D. and LO., respectively. Notice that according to Lemma 1 LO_t, always become a contender. Moreover, if sparing is more profitable than executing in this turn, then it is profitable thereafter by similar logic as provided above. Thus, here a kind of Markov process takes place. Solving for PV_2, PV_3, and $U(sp)$, we get:

$$PV_2 = \frac{Y}{1-\delta} - \frac{pY}{(1-\delta)(1-\delta+2p\delta)}$$

$$PV_3 = \frac{pY}{(1-\delta)(1-\delta+2p\delta)}$$

$$U(sp) = \frac{Y}{1-\delta} - \frac{\delta pY}{(1-\delta)(1-\delta+2p\delta)}$$

If player j is executed the winner become opposed to a new player in the next period, who can either “enter” or “stay out”; if she “enters” and win, then the ex-dictator get executed, otherwise, the dictator receives Y. Formally, it takes the form:

$$U(ex) = Y + \delta\{(1-\bar{\theta})U(ex) + \bar{\theta}\big((1-p)U(ex) + p(-D)\big)\}$$

$$U(ex) = PV_4 = \frac{Y - \delta p\bar{\theta}D}{1-\delta+\delta p\bar{\theta}} = Y + \delta PV_1$$

Here we used the fact that there is no sense in sparing “default” player (the one who just joined the active part of the game) for “default” player follow the equilibrium strategy by the initial assumption and, subsequently, executes the ex-dictator for sure, thus, killing “default” player does not affect killer’s reputation, but may lead to one or more “calm” (with no contenders) years.

We assumed:

$$U(sp) > U(ex)$$

$$\frac{Y}{1-\delta} - \frac{\delta pY}{(1-\delta)(1-\delta+2p\delta)} > \frac{Y - \delta p\bar{\theta}D}{1-\delta+\delta p\bar{\theta}}$$

$$\frac{Y((1-\delta)(1-\bar{\theta}) - \delta p\bar{\theta})}{(1-\delta)(1-\delta+2p\delta)} < \bar{\theta}D$$

This contradicts the condition of Case 1. Thus, executing player j at period t is more profitable than sparing even if $p_j(H_t) = 0$. Obviously, it is also true if $p_j(H_t) > 0$, since in this case $U(sp)$ is smaller while $U(ex)$ is the same.

To finish the proof for Case 1 we need to show that the beliefs satisfy Bayes' rule, which is a trivial fact for as it is shown above all the players behave the same way and have the same "enter" threshold for $T(i)_2$, thus players who perform executions cannot be distinguished by types, and the probability of them being bloody type must be assessed as ε.

Case 2: Again, assume that every player follows the equilibrium strategy, and show that no player has incentives to deviate. First, consider conditions under which a player, who has not joined the active part of the game, starts fighting. Notice that this case does not differ from the previous one in this part, since on the equilibrium path Case 1 and Case 2 are the same. Hence, the threshold for $T(i)_2$ is the same as in Case 1:

$$\bar{\theta} = \frac{\sqrt{(1-\delta)^2 + 4p^2\delta \dfrac{Y + (1-\delta)D}{(\alpha Y + D)}} - (1-\delta)}{2p\delta}$$

It is smaller than 1 under the condition of Case 2 as well. To see this one can simply plug 1 instead of $\bar{\theta}$ into the condition and obtain contradiction:

$$0 < \frac{p\delta(1-\varepsilon)}{(1-\delta)(1-\delta+2p\delta)} Y < -(1-\varepsilon)D < 0$$

Now, show that executing player j such that $p_j(H_t) > 0$ is more profitable than sparing. Notice that in this case $p_j(H_t) = \varepsilon$, since both opportunistic and bloody players behave the same way on the equilibrium path. It is also pointless to consider a case $p_{W_t}(H_t) = 0$ because that means W_t spares some player k before period t, that is, finds it more profitable to spare than to execute player k and, thus, finds it so thereafter: the present value of player k being spared is at least that much it is when player k is spared first time, but the present value of execution decreases, since other players follow equilibrium strategy and always "enter" against opportunistic dictator. Moreover, by Lemma 1, player

k always become a contender after being defeated by player W_t. Subsequently, player j is exactly that player spared before, that is, $j = k$. So, if $p_{W_t}(H_t) = 0$, then $A_t = S$.

Consider a case $p_{W_t}(H_t) > 0$, which implies $p_{W_t}(H_t) = \varepsilon$, and assume by contrary that sparing is more preferable than execution. The present value of sparing is:

$$U(sp) = Y + \delta PV_5$$

$$PV_5 = (1-p)(Y + \delta PV_5) + p\{\varepsilon(-D) + (1-\varepsilon)\delta PV_3\}$$

where PV_5 is the present value of D_t given $p_{D_t}(H_t) = 0$ and $p_{LO_t}(H_t) > 0$. By adding term PV_3 we also employed the fact that, first, if $p_{W_t}(H_t) = 0$, then $A_t = S$, second, other players follow equilibrium strategy: spare certainly opportunistic players. Solving for PV_5 and $U(sp)$:

$$PV_5 = \frac{1}{1-\delta+p\delta}\left((1-p)Y + \frac{p^2\delta(1-\varepsilon)Y}{(1-\delta)(1-\delta+2p\delta)} - p\varepsilon D\right)$$

$$U(sp) = \frac{1}{1-\delta+p\delta}\left(Y + \frac{p^2\delta^2(1-\varepsilon)Y}{(1-\delta)(1-\delta+2p\delta)} - \delta p\varepsilon D\right)$$

The present value of execution is:

$$U(ex) = Y + \delta PV_6$$

$$PV_6 = (1-\bar{\theta})(Y + \delta PV_6) + \bar{\theta}\big((1-p)U(sp) + p(-D)\big)$$

PV_6 includes term $U(sp)$, since we assume that sparing is more profitable than execution and consider only one-period deviation. If we consider permanent deviation, then the present value of execution is the same as it is in Case 1

$$\widetilde{U}(ex) = PV_4 = \frac{Y - \delta p\bar{\theta}D}{1-\delta+\delta p\bar{\theta}}$$

$$PV_6 = \frac{(1-\bar{\theta})Y + \bar{\theta}\big((1-p)U(sp) + p(-D)\big)}{1-\delta+\delta\bar{\theta}} = \frac{\bar{\theta}(1-p)}{1-\delta+\delta\bar{\theta}}U(sp) + \frac{(1-\bar{\theta})Y - \bar{\theta}pD}{1-\delta+\delta\bar{\theta}}$$

$$U(ex) = Y + \frac{\delta\bar{\theta}(1-p)}{1-\delta+\delta\bar{\theta}}U(sp) + \delta\frac{(1-\bar{\theta})Y - \bar{\theta}pD}{1-\delta+\delta\bar{\theta}}$$

Our assumption is:

$$U(sp) > U(ex)$$

$$(1-\delta+\delta\bar{\theta}p)U(sp) > Y - \bar{\theta}p\delta D$$

$$U(sp) > \frac{Y-\delta p\bar{\theta}D}{1-\delta+\delta p\bar{\theta}}$$

Not surprisingly we obtain the same expression as if we compare $U(sp)$ with $\widetilde{U}(ex)$. Finally, we obtain:

$$\frac{1}{1-\delta+p\delta}\left(Y+\frac{p^2\delta^2(1-\varepsilon)Y}{(1-\delta)(1-\delta+2p\delta)}-\delta p\varepsilon D\right) > \frac{Y-\delta p\bar{\theta}D}{1-\delta+\delta p\bar{\theta}}$$

$$\left\{\frac{(1-\delta+\delta p\bar{\theta})p\delta(1-\varepsilon)}{(1-\delta)(1-\delta+2p\delta)}-(1-\bar{\theta})\right\}Y > \{(\varepsilon-\bar{\theta})(1-\delta)-\delta p\bar{\theta}(1-\varepsilon)\}D$$

This contradicts the conditions of Case 2. Thus, it is a best response to execute "default player".

Now, check whether it is a best response to spare certainly opportunistic player. Again consider the only "possible" case $p_{L_t}(H_t)=0$ and $p_{W_t}(H_t)>0$. In the previous case we showed that under the condition $\frac{Y((1-\delta)(1-\bar{\theta})-\delta p\bar{\theta})}{(1-\delta)(1-\delta+2p\delta)} < \bar{\theta}D$, $U(sp)$ is greater than $U(ex)$, where $U(sp)$ is the present value of permanent deviation to "sparing" strategy. Notice that beliefs about the winner and the loser are the same in the current and the previous cases. Moreover, equilibria in both cases coincide along the equilibrium path. Hence, there is no need to prove that it is a best response to spare certainly opportunistic player, since it is shown in the previous case.

It is a trivial fact that the beliefs satisfy Bayes' rule, and it can be proved the same way it is done in Case 1.

Case 3: This case is significantly more complicated than the previous ones due to different behavior of the two types of players on the equilibrium path, and, subsequently different "enter" thresholds. Moreover, these thresholds depend on beliefs about a dictator's type, since by contrast to the previous cases on the equilibrium path the dictator in absence of a leader of opposition can be believed to be either of bloody type or of uncertain type. As usual, assume that every player follows the equilibrium strategy, and show that no player has incentives to deviate. First, consider conditions under which a player, who has not joined the active part of the game, starts fighting. Define:

$$Pr(N_t = C_t \mid p_{D_t}(H_t) = 1, T(N_t)_1 = o) = \hat{\theta}_o,$$

that is, $\hat{\theta}_o$ is the "enter" threshold for $T(N_t)_2$ of a player who has not joined the active part of the game given that she is of opportunistic type and the current dictator is certainly bloody.

Similar, define thresholds for bloody player, and the resulting probability of "enter" against bloody dictator:

$$Pr(N_t = C_t \mid p_{D_t}(H_t) = 1, T(N_t)_1 = b) = \hat{\theta}_b$$

$$\hat{\theta} = \varepsilon\hat{\theta}_b + (1-\varepsilon)\hat{\theta}_o$$

Given these notations, the probability of a player who just "entered" being bloody is:

$$\hat{\varepsilon} = \frac{\varepsilon\hat{\theta}_b}{\varepsilon\hat{\theta}_b + (1-\varepsilon)\hat{\theta}_o} = \frac{\varepsilon\hat{\theta}_b}{\hat{\theta}}$$

This is exactly belief of other players about the type of player who "entered" against certainly bloody dictator and either performed no action (sparing or execution) during the history H_t or executed only certainly bloody players.

Define also thresholds and respective probabilities for the case $p_{D_t}(H_t) = \hat{\varepsilon}$. Apparently, on the equilibrium path there is no other belief about dictator given the absence of leader of opposition.

$$Pr(N_t = C_t \mid p_{D_t}(H_t) = \hat{\varepsilon}, T(N_t)_1 = o) = \tilde{\theta}_o$$

$$Pr(N_t = C_t \mid p_{D_t}(H_t) = \hat{\varepsilon}, T(N_t)_1 = b) = \tilde{\theta}_b$$

$$\tilde{\theta} = \varepsilon\tilde{\theta}_b + (1-\varepsilon)\tilde{\theta}_o$$

$$\tilde{\varepsilon} = \frac{\varepsilon\tilde{\theta}_b}{\varepsilon\tilde{\theta}_b + (1-\varepsilon)\tilde{\theta}_o} = \frac{\varepsilon\tilde{\theta}_b}{\tilde{\theta}}$$

To derive these thresholds consider present values of "entering" in all the cases. First, the case of $\hat{\theta}_b$:

$$U(N_t = C_t \mid p_{D_t}(H_t) = 1, T(N_t)_1 = b) = (1-p)(-D) + p(Y + \delta\widehat{PV}_1)$$

$$\widehat{PV}_1 = (1-\tilde{\theta})(Y + \delta\widehat{PV}_1) + \tilde{\theta}\left((1-p)(Y + \delta\widehat{PV}_3) + p\big(\tilde{\varepsilon}(-D) + (1-\tilde{\varepsilon})\delta\widehat{PV}_2\big)\right)$$

$$\widehat{PV}_2 = (1-p)\delta\widehat{PV}_2 + p(Y + \delta\widehat{PV}_3)$$

$$\widehat{PV}_3 = \left(1-\hat{\theta}\right)\left(Y+\delta\widehat{PV}_3\right) + \hat{\theta}\left(p(-D) + (1-p)\left(Y+\delta\widehat{PV}_3\right)\right)$$

Here $\widehat{PV}_1$ and $\widehat{PV}_2$ are the present values of being a bloody dictator and a leader of opposition, respectively, while other players do not know this and following the equilibrium strategy spare the player. $\widehat{PV}_3$ is the present values of being a bloody dictator, while other players know this and following the equilibrium strategy execute the dictator.

$$\widehat{PV}_3 = \frac{(1-p\hat{\theta})Y - p\hat{\theta}D}{1-\delta+\delta p\hat{\theta}}$$

$$\widehat{PV}_2 = \frac{p\left(Y+\delta\widehat{PV}_3\right)}{1-\delta+\delta p} = \frac{p(Y-\delta p\hat{\theta}D)}{(1-\delta+\delta p)(1-\delta+\delta p\hat{\theta})}$$

$$\widehat{PV}_1 = \frac{1}{1-\delta+\delta\tilde{\theta}}\left\{\left(1-\tilde{\theta}\right)Y - \varepsilon\tilde{\theta}_b pD + \tilde{\theta}\,\frac{(1-\delta-p+2p\delta-\tilde{\varepsilon}p^2\delta)(Y-\delta p\hat{\theta}D)}{(1-\delta+\delta p)(1-\delta+\delta p\hat{\theta})}\right\}$$

Finally, the present value of “entering” is $U(N_t = C_t \mid p_{D_t}(H_t) = 1, T(N_t)_1 = b) = \frac{p}{1-\delta+\delta\tilde{\theta}}\left\{Y - \delta\varepsilon\tilde{\theta}_b pD + \delta\tilde{\theta}\,\frac{(1-\delta-p+2p\delta-\tilde{\varepsilon}p^2\delta)(Y-\delta p\hat{\theta}D)}{(1-\delta+\delta p)(1-\delta+\delta p\hat{\theta})}\right\} - (1-p)D$

So, an equation for the threshold takes the form:

$$\hat{\theta}_b(\alpha Y + D) - pD = \frac{p}{1-\delta+\delta\tilde{\theta}}\left\{Y - \delta\tilde{\theta}_b pD + \delta\tilde{\theta}\,\frac{(1-\delta-p+2p\delta-\tilde{\varepsilon}p^2\delta)(Y-\delta p\hat{\theta}D)}{(1-\delta+\delta p)(1-\delta+\delta p\hat{\theta})}\right\}$$

Now, derive the equation for $\hat{\theta}_o$:

$$U(N_t = C_t \mid p_{D_t}(H_t) = 1, T(N_t)_1 = o) = (1-p)(-D) + p(Y+\delta\widehat{PV}_4)$$

$$\widehat{PV}_4 = \left(1-\tilde{\theta}\right)\left(Y+\delta\widehat{PV}_4\right) + \tilde{\theta}\left\{p(\tilde{\varepsilon}(-D) + (1-\tilde{\varepsilon})\delta PV_3) + (1-p)\left(Y+\delta\widehat{PV}_5\right)\right\}$$

$$\widehat{PV}_5 = p(\tilde{\varepsilon}(-D) + (1-\tilde{\varepsilon})\delta PV_3) + (1-p)\left(Y+\delta\widehat{PV}_5\right)$$

Present values $\widehat{PV}_4$ and $\widehat{PV}_5$ are obtained similarly to the previous ones.

$$\widehat{PV}_5 = \frac{p(\tilde{\varepsilon}(-D) + (1-\tilde{\varepsilon})\delta PV_3) + (1-p)Y}{1-\delta+p\delta} =$$

$$= \frac{1}{1-\delta+p\delta}\left\{(1-p)Y + \frac{(1-\tilde{\varepsilon})p^2\delta Y}{(1-\delta)(1-\delta+2p\delta)} - \tilde{\varepsilon}pD\right\}$$

$$\widehat{PV}_4 = \frac{1}{1-\delta+\tilde{\theta}\delta}\left(Y\left(1-\tilde{\theta}\right) + \frac{\tilde{\theta}}{1-\delta+p\delta}\left\{(1-p)Y + \frac{(1-\tilde{\varepsilon})p^2\delta Y}{(1-\delta)(1-\delta+2p\delta)} - \tilde{\varepsilon}pD\right\}\right)$$

Finally, the utility of "entering" is: $U(N_t = C_t \mid p_{D_t}(H_t) = 1, T(N_t)_1 = o) = (1-p)(-D) + \frac{p}{1-\delta+\tilde{\theta}\delta}\left(Y + \frac{\delta\tilde{\theta}}{1-\delta+p\delta}\left\{(1-p)Y + \frac{(1-\tilde{\varepsilon})p^2\delta Y}{(1-\delta)(1-\delta+2p\delta)} - \tilde{\varepsilon}pD\right\}\right)$, and the equation for the threshold is: $\hat{\theta}_o(\alpha Y + D) = p\left(D + \frac{1}{1-\delta+\tilde{\theta}\delta}\left(Y + \frac{\delta\tilde{\theta}}{1-\delta+p\delta}\left\{(1-p)Y + \frac{(1-\tilde{\varepsilon})p^2\delta Y}{(1-\delta)(1-\delta+2p\delta)} - \tilde{\varepsilon}pD\right\}\right)\right)$.

Now, derive the equations for $\tilde{\theta}_o$ and $\tilde{\theta}_b$:

$$U(N_t = C_t \mid p_{D_t}(H_t) = \hat{\varepsilon}, T(N_t)_1 = o) = p\left(Y + \delta\widetilde{PV}_1\right) + (1-p)(\hat{\varepsilon}(-D) + (1-\hat{\varepsilon})PV_3)$$

$$\widetilde{PV}_1 = (1-p)\left(Y + \delta\widetilde{PV}_1\right) + p(\hat{\varepsilon}(-D) + (1-\hat{\varepsilon})PV_3)$$

$$U(N_t = C_t \mid p_{D_t}(H_t) = \hat{\varepsilon}, T(N_t)_1 = b) = p\left(Y + \delta\widehat{PV}_3\right) + (1-p)\left(\hat{\varepsilon}(-D) + (1-\hat{\varepsilon})\delta\widehat{PV}_2\right)$$

Substituting PV_3, $\widehat{PV}_2$, $\widehat{PV}_3$, obtain:

$$\widetilde{PV}_1 = \frac{1}{1-\delta+p\delta}\left\{(1-p)Y + p\left(\hat{\varepsilon}(-D) + (1-\hat{\varepsilon})\frac{pY}{(1-\delta)(1-\delta+2p\delta)}\right)\right\}$$

$$U(N_t = C_t \mid p_{D_t}(H_t) = \hat{\varepsilon}, T(N_t)_1 = o) =$$

$$= \frac{pY}{1-\delta+p\delta} + \frac{1-\delta-p+2p\delta}{1-\delta+p\delta}\left(\hat{\varepsilon}(-D) + (1-\hat{\varepsilon})\frac{pY}{(1-\delta)(1-\delta+2p\delta)}\right)$$

$$U(N_t = C_t \mid p_{D_t}(H_t) = \hat{\varepsilon}, T(N_t)_1 = b) =$$

$$= \frac{(1-\hat{\varepsilon}(1-p)\delta)p}{(1-\delta+\delta p)(1-\delta+\delta p\hat{\theta})}Y - \left(\hat{\varepsilon}(1-p) + \frac{(1-\hat{\varepsilon}(1-p)\delta)\delta p^2\hat{\theta}}{(1-\delta+\delta p)\left(1-\delta+\delta p\hat{\theta}\right)}\right)D$$

and the equations for $\tilde{\theta}_o$ and $\tilde{\theta}_b$ are:

$$\tilde{\theta}_o(\alpha Y + D) - D = \frac{pY}{1-\delta+p\delta} + \frac{1-\delta-p+2p\delta}{1-\delta+p\delta}\left(\hat{\varepsilon}(-D) + (1-\hat{\varepsilon})\frac{pY}{(1-\delta)(1-\delta+2p\delta)}\right)$$

$$\tilde{\theta}_b(\alpha Y + D) - (1-\hat{\varepsilon}(1-p))D = \frac{(1-\hat{\varepsilon}(1-p)\delta)p}{(1-\delta+\delta p)\left(1-\delta+\delta p\hat{\theta}\right)}(Y - \delta p\hat{\theta}D)$$

These equations on $\hat{\theta}_o$, $\hat{\theta}_b$, $\tilde{\theta}_o$, $\tilde{\theta}_b$, $\hat{\varepsilon}$, and $\tilde{\varepsilon}$ uniquely determine the thresholds and beliefs, but unfortunately, they cannot be solved in a closed form analytically, thus, we leave them as it is.

However, it is much easier to show that no player has incentives to deviate from the equilibrium strategy in this case than it is for the first two cases.

First, obviously, it is suboptimal to spare a player of a certainly bloody type for killing bloody player does not affect killer's reputation, but may lead to one or more "calm" (with no contenders) years.

Second, compare the present values of sparing and execution of player j with "uncertain" reputation ($p_j(H_t) = \hat{\varepsilon}$, $\tilde{\varepsilon}$): on the one hand, the present value of sparing is at least as much as it is in the "killing equilibrium" from Case 2: $\tilde{\theta}_b \leq \tilde{\theta}_o$, and $\tilde{\theta}_b \leq \tilde{\theta}_o$, since opportunistic player can behave as if she is a bloody one, thus, her present value of "entering" cannot be lower than of a bloody one, and, hence, the threshold for entering of the opportunistic type is at least as much as it is of the bloody type; subsequently, both $\hat{\varepsilon}$, and $\tilde{\varepsilon}$ are smaller than ε, so, the present value of sparing is at least as much as it is in the "killing equilibrium" from Case 2, that is,

$$U(sparing) \geq \frac{1}{1-\delta+p\delta}\left(Y + \frac{p^2\delta^2(1-\varepsilon)Y}{(1-\delta)(1-\delta+2p\delta)} - \delta p\varepsilon D\right)$$

On the other hand, the present value of execution is always $U(execution) = \widehat{PV}_3$. Notice that $\hat{\theta} > \bar{\theta}$, since both bloody and opportunistic players expect lower probability of being executed, and, thus,

$$0 < \delta pY(\hat{\theta} - \bar{\theta}) + \delta pD(\hat{\theta} - \bar{\theta})(1-\delta)$$

$$\delta p\theta Y - \delta p\hat{\theta}D(1-\delta) < \delta p\hat{\theta}Y - \delta p\theta D(1-\delta)$$

$$(1-\delta)Y + \delta p\theta Y - \delta p\hat{\theta}D(1-\delta) - \theta\delta^2p^2\hat{\theta}D < (1-\delta)Y + \delta p\hat{\theta}Y - \delta p\theta D(1-\delta) - \theta\delta^2p^2\hat{\theta}D$$

$$\left(Y - \delta p\hat{\theta}D\right)(1-\delta+\delta p\theta) < (Y - \delta p\theta D)(1-\delta+\delta p\hat{\theta})$$

$$\frac{Y - \delta p\hat{\theta}D}{1-\delta+\delta p\hat{\theta}} < \frac{Y - \delta p\bar{\theta}D}{1-\delta+\delta p\bar{\theta}}$$

$$U(execution) = \widehat{PV}_3 < PV_3$$

By the condition of the Case,

$$\left\{\frac{(1-\delta+\delta p\bar{\theta})p\delta(1-\varepsilon)}{(1-\delta)(1-\delta+2p\delta)} - (1-\bar{\theta})\right\}Y > \{(\varepsilon-\bar{\theta})(1-\delta) - \delta p\bar{\theta}(1-\varepsilon)\}D$$

$$\frac{1}{1-\delta+p\delta}\left(Y + \frac{p^2\delta^2(1-\varepsilon)Y}{(1-\delta)(1-\delta+2p\delta)} - \delta p\varepsilon D\right) > \frac{Y - \delta p\theta D}{1-\delta+\delta p\theta} = PV_3$$

Thus,

$$U(sparing) \geq \frac{1}{1-\delta+p\delta}\left(Y+\frac{p^2\delta^2(1-\varepsilon)Y}{(1-\delta)(1-\delta+2p\delta)}-\delta p\varepsilon D\right) > \widehat{PV}_3$$

$$U(sparing) > U(execution)$$

This implies that executing a player with "uncertain" reputation is suboptimal. Apparently, executing player of certainly opportunistic type is suboptimal as well.

The fact that beliefs satisfy Bayes' rule was shown above. ■

Proof of Claim 1

Cases 1 and 2 are mutually exclusive by definition as well as Cases 2 and 3. Assume by contrary that the conditions of both Case 1 and Case 3 hold simultaneously, that is,

$$\frac{Y((1-\delta)(1-\bar{\theta})-\delta p\bar{\theta})}{(1-\delta)(1-\delta+2p\delta)} > \bar{\theta}D$$

$$\{\bar{\theta}(1-\delta+\delta p)-\varepsilon(1-\delta+\delta p\bar{\theta})\}D > \left\{(1-\bar{\theta})-\frac{(1-\delta+\delta p\bar{\theta})p\delta(1-\varepsilon)}{(1-\delta)(1-\delta+2p\delta)}\right\}Y$$

First, consider case $\{\bar{\theta}(1-\delta+\delta p)-\varepsilon(1-\delta+\delta p\bar{\theta})\} > 0$ or $\varepsilon < \bar{\theta}\frac{(1-\delta+\delta p)}{(1-\delta+\delta p\bar{\theta})}$. Then, the conditions imply the following inequality:

$$\frac{1}{\bar{\theta}(1-\delta+\delta p)-\varepsilon(1-\delta+\delta p\bar{\theta})}\left\{(1-\bar{\theta})-\frac{(1-\delta+\delta p\bar{\theta})p\delta(1-\varepsilon)}{(1-\delta)(1-\delta+2p\delta)}\right\} < \frac{(1-\delta)(1-\bar{\theta})-\delta p\bar{\theta}}{\bar{\theta}(1-\delta)(1-\delta+2p\delta)}$$

After some derivations the inequality transforms to the following:

$$\bar{\theta}^2\delta\frac{(1-\delta+p\delta)}{(1-\delta+\delta p\bar{\theta})} < \varepsilon(-(1-\delta)+\bar{\theta})$$

Recall that $\varepsilon < \bar{\theta}\frac{(1-\delta+\delta p)}{(1-\delta+\delta p\bar{\theta})}$. Plug it into the inequality and obtain contradiction:

$$0 < (1-\delta)\varepsilon(\bar{\theta}-1) < 0$$

Second, consider the opposite case, $\varepsilon\frac{(1-\delta+\delta p\bar{\theta})}{(1-\delta+\delta p)} > \bar{\theta}$. From the condition of Case 3 we obtain:

$$(1-\bar{\theta})-\frac{(1-\delta+\delta p\bar{\theta})p\delta(1-\varepsilon)}{(1-\delta)(1-\delta+2p\delta)} < 0$$

The inequality transforms to the following:

$$1-\frac{\delta p(1-\varepsilon)}{(1-\delta+2\delta p)}<\bar{\theta}\left(1+\frac{\delta^2 p^2(1-\varepsilon)}{(1-\delta)(1-\delta+2\delta p)}\right)$$

From the condition of Case 3 we obtain, we get $(1-\delta)(1-\bar{\theta})-\delta p\bar{\theta}>0$, or $\bar{\theta}<\frac{1-\delta}{1-\delta+\delta p}$.

Plug it into the previous inequality:

$$1-\frac{\delta p(1-\varepsilon)}{(1-\delta+2\delta p)}<\frac{1-\delta}{1-\delta+\delta p}\left(1+\frac{\delta^2 p^2(1-\varepsilon)}{(1-\delta)(1-\delta+2\delta p)}\right)$$

After some derivations come up with contradiction:

$$\frac{1-\delta+\varepsilon\delta p}{1-\delta+\delta p}<\frac{1-\delta}{1-\delta+\delta p}$$

Thus, Case 1 and Case 3 are mutually exclusive. ■

Proof of Proposition 2

The proof might look too short and even not full due to a fact that it refers to the proof of Proposition 1 and by doing this skips a significant part of derivations.

First, consider condition $\frac{Y(\delta p\bar{\theta}-(1-\delta)(1-\bar{\theta}))}{(1-\delta)(1-\delta+2p\delta)}<\bar{\theta}D$. In the proof of Proposition 1 it is shown that under the condition no matter what the history is it is always optimal to execute any player even if it is known that a particular player is opportunistic (notice that there the fact that the rest of the players follow “execution” strategy is employed, but if they follow any other strategy, then there are even more incentives to execute a player with certainly opportunistic reputation, since payoff of sparing an opportunistic player does not change, while the one of executing may only increase), thus, there does not exist monotone in beliefs and actions sequential equilibrium in which sparing a player with certainly opportunistic reputation is a best response, subsequently, sparing any player is not a best response as well.

Second, consider condition $\left\{\frac{(1-\delta+\delta p\bar{\theta})p\delta(1-\varepsilon)}{(1-\delta)(1-\delta+2p\delta)}-(1-\bar{\theta})\right\}Y>\{(\varepsilon-\bar{\theta})(1-\delta)-\delta p\bar{\theta}(1-\varepsilon)\}D$. In the proof of Proposition 1 it is shown that under the condition no matter what the history is assessment of a player being bloody cannot exceed ε, if it is not equal to 1, and it is a best response to spare a player j at period t with $p_j(H_t) = \varepsilon$, given that in case of execution the winner is executed as soon as her next defeat occurs, and in case of sparing the decision maker is always spared by opportunistic player, and that the assessment of the winner's type being bloody is equal to ε as well, $p_{W_t}(H_t) = \varepsilon$. Hence, by monotonicity in actions, it is a best response to spare any player player j at period t with $p_j(H_t) < \varepsilon$. Thus, in a motone in beliefs and actions sequential equilibrium it is a best response to spare a player j at period t if $p_j(H_t) < \varepsilon$, that is, by Bayes' rule, every player who performed no action or killed only certainly bloody players. ■