РЭШ
Российская
экономическая
школа

МАГИСТЕРСКАЯ ДИССЕРТАЦИЯ

MASTER THESIS

Тема: Рекурсивная устойчивость в модели страхования в отсутствие связывающих обязательств.

Title: Recursive stability in a model of risk pooling with limited commitment.

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Оценка/ Grade:

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Подпись/ Signature:

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Москва 2013

Abstract

We build on a model of informal insurance in communities, that was proposed in Genicot and Ray (2003). In a homogeneous community where agents randomly draw high or low incomes, self-enforced insurance contracts can be devised that allow them to smooth consumption. The distinct feature of the model is a recursive structure of stability, that arises when group deviations are allowed, but only if the subgroup itself is robust to deviations in the future. One of the main results of the GR model is that there is an upper boundary to the size of the stable group. We generalize the model to more than two types of incomes and analyze the set of stable schemes in detail. The new model leads to, with a slip of the tongue, same result as the original one. In addition, we provide a microeconomic foundation to the particular notion of stability. We also justify the two restrictions on the complexity of contracts that the agents can devise, namely, stationarity and deterministicity, by showing that these assumptions do not lead to the loss of generality. Finally, we introduce a perturbation of our model that allows to incorporate features like costs/benefits of scale in a cooperating community, as well as punishing/encouraging stimuli to the deviating agent. We show that for a small (in some sense) perturbation of our model the result on the existence of the upper boundary preserves. Put it differently, to maintain stability of an increasingly large risk-sharing group, the amount of perturbation should explode.

Acknowledgements

I would like to thank my advisor K.Sonin, my professors A.Bremzen, E.Paltseva, O.Evren, S.Weber, my colleagues D.Dagayev, P.Yaskov, D.Anosova, M.Tirskih, for fruitful discussions and helpful ideas.

Contents

1 Introduction

Risk-sharing in underdeveloped rural areas is often characterized by limited access to financial markets and weak legal enforcement. A high volatility of agricultural production pushes households to form certain informal agreements in order to smooth consumption. These contracts, however often fail to reach the ex-ante optimal allocation (which can be characterized by perfect consumption smoothing). This observation is supported by a large body of empirical evidence from various village communities : Townsend (1994), Udry (1994), Jalan and Ravallion (1999), Ligon, Thomas, and Worrall (2002), Grimard (1997), Gertler and Gruber (2002), Foster and Rosenzweig (2001).

Among many factors that can explain this phenomenon, like information asymmetry, or costs of devising and maintaining agreements, one is particularly appealing, namely, the self-enforced nature of the agreements itself. Voluntary participation places certain incentive constraints on the amount of insurance, which makes the first-best allocation infeasible. Several theoretic studies : Kimball (1988), Coate and Ravallion (1993), Kocherlakota (1996), Kletzer and Wright (2000); investigate this question. In a paper by Fafchamps and Lund (2003) a model of mutual insurance with limited commitment is proposed and then tested on rice farmers in the Philippines. Ligon, Thomas, and Worrall (2002) propose and test a similar model on an Indian village data.

All these studies, however, focus on deviations of a single agent as a threat to the insurance agreement. A series of papers : Bloch (1996), Bloch (1997), Ray and Vohra (1997), Ray and Vohra (1999), Ray and Vohra (2001), Genicot and Ray (2003), Bloch, Genicot, and Ray (2007), Bloch, Genicot, and Ray (2008); investigate group deviations in various settings. We focus on one paper by Genicot and Ray (2003) that studies very specific group deviations with regard to informal insurance agreements. The paper models a repeated game, where deviations are allowed only to subgroups that are themselves immune to future deviations. This idea reminds of the Coalition-Proof Nash Equilibrium, but in fact, is slightly more complex, because blocking coalitions are separated in time. This type of deviation leads to a specific type of recursive stability, that we are interested in. The set of stable group sizes turns out to be finite, which is in sharp contrast to other literature on coalition formation Bloch (1996), Ray and Vohra (1997) Ray and Vohra (2001), Acemoglu, Egorov, and Sonin (2006).

In the current paper we construct a model that is very similar to the one in Genicot and Ray (2003), but differs in several small, but important details. The new model is also more general, because we allow for a finite discrete distribution of income for an agent, in contrast to the original model, that had a binary distribution. Luckily, we manage to reach the same result on the finiteness of the set of stable sizes. Apart of that, we try to focus on some other aspects that were given little or no attention in the original paper on the one hand, and provide a deeper insight into some of the questions that were already discussed. We shall briefly state how our approach differs from that of Genicot and Ray's.

In the original model the agents exhibit some level of irrationality when making two types of strategic choices. The first one is the choice of the optimal contract. The trivial insurance contract (with zero transfers) was excluded from consideration in the original model. It seems strange that the agents that can maintain a sophisticated arrangement fail to coordinate on the trivial one. This also makes the set of decisions not compact, which is an undesirable property. We get rid of this assumption. The second one is the decision to deviate to a subgroup. Deviations towards unstable schemes were ruled out. We believe that such behavior deserves at least some microeconomic foundation. In fact, if one tries to specify the payoffs of the agents that fail to get into the deviating group (for example, they get their standalone payoffs to the rest of their days), a deviation towards an unstable group may be a good alternative if the probability of further deviation is fairly small. To justify such irrational fear of further deviations we set the payoffs of those agents to be negative and unbounded (one may think that they get killed). We provide two formal game-theoretic settings, a cooperative and a non-cooperative, that exhibit the exact behavior that was described in the original model.

We pay special effort to develop a system of notations that is convenient for analyzing this setting in all detail. In contrast to the original model, where agents had only two possible levels of income (rich and poor), and then performed unilateral transfers from high-income to low-income ones, we allow for more than two types. As a result, a more convenient description of the insurance contract is a centralized mechanism that first collects the agents incomes and then somehow redistributes them back. The scheme (if it is symmetric and stationary) that governs that mechanism depends on the state of the world only, that is, in turn, determined by the quantity of agents of each type. It turns out that the easiest way to get an insight into the insurance mechanism is to fix a state of the world, but let the type of the agent be a random variable. From this point of view the optimal scheme is a maximizer of a specific value function on a specific budget set. The incentive constraints are just linear constraints on the product of budget sets over all states. Most of the analysis can be conducted in these terms, allowing for simple geometric interpretation of the main ideas.

Three restrictions on the complexity of the distribution scheme are considered in the original model: symmetry, stationarity and deterministicity. The main result is first obtained in the presence of all three assumptions, and then, as a robustness check, obtained when all of the three assumptions are dropped. Though it may seem that the second result is a stronger version of the first one, it in not the case. These two settings are just two independent problems. It turns out, however, that each of these three assumptions has a different impact on the model. If we abstain from analyzing the asymmetric case, it can be shown that the other two restrictions, in some sense, do not lead to a loss of generality. In fact, the optimal symmetric self-enforced scheme can be always found in the class of stationary and deterministic ones. This somehow contradicts the overall trend of considering history-dependent schemes : Kocherlakota (1996), Kletzer and Wright (2000), Ligon, Thomas, and Worrall (2002); but it is the essential property of this particular framework.

We also introduce a totally new extension to the model. We define two types of perturbations, the change of the group's total endowment, and the change of the deviator's payoff. They can be used to introduce a wide amount of real-life phenomena to the model, like: costs of maintaining cooperation, increasing returns to scale, saving and borrowing, uncertainty from deviation, reaping the deviator's assets, e.t.c. These perturbations can be also thought of as policies applied to the group by an external entity or institutions adopted by the groups themselves. We analyze, how these perturbations can change the set of stable schemes, in particular, whether the group of an arbitrary large size could be made self-enforced. We show, however, that for, in a specific sense, small perturbations this is impossible, that is the result of the main theorem continues to hold.

Our results may have several implications. First, we have strengthened some of the existing results, in particular, the existence of the upper boundary on the set of stable sizes. This fundamental result suggests that risk-sharing agreements need not be community-wide, which is important for both further theoretical and empirical studies. We have also built solid microeconomic foundations for the type particular type of recursive stability, and proved the redundancy of the stochasticity and history dependence when considering symmetric schemes, both of these are important to understand the modeling methodology. Finally, the perturbations that we have introduced can qualitatively predict the impact of certain policies or institutions on stability of risk-sharing agreements. One intriguing result is that to maintain stability of a growing risk-sharing community, one should pour in a flow of subsidies that grows linearly in the size of that community.

2 Basic framework

We model a homogeneous community that is involved in production and consumption of a perishable good. At each date agents produce the good independently from each other. The amount of product is stochastic and taken from a distribution that is the same for all agents. The agents are ex-ante identical, however, after the good is produced, they can be distinguished by the amount of good. We say that at a certain date agents are of the same type, if they produce the same amount of good. Types are re-defined at each date. There is no history of agent's previous productivity, so only today's type can be observed.

These agents are the only consumers of this good, and they are risk-averse expected utility maximizers with the same utility function. The only type of agreement that they can form is a stationary and symmetric distribution scheme. Stationary means that the same mechanism is set in action at each date, that can use only information available at that date. Symmetric means that the mechanism can not distinguish among the agents of the same type, it can not randomize among them as well. The distribution scheme works in the following way: at each date agents first give away their amounts of good (that we call private endowments) to form a group endowment that is then distributed back according to a predefined rule. We assume the technology to be costless, so the total amount of good collected equals the total amount of good that was distributed back.

A key element in our model is that the scheme is self-enforced. At any date a unique subgroup of agents, that does not coincide with the group itself, has a chance to deviate from the scheme. Each member of the deviating subgroup can veto that decision, so, the deviation will happen if and only if all members of the subgroup agree to do so. In this case they consume their private endowments, then leave the original group to form a new distribution scheme, that will start operating at the next date. They also permanently lose their ability to cooperate with the rest of the original group.

In this section we do not exactly model what happens to the rest of the group after deviation, we also do not specify how the potential deviating subgroup is chosen. However, we impose one substantial condition on the deviating subgroup. Only deviations to subgroups that are themselves protected from further deviations are considered. It follows that a subgroup that could have agreed to deviate in order to form a new enforced scheme, would not do so given even the slightest probability of a subsequent deviation if their new scheme were self-enforced.

We shall describe the framework formally in detail in the following subsections.

2.1 Types, states and information structure

Let the community consist of n agents, who live infinitely. At each date, each agent can be of one of m types. Types are drawn independently from the distribution that is the same for all agents and does not change from date to date. We do not distinguish between agents of the same type, and there is no history, therefore, at a given date, the state of the world is determined by the quantity of agents of each type.

- Denote T - set of types of agents, $\dim T = m$.
- Denote S - set of states of the world.

Throughout the paper we shall refer to the type of an agent as $t \in T$, to the state of the world as $s \in S$. For convenience, all type dependent variables shall have subscript t, and all state dependent variables shall have subscript s. We shall drop the size of the group as an argument of these variables occasionally.

- Denote p_t - probability of being type $t \in T$ (assume $p_t \neq 0$ for all $t \in T$).
- Denote P_s - probability of state $s \in S$.
- Denote $q_{s,t}$ - quantity of agents of type $t \in T$ at state $s \in S$.

We shall use the following notations for the corresponding vectors.

- Denote $p = \{p_t\}_{t \in T}$ - vector of probabilities.
- Denote $q_s = \{q_{s,t}\}_{t \in T}$ - vector of quantities of each type at state $s \in S$.
- Denote $\frac{q_s}{n} = \{\frac{q_{s,t}}{n}\}_{t \in T}$ - vector of shares of each type at state $s \in S$.

As we have mentioned earlier, state $s \in S$ is fully determined by the vector of quantities q_s (as well as by the vector of shares $\frac{q_s}{n}$), and visa versa. The probabilities P_s of states $s \in S$ can be easily computed as functions of $p_t,\ t \in T$.

Throughout the paper we shall refer to τ, σ as random variables, that determine the type of an agent and the state of the world respectively. We assume that there are two possible information structures in our community:

- ex-ante: agents know the distributions of σ, τ, but not the realizations.

- ex-post: agents know the realizations s, t of σ, τ.

We also define $\tau|s$ as a random variable that determines the type of an agent conditional on the realization s of the state of the world σ. For future analysis we shall need an additional artificial information structure:

- in-state: agents know the realization s of σ, and the distribution of $\tau|s$.

The last information structure reminds of the ex-interim, but it is not the same. In the ex-interim case the realization of τ is known, and the realization of σ is unknown.

Agents are able to take expectations conditional on the current information structure. Let x_t be type dependent, x_s state dependent and $x_{s,t}$ type and state dependent (nonrandom) variables, then:

- Denote $\mathbb{E}_\tau x_t = \sum_{t\in T} p_t x_t$.
- Denote $\mathbb{E}_\sigma x_s = \sum_{s\in S} P_s x_s$.
- Denote $\mathbb{E}_{\tau|s} x_{s,t} = \sum_{t\in T} \frac{q_{s,t}}{n} x_{s,t}$.

All the agent's decisions are taken ex-post, after the types are learned. There are no ex-ante and ex-interim stages, however, as we shall see later, it is extremely convenient to use them in our analysis.

2.2 Production, consumption and utility

Agents produce a perishable good, it can not be stored or traded, and they are the only consumers of this good. They have the same preferences for consumption of the good and are risk averse expected utility maximizers.

- Agents have the same str. increasing, str. concave utility function U.
- Discount factor is δ (assume $\delta \in (0, 1)$).

All agents are ex-ante identical. After agents receive their private endowments of good they learn their types and the types of other agents. Without cooperation an agent would simply consume his private endowment.

- Denote e_t - private endowment of an agent of type $t \in T$ (assume $e_i \neq e_j$ for $i \neq j$).
- Denote $e = \{e_t\}_{t\in T}$ - vector of private endowments.

Two values can be immediately computed as functions of e:

- One-day expected no-insurance utility: $\mathbb{E}_\tau U(e_t)$
- One-day expected full-insurance utility: $U(\mathbb{E}_\tau e_t)$

2.3 Redistribution

Agents can write an agreement, that will govern the redistribution of good among them. Since agents are risk-averse, they would prefer to insure themselves from bad outcomes. We model it by saying that they can devise a distribution scheme d, that is symmetric, stationary and deterministic. Symmetric means that the scheme does not distinguish between the agents of the same type. The scheme is stationary in the sense that it is a function of today's information only and does not depend on the history. A deterministic scheme is the one that does not allow for randomization. The scheme is conducted in three steps:

- step 1: agents give away their private endowments e_t.
- step 2: a group endowment E_s is formed.
- step 3: E_s is redistributed back.

In principle, we could consider various technologies, that make the group endowment out of the private ones, but, in the baseline framework, we assume that E_s is just the sum of all agents' private endowments:

$$E_s = \sum_{t \in T} q_{s,t} e_t \tag{1}$$

A distribution scheme d can be thought of as a $\dim T \times \dim S$ matrix. Each row is a vector of distributions to different types at a particular state. Each column is a vector of distributions to a particular type at different states.

- Denote $d_{s,t}$ - distribution to a $t \in T$ type agent at state $s \in S$.
- Denote $d_s = \{d_{s,t}\}_{t \in T}$, $d = \{d_s\}_{s \in S}$.
- Denote D_s - set of attainable distribution schemes at state $s \in S$, $D = \{D_s\}_{s \in S}$.

The set D_s is simply the set of positive distributions such that the total amount of good distributed does not exceed the total amount of good collected:

$$D_s = \{d_s : \ d_{s,t} \geq 0, \ \sum_{t \in T} q_{s,t} d_{s,t} \leq E_s\} \tag{2}$$

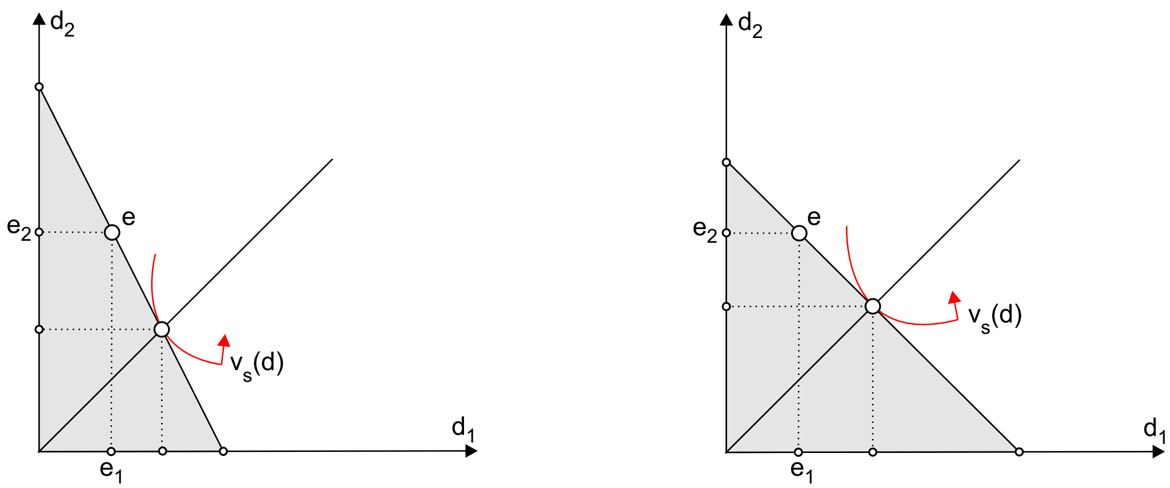
Assume for now that all agents abide to the distribution scheme $d \in D$. We can compute the expected value of one-day participation in that scheme, depending on the information available to the agent.

- ex-post: $v_{s,t}(d) = U(d_{s,t})$
- in-state: $v_s(d) = \mathbb{E}_{\tau|s} v_{s,t}(d)$
- ex-ante: $v(d) = \mathbb{E}_\sigma v_s(d)$

Note that the in-state utility depends only on d_s, just like the ex-post utility depends only on $d_{s,t}$.

2.4 In-state utility maximization problem

Assume that the designer of the scheme knows the state of the world $s \in S$, and all agents abide. His objective is to maximize the ex-interim utility of an agent. We can formulate this as a consumption problem of a single fictive agent.



a) at state s, such that $q_{s,1} > q_{s,2}$ b) at state s, such that $q_{s,1} = q_{s,2}$

Figure 1: In-state utility maximization problem, case $n = 2$

- Let the number of goods be $m = \dim T$.
- Let e_t - be the initial endowment of good $t \in T$.
- Let $q_{s,t}$ - be the price of good $t \in T$.

It follows that E_s - is the total wealth of our agent. He trades his goods at a fictive market at given prices. Intuitively, good t is an act of distributing towards a t type agent, and the price of this good is the quantity of agents of type t at this state of the world. The consumption bundle therefore is d_s, and the budget set is D_s.

- Let $v_s(d)$ - be the utility function.

We maximize $v_s(d)$ (that depends on d_s only) subject to $d_s \in D_s$, where D_s is given by formulas (1), (2). When the number of types is $m = 2$, this problem can be easily illustrated on a 2-dimensional plane, see Figure 1.

Note that the picture does not depend on the number of agents, which is extremely convenient. The budget line passes through e - vector of private endowments and is orthogonal to q_s - vector of quantities. The slope of the budget line corresponds to the state of the world. The level curve of $v_s(d)$ is tangent to the budget line at the point of intersection with the bisector. For states of the world such that some types are missing, the picture degenerates.

2.5 Stability

We shall operate two notions: stable distribution scheme and stable group size.

Stability of a distribution scheme is to be defined by induction by the size of the group. For a singleton group all distribution schemes should be stable. If we can find stable distribution schemes for all group sizes smaller than the size of a given group, then we can tell whether a given distribution scheme is stable or not.

A given distribution scheme is stable iff for all states of the world there does not exist a subgroup of agents for whom the expected utility of abiding to the scheme is smaller than utility from consuming ones private endowment and then forming a new group and abide to a new distribution scheme that will start operating from the next day.

To formalize this we first introduce an individual-level dominance relation on the set of all schemes, and then aggregate it to the group-level. In this section we only postulate this as a definition of stability. The game-theoretic foundation of this exact type of stability will be given in Section "Game".

- Denote $\mathbb{S}$ - set of all schemes (d, n) (assuming $d \in D(n)$).

- Denote $(d', k) \succ_{s,t} (d, n)$, if $k < n$ and

$$\frac{\delta}{1-\delta} v(d') + U(e_t) > \frac{\delta}{1-\delta} v(d) + v_{s,t}(d). \tag{3}$$

The left part of the inequality is the discounted value of consuming ones private endowment today and abiding to the scheme (d', k) starting from tomorrow. The right part is the discounted value of staying in the scheme (d, n). If the inequality holds then the new scheme dominates, that is, preferred at state s from the standpoint of an agent of type t.

For a given state of the world let the set of such types be W_s, with a slight abuse of notations we drop the arguments (d, n) and (d', k). Intuitively, this is the set of types that are willing to deviate from (d, n) to (d', k). If the amount of people that prefer a new scheme to the old one is big enough to form a group of a respective size, then we say that the new scheme dominates from the standpoint of the group.

- Denote $W_s \subset T$ - set of types such that $(d', k) \succ_{s,t} (d, n)$.
- Denote $(d', k) \succ_s (d, n)$, if at state $s \in S(n) : \sum_{t \in W_s} q_{s,t} \geq k$.

We have introduced an aggregated dominance relation $\succ_s$ on $\mathbb{S}$. Now we can introduce the notion of a stable scheme. The set of stable schemes $\mathbb{SS}$ is correctly defined by induction by the size of the group.

- Denote $(d', k) \succ (d, n)$, if there exists $s \in S : (d', k) \succ (d, n)$.
- A scheme (d, n) is stable if there does not exist a stable scheme $(d', k) : (d', k) \succ (d, n)$.
- Denote $\mathbb{SS} \subset \mathbb{S}$ - set of stable schemes.

It is easy to verify that the following two properties are satisfied:

- Internal stability: for any $(d, n), (d', k) \in \mathbb{SS}$: $(d', k) \not\succ (d, n)$.
- External stability: for any $(d, n) \in \mathbb{S} \setminus \mathbb{SS}$ there exists $(d', k) \in \mathbb{SS}$ such that $(d', k) \succ (d, n)$.

This means that $\mathbb{SS}$ is a vNM stable set in $\mathbb{S}$ with respect to dominance relation $\succ$.

Finally, we can define a stable group size.

- A group size n is stable if there exists a stable distribution scheme $d \in D(n)$.

2.6 Value

For a group of size n we point out two particular distribution schemes: $d^{fb}(n)$ - first best, $d^{sb}(n)$ - second best. We shall drop the argument n occasionally.

- Denote $d^{fb}(n) = \arg\max v(d)$ s.t. $(d, n) \in \mathbb{S}$.
- Denote $d^{sb}(n) = \arg\max v(d)$ s.t. $(d, n) \in \mathbb{SS}$.

For these schemes we compute the expected one-day values of participation: $v^{fb}(n)$, $v^{sb}(n)$.

- Denote $v^{fb}(n) = v(d^{fb}(n))$ - value of a group when participation is enforced.
- Denote $v^{sb}(n) = v(d^{sb}(n))$ - value of a group when participation is voluntary.

Using the notion of a second best scheme we can equivalently define a stable distribution scheme.

- A scheme (d, n) is stable iff there does not exist $k < n : (d^{sb}, k) \succ (d, n)$.

To define stability of a group one can focus only on second best schemes in groups of smaller sizes. This means that the second best scheme is the result of maximization subject to a finite number of constraints. We describe these constraints in the next section.

2.7 Incentive constraints

In the previous section we have defined the first best scheme as the result of an unconstrained maximization of the value function $v(d)$ over the set of attainable distribution schemes D.

In contrast, the second best scheme is the result of a constrained maximization. These constraints are called incentive constraints and reflect the self-enforced nature of the group.

For each size of the original group one can distinguish a finite set of incentive constraints.

$$IC = \bigcap_{k<n} \bigcap_{s \in S} \bigcap_{\bar{T}} \bigcup_{t \in \bar{T}} IC(k, s, t) \tag{4}$$

where $\bar{T} \subset T : \sum_{t \in \bar{T}} q_{s,t} \geq k$, and k is a stable size. Each single constraint is just a linear constraint of the form:

$$IC(k, s, t) = \{d \in D : \frac{\delta}{1-\delta} v(d^{sb}(k)) + U(e_t) \leq \frac{\delta}{1-\delta} v(d) + v_{s,t}(d)\} \tag{5}$$

Note that there is a finite amount of non-strict constraints, which means that the maximization set is compact. However, it need not be convex.

3 Analysis

3.1 Probabilities

First, we want to compute the probability of a given state of the world. Recall, that each agent can be of type $t \in T$ with probability p_t. Since the types are drawn independently, the states of the world have a multinomial distribution.

Lemma 1. *The probability P_s of state of the world $s \in S(n)$:*

$$P_s = \frac{n!}{q_{s,1}! \dots q_{s,m}!} p_1^{q_{s,1}} \dots p_m^{q_{s,m}}$$

These probabilities define the distribution of σ. The first trivial property is that the expectation of shares of agents of each type equals to the probabilities of these types. The second property is that the variance of these shares goes to zero as n goes to infinity.

Lemma 2. $\mathbb{E}_\sigma \frac{q_{s,t}}{n} = p_t$, $\mathbb{V}_\sigma \frac{q_{s,t}}{n} = \frac{p_t(1-p_t)}{n}$

It follows that as the number of agents increases to infinity, the distribution of shares of agents concentrates around its expected values.

3.2 Universal space of states

By now we have one difficulty, the random variable $\sigma(n)$ depends on n, and for each n the set of states $S(n)$ is different. To cope with it we define a universal for all n set of outcomes X and a sequence of random variables with finite support that will play the role of $\sigma(n)$ for each given n.

- Denote the universal space of states $X = \{x \in \mathbb{R}^m : \; x_i \geq 0, \; \sum\limits_{i=1}^{m} x_i = 1\}$

Note that for any given n and $s \in S(n)$, the vector of shares $\frac{q_s}{n}$ belongs to X as well as the vector of probabilities p. We induce the standard vector norm $\mathbb{R}^m$ to measure distance in X.

- Define a closed ε-neighborhood $U_\varepsilon(x)$ of point $x \in X$ as :

$$U_\varepsilon(x) = \{y \in X : ||y - x|| \leq \varepsilon\}, \quad ||x|| = (\sum_{i=1}^{m} x_i^2)^{1/2}$$

We can treat p as a random variable on X that is constant. Clearly, by the Law of Large Numbers, $\sigma(n)$ converges to p in probability limit, and, moreover the following two properties hold:

Lemma 3. *There exist* $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{i=n}^{\infty}$*, monotonically decreasing to zero:* $Prob(\frac{q_s}{n} \notin U_{\alpha_n}(p)) \leq \beta_n$.

Lemma 4. *For a continuous function* f*:* $\lim_{n\to\infty} \mathbb{E}_{\sigma(n)} f(\frac{q_s}{n}) = f(p)$.

In both lemmas s denotes the realization of $\sigma(n)$ for every given n.

3.3 First best

Our first goal is to characterize the first best distribution scheme. Recall, that $U(.)$ is a strictly concave and a strictly increasing function, and the ex-ante value $v(d)$ of a distribution scheme d is just a linear combination of ex-post values $v_{s,t}(d) = U(d_{s,t})$.

Lemma 5. $v(d)$ *is a strictly concave and strictly increasing in* d.

It follows that the first-best distribution scheme is to divide E_s equally among all agents.

Lemma 6. $d^{fb}(n) = \{d : d_{s,t} = E_s/n\}$.

In the baseline model the first best value of a group of size n monotonically increases up to the full insurance utility of a single agent as n goes to infinity.

Lemma 7. $v^{fb}(n)$ *is increasing in* n.

Lemma 8. $\lim_{n\to\infty} v^{fb}(n) = U(\mathbb{E}_\tau e_t)$.

3.4 Second best

Our next goal is to characterize the second best scheme.

Lemma 9. *If a stable scheme* $d \in S(n)$ *exists, then there exists* $d^{sb}(n)$

Fix a state of the world $s \in S(n)$. The first observation is that the optimal scheme should belong to the budget hyperplane.

Lemma 10. *If* $d^{sb}(n)$ *exists, then* $\sum q_{s,t} d_{s,t}^{sb} = E_s$ *for all* $s \in S(n)$.

Let k, n be stable group sizes and $k < n$. We can derive a simple necessary condition for this to be true.

Lemma 11. *If two group sizes $k < n$ are stable, then for any $s \in S(n)$:*

$$\sum_{t \in W_s(k,n)} q_{s,t}(n) < k, \qquad W_s(k,n) = \{t \in T \ : \ \frac{\delta}{1-\delta} v^{sb}(k) + U(e_t) > \frac{\delta}{1-\delta} v^{sb}(n) + v_{s,t}(d^{sb}(n))\}. \tag{6}$$

Here $W_s(k,n) \subset T$ is the set of types that are willing to deviate from a group of size n to a group of size k at state $s \in S(n)$. For the types $t \in T \setminus W_s(k,n)$ the incentive constraint $IC(k,s,t)$ must hold:

$$\frac{\delta}{1-\delta} v^{sb}(k) + U(e_t) \leq \frac{\delta}{1-\delta} v^{sb}(n) + v_{s,t}(d) \tag{7}$$

The left part of the inequality is the marginal utility of consuming ones private endowment instead of abiding to the scheme. The right part is the discounted marginal utility of staying in the original group.

Condition (7) is equivalent to:

$$d^{sb}_{s,t}(n) \geq e_t - \gamma_t(k,n), \tag{8}$$

where $\gamma_t(k,n)$ solves the equation:

$$U(e_t) - U(e_t - \gamma_t(k,n)) = \frac{\delta}{1-\delta}(v^{sb}(n) - v^{sb}(k)) \tag{9}$$

Lemma 12. *For stable n, k such that $k < n$: $v^{sb}(n) - v^{sb}(k) \geq 0$, and $\gamma_t(k,n) \geq 0$.*

Lemma 13. *For two stable sequences $\{n_i\}_{i=1}^{\infty}, \{k_i\}_{i=1}^{\infty}$ such that $k_i > n_i$:*

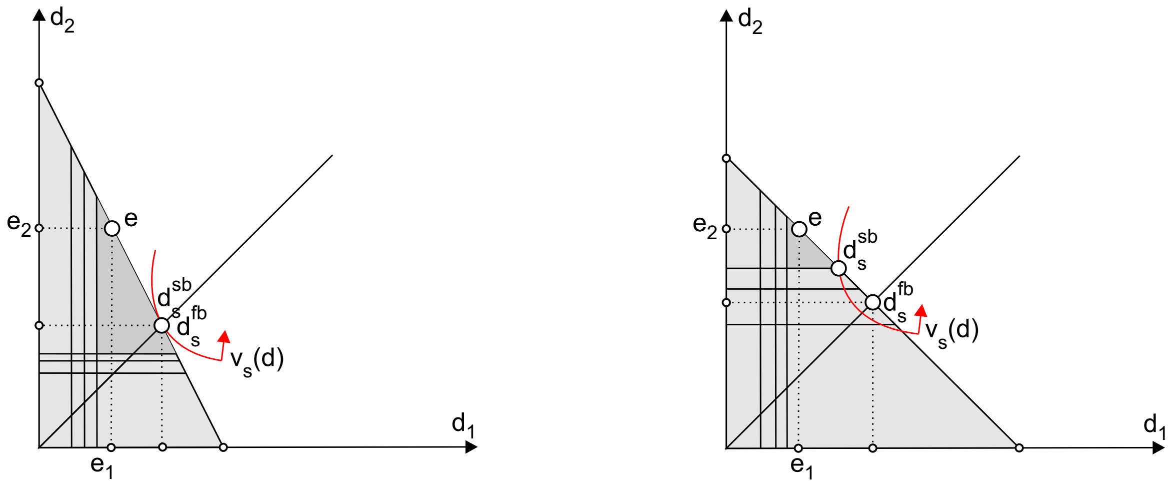
$$\lim_{i\to\infty} (v^{sb}(n_i) - v^{sb}(k_i)) = 0 \quad \Longrightarrow \quad \lim_{i\to\infty} \gamma_t(k_i, n_i) = 0.$$

At a given state of the world, in a second best scheme, either one of the participation constraints binds, or the scheme coincides with the first-best.

Lemma 14. *If $d^{sb}(n)$ exists, then for any $s \in S(n)$ either (i) or (ii) holds:*

(i) $d^{sb}_s = d^{fb}_s$

(ii) $d^{sb}_{s,t}(n) = e_t + \gamma_t(k, n)$ *for some stable* $k < n$ *and* $t \in T$.



a) participation constraint does not bind b) participation constraint binds

Figure 2: Second best schemes

Though for $m = 2$ the second best scheme is obviously unique, this need not be the case for greater m. Recall that at a given state of the world only a subset of types is to be incentivized. For a given subset of types the participation constraints determine a convex subset in D, and hence the strictly concave function $v(d)$ is maximized in a unique point by the separating hyperplane theorem. In our case, however, different subsets of T can be chosen, and hence the true maximization set is not necessarily convex.

We informally provide a possible scenario for a non-unique second best scheme. Let there be three types $\{t_1, t_2, t_3\}$ and a group of size n. Consider two schemes that coincide in all but one state s. Let in that state there exist a stable size $k < n : k > q_{s,t_1} + q_{s,t_2}$, $k > q_{s,t_1} + q_{s,t_3}$ and $k > q_{s,t_2} + q_{s,t_3}$. This means that to block a deviation towards a subgroup of size k it is sufficient to incentivize only one of the three possible types. On Figure 3 we show that the value function $v_s(d)$ can be maximized over the constraint $\bigcup_{t \in \{t_1,t_2,t_3\}} IC(k, s, t)$ in two distinct points simultaneously. If the rest of the constraints are satisfied in these points, then each of the two schemes will be second best.

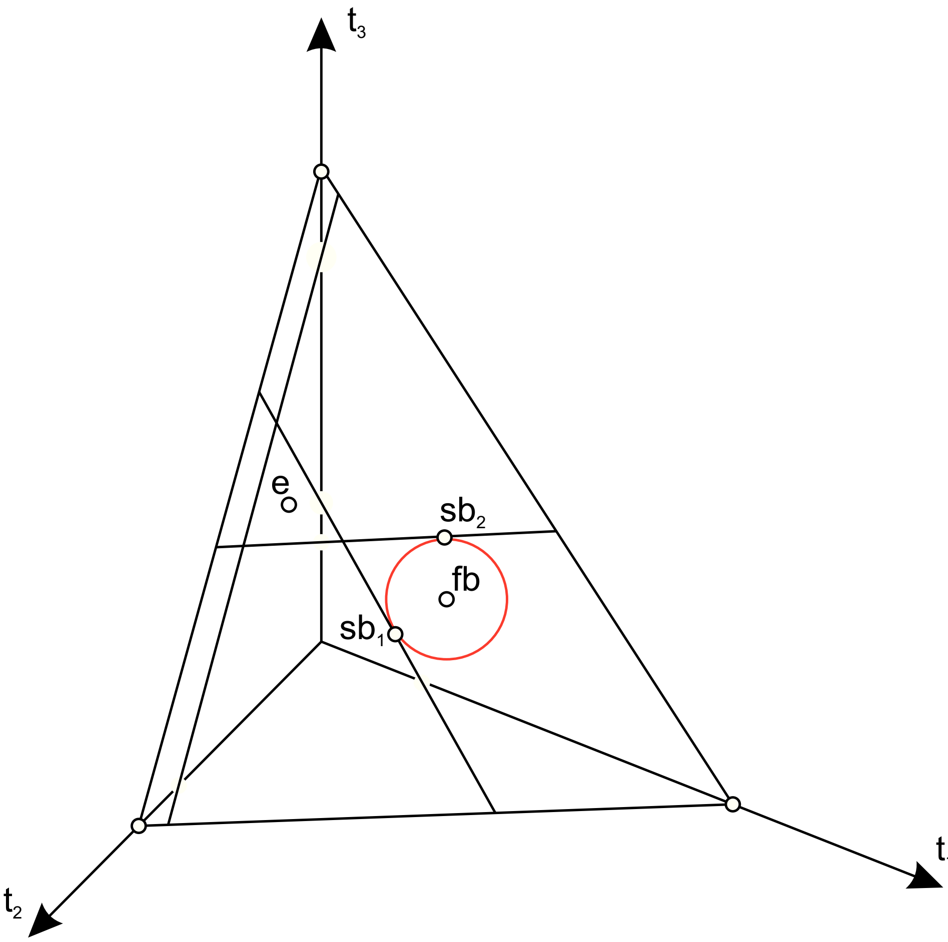


Figure 3: Two second-best schemes

We shall refer to d^{sb} as one of the second best schemes, or the collection of all second best schemes.

3.5 Stability in the basic framework

Now we are ready to answer, whether the set of stable sizes is finite or not.

Consider first the case $m = 1$, when there is no uncertainty, and all agents have identical private endowments. Obviously, the second best scheme coincides with first best, and, hence, the set of stable sizes is infinite.

Assume now that $m \geq 2$ there is some nontrivial uncertainty:

- $p_t \neq 0$ for all $t \in T$
- $e_i \neq e_j$ for some $i \neq j$

Lemma 15. *$v^{sb}(k) \in [\mathbb{E}_\tau U(e_t), U(\mathbb{E}_\tau e_t)]$ for any stable k.*

Theorem 1. *In the assumptions of the baseline model either $v^{sb}(n) = \mathbb{E}_\tau U(e_t)$ for all stable n, or the set of stable sizes is finite.*

Define $v^*(\delta)$ - the second best value of the biggest stable group (the biggest value of a stable group) as a function of the discount factor δ.

Lemma 16. $\lim_{\delta \to 1} v^*(\delta) = U(\mathbb{E}_\tau e_t)$.

Lemma 17. $\lim_{\delta \to 0} v^*(\delta) = \mathbb{E}_\tau U(e_t)$.

4 Game-theoretic foundations

The aim of this section is to provide a game-theoretic reasoning to the notion of stability described above.

4.1 Cooperative setting

To describe this setting we need two things: payoffs and a decision rule.

At each date each agent can either abide to the scheme, or deviate, or stay while others deviate. For an agent of type t at state s and a scheme $d \in D$ the payoffs are:

- if abide the payoff is $U(d_{s,t})$
- if deviate the payoff is $U(e_t)$
- if stay while others deviate the payoff is $-\infty$.

Assume that a group agents operate a predefined scheme (d, n). Now consider a scheme (d', k) for some $k < n$ (not necessarily stable). There are two alternatives for the group: stay together or form a deviating subgroup that will start operating (k, n) from the next period. We first define the individual and group preferences over the set of this two alternatives.

- Denote $(d', k) \succ_{s,t} (d, n)$ if at state $s \in S(n)$ an agent of type $t \in T$ strictly prefers to get into the deviating group rather then no deviation at all.
- Denote $(d', k) \succ_s (d, n)$ if at state $s \in S(n)$ there are at least k agents such that $(d', k) \succ_{s,t} (d, n)$.

We want to define a decision rule that will transfer these preferences into the final decision of the group.

(i) if at state $s \in S(n)$ there exist $(d', k) : (d', k) \succ_s (d, n)$ then some deviation will necessarily occur.

(ii) the deviating group is formed from those agents for whom $(d', k) \succ_{s,t} (d, n)$.

(iii) only deviations to second best schemes are considered.

(iv) if there are several potential deviating groups, a unique is chosen randomly.

Using this decision rule we can determine which kind of deviation (or a lottery over potential deviations) is chosen in each state of the world, recursively by the size of the group. It turns out that a deviation towards a group that poses a threat of future deviation will never occur since the expected utility of this choice is $-\infty$ for each participant.

It follows that the individual preferences $\succ_{s,t}$ (and hence $\succ_s$) take exactly the same form as in Section 2.5.

Theorem 2. *The set of stable schemes* $\mathbb{SS}$ *coincides with the set of schemes that are proof to deviations given described payoffs and decision rule.*

4.2 Non-cooperative setting

To describe this setting we need payoffs, timing, strategies and an equilibrium concept. Let the payoffs be the same as in the cooperative setting.

At each date there is a unique agenda setter chosen randomly among all agents. The agenda setter proposes a subgroup of size $k < n$ and a scheme $d' \in D(k)$ (or he does nothing). The agents in that subgroup then vote for or against that deviation, and if all of them vote for, that group deviates.

Theorem 3. *There exists a subgame perfect Nash equilibrium in which agents vote positively iff* $(d', k) \succ_{s,t} (d, n)$ *and the agenda setter proposes a second best scheme that yields the highest value among those for which* $(d', k) \succ_s (d, n)$ *(and nothing if such scheme does not exist).*

This means that there exists a subgame perfect Nash equilibrium such that the described voting procedure bears exactly the same properties as the decision rule in the cooperative setting. Note that the strategies are symmetric and stationary. Moreover, this kind of voting is sincere, that is, agents vote positively for the alternative iff they truly prefer it.

Unfortunately, there are other subgame perfect Nash equilibria. For example, agents always vote negatively except for the case when the subgroup consists of the agenda setter himself, and all agenda setters propose deviations towards singleton groups. This equilibrium is also symmetric and stationary and involves sincere voting, the difference is that it supports a different structure of stability, namely, when only individual deviations are considered.

5 On stationarity and deterministicity

In the basic framework schemes were stationary and deterministic. One may ask how non-stationary and stochastic schemes affect stability. A non-stationary scheme is a function of history. A stochastic scheme is a lottery over the space of attainable deterministic schemes. In the most general case a non-stationary stochastic scheme maps the space of histories into the set of lotteries over attainable simple (deterministic and stationary) schemes.

At a first glance this setting seems to be a generalization of the baseline model, but this is a deceptive feeling. Due to the recursive structure of stability, relaxing the set of contracts that the agents can use affects the incentive constraints for all group sizes simultaneously.

There is one peculiar feature of the assumptions of stationarity and deterministicity though. It turns out that they do not really change the incentive constraints. It is possible to show that allowing for history dependence and randomization, separately or simultaneously, in some sense, does not change the set of stable schemes at all, if symmetry is maintained. It follows that these two assumptions do not lead to a loss of generality in the basic framework.

5.1 Histories and lotteries

We first modify the notations to incorporate the new features of the distribution schemes.

- Denote I_j - information on date j only.
- Denote $h = \{I_1, \ldots, I_j\}$ – a history up to date j.
- Denote H – the space of all finite histories.

At the end of each day the latest realizations of the state of the world as well as the outcome of the lottery are added to the history. This history is then used to determine the scheme (that can be stochastic) for the next day. Note that the lottery itself need not be added to the history, since it is already a function of the part of that history.

- Recall $D = \{D_s\}_{s \in S(n)}$ is the set of attainable deterministic stationary schemes.
- Denote $M(D)$ is the set of probability measures (lotteries) over D.
- A non-stationary stochastic scheme d is an element in $H^{M(D)}$ (it maps H into $M(D)$).

Note that a stationary scheme is a particular case of a non-stationary scheme that is constant over H, and a deterministic scheme is a particular case of a stochastic scheme where there is only one outcome of the lottery, or the schemes coincide for all outcomes.

- Denote λ - the probability measure (lottery) on D.
- Denote l - realization of the lottery.
- Denote $L(d, h)$ - support of λ for the scheme d and history h.

5.2 Value

Second, we need to introduce a proper value function that can order our new class of schemes. For a non-stationary stochastic scheme d let $d(h, l)$ denote a particular point in D that corresponds to history h and the outcome l of the lottery.

- Denote $v_{l,s,t}(d, h) = v_{s,t}(d(h, l))$ – ex-post one-day utility from a scheme d given h, s, l, t.
- Denote $v_l(d, h) = \mathbb{E}_\sigma \mathbb{E}_{\tau|s} v_{l,s,t}(d, h)$
- Denote $v(d, h) = \mathbb{E}_\lambda v_l(d, h)$ – ex-ante one-day utility from scheme d given history h.
- Denote $\bar{v}(d, h)$ – discounted future utility from scheme d, given history h.

Let h be the history of length j, and let $h \cup I_{j+1}(s, l)$ denote one of the possible histories that follow from h after the realization of s - state of the world and l - outcome of the lottery.

$$\bar{v}(d, \ \cdot \ ) : H \to \mathbb{R}, \quad \bar{v}(d, h) = v(d, h) + \delta \mathbb{E}_\lambda \mathbb{E}_\sigma \bar{v}(d, h \cup I_{j+1}(s, l)) \tag{10}$$

Finally we can order the new set of schemes by means of function $\bar{v}(d, \emptyset)$.

5.3 Incentive constraints

Skipping several steps, the set of incentive constraints now takes the following form:

$$IC = \bigcap_{h \in H} \bigcap_{k<n} \bigcap_{\lambda} \bigcap_{s \in S} \bigcap_{\bar{T}} \bigcup_{t \in \bar{T}} IC(k, s, h, \lambda, \bar{T}, t) \tag{11}$$

$$IC(k, s, h, l, \bar{T}, t) = \{d \in D : \delta \bar{v}(d^{sb}(k), \emptyset) + U(e_t) \leq \delta \bar{v}(d, h \cup I_{j+1}(s, l)) + v_{l,s,t}(d)\} \tag{12}$$

Again, $\bar{T}$ is such that $\sum_{t \in \bar{T}} q_{s,t} \geq k$, and k is the stable group size. We here assume (rightfully, as we shall later see) that the second best scheme is correctly defined for all $k < n$. Finally we can say what a second best scheme is:

- $d^{sb} = \arg\max \bar{v}(d, \emptyset)$ subject to IC.

Note that at this point we can non even guarantee existence of the second best scheme, because the maximization set is not finite-dimensional, however, we are able to show that maximization set can be reduced to the subset of stationary and deterministic schemes, which is compact and finite-dimensional, and, hence maximum is attained.

5.4 Equivalence to the basic framework

Our first observation is that for any stable stochastic scheme d there exists a deterministic stable scheme d' that yields at least the same expected value: $\bar{v}(d', \emptyset) \geq \bar{v}(d, \emptyset)$.

Let for some history h a scheme d assigns a stochastic outcome, that is, there be at least two outcomes l_1, l_2 (with nonzero probability) of the lottery such that $d(h, l_1) \neq d(h, l_2)$. Consider the closure of the support $L(d, h)$ of the lottery λ and pick the one $l^\star$ that yields the highest value $v_l(d, h)$ (the incentive constraints hold for it due to continuity). Then a new scheme can be constructed that coincides with d at each point in H except for h. At point h the new scheme assigns the deterministic outcome that coincides with $l^\star$. Repeating the procedure for all $h \in H$ we can come up with a deterministic scheme d'. At each step the incentive constraints continue to hold, because $\bar{v}(d, h)$ does not decrease. Hence the scheme d' is stable.

Our second observation is that for any stable history-dependent scheme d there exists a stationary stable scheme d'' that yields at least the same expected value.

Note that first we can replace d with a deterministic scheme d' that will yield at least the same value. Consider the closure of the set of all schemes that are used in d' and pick a point that yields the highest value (it exists by W.Th). Construct a new stationary scheme d'' that corresponds to that point. The incentive constraints will hold and hence the scheme will be stable.

Theorem 4. *The function $\bar{v}(d, \emptyset)$ attains maximum in the set of stochastic and history-dependent schemes, and there exists a stationary and deterministic scheme that maximizes it.*

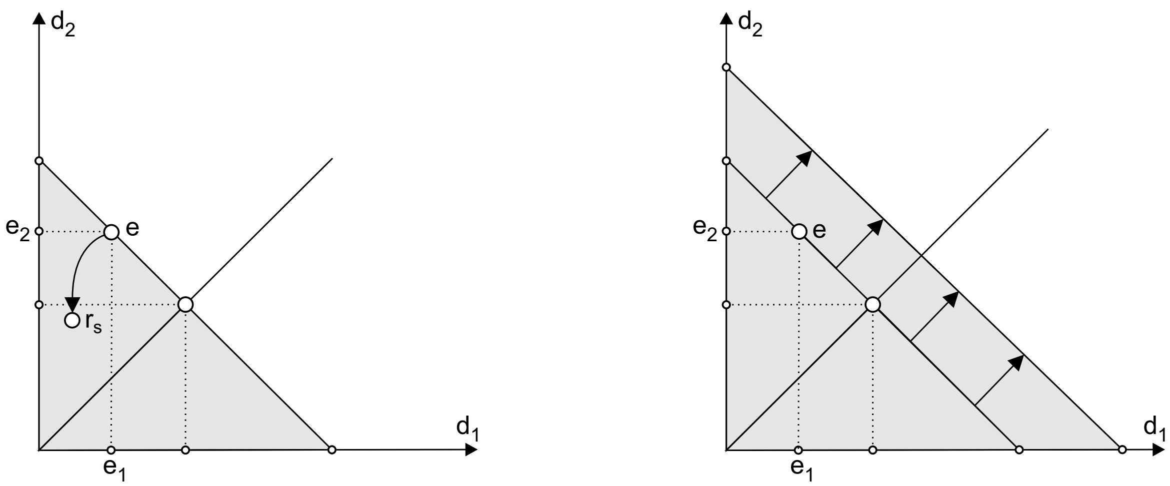
Applying this theorem iteratively by induction by size of the group we can prove that the structure of stability does not change under the assumptions of stationarity and deterministicity. The only difference comes from non-uniqueness of second best schemes, that can generate stochastic history-dependent schemes, but they will yield the same value.

6 Adding perturbations

We can think of two types of natural perturbations of the model that can promote stability. The first one is punishing the deviating agent. Intuitively, a strong enough punishment can stop an agent from leaving his group. The second one is encouraging the agents that stay in the group. Since their wealth is optimally redistributed, we can consider just a subsidy to the group endowment. Again, it is intuitive that by subsidizing a group with a big enough amount of good, one can incentivize agents to stay. We are interested in whether the results of the basic framework are preserved under, in some sense, small perturbations of these two types.

6.1 Policies and institutions

These perturbations can be thought of in two ways. If punishing or encouraging is applied to a single group, but not to any of its potential succeeding deviating subgroups, we call it a policy, meaning that it is applied by some external entity to a particular group, in order to promote stability. An institution, in contrast, is an endogenous property of the group, hence, it is natural to assume that it is available to all groups. We shall call an institution a collection of perturbations of groups of all sizes.



a) punishing b) encouraging

Figure 4: Policies

To model a punishing policy we modify the structure of deviation, by changing the incentive constraints in the following way:

$$IC(k,s,t) = \{d \in D : \frac{\delta}{1-\delta} v(d^{sb}(k)) + U(r_{s,t}) \le \frac{\delta}{1-\delta} v(d) + v_{s,t}(d)\}, \quad r_{s,t} = e_t - \varphi_{s,t}(n) \tag{13}$$

Here $r_{s,t}$ is the (reserved) amount of good that the deviating agent of type $t \in T$ can take with him if he chooses to deviate.

- $r_s = \{r_{s,t}\}_{t \in T}$ - vector of reserved amounts of good.

In other words, $\varphi_{s,t}(n)$ is the amount of punishment (in terms of good) applied to an agent of type t at state s if he deviates. As applied to Figure 1, a point r_s appears, see Figure 4 a). This point is used in the modified incentive constraint (13) instead of e. In other words, r_s is the vector of private endowments as seen by the deviator.

- A punishing policy is a profile $\{\varphi_{s,t}(n)\}_{s \in S(n), t \in T}$ of punishments applied to a group of size n.
- A punishing institution is a sequence of profiles $\{\varphi_{s,t}(n)\}_{s \in S(n), t \in T}^{n \in \mathbb{N}}$ applied to all groups.

To model an encouraging policy we modify the definition of group endowment by changing the formula (1) in the following way:

$$E_s = \sum_{t \in T} q_{s,t} e_t + \psi_s(n) \tag{14}$$

Here $\psi_s(n)$ is the group premium (in terms of good), that can reflect either a subsidy by an external entity, or a property of a redistributing technology. As applied to Figure 1, the group premium simply shifts the budget line outwards, see Figure 4 b).

- An encouraging policy is a profile $\{\psi_s(n)\}_{s \in S(n)}$ of punishments applied to a group of size n.
- An encouraging institution is a sequence of profiles $\{\psi_s(n)\}_{s \in S(n)}^{n \in \mathbb{N}}$ applied to all groups.

We would like to impose some reasonable restrictions on the magnitude of these perturbations. Intuitively, the perturbations are considered small if they are per-capita negligible when $n \to \infty$. There are at least three forms of this restriction:

(i) $\psi_s(n)$ $(\sum_{t \in T} q_{s,t}\varphi_s(n))$ is uniformly bounded.

(ii) $\frac{\psi_s(n)}{n}$ $(\sum_{t \in T} \frac{q_{s,t}\varphi_s(n)}{n})$ converges to zero uniformly over $s \in S(n)$.

(iii) $\mathbb{E}_\sigma \frac{\psi_s(n)}{n}$ ($\mathbb{E}_\sigma \sum_{t \in T} \frac{q_{s,t}\varphi_s(n)}{n}$) converges to zero.

Condition (i) says that there is just a physical restriction to the amount of the policy, however, we believe that this condition is too strict. Condition (ii) says that the per capita amount of policy converges to zero uniformly across states, and it follows from (i). The weakest assumption is (iii), that says that the average of per capita amount of policy converges to zero.

6.2 Punishing policy

Consider a punishing policy or institution. The new participation constraint is given by equation (13). The analog of Lemma 11 states:

Lemma 18. *If two group sizes $k < n$ are stable, then for any $s \in S(n)$:*

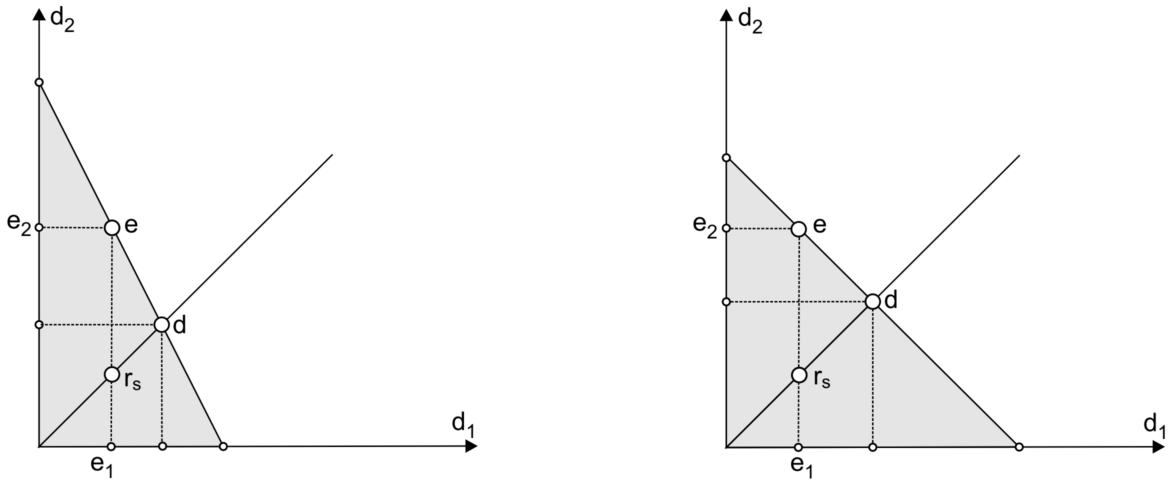
$$\sum_{t \in W_s(k,n)} q_{s,t}(n) < k, \qquad W_s(n,k) = \{t \in T \,:\, \frac{\delta}{1-\delta} v^{sb}(k) + U(e_t - \varphi_{s,t}(n)) > \frac{\delta}{1-\delta} v^{sb}(n) + v_{s,t}(d^{sb}(n))\}.$$

Our first observation is that for an arbitrary punishing institution, if the amount of punishment is non-positive, that is, deviating agents consume at least their private endowments at the date of deviation, then the result of Theorem 1 holds.

Lemma 19. *For a non-positive sequence of punishing profiles $\{\varphi_{s,t}(n)\}_{s \in S(n), t \in T}^{n \in \mathbb{N}}$, Theorem 1 holds.*

Clearly, to promote stability, the amount of punishment should be positive at least for some states. From this moment we will consider only nonnegative punishing profiles. The second observation is that there exists a punishing policy such that a given group is stable. To construct such a policy it is sufficient to set $r_{s,t}$ in such a way that the first-best allocation satisfies all participation constraints at all states.

Our first example is such that $r_{s,t} = \min(e_t)$, see Figure 5.



a) at state s, such that $q_{s,1} > q_{s,2}$ b) at state s, such that $q_{s,1} = q_{s,2}$

Figure 5: First example of a punishing policy.

A rule of thumb for this example is: *to promote stability in a group, a deviating agent should consume no more than the smallest income in the group.*

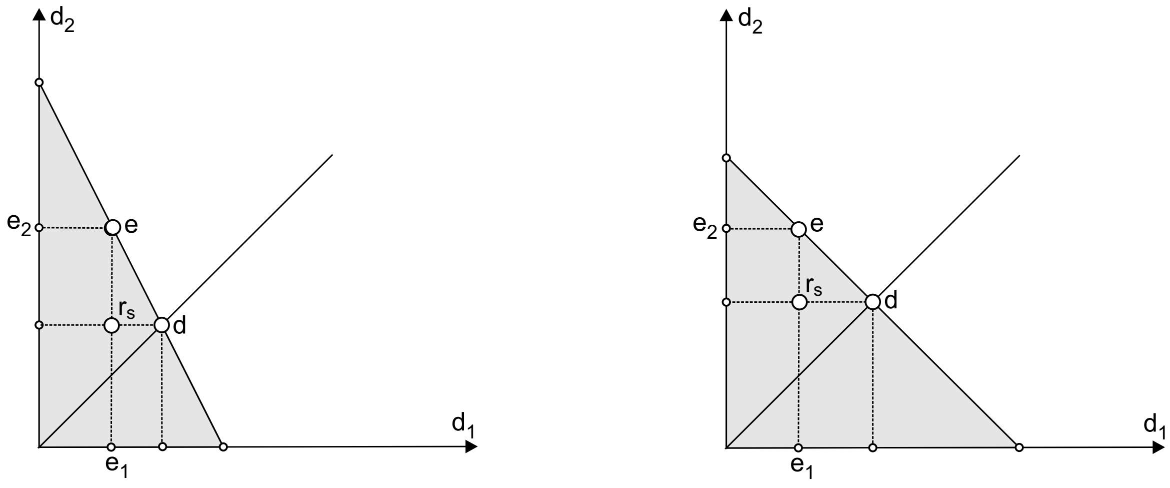
Lemma 20. *With a punishing profile* $\{\varphi_{s,t}(n)\}_{s\in S(n),t\in T}$ *such that* $r_{s,t} = \min(e_t)$, *the group is stable.*

The example above is an extreme case of punishment. Intuitively, the uncertainty in the group is implicitly reduced to zero, and hence the first best distribution is attainable. Moreover, for a sequence of profiles constructed in this way, all groups will be stable. In other words, under this institution, the set of stable groups is infinite, and all of them reach the first-best distribution. Unfortunately, the capacity of punishment needed for this institution, explodes as n increases to infinity.

Lemma 21. *With a punishing profile* $\{\varphi_{s,t}(n)\}_{s\in S(n),t\in T}$ *such that* $r_{s,t} = \min_{t\in T}(e_t)$,

$$\lim_{n\to\infty} \mathbb{E}_\sigma \sum_{t\in T} \frac{q_{s,t}(n)\varphi_{s,t}(n)}{n} = \sum_{t\in T} p_t(e_t - \min_{t\in T} e_t) > 0.$$

Our second example is such that $r_{s,t} = \min(e_t, \mathbb{E}_{\tau|s} e_t)$, see Figure 6.



a) at state s, such that $q_{s,1} > q_{s,2}$ b) at state s, such that $q_{s,1} = q_{s,2}$

Figure 6: Second example of a punishing policy.

A rule of thumb for this example is: *to promote stability in a group, a deviating agent should consume no more than the average income in the group.*

Lemma 22. *With a punishing profile $\{\varphi_{s,t}(n)\}_{s\in S(n),t\in T}$ such that $r_{s,t} = \min(e_t, \mathbb{E}_\tau e_t)$, the group is stable.*

Note that this example is more sophisticated that the first one, since the point r_s depends on the state. However, the capacity needed for punishment still grows linearly in n.

Lemma 23. *With a punishing profile $\{\varphi_{s,t}(n)\}_{s\in S(n),t\in T}$ such that $r_{s,t} = \min(e_t, \mathbb{E}_\tau e_t)$,*

$$\lim_{n\to\infty} \mathbb{E}_\sigma \sum_{t\in T} \frac{q_{s,t}(n)\varphi_{s,t}(n)}{n} = \sum_{t\in T} p_t(e_t - \min(e_t, \mathbb{E}_\tau e_t)) > 0.$$

6.3 Encouraging policy

Consider an encouraging policy or institution. The new formula for the group endowment is (14). It follows that the set of attainable distributions is given by formula:

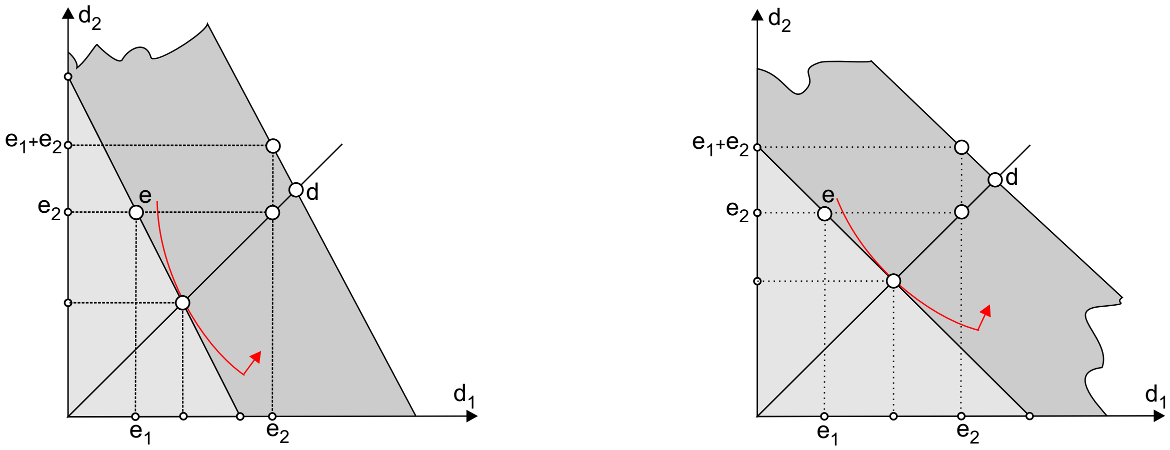
$$D_s = \{d_s : \ d_{s,t} \geq 0, \ \sum_{t\in T} q_{s,t} d_{s,t} \leq \sum_{t\in T} e_t d_{s,t} + \psi_s(n)\}$$

The first observation is very similar to the one for the punishing institution: if the group premium $\psi_s(n)$ is non-positive for all $s \in S(n)$, that is, there is less good distributed than it was collected, then the result of Theorem 1 holds.

Lemma 24. *For a non-positive sequence of encouraging profiles $\{\psi_s(n)\}_{s\in S(n)}^{n\in\mathbb{N}}$, Theorem 1 holds.*

Clearly, to promote stability, the group premium should be positive at least in some states. From now on, we shall consider only nonnegative encouraging profiles. The second observation is that there exists an encouraging policy such that a given group is stable. To construct such a policy it is sufficient to shift the budget hyperplane in such a way that the first-best allocation satisfies all participation constraints at all states.

Our first example is such that $\psi_s(n) = n(\max_{t\in T}(e_t) - \min_{t\in T}(e_t))$, see Figure 7.



a) at state s, such that $q_{s,1} > q_{s,2}$ b) at state s, such that $q_{s,1} = q_{s,2}$

Figure 7: First example of an encouraging policy.

A rule of thumb for this example is: *to promote stability in a group, one should subsidize it in such a way that all agents can get at least as much as their highest income.*

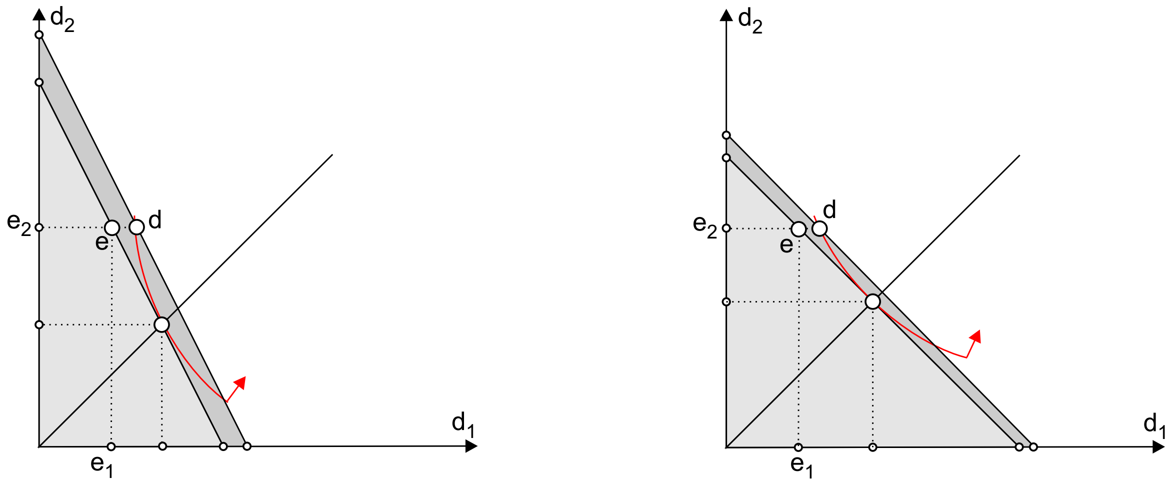
Lemma 25. *With an encouraging profile* $\{\psi_s(n)\}_{s\in S(n)}$ *such that* $\psi_s(n) = n(\max_{t\in T}(e_t) - \min_{t\in T}(e_t))$, *the group is stable.*

The example above is an extreme case of encouraging. Intuitively, the uncertainty in the group is implicitly reduced to zero, and hence the first best distribution is attainable. Just like in the case of a punishing policy, the amount of good needed for this policy explodes as n increases to infinity.

Lemma 26. *With an encouraging profile* $\{\psi_s(n)\}_{s\in S(n)}$ *such that* $\psi_s(n) = n(\max_{t\in T}(e_t) - \min_{t\in T}(e_t))$,

$$\lim_{n\to\infty} \mathbb{E}_\sigma \frac{\psi_s(n)}{n} = \max_{t\in T}(e_t) - \min_{t\in T}(e_t) > 0.$$

Our second example is more tricky. At each state $s \in S(n)$ there are types that are relatively rich, or $e_t \geq \mathbb{E}_{\tau|s} e_t$, and there are types that are relatively poor, that is, $e_t < \mathbb{E}_{\tau|s} e_t$. We want to insure the poor so that they are ex-ante as good as with the first-best scheme, but leave the rich with the same level of good so that they have no incentives to deviate, see Figure 8.



a) at state s, such that $q_{s,1} > q_{s,2}$ b) at state s, such that $q_{s,1} = q_{s,2}$

Figure 8: Second example of an encouraging policy.

A rule of thumb for this example is: *to promote stability in a group, one should subsidize it in such a way that all rich agents can get what they have, and the poor are insured up to the first-best level.*

Lemma 27. *With an encouraging profile $\{\psi_s(n)\}_{s\in S(n)}$ such that there exists a scheme $d \in D$ satisfying $v_s(d) = U(\mathbb{E}_{\tau|s}e_t)$, but if $e_t \geq \mathbb{E}_{\tau|s}e_t$ then $d_{s,t} = e_t$; the group is stable.*

However, as the size of the group increases, the subsidy still explodes.

Lemma 28. *With an encouraging profile $\{\psi_s(n)\}_{s\in S(n)}$ such that there exists a scheme $d \in D$ satisfying $v_s(d) = U(\mathbb{E}_{\tau|s}e_t)$, but if $e_t \geq \mathbb{E}_{\tau|s}e_t$ then $d_{s,t} = e_t$;*

$$\lim_{n\to\infty} \mathbb{E}_\sigma \frac{\psi_s(n)}{n} > 0.$$

6.4 Saving and borrowing

In the previous section we considered cases of non-negative and non-positive encouraging policies. We can model saving and borrowing by assuming that a group can purchase a risky asset. It will be an encouraging policy that is positive or negative depending on the state of the world.

Consider an example, where the group is subsidized when the group endowment is below average and it is taxed when the group endowment is above average, so that it gets its own average at all states of the world. Clearly, under this policy the group is stable.

Lemma 29. *With an encouraging profile* $\{\psi_s(n)\}_{s\in S(n)}$ *such that* $\psi_s(n) = \mathbb{E}_\sigma E_s - E_s$*, the group is stable.*

Moreover, under such institution (if all groups have access to this asset), there will be an infinity of stable groups. Though the average amount of policy is always zero, it explodes at some states of the world.

Lemma 30. *With an encouraging profile* $\{\psi_s(n)\}_{s\in S(n)}$ *such that* $\psi_s(n) = \mathbb{E}_\sigma E_s - E_s$*,* $\frac{\psi_s}{n}$ *does not converge uniformly to zero.*

6.5 Stability in the perturbed framework

Assume a sequence of punishing and encouraging profiles $\{\varphi_{s,t}(n), \psi_s(n)\}_{s\in S(n),t\in T}^{n>1}$. We let them be almost arbitrary, the only restrictions that we impose are that they are nonnegative on they are bounded in magnitude (in fact we make a weaker assumption):

- $\sum_{t\in T} \frac{q_{s,t}\varphi_{s,t}(n)}{n} \to 0$ uniformly on $s \in S(n)$.
- $\frac{\psi_s(n)}{n} \to 0$ uniformly on $s \in S(n)$.

Assume again that there is some nontrivial uncertainty:

- $p_t \neq 0$ for all $t \in T$
- $e_i \neq e_j$ for some $i \neq j$

Recall the assumptions of the extended model:

- Group endowment : $E_s = \sum_{t\in T} q_{s,t}e_t + \psi_s(n)$
- Set of attainable distributions : $D_s = \{d_s : \ d_{s,t} \geq 0, \ \sum_{t\in T} q_{s,t}d_{s,t} \leq E_s\}$
- Participation constraint : $U(r_{s,t}) - v_{s,t}(d^{sb}(n)) \leq \frac{\delta}{1-\delta}(v^{sb}(n) - v^{sb}(k))$
- Reserved value : $r_{s,t} = e_t - \varphi_{s,t}(n)$

Lemma 31. *In the assumptions of the perturbed model,* $v^{sb}(k)$ *is bounded for all stable* k*.*

Lemma 32. *For any increasing sequence* $\{k_i\}_{i=1}^\infty$ *of stable sizes such that* $\{v^{sb}(k_i)\}_{i=1}^\infty$ *is monotonic,*

$$\lim_{i\to\infty} v^{sb}(k_i) = \mathbb{E}_\tau U(e_t).$$

Lemma 33. *For any increasing sequence $\{k_i\}_{i=1}^{\infty}$ of stable sizes such that $\{v^{sb}(k_i)\}_{i=1}^{\infty}$ converges,*

$$\lim_{i\to\infty} v^{sb}(k_i) \geq v^{sb}(k_j) \text{ for all } j.$$

Theorem 5. *In the assumptions of the perturbed model either $v^{sb}(n) = \mathbb{E}_\tau U(e_t)$ for all stable n, or the set of stable sizes is finite.*

7 Proofs

Lemmas 1, 2 are simply algebraic properties of the multinominal distribution. Lemmas 3, 4 are less trivial.

Proof of Lemma 3:

Apply the multivariate case of the Chebyshev inequality :

$$Prob(||\frac{q_s}{n} - p|| > \frac{1}{\ln(n)}) \leq (\sum_{t\in T} \mathbb{V}_\sigma \frac{q_{s,t}}{n}) \ln^2(n) = const \cdot \frac{\ln^2(n)}{n}$$

Proof of Lemma 4:

Step 1: By the Law of Large Numbers:

$$\frac{q_s}{n} \xrightarrow{p} p$$

Step 2: By the Continuous Mapping Theorem it follows that for a continuous function f:

$$f(\frac{q_s}{n}) \xrightarrow{p} f(p).$$

Step 3: Apply the Lebesgue's Dominated Convergence Theorem to show that:

$$\mathbb{E}_\sigma f(\frac{q_s}{n}) \longrightarrow f(p).$$

Lemma 5 is obvious, since $v(d)$ is a linear combination of $U(d_{s,t})$.

Proof of Lemma 6:

We can show that the first best distribution maximizes $v_s(d)$ separately in each state $s \in S(n)$:

$$v_s(d) = \mathbb{E}_{\tau|s} U(d_{s,t}) \leq U(\mathbb{E}_{\tau|s} d_{s,t}) = U(E_s/n)$$

Proof of Lemma 7

Step 1: Let $\{\xi_i\}_{i=1}^{\infty}$ be a sequence of identically distributed random variables.

Denote $S_n = \sum_{i=1}^{n} \xi_i$ and let f be a concave function.

Due to symmetry:

$$\mathbb{E}(\xi_1|S_n, S_{n+1}) = \cdots = \mathbb{E}(\xi_n|S_n, S_{n+1}) = \frac{S_n}{n}$$

on the other hand

$$\mathbb{E}(\xi_1|S_{n+1}) = \cdots = \mathbb{E}(\xi_{n+1}|S_{n+1}) = \frac{S_{n+1}}{n+1}$$

Step 2: By the law of iterated expectations and by conditional Jensen inequality

$$\mathbb{E}(f(\frac{S_n}{n})) = \mathbb{E}(\mathbb{E}(f(\frac{S_n}{n})|S_{n+1})) = \mathbb{E}(\mathbb{E}(f(\mathbb{E}(\xi_1|S_n, S_{n+1}))|S_{n+1})) \leq \mathbb{E}(f(\mathbb{E}(\xi_1|S_{n+1}))) = \mathbb{E}(f(\frac{S_{n+1}}{n+1}))$$

Step 3: The first best distribution $d_{s,t}^{fb} = \frac{E_s}{n}$ is the average of all private endowments that are identically distributed. It then follows that $v(d^{fb}(n)) \leq v(d^{fb}(n+1))$.

Proof of Lemma 8

We use Lemma 4 to show:

$$\lim_{n\to\infty} v^{fb}(n) = \lim_{n\to\infty} \mathbb{E}_\sigma U(E_s/n) = \lim_{n\to\infty} \mathbb{E}_\sigma U(\sum_{t\in T} \frac{q_{s,t}}{n} e_t) = U(\sum_{t\in T} p_t e_t) = U(\mathbb{E}_\tau e_t)$$

Proof of Lemma 9

Since we maximize a continious function on an intersection of a compact set (D) with a finite colloction of closed sets (participation constraints), a maximum is always attained.

Proof of Lemma 10

Assume that $\sum q_{s,t} d_{s,t}^{sb} < E_s$ for some states $s \in S(n)$. Then there exists a distribution sheme $\tilde{d}^{sb}$ such that $\tilde{d}_{s,t}^{sb} > d_{s,t}^{sb}$ for all t and if a constraint is satisfied for d^{sb} then it is satisfied for $\tilde{d}^{sb}$, hence $\tilde{d}^{sb}$ is stable and gives a higher value. This contradicts the fact that d^{sb} maximises $v(d)$ over $\mathbb{SS}$.

Proof of Lemma 11

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Proof of Lemma 12

Pick a state $\hat{s} \in S(n)$ such that all agents are of the same type $\hat{t}$. At this state $d^{sb}_{\hat{s},\hat{t}} \leq e_{\hat{t}}$, and, hence, $v_{\hat{s},\hat{t}}(d^{sb}) \leq U(e_{\hat{t}})$. On the other hand, $\hat{t}$ should not belong to $W_{\hat{s}}(n,k)$, otherwise the deviation will occur. It follows that $v^{sb}(n) - v^{sb}(k) \geq 0$ and, hence $\gamma_t(k,n) \geq 0$.

Proof of Lemma 13

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Proof of Lemma 14

If $d^{sb}_s \neq d^{fb}_s$ and none of the participation constraints binds, then, since $v(d)$ is a continuous function, one can choose $\tilde{d}^{sb}_s$ arbitrarily close to d^{sb}_s such that all the participation constraints hold and $v(\tilde{d}^{sb}) > v(d^{sb})$. This contradicts the fact that d^{sb} maximizes $v(d)$ over $\mathbb{SS}$.

Proof of Lemma 15

Assume that $v^{sb}(n) < \mathbb{E}_\tau U(e_t)$. In the state of the world s such that all agents are of the same type the participation constraint then will certainly fail: $v^{sb}(n) < v^{sb}(1)$. The upper bound comes from the upper bound of $v^{fb}(n)$.

Proof of Lemma 16

Notice that since all value functions are bounded for a given group size n, the incentive constraints degenerate to $v^{sb}(n) \geq v^{sb}(k)$ for all stable $k < n$, as $\delta \to 1$. Consequently, for δ close enough to 1, any group can become stable and the second best scheme approaches to the first best one. It then follows that $\lim_{\delta\to 1} v^\star(\delta) = U(\mathbb{E}_\tau e_t)$.

Proof of Lemma 17

Notice that since all value functions are bounded for a given group size n, the incentive constraints degenerate to $v(d_{s,t}) \geq U(e_t)$ for all s, t, as $\delta \to 0$. It then follows that $\lim_{\delta\to 0} v^\star(\delta) = \mathbb{E}_\tau U(e_t)$.

Proof of Lemma 31

Since $\frac{\psi_s(n)}{n} \to 0$ uniformly on $s \in S(n)$, it is also bounded uniformly on s, n. The second best value $v^{sb}(n)$ therefore is smaller than $U(\mathbb{E}_\tau e_t + \sup \frac{\psi_s(n)}{n})$.

Proof of Lemma 32

Let $\{k_i\}_{i=1}^{\infty}$ be an increasing sequence of stable sizes such that $\{v(k_i)\}_{i=1}^{\infty}$ is monotonic.

Step 1: Pick a small ε-neighborhood of point p such that for any point x in this neighborhood all coordinates x_i are strictly positive.

$$\varepsilon = \frac{\min_t p_t}{2} \quad \Longrightarrow \quad \forall x \in U_{\varepsilon}(p) : \forall t \ x_t > 0$$

It then follows that there exist two positive constants c_1, c_2:

$$c_1 \leq \frac{q_{s,t}}{n} \leq c_2, \ \forall t \in T, \ s \in S(n) \cap U_{\varepsilon}(p)$$

In other words, the slope of the budget hyperplane is uniformly bounded in the neighborhood of p.

Step 2: For any k define $n_{\varepsilon}(k)$ such that for any $n > n_{\varepsilon}(k)$ and a state $s \in S(n)$ that appears in the ε-neighborhood of point p, the minimal amount of agents of the same type is greater or equal than k.

$$n_{\varepsilon}(k) = \left[\frac{k}{\min\limits_{x \in U_{\varepsilon}(p)} \min\limits_{t} x_t} \right] + 1, \ n > n_{\varepsilon}(k) \quad \Longrightarrow \quad \min_{s \in S(n), \frac{q_s}{n} \in U_{\varepsilon}(p)} \min_t q_{s,t} \geq n \cdot \min_{x \in U_{\varepsilon}(p)} \min_t x_t \geq k$$

Step 3: Construct a subsequence $\{n_i\}_{i=1}^{\infty}$ of the original sequence $\{k_i\}_{i=1}^{\infty}$, such that $n_i > k_i$ and the set of types of agents who are willing to deviate from $d^{sb}(n_i)$ to $d^{sb}(k_i)$ is empty.

$$n_i = \min(n \in \{k_i\}_{i=1}^{\infty} : n > n_{\varepsilon}(k_i), \ n_i > n_{i-1}) \quad \Longrightarrow \quad W_s(k_i, n_i) = \emptyset, \ \forall s \in S(n_i) \cap U_{\varepsilon}(p)$$

If at a certain state of the world the minimal amount of agents of the same type is greater than k, then the set of types that are willing to deviate to a group of size k at this state of the world is nesessarily empty, otherwise a deviation will occur.

Step 4: Since for the states that appear in the ε-neighborhood of p the participation constraints when deviating from $d^{sb}(n_i)$ to $d^{sb}(k_i)$ hold for all types, it follows that

$$\forall t \in T, \ \forall s \in S(n_i) \cap U_{\varepsilon}(p) : \ d^{sb}_{s,t}(n_i) \geq (e_t + \varphi_{s,t}(n_i)) - \gamma_t(k_i, n_i)$$

Step 5: Since $\sum_{t \in T} \frac{q_{s,t}\varphi_{s,t}(n)}{n} \geq c_1 \sum_{t \in T} \varphi_{s,t}(n)$ in the ε-neighborhood of p :

$$\forall t \in T: \ \varphi_{s,t}(n_i) \to 0 \text{ uniformly on } s \in S(n_i) \cap U_\varepsilon(p)$$

That is, the distance from the point e to the point p shrinks.

Step 6: Since $\frac{\psi_s(n)}{n} \geq \frac{c_2 \psi_s(n)}{\sqrt{\sum q_{s,t}^2}}$ in the ε-neighborhood of p :

$$\forall t \in T: \ \frac{\psi_s(n_i)}{\sqrt{\sum q_{s,t}^2}} \to 0 \text{ uniformly on } s \in S(n_i) \cap U_\varepsilon(p)$$

That is, the distance from the point e to the budget hyperplane shrinks.

Step 7: Since $v^{sb}(k_i)$ is monotonic by assumption and bounded by Lemma,

$$\lim(v^{sb}(n_i) - v^{sb}(k_i)) = 0,$$

and hence

$$\forall t \in T: \ \lim \gamma_t(k_i, n_i) = 0$$

Step 8: Combining steps 4-7 and keeping in mind that the slope of the budget hyperplane is bounded in the ε-neighborhood of point p, we obtain that

$$\forall t \in T: \ d_{s,t}^{sb}(n_i) \to e_t \text{ uniformly on } s \in S(n_i) \cap U_\varepsilon(p)$$

Step 9: By Lemma 3 there exist two sequences $\{\alpha_i\}_{i=1}^n$, $\{\beta_i\}_{i=1}^n$ decreasing to zero such that

$$Prob\left(\frac{q_s}{n} \notin U_{\alpha_i}(p)\right) \leq \beta_i.$$

Write down the ex-post value $v^{sb}(n_i) = V_1(i) + V_2(i)$:

$$V_1(i) = \sum_{s \in S(n), \ s \in U_{\alpha_i}(p)} P_s v_s(d^{sb}(n_i)), \quad V_2(i) = \sum_{s \in S(n), \ s \notin U_{\alpha_i}(p)} P_s v_s(d^{sb}(n_i))$$

Step 10: On the one hand, since v_s is a continuous function, it is bounded on a compact, and hence

$\lim V_2(i) = 0$. On the other hand, from step 8 it follows that $\lim V_1(i) = \mathbb{E}_\tau U(e_t)$. Hence

$$\lim v^{sb}(n_i) = \mathbb{E}_\tau U(e_t)$$

and hence

$$\lim v^{sb}(k_i) = \mathbb{E}_\tau U(e_t).$$

Proof of Lemma 33

Let $\{k_i\}_{i=1}^{\infty}$ be an increasing sequence of stable sizes such that $\{v(k_i)\}_{i=1}^{\infty}$ converges.

Fix a stable size k_j and let $\lim v^{sb}(k_i) < v^{sb}(k_j)$.

Step 1: Repeat steps 1,2 from Lemma 32.

Step 2: Construct a subsequence $\{n_i\}_{i=1}^{\infty}$ of the original sequence $\{k_i\}_{i=1}^{\infty}$, such that $n_i > k_j$ and the set of types of agents who are willing to deviate from $d^{sb}(n_i)$ to $d^{sb}(k_j)$ is empty.

$$n_i = \min(n \in \{k_i\}_{i=1}^{\infty} : n > n_\varepsilon(k_j),\ n_i > n_{i-1}) \quad \Longrightarrow \quad W_s(k_j, n_i) = \emptyset \ \forall s \in S(n_i) \cap U_\varepsilon(p)$$

If at a certain state of the world the minimal amount of agents of the same type is greater than k, then the set of types that are willing to deviate to a group of size k at this state of the world is necessarily empty, otherwise a deviation will occur.

Step 3: Repeat steps 4,5,6 from Lemma 32

Step 4: By construction $\lim(v^{sb}(n_i) - v^{sb}(k_j)) < 0$, and hence $\forall t \in T: \ \lim \gamma_t(k_j, n_i) < 0$.

Step 5: For i large enough the incentive constraints together with the budget constraint, at some state s, will form a system of inequalities that have no solution. Hence $\lim v^{sb}(k_i) \geq v^{sb}(k_j)$.

Proof of Theorem 1

Follows from Theorem 5.

Proof of Theorem 5

The proof follows from Lemmas 31, 32 and 33.

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