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**Abstract**

I study markets in which the consumers exhibit "compromise effect", which refers to a tendency to choose intermediate options. The central problem is that in the presence of the compromise effect, a firm might benefit from introducing some products solely to manipulate the consumers' demand for other products. I consider the cases of monopolistic and competitive markets and show that an equilibrium exists in both cases. Moreover, it is shown that with the increase of the degree of the compromise effect, an equilibrium product line is getting less discriminative. In the monopolistic model, I find that if there are two types of consumers, it is optimal for the firm to introduce either one product, or at least three products, with only two of these products are actually sold. An unsold product has either excessively high quality and price, or excessively low quality and price. Such additional products serve only to make intermediate the other two goods that the consumers actually buy. In turn, for competitive markets, I show that under suitable assumptions on parameters, at least one unsold product is present in any equilibrium.

# Contents

# 1 Introduction

In the standard monopolistic screening model with vertically differentiated consumers and private valuations (e.g., Maskin and Riley (1984), Moorthy (1984)), the agents are assumed to be rational. That is, agents' choice behavior satisfies the principle of independence of irrelevant alternatives. This principle is violated when the choice between two alternatives depends on other available options (Luce and Raiffa (1957), Huber, Payne, and Puto (1982), Simonson and Tversky (1992), Sen (1993), Mellers and Cooke (1996), Hsee and Leclerc (1998), Brenner, Rottenstreich, and Sood (1999), Drolet, Simonson, and Tversky (2000), Shafir (2002)). For example, Simonson and Tversky (1992) observe that the sales of a given brand may increase when another brand is introduced. A real world example that they discuss is the case of Williams-Sonoma, a firm that produces bread-baking appliances, which has found that the introduction of an expensive appliance has increased the sales of a less expensive product of the same firm almost twice, although the firm could not sell many units of the expensive appliance itself. This observation tells us that managers should take into account such context effects while designing their product lines.

One of the most robust context effects is the compromise effect (Simonson, 1989), which refers to a tendency to choose intermediate options in a given choice set. Despite the mounting evidence on the compromise effect (e.g., McFadden (1999), Herne (1997), Benartzi and Thaler (2002)), potential implications of this phenomenon for the product-line design problem have not been studied thoroughly.

In this paper, I study product-line design problems that incorporate both discrimination motives and context management considerations due to the compromise effect. I focus on vertically differentiated consumers with private valuations of the quality dimension.

To model the compromise effect, I propose a modification of the utility functions in classical models with vertically differentiated consumers. The utility functions in my model consist of two components. The first component is a standard linear utility function that increases with quality and decreases with price. The second component focuses on the distance between a given product and a hypothetical middle option. This middle option does not necessarily belong to the choice set that the consumer faces, rather it acts as a reference point. The quality and price dimensions of the middle option are the mid points of the best and worst available alternatives in quality and price dimensions, respectively. The second component of the utility associated with a given product decreases with the distance between that product and the middle option in both dimensions. Consequently, the second component may or may not increase with quality or price, depending on the location of the product in question relative to the middle option. However, the overall utility function, i.e., the sum of two components, increases in quality and decreases in price. The reference dependent component of consumers' utility function acts as an adjustment factor that makes the model compatible with the empirical evidence on compromise effect.

I consider the cases of monopolistic and competitive markets and show that an equilibrium exists in both cases. Moreover, it is shown that as the weight of the second component of the utility function gets bigger, an equilibrium product line becomes less discriminative. In the monopolistic model, I find that if there are two types of consumers, it is optimal for the firm to introduce either one product, or at least three products, with only two of these products are actually sold. An unsold product has either excessively high quality and price, or excessively low quality and price. Such additional products serve only to make intermediate the other two goods that the consumers actually buy. In turn, for competitive markets, I show that under suitable assumptions on parameters, at least one unsold product is present in any equilibrium.

There are several other papers which utilize reference dependent utility functions to model context effects (see, among others, Tversky and Kahneman (1991), Tversky and Simonson (1993), Bodner and Prelec (2001), and Orhun (2009)). In this literature, the most closely related paper to mine is that of Orhun (2009), who studies a monopolistic screening model. Just as I do here, Orhun proposes a reference dependent modification of the utility functions in the classical monopolistic screening models. Orhun's approach is based on the notion of loss aversion. The reference dependent utility functions are compatible with the compromise effect but they are not particularly designed to model this phenomenon. In fact, unfortunately, Orhun's approach is of limited use, for in that set-up the firm's optimal product-line design problem has no solution; the firm has an incentive to introduce additional products of arbitrarily large prices to create a utility pump that makes the consumers increasingly happy with (and willing to pay more for) their product choices. Moreover, Orhun studies product lines with only two products, presumably because of this existence problem. By contrast, as I noted earlier, in my model an optimal product line exists, and it involves either one product, or more than two products.

Alternative models in the related literature include Dhar and Glazer (1996), who propose a perceptual-based explanation of context effects. In their model, the structure of the entire choice set that a consumer faces influences her judgments on the similarity between any two given options. Wernerfelt (1995) and Kamenica (2008) focus on asymmetric information as a potential source of compromise effect. In their set-up, the set of available options informs the consumer about which product may be most suitable for her. While these models try to "rationalize" the compromise effect, I take this phenomenon as a boundedly rational mode of behavior driven by the difficulties associated with aggregating multiple attributes into a single ranking of alternatives, as suggested by Simonson (1989), Tversky and Shafir (1992), and Shafir et al. (1993), among others.

The remainder of the paper is structured as follows. In Section 2, I introduce my model of monopolistic product-line design problem and analyze it. In Section 3, I present a competitive analog of the model and describe the properties of equilibrium. Section 4 concludes. All proofs are relegated to Appendix.

# 2 Monopoly

There is only one firm in the market. The maximum quality that the firm can produce is $\bar{q} > 0$. In principle, the firm can charge any nonnegative price for a given quality. Thus, the set of possible products that this firm can produce are

$$X := \{x = (p, q) : p \geqslant 0, 0 \leqslant q \leqslant \bar{q}\},$$

where $p$ is the price and $q$ is the quality of a product.

The firm can produce one product of some quality $q$ at constant unit variable cost. I assume that the unit variable cost function $c(q)$ is continuous, twice differentiable, and strictly increasing. Moreover, I assume $c'(0) = 0$, $c''(q) > 0$, $\forall q$.

There are $n$ types of consumers who have reference dependent utility functions, $\nu_i$ is the measure of consumers of type $i$, $\sum\limits_{i=1}^{n} \nu_i = 1$.

The utility of a type $i$ consumer is 0 if she does not buy anything, and it is

$$u_i(p, q; r_p, r_q) := \theta_i q - p - \alpha|p - r_p| - \beta|q - r_q|,$$

where $0 < \beta < \theta_1 < \theta_2 < \ldots < \theta_n$ and $0 < \alpha < 1$, if she chooses $x = (p, q)$ from $A$.

The reference point for the finite menu $A$ is

$$r(A) \equiv (r_p, r_q) = \frac{1}{2}\left(\max_{(p,q)\in A} p + \min_{(p,q)\in A} p,\ \max_{(p,q)\in A} q + \min_{(p,q)\in A} q\right).$$

This reference point represents a middle, or compromise, point for the consumer in question, who exhibits the compromise effect.

Assume that whenever a consumer is indifferent among some options, she chooses the best for the firm.

The profit from a consumer of type $i$ is given by

$$\pi_i(A) := \begin{cases} \max\{p - c(q) \mid (p,q) \in C_i(A)\}, & \text{if } \max_{x\in A} u_i(x, r(A)) > 0, \\ 0, & \text{if } \max_{x\in A} u_i(x, r(A)) < 0, \\ \max\{0, \max\{p - c(q) \mid (p,q) \in C_i(A)\}\}, & \text{if } \max_{x\in A} u_i(x, r(A)) = 0, \end{cases}$$

where

$$C_i(A) := \arg\max_{x\in A} u_i(x, r(A)).$$

The total profit of the firm is equal to

$$\pi(A) := \nu_1\pi_1(A) + \ldots + \nu_n\pi_n(A).$$

The firm's problem is to offer a finite nonempty menu $A^* \in \mathcal{F} := \{A \subset X \mid 0 < |A| < \infty\}$ such that

$$A^* \in \arg\max_{A\in\mathcal{F}}\{\pi(A)\}.$$

**Theorem 2.1.** *The firm's problem has a solution.*

*Proof.* See A.1. $\square$

Note that since

$$\forall A \in \mathcal{F}\ \exists A' \subseteq A:\ |A'| \leqslant 4,\ r(A) = r(A'),\ \forall i,$$

for all $A^* \in \arg\max_{A\in\mathcal{F}}\{\pi(A)\}$ there always exists $A'$ such that $|A'| \leqslant n + 4$ and $\pi(A') = \pi(A^*)$.

**Special case of two types of consumers**.

First, consider the case when the firm wants to serve only one segment. Since $u_1(p, q; r_p, r_q) \leqslant u_2(p, q; r_p, r_q)$, if the firm serves only one segment, it is the high type consumers. So, the firm's problem is the following:

$$\begin{cases} \max\limits_{A} & \{\nu_2(\tilde{p}-c(\tilde{q}))\} \\ \text{s.t. } (IR_1) & \theta_1 q - p - \alpha|p-r_p| - \beta|q-r_q| \leqslant 0,\ \forall (p,q)\in A \\ (IR_2) & \theta_2\tilde{q} - \tilde{p} - \alpha|\tilde{p}-r_p| - \beta|\tilde{q}-r_q| \geqslant 0, \\ (IC_2) & \theta_2\tilde{q} - \tilde{p} - \alpha|\tilde{p}-r_p| - \beta|\tilde{q}-r_q| \geqslant \theta_2 q - p - \alpha|p-r_p| - \beta|q-r_q|,\ \forall (p,q)\in A \\ & r_p = \frac{1}{2}\left(\max\limits_{(p,q)\in A} p + \min\limits_{(p,q)\in A} p\right), \quad r_q = \frac{1}{2}\left(\max\limits_{(p,q)\in A} q + \min\limits_{(p,q)\in A} q\right). \end{cases}$$

**Claim 2.1.** *The solution to this problem is $A = \{\tilde{p}, \tilde{q}\}$, where $\tilde{p} = \theta_2\tilde{q}$ and $c'(\tilde{q}) = \theta_2$. The profit is equal to $\nu_2(\theta_2\tilde{q} - c(\tilde{q}))$.*

*Proof.* See A.2. □

Now, consider the case when monopoly serves both segments. Assume that the firm wants to sell $(p_1, q_1)$ to the first type consumer and $(p_2, q_2)$ to the second type consumer. Then the firm's problem is the following:

$$\begin{cases} \max\limits_{A} & \{\nu_1(p_1-c(q_1)) + \nu_2(p_2-c(q_2))\} \\ \text{s.t. } (IR_1) & \theta_1 q_1 - p_1 - \alpha|p_1-r_p| - \beta|q_1-r_q| \geqslant 0, \\ (IR_2) & \theta_2 q_2 - p_2 - \alpha|p_2-r_p| - \beta|q_2-r_q| \geqslant 0, \\ (IC_{12}) & \theta_1 q_1 - p_1 - \alpha|p_1-r_p| - \beta|q_1-r_q| \geqslant \theta_1 q_2 - p_2 - \alpha|p_2-r_p| - \beta|q_2-r_q| \\ (IC_{21}) & \theta_2 q_2 - p_2 - \alpha|p_2-r_p| - \beta|q_2-r_q| \geqslant \theta_2 q_1 - p_1 - \alpha|p_1-r_p| - \beta|q_1-r_q| \\ (IC_1) & \theta_1 q_1 - p_1 - \alpha|p_1-r_p| - \beta|q_1-r_q| \geqslant \theta_1 q - p - \alpha|p-r_p| - \beta|q-r_q|, \\ & \qquad \forall (p,q)\in A\setminus\{(p_1,q_1),(p_2,q_2)\} \\ (IC_2) & \theta_2 q_2 - p_2 - \alpha|p_2-r_p| - \beta|q_2-r_q| \geqslant \theta_1 q - p - \alpha|p-r_p| - \beta|q-r_q|, \\ & \qquad \forall (p,q)\in A\setminus\{(p_1,q_1),(p_2,q_2)\} \\ & r_p = \frac{1}{2}\left(\max\limits_{(p,q)\in A} p + \min\limits_{(p,q)\in A} p\right), \quad r_q = \frac{1}{2}\left(\max\limits_{(p,q)\in A} q + \min\limits_{(p,q)\in A} q\right) \end{cases}$$

**Claim 2.2.** *For any menu $A$ there exists $A^* = \{(p_1, q_1), (p_2, q_2), (p_h, q_h), (p_l, q_l)\}$ such that it does not yield less profit and $p_l \leqslant p_1 \leqslant p_2 \leqslant p_h$, $q_l \leqslant q_1 \leqslant q_2 \leqslant q_h$. Moreover, $(IR_1)$ is binging and $(IR_2)$ follows from $(IC_{21})$ and $(IR_1)$.*

*Proof.* See A.3. □

Thus, the firm's problem can be rewritten:

$$\begin{cases} \max\limits_{p_1,p_2,p_l,p_h,\, q_1,q_2,q_l,q_h} & \{\nu_1(p_1-c(q_1))+\nu_2(p_2-c(q_2))\} & & \\ \text{s.t. } (IR_1) & \theta_1 q_1 - p_1 - \alpha|r_p - p_1| - \beta|q_1 - r_q| & = & 0, \\ (IC_{12}) & \theta_1 q_2 - p_2 - \alpha|p_2 - r_p| - \beta|q_2 - r_q| & \leqslant & 0, \\ (IC_{21}) & \theta_2 q_2 - p_2 - \alpha|p_2 - r_p| - \beta|q_2 - r_q| & \geqslant & (\theta_2-\theta_1)q_1, \\ (IC_1^h) & \theta_1 q_h - p_h - \frac{\alpha}{2}(p_h - p_l) - \frac{\beta}{2}(q_h - q_l) & \leqslant & 0, \\ (IC_1^l) & \theta_1 q_l - p_l - \frac{\alpha}{2}(p_h - p_l) - \frac{\beta}{2}(q_h - q_l) & \leqslant & 0, \\ (IC_2^h) & \theta_2 q_2 - p_2 - \alpha|p_2 - r_p| - \beta|q_2 - r_q| & \geqslant & \theta_2 q_h - p_h - \frac{\alpha}{2}(p_h - p_l) - \frac{\beta}{2}(q_h - q_l), \\ (IC_2^l) & \theta_2 q_2 - p_2 - \alpha|p_2 - r_p| - \beta|q_2 - r_q| & \geqslant & \theta_2 q_l - p_l - \frac{\alpha}{2}(p_h - p_l) - \frac{\beta}{2}(q_h - q_l), \\ & r_p = \frac{1}{2}\left(p_h + p_l\right), & & r_q = \frac{1}{2}\left(q_h + q_l\right) \\ & 0 \leqslant p_l \leqslant p_1 \leqslant p_2 \leqslant p_h, & & 0 \leqslant q_l \leqslant q_1 \leqslant q_2 \leqslant q_h \leqslant \bar{q} \end{cases}$$

Due to the complexity of this problem, I had to use `MatLab` to solve it. I got analytical solution but it is too complicated to analyze. Thus, I present numerical solution for certain values of parameters and describe the algorithm that I used to get analytical expressions for the optimal product line[1].

1. Analytical Solution.

   First of all, note that the following properties can be easily proved:

   **Claim 2.3.** *(a)* $p_1 \leqslant r_p$*;* $q_2 \geqslant r_q$*;* $q_1 > 0$*;*

   *(b)* $(IC_2^h)$, $(IC_{12}) \implies (IC_1^h)$*;* $(IC_1^l)$, $(IC_{21}) \implies (IC_2^l)$*.*

   *(c) Both* $(IC_{12})$ *and* $(IC_{21})$ *cannot be binding simultaneously unless* $q_1 = q_2$*.*

   *(d) If* $q_h < \bar{q}$ *and either* $p_1 = p_2$ *or* $q_1 = q_2$*, then the solution is* $p_1 = p_2 = p_h = p_l = \theta_1 q_1$*,* $q_1 = q_2 = q_h = q_l$*,* $c'(q_1) = \theta_1$*.*

   *(e) If* $p_2 = p_h$*, then* $q_2 = q_h$*. If* $q_1 = q_l$*, then* $p_1 = p_l$*.*

   *(f) If a menu* $A = \{(p_1,q_1),(p_2,q_2),(p_h,q_h),(p_l,q_l)\}$ *is such that* $p_l > 0$*,* $q_l > 0$*, and* $q_h < \bar{q}$*, then there exists the menu* $A' = \{(p_1,q_1),(p_2,q_2),({p'}_h,{q'}_h),({p'}_l,{q'}_l)\}$ *that yields the same profit and at least one of the following holds:*

   *i.* ${p'}_l = 0$*,*

   *ii.* ${q'}_l = 0$*,*

[1] Full solution is not presented here to conserve space and is available upon request.

*iii.* $q'_h = \bar{q}$.

(g) *If* $q_l = 0$*, then* $(IC_1^l)$ *is not binding.*

(h) *At least one of the constraints* $(IC_{21})$ *and* $(IC_2^h)$ *is binding.*

(i) *If* $(IC_1^l)$ *is binding, then either* $p_1 = p_l$ *and* $q_1 = q_l$*, or* $p_1 > p_l$ *and* $q_1 > q_l$*. If* $(IC_2^h)$ *is binding, then either* $p_2 = p_h$ *and* $q_2 = q_h$*, or* $p_2 < p_h$ *and* $q_2 < q_h$*.*

*Proof.* See A.4. □

It is worth noting that the optimal bundle is not unique, mainly because there are different ways of selecting the "additional" goods, $(p_h, q_h)$ and $(p_l, q_l)$, without changing the implied reference point. Thus, the solution I found is only one out of many possible equilibria.

My method is rather straightforward and purely technical. To begin with, note that there are four cases that require consideration:

(a) $p_2 \geqslant r_p$, $r_q \geqslant q_1$;

(b) $r_p \geqslant p_2$, $r_q \geqslant q_1$;

(c) $p_2 \geqslant r_p$, $q_1 \geqslant r_q$;

(d) $r_p \geqslant p_2$, $q_1 \geqslant r_q$.

In each case, the problem is to maximize $\nu_1(p_1 - q_1^2) + \nu_2(p_2 - q_2^2)$ subject to 18 *linear* constraints: $(IR_1)$, $(IC_{12})$, $(IC_{21})$, $(IC_2^h)$, $(IC_1^l)$, $p_l \geqslant 0$, $p_1 \geqslant p_l$, $p_2 \geqslant p_1$, $p_h \geqslant p_2$, $q_l \geqslant 0$, $q_1 \geqslant q_l$, $q_2 \geqslant q_1$, $q_h \geqslant q_2$, $q_h \leqslant \bar{q}$, $p_1 \leqslant r_p$, $q_2 \geqslant r_q$, and two inequalities that characterize the case under consideration.

I found the solution using `MatLab`, under two additional assumptions.

**Assumption 2.1.** $c(q) = q^2$.

**Assumption 2.2.** $\bar{q}$ *is sufficiently large, i.e. the constraint* $q_h \leqslant \bar{q}$ *does not bind.*

My method of solution consists of two stages, which I describe below.

(a) For all four cases, I consider all possible[2] sets of binding constraints. Each set generates a system of linear equations. The solution of this system is plugged to the objective function. If the objective function is linear in some variables, at any given point it can be increased locally without violating the binding or non-binding constraints. This, in turn, implies that the solution of the firm's problem cannot lie in the region characterized by the set of binding constraints in question. If this is not the case, I solve the system generated by the first order conditions[3]. This gives me expressions for $p_1, p_2, p_l, p_h,\ q_1, q_2, q_l, q_h$. Plugging these expressions into the initial constraints, I get the constraints for parameters under which such bundle is possible. At the end of this stage I have a set of cases, and each case can be described by the following three structures:

Profit: the expression for the profit of the firm;

Variables: 8 expressions for $p_1, p_2, p_l, p_h,\ q_1, q_2, q_l, q_h$;

Constraints: a set of constraints under which the firm can choose this bundle.

This gives rise to 312 cases.

(b) At the second stage, I use Monte-Carlo method to determine which cases can appear in equilibrium. On each iteration, I randomly choose values for parameters $\alpha$, $\beta$, $\theta_1$, $\theta_2$, and $\nu_1$ and calculate which case corresponds to equilibrium for such values.

At the end of this stage I got 36 cases, which characterize the solution under all possible values of parameters.

I got that the solution is one of the following types:

(a) Only one product is offered, i.e. no discrimination:

$$p_1 = p_2 = p_h = p_l = \frac{1}{2}\theta_1^2,$$
$$q_1 = q_2 = q_h = q_l = \frac{1}{2}\theta_1.$$

[2]Claim 2.3 helps me to eliminate a sufficiently large number of cases. Moreover, one can easily prove that the number of independent binding constraints must be more than 5 (otherwise the objective function can be done arbitrarily large) and less than 9 (since the number of variables is 8).

[3]It can be easily proved that the objective function is strictly concave and thus the first order conditions imply unique solution.

Profit: $\pi = \frac{1}{4}\theta_1^2$.

(b) Three or four products are offered.

Thus, the following proposition is proved:

**Proposition 2.1.** *It is never optimal to offer two products. In other words, in equilibrium, if more than one product are presented, there is at least one product that is unsold.*

2. Numerical Solution.

As a cost function I use $c(q) = q^2$. I also assume that the constraint $q_h \leqslant \bar{q}$ is not binding. Thus, I use $\bar{q} = 1000$. All pictures are in the appendix (A.5).

(a) $\nu_1 = \nu_2 = 0.5$, $\theta_1 = 1$, $\theta_2 = 2$. If $\alpha$ and $\beta$ are rather small, the firm finds optimal to introduce additional third product of the lowest quality and the lowest price to make $(p_1, q_1)$ the middle option. As $\alpha$ and $\beta$ increase, the absolute spread in quality and price gets lower. With $\alpha$ and $\beta$ big enough, the firm does not discriminate at all and offers only one product $(p, q)$ such that $p = \theta_1 q$, $c'(q) = \theta_1$.

(b) $\nu_1 = 0.2$, $\nu_2 = 0.8$, $\theta_1 = 1$, $\theta_2 = 2$. For small $\alpha$ and big $\beta$, it is optimal to introduce the third product of the highest quality and the highest price. For small $\beta$ and large $\alpha$, the firm will offer the additional product of the lowest quality and the lowest price. For large $\alpha$ and $\beta$, the firm again offers only one product $(p, q)$ such that $p = \theta_1 q$, $c'(q) = \theta_1$.

(c) $\nu_1 = \nu_2 = 0.5$, $\theta_1 = 1$, $\theta_2 = 1.1$. The firm will choose to offer only one product $(p, q)$ $(p = \theta_1 q,\ c'(q) = \theta_1)$ unless $\alpha$ and $\beta$ are very small (in which case it prefers to introduce the third product of the lowest quality and the lowest price).

(d) $\nu_1 = \nu_2 = 0.5$, $\theta_1 = 1$, $\theta_2 = 20$. Compare to the first case, the firm always finds profitable to offer three products.

(e) $\nu_1 = 0.8$, $\nu_2 = 0.2$, $\theta_1 = 1$, $\theta_2 = 2$. Compare to the first case, the firm will offer $(p_1, q_1)$ of higher quality (closer to the optimal for the low type) and higher price.

To sum up,

(a) As $\alpha$ and $\beta$ increase, the absolute spread in quality and price gets lower. Indeed, suppose we are in equilibrium for fixed $\alpha$ and $\beta$. As $\alpha$ and $\beta$ increase, the constraints $(IR_1)$ and $(IC_{21})$ are more likely to fail for given prices and qualities. Thus, as $\alpha$ and $\beta$ increase, the firm should compensate the decreases in consumers' utilities by moving $p_1$ and $p_2$ closer to $r_p$, as well as moving $q_1$ and $q_2$ closer to $r_q$. Intuitively, the more sensitive consumers are to the compromise effect, the more profitable is to offer more similar products. In other words, if the compromise effect is rather high, it is better for the firm to mitigate this effect by offering similar products. Indeed, if the firm wants to separate consumers, at least one type will exhibit loss due to choosing not the middle option. By offering similar products, the firm can alleviate this loss.

(b) If the firm introduces more than one product, it prefers to offer an additional product to make one of the other two products the middle option. This additional product is of the lowest quality and the lowest price unless $\nu_2$ and $\beta$ is big enough, in which case the third product will be of the highest quality and the highest price. This follows from the fact that the firm will always extract all surplus from the low type consumers whereas the high type might getting a positive rent.

(c) As $\frac{\theta_2}{\theta_1}$ gets smaller, the firm has less incentive to offer several products. In other words, if two types are very similar, the firm's benefits from discrimination is smaller than the firm's loss due to the compromise effect.

(d) The higher $\nu_1$, the closer $q_1$ to the optimal for the low type.

# 3 Competition

In this section I consider the implications of my model for competitive market. To remind, the utility function of consumer $i \in [1, n]$ is

$$u_i(p, q; r_p, r_q) := \theta_i q - p - \alpha|p - r_p| - \beta|q - r_q|,$$

where $0 < \beta < \theta_1 < \theta_2 < \ldots < \theta_n$ and $0 < \alpha < 1$. The reference point for the finite menu $A$ is

$$r(A) \equiv (r_p, r_q) = \frac{1}{2}\left(\max_{(p,q)\in A} p + \min_{(p,q)\in A} p, \max_{(p,q)\in A} q + \min_{(p,q)\in A} q\right).$$

The market is competitive in the sense that there are more than one identical firms that can choose arbitrary number of products $(p, q)$. I assume that if more than one firm offer a product, than the profit from this product is divided equally.

**Claim 3.1.** *If a good $(p, q)$ is sold in equilibrium, then $p = c(q)$.*

*Proof.* See B.1. $\square$

In particular, this means that if a consumer chooses $(p = c(q), q)$ from menu $A$, then she strictly prefers it to any $(p \neq c(q), q)$ in that menu.

**Corollary 3.1.** *The profit of each firm in the market is zero.*

**Proposition 3.1.** *For given reference point $(r_p, r_q)$, consumer of type $i$ either buys nothing, or she buys the good $(p_i, q_i)$ such that*

$$p_i = c(q_i^*),\ q_i = q_i^* = \arg \max_{0 \leqslant \tilde{q} \leqslant \bar{q}} \{u_i(c(\tilde{q}), \tilde{q}; r_p, r_q)\},$$

*where $q_i^*$ is defined as follows*[4]*:*

$$q_i^* = \begin{cases} \bar{q}, & \bar{q} \leqslant (c')^{-1}\left(\frac{\theta_i - \beta}{1+\alpha}\right) \text{ or } \bar{q} \leqslant \min\left\{c^{-1}(r_p), (c')^{-1}\left(\frac{\theta_i-\beta}{1-\alpha}\right)\right\}, \\ (c')^{-1}\left(\frac{\theta_i-\beta}{1+\alpha}\right), & \max\{c^{-1}(r_p), r_q\} \leqslant (c')^{-1}\left(\frac{\theta_i-\beta}{1+\alpha}\right) \leqslant \bar{q}, \\ r_q, & \max\left\{(c')^{-1}\left(\frac{\theta_i-\beta}{1+\alpha}\right), c^{-1}(r_p)\right\} \leqslant r_q \leqslant (c')^{-1}\left(\frac{\theta_i+\beta}{1+\alpha}\right) \\ & \quad \text{or } (c')^{-1}\left(\frac{\theta_i-\beta}{1-\alpha}\right) \leqslant r_q \leqslant \min\left\{c^{-1}(r_p), (c')^{-1}\left(\frac{\theta_i+\beta}{1-\alpha}\right)\right\}, \\ (c')^{-1}\left(\frac{\theta_i+\beta}{1+\alpha}\right), & c^{-1}(r_p) \leqslant (c')^{-1}\left(\frac{\theta_i+\beta}{1+\alpha}\right) \leqslant r_q, \\ c^{-1}(r_p), & (c')^{-1}\left(\frac{\theta_i+\beta}{1+\alpha}\right) \leqslant c^{-1}(r_p) \leqslant \min\left\{r_q, (c')^{-1}\left(\frac{\theta_i+\beta}{1-\alpha}\right)\right\} \\ & \quad \text{or } \max\left\{r_q, (c')^{-1}\left(\frac{\theta_i-\beta}{1+\alpha}\right)\right\} \leqslant c^{-1}(r_p) \leqslant \min\left\{(c')^{-1}\left(\frac{\theta_i-\beta}{1-\alpha}\right), \bar{q}\right\}, \\ (c')^{-1}\left(\frac{\theta_i-\beta}{1-\alpha}\right), & r_q \leqslant (c')^{-1}\left(\frac{\theta_i-\beta}{1-\alpha}\right) \leqslant \min\{c^{-1}(r_p), \bar{q}\}, \\ (c')^{-1}\left(\frac{\theta_i+\beta}{1-\alpha}\right), & (c')^{-1}\left(\frac{\theta_i+\beta}{1-\alpha}\right) \leqslant \min\{c^{-1}(r_p), r_q\}. \end{cases}$$

*Proof.* See B.2. $\square$

So, all surplus goes to consumers, as it usually occurs in competitive markets. Intuitively, consumers get all bargaining power and therefore get the maximum utility (if this maximum is nonnegative, otherwise a consumer buys nothing).

[4]Note that $u_i(c(q), q; r_p, r_q)$ is strictly increasing in $q \in [0, q_i^*]$ and strictly decreasing in $q \in [q_i^*, \bar{q}]$.

Thus, the only thing to determine is the reference point. Each firm tries to manipulate the reference point to get more profit. Intuitively, if firm can shift the reference point such that at least one consumer does not find his "first best" product $(c(q^*), q^*)$, then there is a room for the firm to get positive profit. For example, the easiest way to shift the reference point is to offer very expensive product. Therefore, firms compete by moving the reference point.

**Definition 3.1.** *An equilibrium is called* SYMMETRIC *if each firm offers all products presented in the market.*

**Definition 3.2.** *An equilibrium is called* STABLE *if a new firm cannot offer a profitable product line.*

**Observation 3.1.** *If menu $A$ occurs in equilibrium that is stable, then there exists a symmetric equilibrium with the same menu $A$.*

*Proof.* If $(p, q) \in A$, then either $p = c(q)$, or it is not sold. Therefore, adding this product to the product line of any firm does not change its profit. Suppose that after each firm adds all missing products to its product line, there appears a profitable deviation for any firm. To specify, let it be profitable to offer $A'$. Then a new firm can enter to the initial market and profitably offer $A'$. The contradiction. □

Note that an equilibrium does not need to be stable. However, if it is not stable, then for any firm F there exists a nonempty set of products $S \subset A$ such that (1) all products in $S$ are offered by only that firm, (2) all products in $S$ are not sold, (3) $p < c(q)$ for any $(p, q) \in S$, and (4) this set serves as a "commitment device" for firm F in the sense that if for any product in $S$ there existed a firm that also offered this product, then firm F could profitably deviate.

From this point I will consider only symmetric equilibria.

Suppose menu $A$ is offered by all firms and each consumer buys her first best product given the reference point $r(A)$. Then, roughly speaking, this is an equilibrium if and only if no firm can change reference point such that (1) there exists consumer $i$ whom first best product has changed to the one which is not presented in the menu $A$, (2) there is no product in menu $A$ which is not worse for consumer $i$ than her new first best, and (3) the new products that deviating firm offers to change the reference point do not give too much loss.

It is obvious that if consumer $i < n$ buys something, then consumer $i+1$ also buys something and gets more utility than consumer $i$. Moreover, if consumer $i < n$ buys $(c(q_i), q_i)$, then consumer $i+1$ buys $(c(q_{i+1}), q_{i+1})$, where $q_{i+1} \geqslant q_i$.

**Observation 3.2.** *In standard model without compromise effect, i.e. when $\alpha = 0$ and $\beta = 0$, all consumers are served and consumer i buys $(p_i^{FB} = c(q_i^{FB}), q_i^{FB})$, where $q_i^{FB} = (c')^{-1}(\theta_i)$. Now, when $\alpha > 0$ and/or $\beta > 0$, consumers that buy a product with price and quality both higher (lower) than the reference ones, chooses the quality lower (higher) than their $q^{FB}$.*

**Claim 3.2.** *For sufficiently large $p$, there always exists a symmetric equilibrium where only $(0,0)$ and $(p, \bar{q})$ are offered (and nothing is sold).*

*Proof.* Note that in this case $r_q = \frac{\bar{q}}{2}$ and $r_p = \frac{p}{2}$. Suppose that $p$ is such that

- $\bar{q} < c^{-1}(\frac{p}{2})$,
- $(2\theta_n + \beta)\bar{q} \leqslant \alpha p \Rightarrow$ if $(c(q), q) \in A$ is sold, then $q \geqslant c^{-1}(\frac{p}{2})$.

Therefore, no product $(c(q), q)$ can be sold and this is an equilibrium because no firm can profitably change the reference point. □

**Claim 3.3.** *Menu $A$ constitutes a symmetric equilibrium with all products that offered are sold if and only if there exist such $i_{min}$, $i_{mid}$, and $i_{max}$ that $1 \leqslant i_{min} < i_{mid} \leqslant i_{max} \leqslant n$ and the following conditions hold:*

1. $(\theta_{i_{min}} + \frac{\beta}{2})(c')^{-1}(\frac{\theta_{i_{min}}+\beta}{1-\alpha}) \geqslant (1-\frac{\alpha}{2})c((c')^{-1}(\frac{\theta_{i_{min}}+\beta}{1-\alpha})) + \frac{\alpha c(\bar{q})}{2} + \frac{\beta\bar{q}}{2}$;
2. *if* $i_{min} > 1$, *then* $(\theta_{i_{min}-1} + \beta)(c')^{-1}(\frac{\theta_{i_{min}-1}+\beta}{1-\alpha}) - (1-\alpha)c((c')^{-1}(\frac{\theta_{i_{min}-1}+\beta}{1-\alpha})) \leqslant \frac{\alpha c(\bar{q})}{2} + \frac{\beta\bar{q}}{2}$;
3. $(c')^{-1}(\frac{\theta_{i_{mid}-1}+\beta}{1-\alpha}) \leqslant \min\left\{c^{-1}\left(\frac{c(\bar{q})}{2}\right), \frac{\bar{q}}{2}\right\}$;
4. *if* $i_{mid} < i_{max}$, *then* $\frac{1}{2}(c')^{-1}(\frac{\theta_{i_{min}}+\beta}{1-\alpha}) + \frac{\bar{q}}{2} \leqslant (c')^{-1}(\frac{\theta_{i_{mid}}-\beta}{1-\alpha})$ *and* $(c')^{-1}(\frac{\theta_{i_{max}-1}-\beta}{1-\alpha}) \leqslant c^{-1}\left(\frac{c(\bar{q})}{2}\right)$;
5. $(c')^{-1}(\frac{\theta_{i_{max}}-\beta}{1+\alpha}) \geqslant \bar{q}$.

*Moreover, in this equilibrium*

1. *consumer of type $i \in [1, i_{min}-1]$ buys nothing,*

2. *consumer of type* $i \in [i_{min}, i_{mid}-1]$ *buys* $(c((c')^{-1}(\frac{\theta_i+\beta}{1-\alpha})), (c')^{-1}(\frac{\theta_i+\beta}{1-\alpha}))$,

3. *consumer of type* $i \in [i_{mid}, i_{max}-1]$ *buys* $(c((c')^{-1}(\frac{\theta_i-\beta}{1-\alpha})), (c')^{-1}(\frac{\theta_i-\beta}{1-\alpha}))$, *and*

4. *consumer of type* $i \in [i_{max}, n]$ *buys* $(c(\bar{q}), \bar{q})$.

*Proof.* See B.3. □

Therefore,

**Proposition 3.2.** *If $\bar{q}$ is large enough, then in any symmetric equilibrium there exists at least one product that is offered but not sold.*

This conclusion raises the question of the existence of equilibria with at least one consumer buys something, for large $\bar{q}$. That is, assuming $\bar{q}$ large enough, does there exist an equilibrium where the highest type buys something?

**Proposition 3.3.** *If*

$$c\left((c')^{-1}\left(\frac{\theta_n+\beta}{1-\alpha}\right)\right) \leqslant \theta_n (c')^{-1}\left(\frac{\theta_n+\beta}{1-\alpha}\right) \leqslant \frac{\theta_n \bar{q}}{2},$$

*then there exists a symmetric equilibrium where exactly two products (one is of the lowest price and the lowest quality, the other is of the highest price and the highest quality) are unsold and at least one product is sold.*

*Proof.* Obviously, if the product chosen by the highest type has the maximum quality and the maximum price, it will be above the reference point, and thus any firm can profitably deviate by increasing both components of the reference point and offering this reference point as product for the highest type. Therefore, if $\bar{q}$ is large, there should be at least one product with either maximum price or maximum quality (or both), that is not sold.

Consider a menu where all products that are sold have price and quality below the reference values. Therefore, there is no sense for any firm to increase the reference price or/and the reference quality. To exclude profitable deviations when a firm lowers the reference point, add the product of the lowest possible price and the lowest possible quality to the menu[5].

For formal proof see B.4. □

[5]Product $(0,0)$ will never be sold since it brings negative utility to all types of consumers unless $r_p = 0$ and $r_q = 0$.

**Comparative Statics**

Let menu $A$ be a set of products that offered in a certain symmetric equilibrium. Suppose that at $t = 0$ the market is in this equilibrium. I'm interested in dynamic evolution of the set of offered products as parameters of the model start to change. To specify, the question under focus is how the outcome will change at $t = 1$ if parameters $\alpha$ and $\beta$ change a bit.

Assume that the timing is as follows:

1. change of parameters from $(\alpha, \beta)$ to $(\alpha', \beta')$;

2. convergence to an equilibrium with restriction that each time a firm decides to change its product line, it chooses a deviation that brings the maximum possible profit and, given this profit, minimizes the Hausdorff distance between the old and the new product lines[6] (if there is more than one such deviation, the firm chooses any).

**Definition 3.3.** *Given fixed* $(\alpha, \beta)$, THE TYPE OF EQUILIBRIUM DOES NOT CHANGE AS PARAMETER $a$ INCREASES (DECREASES), *if for all* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that the change from* $a$ *to* $a'$ *for any* $a' \in (a, a + \delta)$ $(a' \in (a - \delta, a))$ *leads to the change of equilibrium from* $A$ *to* $A'$*, where the Hausdorff distance between the old and the new menus is less than* $\varepsilon$.

It is easy to see, that the type of equilibrium does not change as parameter $a$ increases (decreases), if and only if for all $\varepsilon > 0$ there exist an menu $A'$ and $a' > a$ $(a' < a)$ such that

1. $A'$ is an equilibrium menu for new $a'$,

2. the Hausdorff distance between $A$ and $A'$ is less than $\varepsilon$.

Define

- $N_h = \{(p, q) \in A \mid (p, q) \text{ is sold and } p > r_p,\ q > r_q\}$,
- $N_l = \{(p, q) \in A \mid (p, q) \text{ is sold and } p < r_p,\ q < r_q\}$,
- $N_m = \{(p, q) \in A \mid (p, q) \text{ is sold}\} \setminus (N_l \cup N_h)$.

[6]This requirement allows to consider "continuous" dynamics, that is when firms switch to another type of equilibria only when the old one does not exist for new values of parameters, and can be interpreted as the managers' tendency to follow the most "conservative" strategy.

Note that

- $(p, q) \in N_h \Rightarrow p = c(q)$ and either $q = \bar{q}$, or $q = (c')^{-1}\left(\frac{\theta_i - \beta}{1+\alpha}\right)$ for some $i$;
- $(p, q) \in N_l \Rightarrow p = c(q)$ and $q = (c')^{-1}\left(\frac{\theta_i + \beta}{1-\alpha}\right)$ for some $i$.

Therefore,

1. if the type of equilibrium does not change as parameter $\alpha$ ($\beta$) increases, then
   (a) each product in $N_h$ either shifts to the lower quality and price, or does not change,
   (b) each product in $N_l$ either shifts to the higher quality and price, or does not change;
2. if the type of equilibrium does not change as parameter $\alpha$ ($\beta$) decreases, then
   (a) each product in $N_h$ either shifts to the higher quality and price, or does not change,
   (b) each product in $N_l$ either shifts to the lower quality and price, or does not change.

This result can be summarized in the following way: as compromise effect gets higher, the set of products that are sold gets less discriminative.

# 4 Conclusion

I presented a model of monopolistic and competitive markets that incorporates the compromise effect. This model is based on the standard model with vertically differentiated consumers and private valuations. The only difference is that now a utility function consists of two components, one is standard and the other reflects the tendency to choose the middle option. The existence of the equilibrium in this model was proved. Moreover, it was shown that with the increase of the degree of the compromise effect (i.e. when the weight of the second component of the utility function gets bigger), an equilibrium product line is getting less discriminative. Finally, in the monopolistic model, I found that if there are two types of consumers, at least one product is unsold unless the optimal product line consists of a unique product.

In turn, for competitive markets, I showed that under quite general assumption, at least one unsold product is present in any equilibrium.

I proposed only one model that incorporates the compromise effect to optimal product line problem. Theoretically, there exist many other ways how to model this effect. For example, one might try to use asymmetric information approach (like in Kamenica (2008)). I would also suggest to try the following iterated procedure for the consumer decision-making process. At first stage, the consumer evaluates the utility of each option in the alternative set (it may be a reference-dependent utility function or whatever which is consistent with the compromise effect). Then she removes the worst option from the consideration and remembers the order of the rest options. At each next stage, the consumer calculates the utility from each of the rest products using the same method as at the first stage but also takes into account the previous results (i.e. her memory) with a certain weight. The consumer continues until she faces only one product. I think this procedure has the following advantage. If we forget about the "memory part", at the last stage the consumer behaves rationally as she has to compare two products. So, her final decision is supposed to be "nearly" rational provided that the memory effects are low enough. This feature may help to prove the existence of equilibrium. In sum, this is an open topic for future research.

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# A Monopoly

## A.1 Proof for Theorem 2.1

The main idea of the existence result is to apply the general version of Weierstrass theorem.

**Theorem A.1.** *An upper semicontinuous function on a compact set attains a maximum value.*

Note that the model has the following four properties.

**Property A.1.** *There exists a natural number $k$ such that for every $A \in \mathcal{F}$, $r(A) = r(A')$ for a set $A' \subseteq A$ with $|A'| \leqslant k$.*

This property tells that any reference point depends on at most $k$ of the existing products.

**Property A.2.** *For each $i$, there exists a sufficiently high price level $\bar{p}_i$ such that $u_i(x, r) < 0$ for every $(x, r) \in X \times X$ with $r_p > \bar{p}_i$.*

To understand the content of property A.2, first note that the maximum quality is bounded from above. Consider $x = (p, q) \in X$ and take any $r$ with a "very high" price component $r_p$. If $p$ is also high, the consumption utility of the consumer will be low because the best quality is finite (albeit the compromise part of the utility may be relatively large). In turn, if $p$ is reasonably low, then $x$ will be distant from the reference point, and hence, the compromise part of the utility will be low. The property requires that for sufficiently large values of $r_p$ these negative effects on the utility become dominant.

**Property A.3.** *For any sequence $\{A_n\}_{n=1}^{\infty}$ in $\mathcal{F}$, if $\max\limits_{(p,q) \in A_n} p \to \infty$, then $r_p(A_n) \to \infty$.*

Property A.3 says that if the firm offers very high prices, then the price component of the reference point would also be high. Combined with property A.2, this implies that the firm has no incentive to offer products with arbitrarily large prices.

**Property A.4.** *The function $r(\cdot)$ is continuous on $\mathcal{F}$. That is, if $A_n \to A$ in the Hausdorff metric, then $r(A_n)$ converges to $r(A)$ in Euclidean norm.*

This property is used in the proof of upper semicontinuity of the profit function. Note that this property typically rules out reference points which only depend on the Pareto frontier of $A$.

1. Without loss of generality, I can focus on a compact subset of $\mathcal{F}$ which has the following form:

$$\{A \in \mathcal{F}: \ |A| \leqslant n+k, \ (p,q) \in A \ \Rightarrow \ p \leqslant b\}.$$

   (a) From property A.1, there exists a natural number $k$ such that for every $A \in \mathcal{F}$, $r(A) = r(A')$ for a set $A' \subseteq A$ with $|A'| \leqslant k$. Combining with the fact that for every $A \in \mathcal{F}$ consumers buy no more than $n$ products, I get that for all $A \in \mathcal{F}$ there exists $A' \subseteq A$ such that $\pi(A') = \pi(A)$ and $|A'| \leqslant n+k$. Hence, it is sufficient to consider only $\{A \in \mathcal{F}: \ |A| \leqslant n+k\}$.

   (b) Let there does not exist such $b$ that for all $\bar{A} \in \mathcal{F}$ such that $|\bar{A}| \leqslant n+k$, the following is true:

$$\pi(\bar{A}) \leqslant \max\{\pi(A) \mid A \in \mathcal{F}: \ |A| \leqslant n+k, \ (p,q) \in A \ \Rightarrow \ p \leqslant b\} \equiv \tilde{\pi}(b).$$

   Then, for all $b_n$ there exists $A_n \in \mathcal{F}$ such that $|A_n| \leqslant n+k$ and $\pi(A_n) > \tilde{\pi}(b_n)$. Hence, there exist $p_n > b_n$ and $q_n$ such that $(p_n, q_n) \in A_n$. Assume $b_n \to \infty$ as $n \to \infty$. Then $\max\limits_{(p,q)\in A_n} p \leqslant p_n > b_n \to \infty$. Thus, by property A.3, $r_p(A_n) \to \infty$. According to property A.2, $\pi(A_n) \to 0$, which leads to the contradiction.

2. The profit function $\pi(\cdot)$ is upper semicontinuous.

   Obviously, it is sufficient to prove that $\pi_i(\cdot)$ is upper semicontinuous, that is

$$\forall A_n \to A \quad \limsup_{n\to\infty} \pi_i(A_n) \leqslant \pi_i(A)$$

   Suppose this is not true. Then there exist $A_n \to A$ such that $\limsup\limits_{n\to\infty} \pi_i(A_n) > \pi_i(A)$. Then there exists a sequence $(p_n, q_n) \in C_i(A_n)$ such that $(p_n, q_n) \to (\bar{p}, \bar{q}) \notin C_i(A)$. Note that $(\bar{p}, \bar{q}) \in A$ since $(p_n, q_n) \in A_n$ and $A_n \to A$. Hence, there exists a sequence $(p_n, q_n) \to (\bar{p}, \bar{q})$ such that $(p_n, q_n) \in A_n$, $(\bar{p}, \bar{q}) \in A$, $u_i(p_n, q_n; r(A_n)) \geqslant u_i(p, q; r(A_n))$ for all $(p,q) \in A_n$, and $u_i(\bar{p}, \bar{q}; r(A)) < u_i(p^*, q^*; r(A))$ for a certain $(p^*, q^*) \in A$. Since $A_n \to A$, there exists a sequence $(p_n^*, q_n^*) \in A_n$ that converges to $(p^*, q^*) \in A$. Thus, using the continuity of the utility function and property A.4, I get $u_i(p_n, q_n; r(A_n)) \to u_i(\bar{p}, \bar{q}; r(A))$ and $u_i(p_n^*, q_n^*; r(A_n)) \to u_i(p^*, q^*; r(A))$. That leads to the contradiction.

Using the general version of Weierstrass theorem, I get the statement.

## A.2 Proof for Claim 2.1

1. Show that $(IR_2)$ is binding. Suppose

$$\theta_2\tilde{q} - \tilde{p} - \alpha|\tilde{p} - r_p| - \beta|\tilde{q} - r_q| > 0$$

   in equilibrium. Then there exists $\varepsilon > 0$ such that if all prices increase by $\varepsilon$, the reference point $r_p$ also increases by $\varepsilon$, all constrains still hold, and the profit increases. That leads to the contradiction.

2. Now let's prove that the constraint $(IR_1)$ is not binding in equilibrium. Since $(IR_2)$ is binding, $(IC_2)$ can be rewritten as

$$0 \geqslant u_2(p, q; r_p, r_q),\ \forall(p, q) \in A$$

   Together with $u_1(p, q; r_p, r_q) \leqslant u_2(p, q; r_p, r_q)$, it means

$$0 \geqslant u_1(p, q; r_p, r_q),\ \forall(p, q) \in A.$$

   That is essentially $(IR_1)$.

3. The last step is obvious. The problem can be rewritten as

$$\begin{cases} \max\limits_{A} \quad \{\theta_2\tilde{q} - \alpha|\tilde{p} - r_p| - \beta|\tilde{q} - r_q| - c(\tilde{q})\} \\ \text{s.t.} \qquad \theta_2\tilde{q} - \tilde{p} - \alpha|\tilde{p} - r_p| - \beta|\tilde{q} - r_q| \geqslant \ \theta_2 q - p - \alpha|p - r_p| - \beta|q - r_q|,\ \forall(p, q) \in A \\ \qquad\qquad r_p = \frac{1}{2}\left(\max\limits_{(p,q)\in A} p + \min\limits_{(p,q)\in A} p\right), \qquad\qquad r_q = \frac{1}{2}\left(\max\limits_{(p,q)\in A} q + \min\limits_{(p,q)\in A} q\right), \end{cases}$$

   which means that without the constraint the firm will choose

$$r_p = \tilde{p},\ r_q = \tilde{q}.$$

   This can be achieved by choosing $A = \{\tilde{p}, \tilde{q}\}$.

## A.3 Proof for Claim 2.2

1. $(IC_{21})$, $(IR_1) \implies (IR_2)$.

2. $(IR_1)$ is binding.

   Suppose $\theta_1 q_1 - p_1 - \alpha|p_1 - r_p| - \beta|q_1 - r_q| > 0$ in equilibrium. Then there exists $\varepsilon > 0$ such that if all prices increase by $\varepsilon$, the reference point $r_p$ also increases by $\varepsilon$, all constrains still hold, and the profit increases. That leads to the contradiction.

3. If $p_2 = 0$, then $q_2 > 0$. $q_1 > 0$. This follows from $(IR_1)$, $(IR_2)$ and

$$q > 0 \quad \Leftrightarrow \quad \theta_i q - \beta|q - r_q| > -\beta r_q.$$

4. Both $(IC_{12})$ and $(IC_{21})$ cannot be binding simultaneously unless $q_1 = q_2$, because $(IC_{12})$ and $(IC_{21})$ together are equivalent to

$$\theta_1(q_2 - q_1) \leqslant p_2 + \alpha|p_2 - r_p| + \beta|q_2 - r_q| - p_1 - \alpha|p_1 - r_p| - \beta|q_1 - r_q| \leqslant \theta_2(q_2 - q_1).$$

5. $q_1 \leqslant q_2$, $p_1 \leqslant p_2$; $q_1 = q_2 \Longleftrightarrow p_1 = p_2$.

   From the previous expression, $\theta_1(q_2 - q_1) \leqslant \theta_2(q_2 - q_1)$. Thus, since $\theta_1 < \theta_2$, $q_1 \leqslant q_2$.

   Assume that $p_1 > p_2$. Then

$$\begin{aligned} q_1 \leqslant q_2 \quad &\Longrightarrow \quad \theta_i q_1 - \beta|q_1 - r_q| \leqslant \theta_i q_2 - \beta|q_2 - r_q|, \ \forall i, r_q \\ p_1 > p_2 \quad &\Longrightarrow \quad -p_1 - \alpha|p_1 - r_p| < -p_2 - \alpha|p_2 - r_p|, \ \forall r_p \end{aligned}$$

$$\Downarrow$$

$$\theta_1 q_1 - \beta|q_1 - r_q| - p_1 - \alpha|p_1 - r_p| < \theta_1 q_2 - \beta|q_2 - r_q| - p_2 - \alpha|p_2 - r_p|,$$

   which is the contradiction with $(IC_{12})$. Thus, $p_1 \leqslant p_2$.

   Assume $q_1 = q_2$. Then

$$\theta_i q_1 - \beta|q_1 - r_q| = \theta_i q_2 - \beta|q_2 - r_q|, \ \forall i, r_q$$

$$\Downarrow$$

$$-p_1 - \alpha|p_1 - r_p| = -p_2 - \alpha|p_2 - r_p|, \ \forall r_p$$

$$\Downarrow$$

$$p_1 = p_2.$$

Assume $p_1 = p_2$. Then

$$-p_1 - \alpha|p_1 - r_p| = -p_2 - \alpha|p_2 - r_p|,\ \forall r_p$$
$$\Downarrow$$
$$\theta_1(q_2 - q_1) \leqslant \beta|q_2 - r_q| - \beta|q_1 - r_q| \leqslant \theta_2(q_2 - q_1).$$

Thus,

$$|q_2 - r_q| - |q_1 - r_q| \leqslant q_2 - q_1$$
$$|q_2 - r_q| - |q_1 - r_q| \geqslant \frac{\theta_1}{\beta}(q_2 - q_1) \geqslant q_2 - q_1$$
$$\Downarrow$$
$$q_1 = q_2.$$

6. If $p_2 < \max\limits_{(p,q)\in A} p$, then either $(IC_2)$ is binding for at least one pair $(p,q) \in A \setminus \{(p_1, q_1), (p_2, q_2)\}$, or $(IC_{21})$ is binding, or both.

   Suppose $p_2 < \max\limits_{(p,q)\in A} p$ and $(IC_{21})$ is not binding. Then there exists $\varepsilon > 0$ such that if $p_2$ increases by $\varepsilon$, the reference point does not change (since $p_1 \leqslant p_2$), all constraints still hold, and the profit increases. That leads to the contradiction.

7. $p_1 \leqslant r_p$; if $\min\limits_{(p,q)\in A} q > 0$, then $q_2 \geqslant r_q$.

   Suppose $p_1 > r_p$. Then $p_2 \geqslant p_1 > \min\limits_{(p,q)\in A} p = p_{min}$ and there exists $\varepsilon > 0$ such that if all prices but those which are equal to $p_{min}$ increase by $\frac{\alpha\varepsilon}{2}$ and the rest increase by $\frac{(2+\alpha)\varepsilon}{2}$, all constraints still hold, and the profit increases. That leads to the contradiction.

   Suppose $q_2 < r_q$ and $\min\limits_{(p,q)\in A} q > 0$. Then $q_1 \leqslant q_2 < \max\limits_{(p,q)\in A} q = q_{max}$ and there exists $\varepsilon > 0$ such that if all qualities but those which are equal to $q_{max}$ decrease by $\frac{\beta\varepsilon}{2\theta_1}$ and the rest decrease by $\frac{(2\theta_1+\beta)\varepsilon}{2\theta_1}$, all constraints still hold, and the profit increases. That leads to the contradiction.

8. If $(IC_1)$ is binding for $(p,q)$, then either $q_1 > q$ and $p_1 > p$ or $q_1 < q$ and $p_1 < p$. The same is true for $(IC_2)$.

   This follows from

$$\begin{aligned} q_1 < (>) q \quad &\Longleftrightarrow \quad \theta_1 q_1 - \beta|q_1 - r_q| < (>) \theta_1 q - \beta|q - r_q| \\ p_1 < (>) p \quad &\Longleftrightarrow \quad -p_1 - \alpha|p_1 - r_p| > (<) - p - \alpha|p - r_p| \end{aligned}$$

9. If $(p,q) \in A$ is such that $p = \max\limits_{(p,q)\in A} p$ and $q < \max\limits_{(p,q)\in A} q$, then $(p,q) \in A \setminus \{(p_1,q_1),(p_2,q_2)\}$ and $(IC_1)$ and $(IC_2)$ are not binding for $(p,q)$, because

$$\forall (p',q') \in A: \; q' = \max_{(p,q)\in A} q \;\Rightarrow$$

$$q' > q,\; p' \leqslant p \;\Rightarrow\; \theta_i q - \beta|q - r_q| - p - \alpha|p - r_p| < \theta_i q' - \beta|q' - r_q| - p' - \alpha|p' - r_p|,\; i = 1,2$$

$$\Downarrow$$

$$(p,q) \in A \setminus \{(p_1,q_1),(p_2,q_2)\}$$

10. If $(p,q) \in A$ is such that $p = \max\limits_{(p,q)\in A} p$ and $\min\limits_{(p,q)\in A} q < q < \max\limits_{(p,q)\in A} q$, then the menu $A \setminus \{(p,q)\} \cup \{(\max\limits_{(p,q)\in A} p, \max\limits_{(p,q)\in A} q)\}$ gives to the firm the same profit as the menu $A$ (this follows from the previous point).

11. If $p_2 = \max\limits_{(p,q)\in A} p$, then $q_2 = \max\limits_{(p,q)\in A} q$ (this follow from 9).

12. If $(p,q) \in A$ is such that $p = \min\limits_{(p,q)\in A} p < p_1$, then the menu $A \setminus \{(p,q)\} \cup \{(\min\limits_{(p,q)\in A} p, \min\limits_{(p,q)\in A} q)\}$ gives to the firm the same profit as the menu $A$.

13. If $(p,q) \in A$ is such that $q = \min\limits_{(p,q)\in A} q$ and $p > \min\limits_{(p,q)\in A} p$, then $(p,q) \in A \setminus \{(p_1,q_1),(p_2,q_2)\}$ and $(IC_1)$ and $(IC_2)$ are not binding for $(p,q)$, because

$$\forall (p',q') \in A: \; p' = \min_{(p,q)\in A} p \;\Rightarrow$$

$$q' \geqslant q,\; p' < p \;\Rightarrow\; \theta_i q - \beta|q - r_q| - p - \alpha|p - r_p| < \theta_i q' - \beta|q' - r_q| - p' - \alpha|p' - r_p|,\; i = 1,2$$

$$\Downarrow$$

$$(p,q) \in A \setminus \{(p_1,q_1),(p_2,q_2)\}$$

14. If $(p,q) \in A$ is such that $q = \min\limits_{(p,q)\in A} q$ and $\max\limits_{(p,q)\in A} p > p > \min\limits_{(p,q)\in A} p$, then the menu $A \setminus \{(p,q)\} \cup \{(\min\limits_{(p,q)\in A} p, \min\limits_{(p,q)\in A} q)\}$ gives to the firm the same profit as the menu $A$ (this follows from the previous point).

15. If $q_1 = \min\limits_{(p,q)\in A} q$, then $p_1 = \min\limits_{(p,q)\in A} p$ (this follows from 13).

16. If $(p,q) \in A$ is such that $q = \max\limits_{(p,q)\in A} q > q_2$, then the menu $A \setminus \{(p,q)\} \cup \{(\max\limits_{(p,q)\in A} p, \max\limits_{(p,q)\in A} q)\}$ gives to the firm the same profit as the menu $A$.

17. If a menu $A$ is such that $p_1 = \min\limits_{(p,q)\in A} p$, $q_2 = \max\limits_{(p,q)\in A} q$, then the menu $\{(p_1,q_1),(p_2,q_2),(\max\limits_{(p,q)\in A} p, \min\limits_{(p,q)\in A} q)\}$ yields the same profit (this follows from the observation that the reference point is the same for both menus).

18. $q > (\geqslant) q_1 \Rightarrow p > (\geqslant) p_1$, $p < (\leqslant) p_1 \Rightarrow q < (\leqslant) q_1$. The same is true for $(p_2, q_2)$.

19. If a menu $A$ is such that $p_1 = \min\limits_{(p,q)\in A} p$, $\max\limits_{(p,q)\in A} q > q_2$, $\min\limits_{(p,q)\in A} q < q_1$, then there exists the menu $A'$ that yields the same profit and $|A'| < |A|$ and either $\max\limits_{(p,q)\in A'} q = q_2$, or $\min\limits_{(p,q)\in A'} q = q_1$, or both.
Indeed, if a menu $A$ is such that $p_1 = \min\limits_{(p,q)\in A} p$, $\tilde{q} = \max\limits_{(p,q)\in A} q > q_2$, $\underline{q} = \min\limits_{(p,q)\in A} q < q_1$, $\tilde{q} - q_2 > q_1 - \underline{q}$ ($\tilde{q} - q_2 \leqslant q_1 - \underline{q}$), then the menu $A' = A \setminus \{(p,q) \in A : q < q_1 \text{ or } q > \tilde{q} - q_1 + \underline{q}\} \cup \{(\max\limits_{(p,q)\in A} p, \tilde{q} - q_1 + \underline{q})\}$ ($A' = A \setminus \{(p,q) \in A : q < \underline{q} - q_2 + \tilde{q} \text{ or } q > q_2\} \cup \{(\max\limits_{(p,q)\in A} p, \underline{q} - q_2 + \tilde{q})\}$) yields the same profit. Note that $|A'| < |A|$ and either $\max\limits_{(p,q)\in A'} q = q_2$, or $\min\limits_{(p,q)\in A'} q = q_1$, or both.

20. If a menu $A$ is such that $q_2 = \max\limits_{(p,q)\in A} q$, $\max\limits_{(p,q)\in A} p > p_2$, $\min\limits_{(p,q)\in A} p < p_1$, then there exists the menu $A'$ that yields the same profit and $|A'| < |A|$ and either $\max\limits_{(p,q)\in A'} p = p_2$, or $\min\limits_{(p,q)\in A'} p = p_1$, or both.
That is because if a menu $A$ is such that $q_2 = \max\limits_{(p,q)\in A} q$, $\bar{p} = \max\limits_{(p,q)\in A} p > p_2$, $\underline{p} = \min\limits_{(p,q)\in A} p < p_1$, $\bar{p} - p_2 > p_1 - \underline{p}$ ($\bar{p} - p_2 \leqslant p_1 - \underline{p}$), then the menu $A' = A \setminus \{(p,q) \in A : p < p_1 \text{ or } p > \bar{p} - p_1 + \underline{p}\} \cup \{(\bar{p} - p_1 + \underline{p}, \min\limits_{(p,q)\in A} q)\}$ ($A' = A \setminus \{(p,q) \in A : p < \underline{p} - p_2 + \bar{p} \text{ or } p > p_2\} \cup \{(\underline{p} - p_2 + \bar{p}, \min\limits_{(p,q)\in A} q)\}$) yields the same profit. Note that $|A'| < |A|$ and either $\max\limits_{(p,q)\in A'} p = p_2$, or $\min\limits_{(p,q)\in A'} p = p_1$, or both.

21. From the previous points it follows that for any menu $A$ there exists $A^*$ such that it does not yield less profit and one of the following is satisfied:

    (a) $A^* = \{(p_1, q_1), (p_2, q_2)\}$

    (b) $A^* = \{(p_1, q_1), (p_2, q_2), (p_h, q_h)\}$, $p_h \geqslant p_2$, $q_h \geqslant q_2$

    (c) $A^* = \{(p_1, q_1), (p_2, q_2), (p_h, q_h)\}$, $p_h \geqslant p_2$, $q_h \leqslant q_1$

    (d) $A^* = \{(p_1, q_1), (p_2, q_2), (p_h, q_h)\}$, $p_h \leqslant p_1$, $q_h < q_1$

    (e) $A^* = \{(p_1, q_1), (p_2, q_2), (p_h, q_h), (p_l, q_l)\}$, $p_h \geqslant p_2$, $q_h \geqslant q_2$, $p_l \leqslant p_1$, $q_l \leqslant q_1$

## A.4 Proof for Claim 2.3

1. $p_1 \leqslant r_p$; $q_2 \geqslant r_q$; $q_1 > 0$;

Suppose $q_2 < r_q$ and $(IC_1^l)$ is not binding. Then $q_h > q_2$ and the firm can increase profit by decreasing $q_h$. Thus, if $q_l = 0$, then $q_2 \geqslant r_q$. On the other hand, from A.3, it follows that if $q_l > 0$, then $q_2 \geqslant r_q$. The rest follows from A.3.

2. $(IC_2^h)$, $(IC_{12}) \Longrightarrow (IC_1^h)$; $(IC_1^l)$, $(IC_{21}) \Longrightarrow (IC_2^l)$.

   (a)

$$\begin{gathered}
\theta_1 q_h - p_h - \frac{\alpha}{2}(p_h - p_l) - \frac{\beta}{2}(q_h - q_l) = \\
\theta_2 q_h - p_h - \frac{\alpha}{2}(p_h - p_l) - \frac{\beta}{2}(q_h - q_l) - (\theta_2 - \theta_1) q_h \leqslant \\
\theta_2 q_2 - p_2 - \alpha|p_2 - r_p| - \beta|q_2 - r_q| - (\theta_2 - \theta_1) q_h = \\
(\theta_2 - \theta_1) q_2 + \theta_1 q_2 - p_2 - \alpha|p_2 - r_p| - \beta|q_2 - r_q| - (\theta_2 - \theta_1) q_h \leqslant \\
-(\theta_2 - \theta_1)(q_h - q_2) \leqslant 0
\end{gathered}$$

   (b)

$$\begin{gathered}
\theta_2 q_l - p_l - \frac{\alpha}{2}(p_h - p_l) - \frac{\beta}{2}(q_h - q_l) \leqslant (\theta_2 - \theta_1) q_l \leqslant (\theta_2 - \theta_1) q_1 \leqslant \\
\theta_2 q_2 - p_2 - \alpha|p_2 - r_p| - \beta|q_2 - r_q|
\end{gathered}$$

3. Both $(IC_{12})$ and $(IC_{21})$ cannot be binding simultaneously unless $q_1 = q_2$.

   This follows from A.3.

4. If $q_h < \bar{q}$ and either $p_1 = p_2$ or $q_1 = q_2$, then the solution is $p_1 = p_2 = p_h = p_l = \theta_1 q_1$, $q_1 = q_2 = q_h = q_l$, $c'(q_1) = \theta_1$.

   If $q_1 = q_2 \equiv q$, $p_1 = p_2 \equiv p$, the constraints $(IC_{12})$ and $(IC_{21})$ both follow from $(IR_1)$. Thus, the problem can be rewritten:

$$\left\{\begin{array}{rrcl}
\max\limits_{p,p_h,p_l,q,q_h,q_l} & \{p - c(q)\} & & \\
\text{s.t. } (IR_1) & \theta_1 q - p - \alpha(r_p - p) - \beta(q - r_q) & = & 0, \\
(IC_2^h) & \theta_2 q - p - \alpha(r_p - p) - \beta(q - r_q) & \geqslant & \theta_2 q_h - p_h - \frac{\alpha}{2}(p_h - p_l) - \frac{\beta}{2}(q_h - q_l), \\
(IC_1^l) & \theta_1 q_l - p_l - \frac{\alpha}{2}(p_h - p_l) - \frac{\beta}{2}(q_h - q_l) & \leqslant & 0, \\
& r_p = \frac{1}{2}(p_h + p_l), & & r_q = \frac{1}{2}(q_h + q_l) \\
& 0 \leqslant p_l \leqslant p \leqslant p_h,\ r_p \geqslant p, & & 0 \leqslant q_l \leqslant q \leqslant q_h \leqslant \bar{q},\ r_q \leqslant q.
\end{array}\right. \tag{1}$$

Note that the maximization of $p - c(q)$ subject to only $(IR_1)$ constraint leads to $r_p = p$ and $r_q = q$. Since $p_h = p_l = p$ and $q_h = q_l = q$ are consistent with all necessary constraints (when $\bar{q}$ is sufficiently large).

5. If $p_2 = p_h$, then $q_2 = q_h$. If $q_1 = q_l$, then $p_1 = p_l$.

   This follows from A.3.

6. If a menu $A = \{(p_1, q_1), (p_2, q_2), (p_h, q_h), (p_l, q_l)\}$ is such that $p_l > 0$, $q_l > 0$, and $q_h < \bar{q}$, then there exists the menu $A' = \{(p_1, q_1), (p_2, q_2), ({p'}_h, {q'}_h), ({p'}_l, {q'}_l)\}$ that yields the same profit and at least one of the following holds:

   (a) ${p'}_l = 0$,

   (b) ${q'}_l = 0$,

   (c) ${q'}_h = \bar{q}$.

   This follows from the fact that there exist $\delta > 0$ and $\varepsilon > 0$ such that $\tilde{p}_h = p_h + \delta$, $\tilde{p}_l = p_l - \delta \geqslant 0$, $\tilde{q}_h = q_h + \varepsilon \leqslant \bar{q}$, $\tilde{q}_l = q_l - \varepsilon \geqslant 0$:

$$\begin{cases} \theta_i q_h - p_h - \frac{\alpha}{2}(p_h - p_l) - \frac{\beta}{2}(q_h - q_l) \geqslant \theta_i \tilde{q}_h - \tilde{p}_h - \frac{\alpha}{2}(\tilde{p}_h - \tilde{p}_l) - \frac{\beta}{2}(\tilde{q}_h - \tilde{q}_l), \; i = 1, 2 \\ \theta_i q_l - p_l - \frac{\alpha}{2}(p_h - p_l) - \frac{\beta}{2}(q_h - q_l) \geqslant \theta_i \tilde{q}_l - \tilde{p}_l - \frac{\alpha}{2}(\tilde{p}_h - \tilde{p}_l) - \frac{\beta}{2}(\tilde{q}_h - \tilde{q}_l), \; i = 1, 2 \end{cases}$$

$$\Updownarrow$$

$$\frac{\theta_1 - \beta}{1 + \alpha}\varepsilon \leqslant \delta \leqslant \frac{\theta_2 + \beta}{1 - \alpha}\varepsilon$$

7. If $q_l = 0$, then $(IC_1^l)$ is not binding.

   This follows from the observation that the case $p_h = q_h = 0$ is worse than $q_1 = q_2 = q_h = q_l = q$ and $p_1 = p_2 = p_h = p_l = \theta_1 q$, where $q > 0$ is such that $c'(q) = \theta_1$.

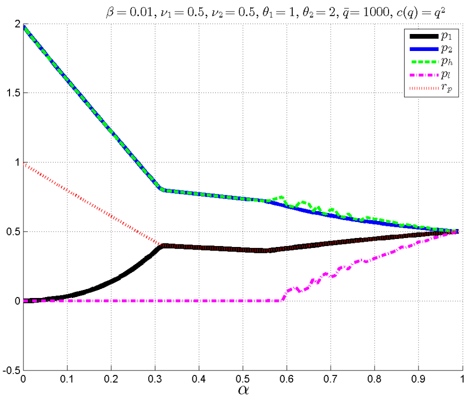
8. At least one of the constraints $(IC_{21})$ and $(IC_2^h)$ is binding.

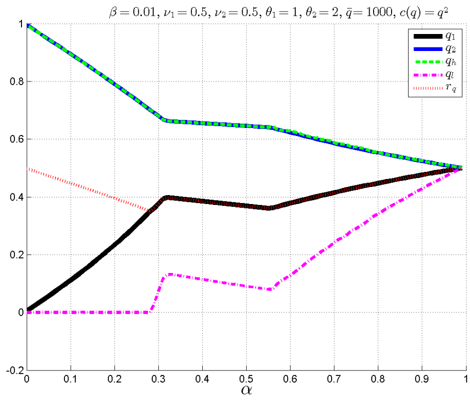
   Suppose $q_2 > q_1$. If both $(IC_2^h)$ and $(IC_{21})$ are not binding, the firm can decrease $q_2$, thus getting more profit $\Longrightarrow$ contradiction. Thus, $q_2 = q_1$. But in this case $p_2 = p_1$ and $(IC_{21})$ is binding.

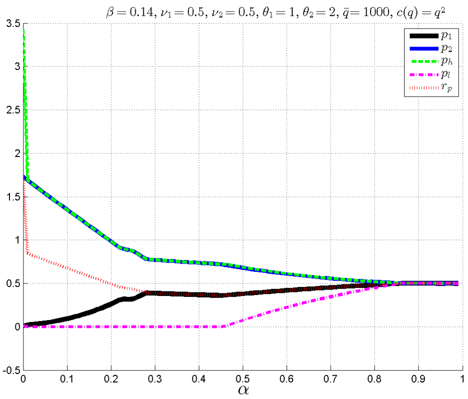
9. If $(IC_1^l)$ is binding, then either $p_1 = p_l$ and $q_1 = q_l$, or $p_1 > p_l$ and $q_1 > q_l$. If $(IC_2^h)$ is binding, then either $p_2 = p_h$ and $q_2 = q_h$, or $p_2 < p_h$ and $q_2 < q_h$.

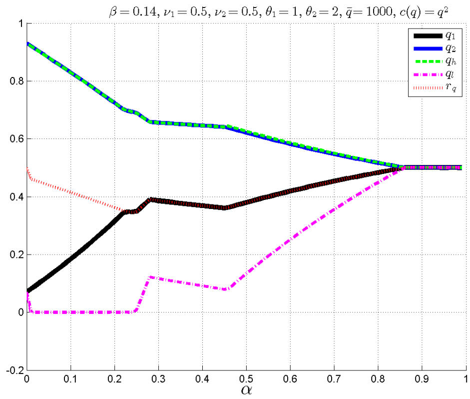
   This follows from A.3.

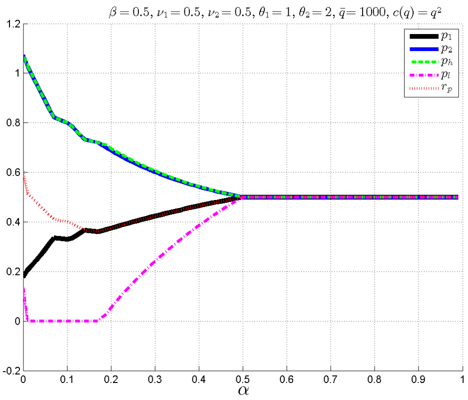
## A.5 Illustrations

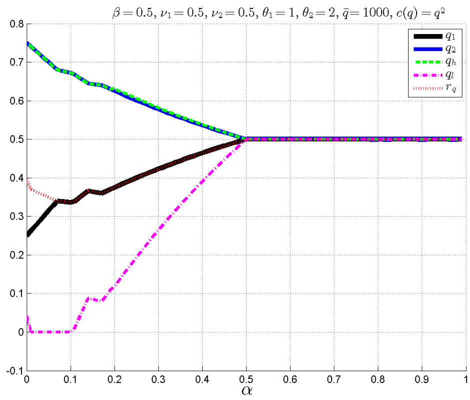


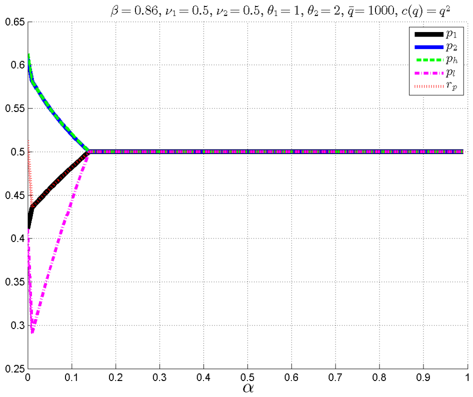

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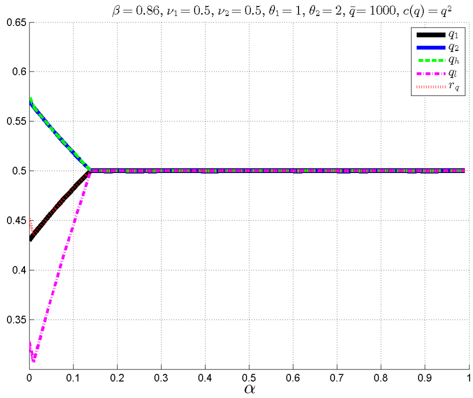
$p_1$
$p_2$
$p_h$
$p_l$
$r_p$
$\alpha$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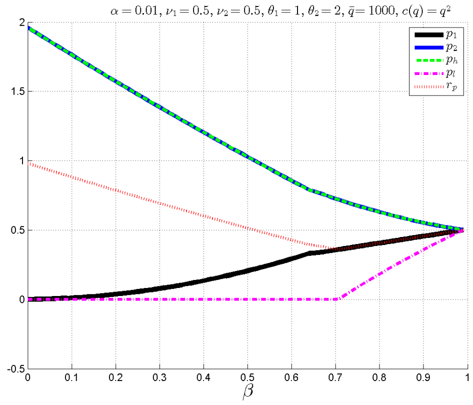
$q_2$
$q_h$
$q_l$
$r_q$
$\alpha$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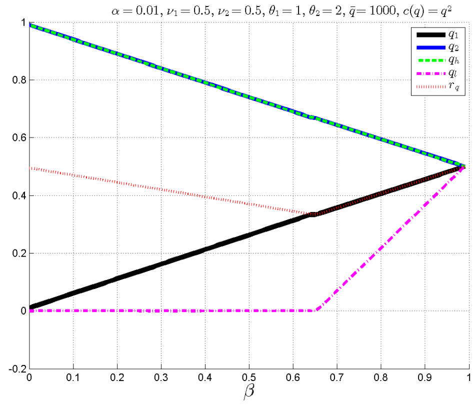
$r_p$
$\beta$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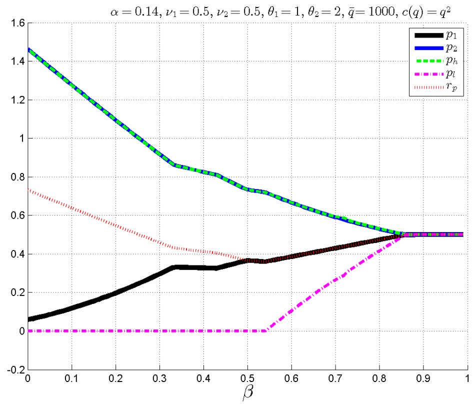
$r_q$
$\beta$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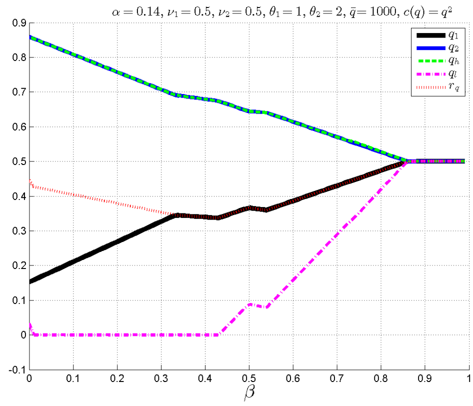
$r_p$
$\beta$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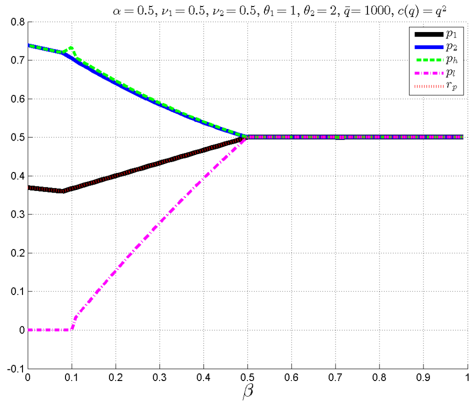
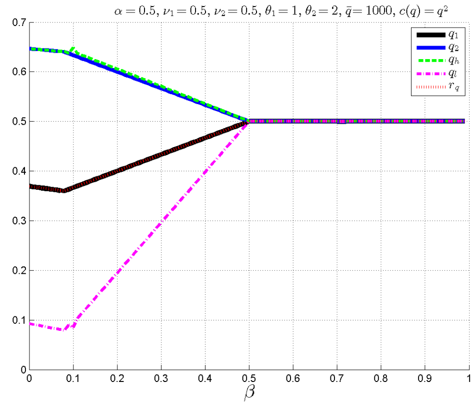
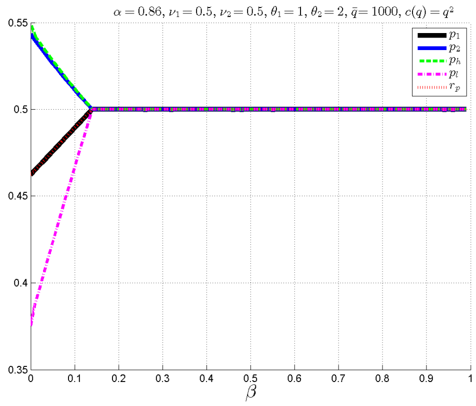
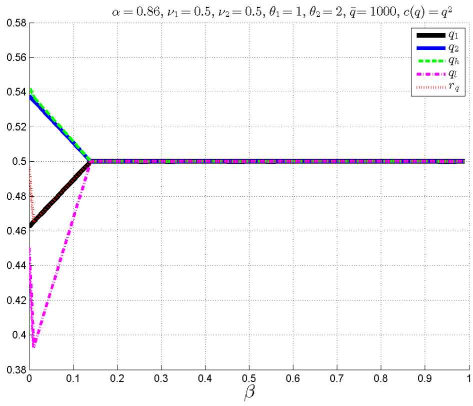
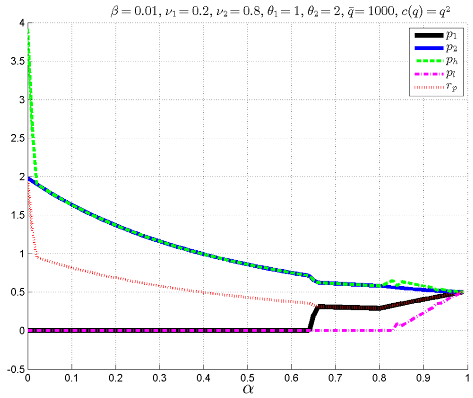
$r_q$
$\beta$

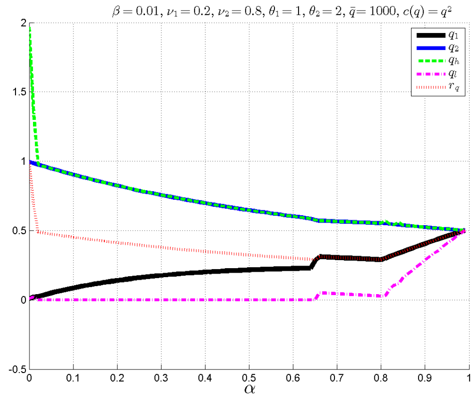

$\alpha=0.5,\ \nu_1=0.5,\ \nu_2=0.5,\ \theta_1=1,\ \theta_2=2,\ \bar{q}=1000,\ c(q)=q^2$
$p_1$
$p_2$
$p_h$
$p_l$
$r_p$
$\beta$

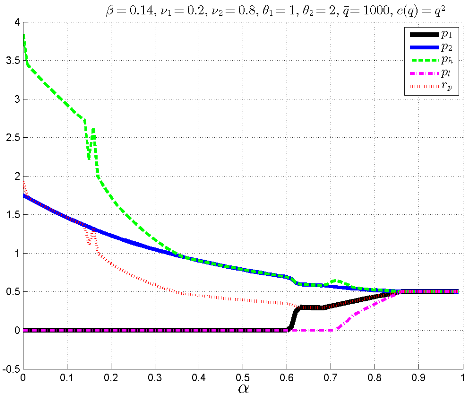

$\alpha=0.5,\ \nu_1=0.5,\ \nu_2=0.5,\ \theta_1=1,\ \theta_2=2,\ \bar{q}=1000,\ c(q)=q^2$
$q_1$
$q_2$
$q_h$
$q_l$
$r_q$
$\beta$

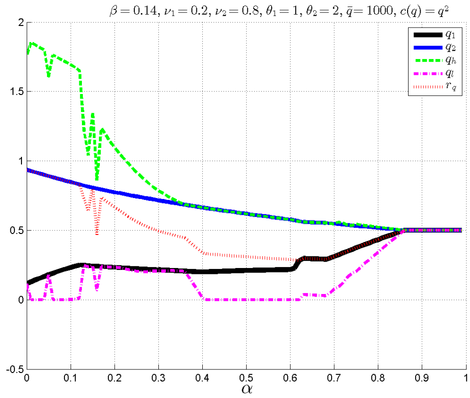

$\alpha=0.86,\ \nu_1=0.5,\ \nu_2=0.5,\ \theta_1=1,\ \theta_2=2,\ \bar{q}=1000,\ c(q)=q^2$
$p_1$
$p_2$
$p_h$
$p_l$
$r_p$
$\beta$

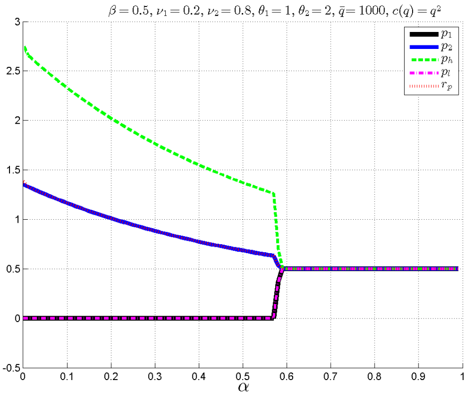

$\alpha=0.86,\ \nu_1=0.5,\ \nu_2=0.5,\ \theta_1=1,\ \theta_2=2,\ \bar{q}=1000,\ c(q)=q^2$
$q_1$
$q_2$
$q_h$
$q_l$
$r_q$
$\beta$

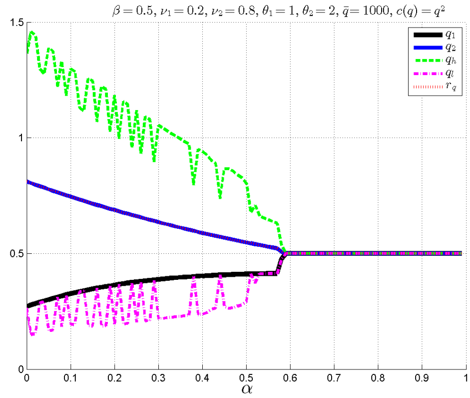

$\beta=0.01,\ \nu_1=0.2,\ \nu_2=0.8,\ \theta_1=1,\ \theta_2=2,\ \bar{q}=1000,\ c(q)=q^2$
$p_1$
$p_2$
$p_h$
$p_l$
$r_p$
$\alpha$

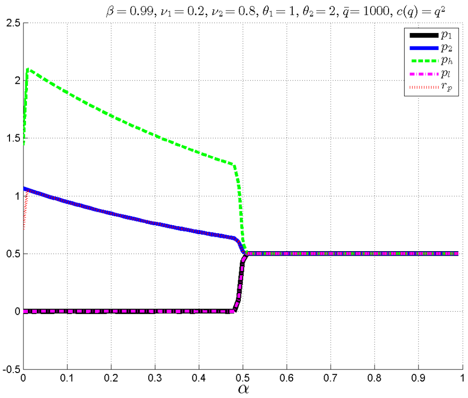

$\beta=0.01,\ \nu_1=0.2,\ \nu_2=0.8,\ \theta_1=1,\ \theta_2=2,\ \bar{q}=1000,\ c(q)=q^2$
$q_1$
$q_2$
$q_h$
$q_l$
$r_q$
$\alpha$

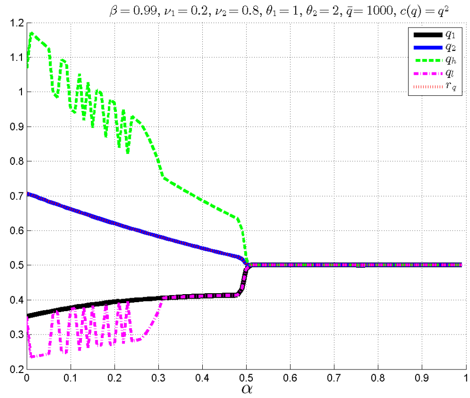

$\beta = 0.14$, $\nu_1 = 0.2$, $\nu_2 = 0.8$, $\theta_1 = 1$, $\theta_2 = 2$, $\bar{q} = 1000$, $c(q) = q^2$
$p_1$
$p_2$
$p_h$
$p_l$
$r_p$
$\alpha$


$\beta = 0.14$, $\nu_1 = 0.2$, $\nu_2 = 0.8$, $\theta_1 = 1$, $\theta_2 = 2$, $\bar{q} = 1000$, $c(q) = q^2$
$q_1$
$q_2$
$q_h$
$q_l$
$r_q$
$\alpha$


$\beta = 0.5$, $\nu_1 = 0.2$, $\nu_2 = 0.8$, $\theta_1 = 1$, $\theta_2 = 2$, $\bar{q} = 1000$, $c(q) = q^2$
$p_1$
$p_2$
$p_h$
$p_l$
$r_p$
$\alpha$


$\beta = 0.5$, $\nu_1 = 0.2$, $\nu_2 = 0.8$, $\theta_1 = 1$, $\theta_2 = 2$, $\bar{q} = 1000$, $c(q) = q^2$
$q_1$
$q_2$
$q_h$
$q_l$
$r_q$
$\alpha$


$\beta = 0.99$, $\nu_1 = 0.2$, $\nu_2 = 0.8$, $\theta_1 = 1$, $\theta_2 = 2$, $\bar{q} = 1000$, $c(q) = q^2$
$p_1$
$p_2$
$p_h$
$p_l$
$r_p$
$\alpha$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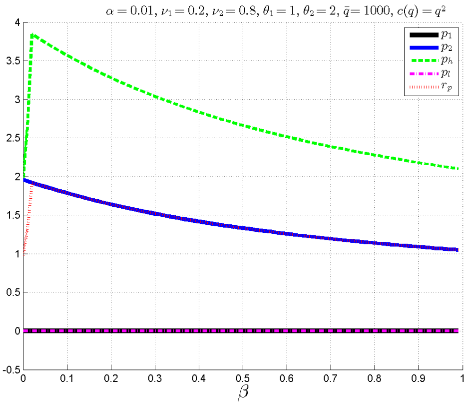
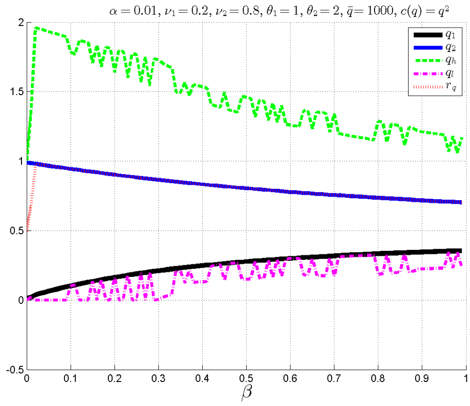
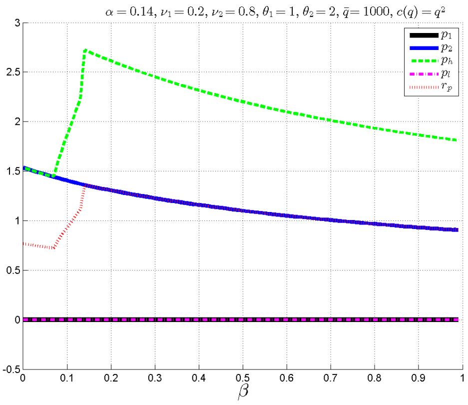
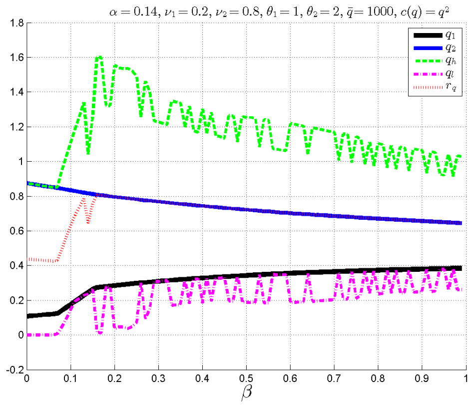
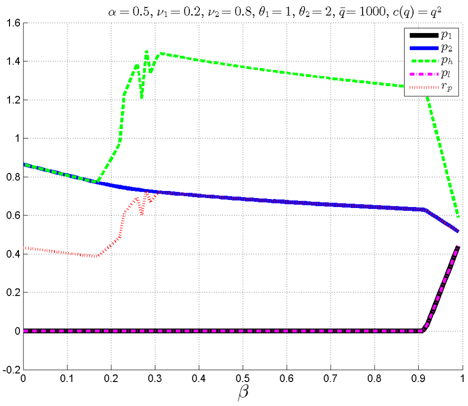
$q_1$
$q_2$
$q_h$
$q_l$
$r_q$
$\alpha$

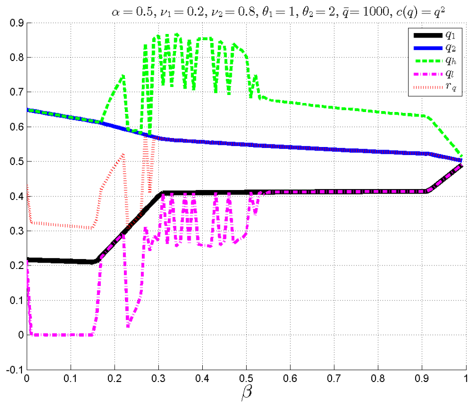

$\alpha = 0.01$, $\nu_1 = 0.2$, $\nu_2 = 0.8$, $\theta_1 = 1$, $\theta_2 = 2$, $\bar{q} = 1000$, $c(q) = q^2$
$p_1$
$p_2$
$p_h$
$p_l$
$r_p$
$\beta$

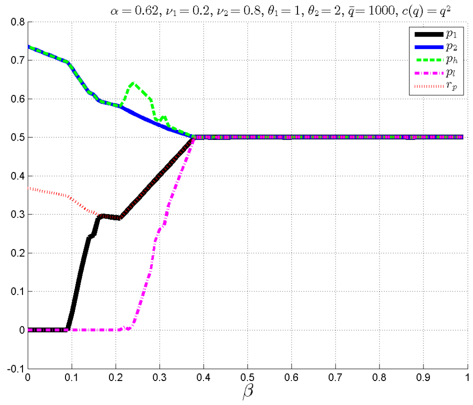

$\alpha = 0.01$, $\nu_1 = 0.2$, $\nu_2 = 0.8$, $\theta_1 = 1$, $\theta_2 = 2$, $\bar{q} = 1000$, $c(q) = q^2$
$q_1$
$q_2$
$q_h$
$q_l$
$r_q$
$\beta$

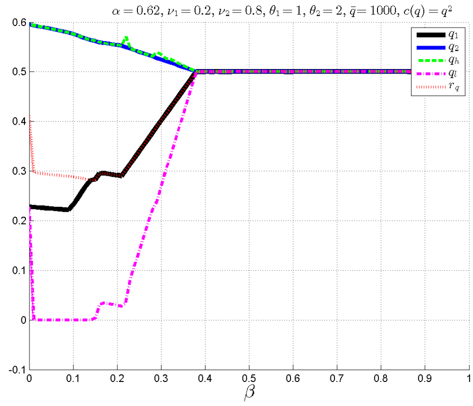

$\alpha = 0.14$, $\nu_1 = 0.2$, $\nu_2 = 0.8$, $\theta_1 = 1$, $\theta_2 = 2$, $\bar{q} = 1000$, $c(q) = q^2$
$p_1$
$p_2$
$p_h$
$p_l$
$r_p$
$\beta$

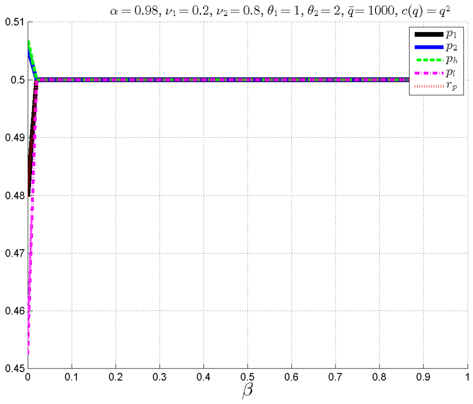

$\alpha = 0.14$, $\nu_1 = 0.2$, $\nu_2 = 0.8$, $\theta_1 = 1$, $\theta_2 = 2$, $\bar{q} = 1000$, $c(q) = q^2$
$q_1$
$q_2$
$q_h$
$q_l$
$r_q$
$\beta$

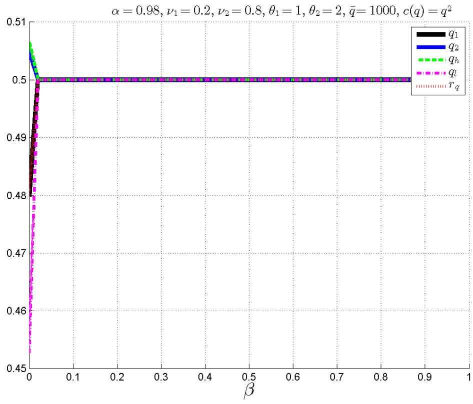

$\alpha = 0.5$, $\nu_1 = 0.2$, $\nu_2 = 0.8$, $\theta_1 = 1$, $\theta_2 = 2$, $\bar{q} = 1000$, $c(q) = q^2$
$p_1$
$p_2$
$p_h$
$p_l$
$r_p$
$\beta$

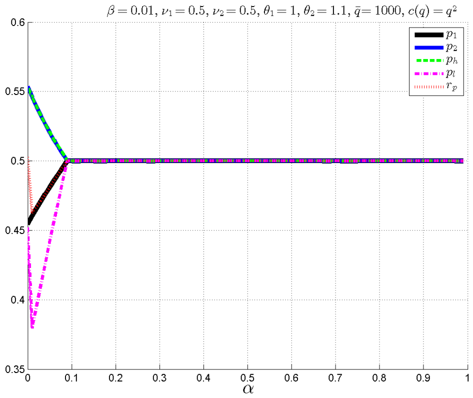

$\alpha = 0.5$, $\nu_1 = 0.2$, $\nu_2 = 0.8$, $\theta_1 = 1$, $\theta_2 = 2$, $\bar{q} = 1000$, $c(q) = q^2$
$q_1$
$q_2$
$q_h$
$q_l$
$r_q$
$\beta$

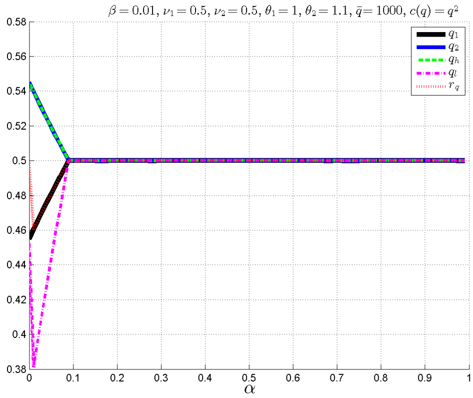

α = 0.62, ν1 = 0.2, ν2 = 0.8, θ1 = 1, θ2 = 2, q̄ = 1000, c(q) = q²
p1
p2
ph
pl
rp
β

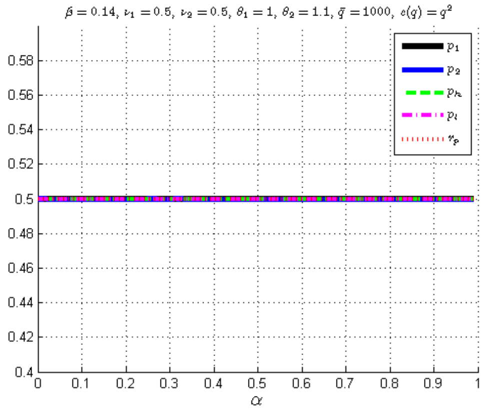

α = 0.62, ν1 = 0.2, ν2 = 0.8, θ1 = 1, θ2 = 2, q̄ = 1000, c(q) = q²
q1
q2
qh
ql
rq
β

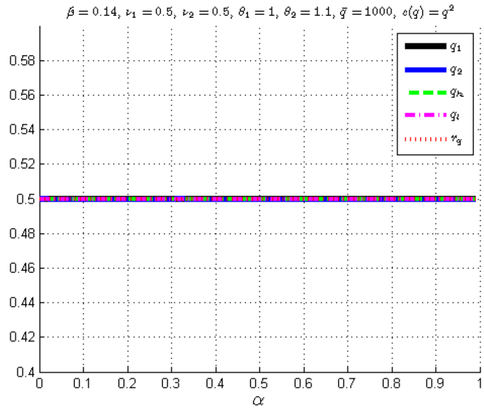

α = 0.98, ν1 = 0.2, ν2 = 0.8, θ1 = 1, θ2 = 2, q̄ = 1000, c(q) = q²
p1
p2
ph
pl
rp
β

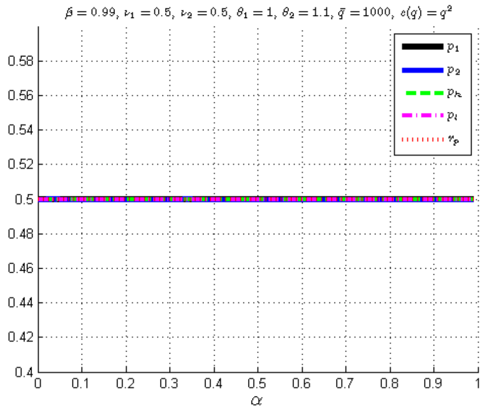

α = 0.98, ν1 = 0.2, ν2 = 0.8, θ1 = 1, θ2 = 2, q̄ = 1000, c(q) = q²
q1
q2
qh
ql
rq
β

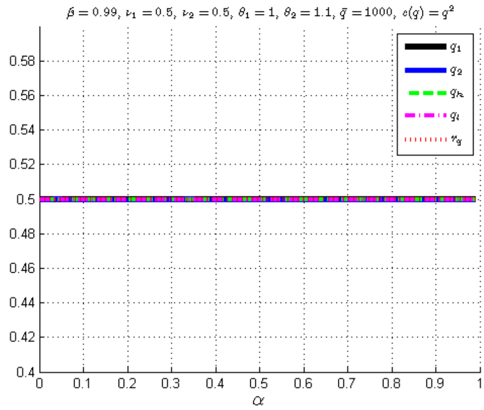

β = 0.01, ν1 = 0.5, ν2 = 0.5, θ1 = 1, θ2 = 1.1, q̄ = 1000, c(q) = q²
p1
p2
ph
pl
rp
α

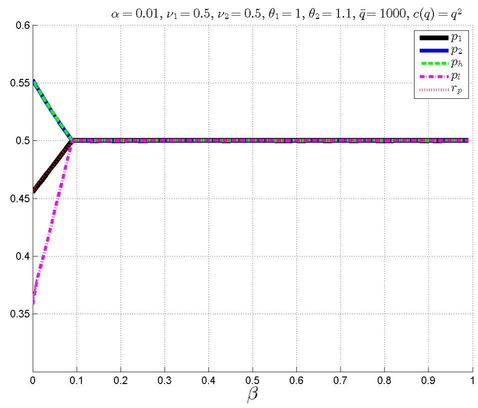

β = 0.01, ν1 = 0.5, ν2 = 0.5, θ1 = 1, θ2 = 1.1, q̄ = 1000, c(q) = q²
q1
q2
qh
ql
rq
α

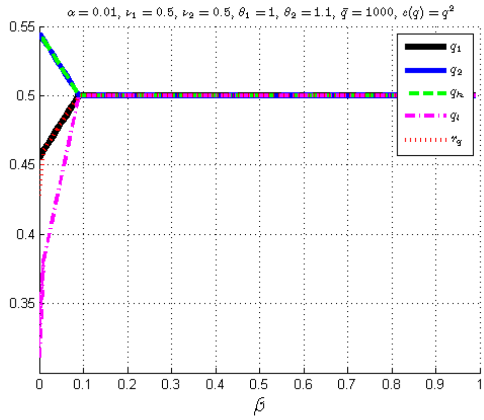

$\beta = 0.14,\ \nu_1 = 0.5,\ \nu_2 = 0.5,\ \theta_1 = 1,\ \theta_2 = 1.1,\ \bar{q} = 1000,\ c(q) = q^2$
$p_1$
$p_2$
$p_h$
$p_l$
$r_p$
$\alpha$

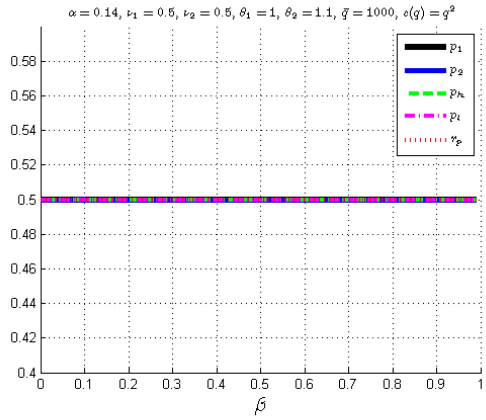

$\beta = 0.14,\ \nu_1 = 0.5,\ \nu_2 = 0.5,\ \theta_1 = 1,\ \theta_2 = 1.1,\ \bar{q} = 1000,\ c(q) = q^2$
$q_1$
$q_2$
$q_h$
$q_l$
$r_q$
$\alpha$

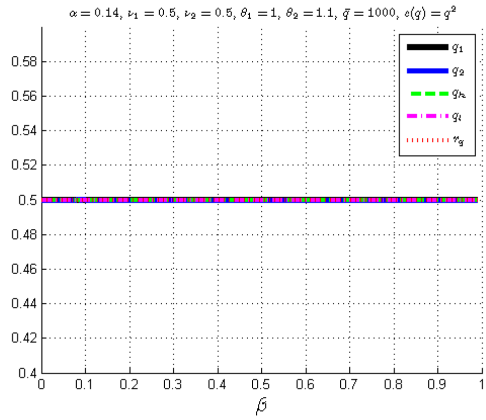

$\beta = 0.99,\ \nu_1 = 0.5,\ \nu_2 = 0.5,\ \theta_1 = 1,\ \theta_2 = 1.1,\ \bar{q} = 1000,\ c(q) = q^2$
$p_1$
$p_2$
$p_h$
$p_l$
$r_p$
$\alpha$

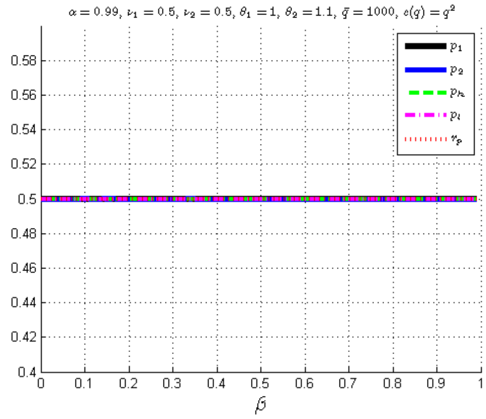

$\beta = 0.99,\ \nu_1 = 0.5,\ \nu_2 = 0.5,\ \theta_1 = 1,\ \theta_2 = 1.1,\ \bar{q} = 1000,\ c(q) = q^2$
$q_1$
$q_2$
$q_h$
$q_l$
$r_q$
$\alpha$

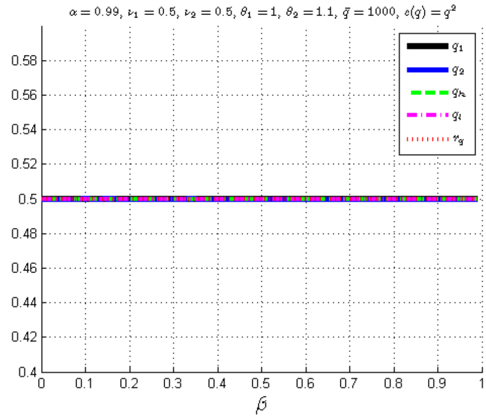

$\alpha = 0.01,\ \nu_1 = 0.5,\ \nu_2 = 0.5,\ \theta_1 = 1,\ \theta_2 = 1.1,\ \bar{q} = 1000,\ c(q) = q^2$
$p_1$
$p_2$
$p_h$
$p_l$
$r_p$
$\beta$

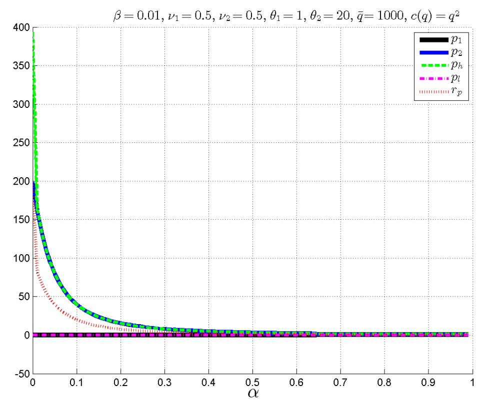

$\alpha = 0.01,\ \nu_1 = 0.5,\ \nu_2 = 0.5,\ \theta_1 = 1,\ \theta_2 = 1.1,\ \bar{q} = 1000,\ c(q) = q^2$
$q_1$
$q_2$
$q_h$
$q_l$
$r_q$
$\beta$

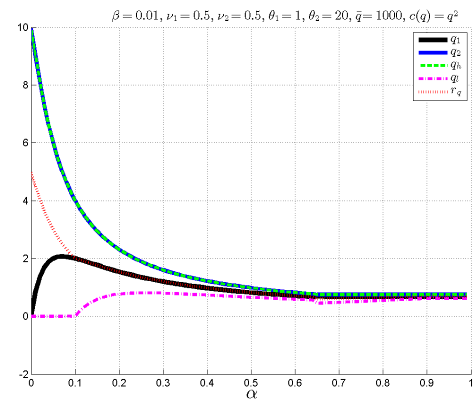

$\alpha = 0.14,\ \nu_1 = 0.5,\ \nu_2 = 0.5,\ \theta_1 = 1,\ \theta_2 = 1.1,\ \bar{q} = 1000,\ c(q) = q^2$
$p_1$
$p_2$
$p_h$
$p_l$
$r_p$
$\beta$

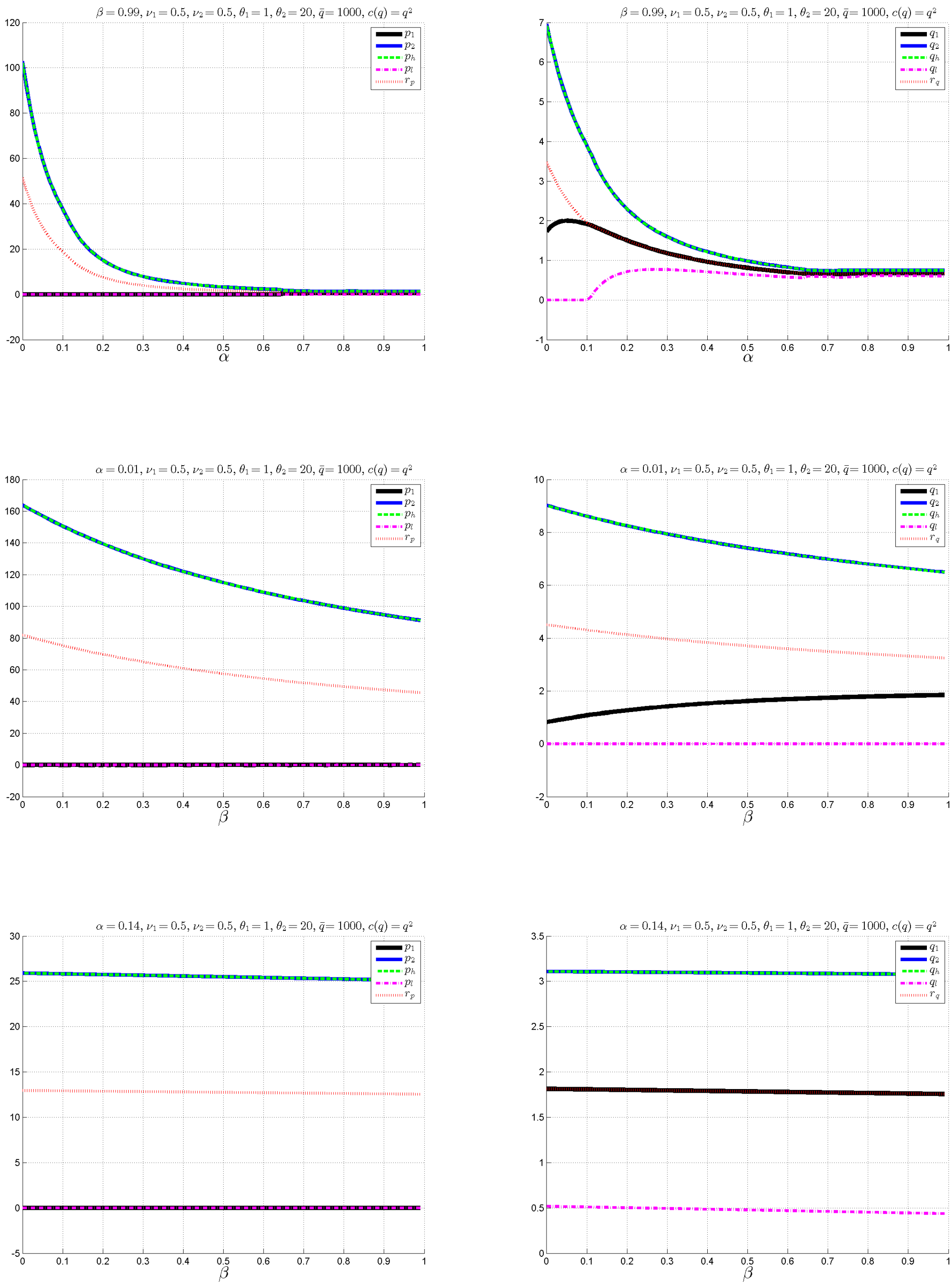

$\alpha = 0.14,\ \nu_1 = 0.5,\ \nu_2 = 0.5,\ \theta_1 = 1,\ \theta_2 = 1.1,\ \bar{q} = 1000,\ c(q) = q^2$
$q_1$
$q_2$
$q_h$
$q_l$
$r_q$
$\beta$


$\alpha = 0.99,\ \nu_1 = 0.5,\ \nu_2 = 0.5,\ \theta_1 = 1,\ \theta_2 = 1.1,\ \bar{q} = 1000,\ c(q) = q^2$
$p_1$
$p_2$
$p_h$
$p_l$
$r_p$
$\beta$

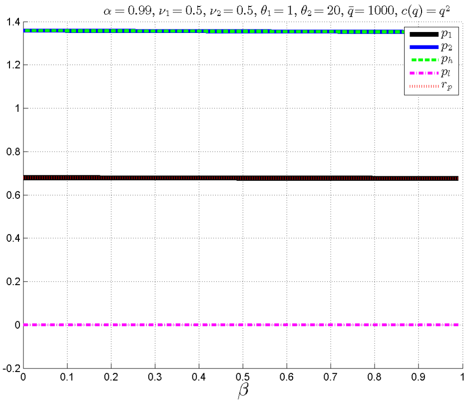

$\alpha = 0.99,\ \nu_1 = 0.5,\ \nu_2 = 0.5,\ \theta_1 = 1,\ \theta_2 = 1.1,\ \bar{q} = 1000,\ c(q) = q^2$
$q_1$
$q_2$
$q_h$
$q_l$
$r_q$
$\beta$

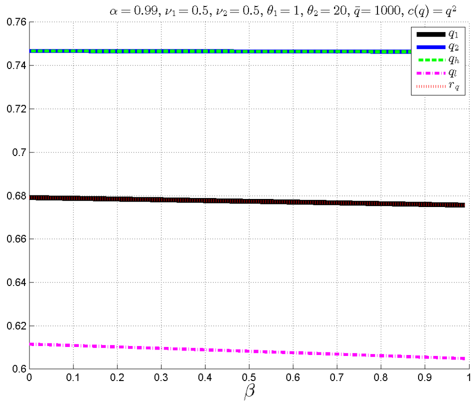

$\beta = 0.01,\ \nu_1 = 0.5,\ \nu_2 = 0.5,\ \theta_1 = 1,\ \theta_2 = 20,\ \bar{q} = 1000,\ c(q) = q^2$
$p_1$
$p_2$
$p_h$
$p_l$
$r_p$
$\alpha$


$\beta = 0.01,\ \nu_1 = 0.5,\ \nu_2 = 0.5,\ \theta_1 = 1,\ \theta_2 = 20,\ \bar{q} = 1000,\ c(q) = q^2$
$q_1$
$q_2$
$q_h$
$q_l$
$r_q$
$\alpha$



$\beta = 0.99$, $\nu_1 = 0.5$, $\nu_2 = 0.5$, $\theta_1 = 1$, $\theta_2 = 20$, $\bar{q} = 1000$, $c(q) = q^2$
$p_1$
$p_2$
$p_h$
$p_l$
$r_p$
$\alpha$
$q_1$
$q_2$
$q_h$
$q_l$
$r_q$
$\alpha = 0.01$, $\nu_1 = 0.5$, $\nu_2 = 0.5$, $\theta_1 = 1$, $\theta_2 = 20$, $\bar{q} = 1000$, $c(q) = q^2$
$\beta$
$\alpha = 0.14$, $\nu_1 = 0.5$, $\nu_2 = 0.5$, $\theta_1 = 1$, $\theta_2 = 20$, $\bar{q} = 1000$, $c(q) = q^2$


$\alpha = 0.99$, $\nu_1 = 0.5$, $\nu_2 = 0.5$, $\theta_1 = 1$, $\theta_2 = 20$, $\bar{q} = 1000$, $c(q) = q^2$
$p_1$
$p_2$
$p_h$
$p_l$
$r_p$
$\beta$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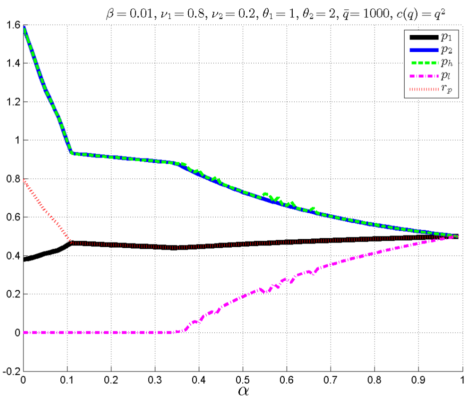
$q_1$
$q_2$
$q_h$
$q_l$
$r_q$
$\beta$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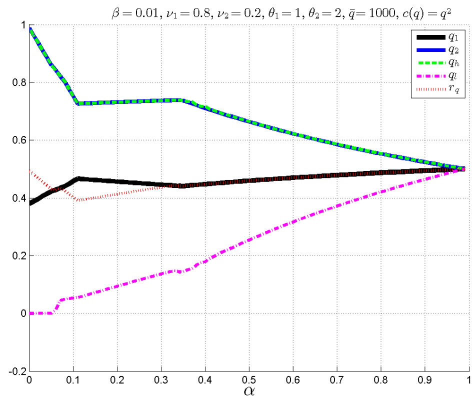
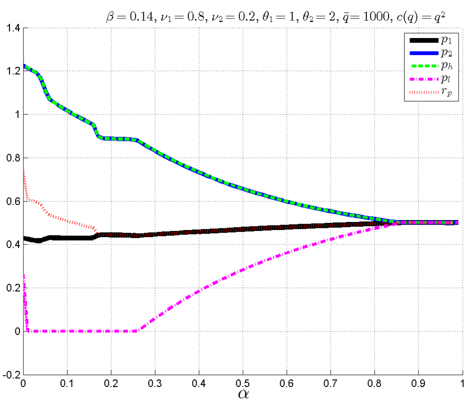
$p_1$
$p_2$
$p_h$
$p_l$
$r_p$
$\alpha$


$\beta = 0.01$, $\nu_1 = 0.8$, $\nu_2 = 0.2$, $\theta_1 = 1$, $\theta_2 = 2$, $\bar{q} = 1000$, $c(q) = q^2$
$q_1$
$q_2$
$q_h$
$q_l$
$r_q$
$\alpha$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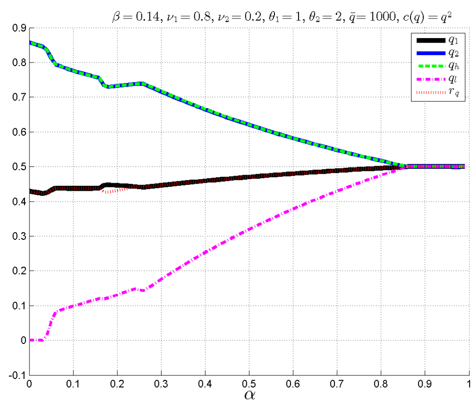
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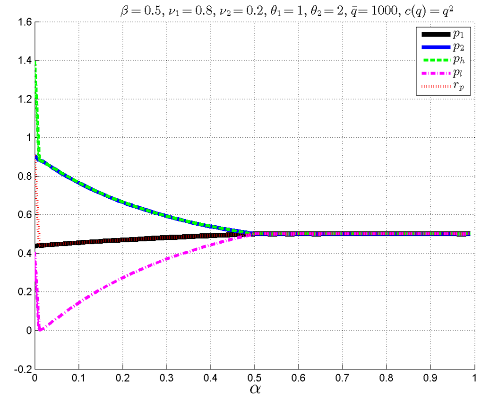
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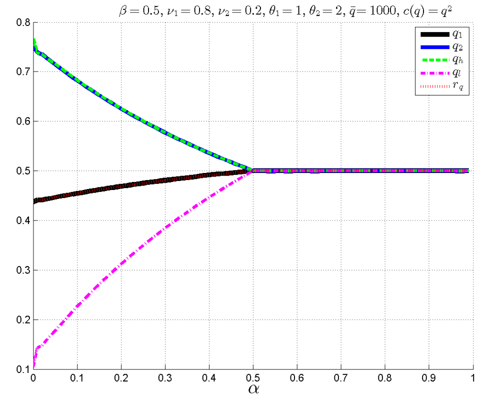
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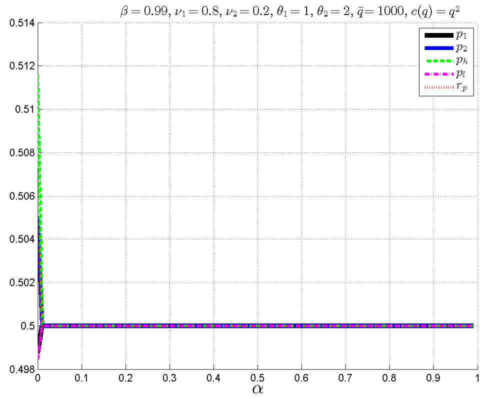
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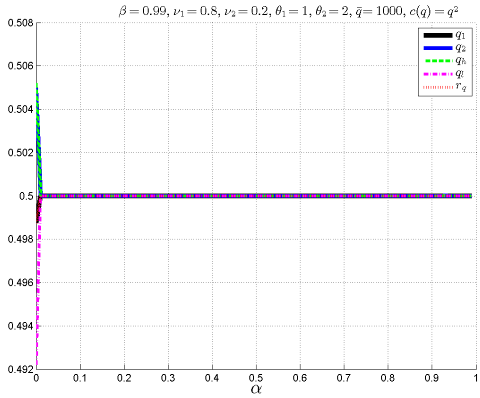
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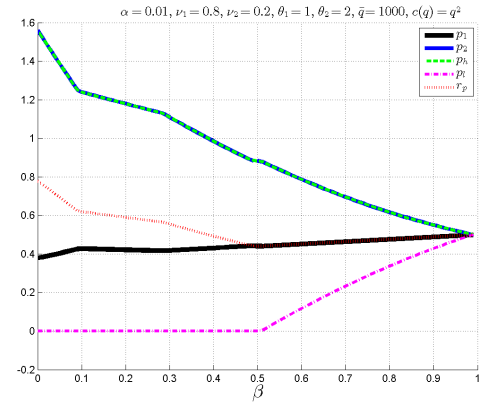
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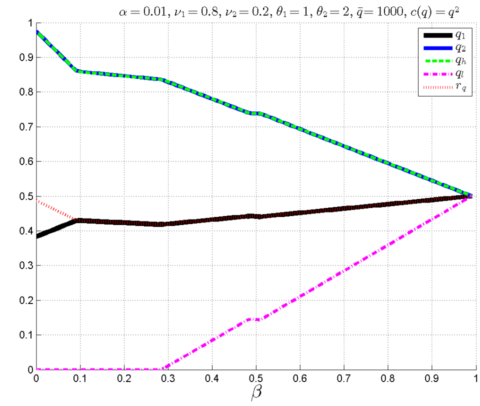
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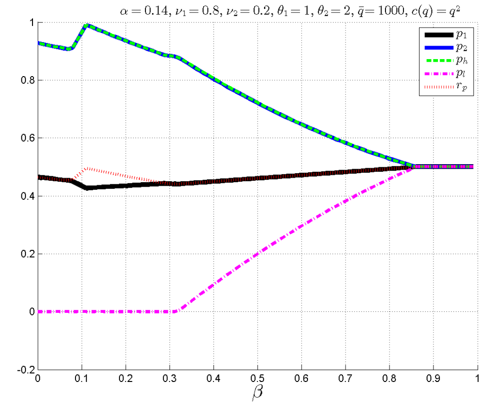
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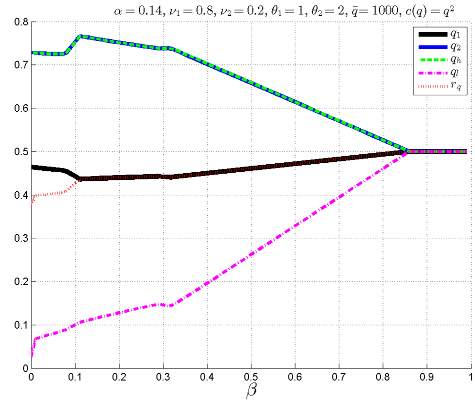
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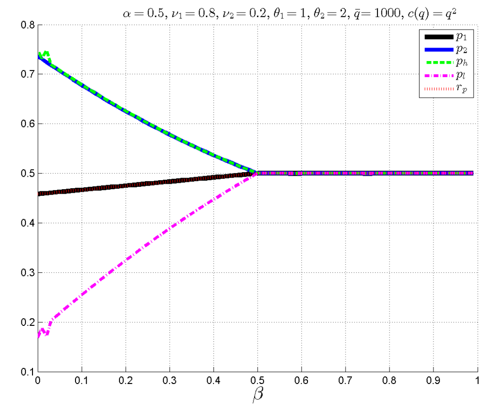
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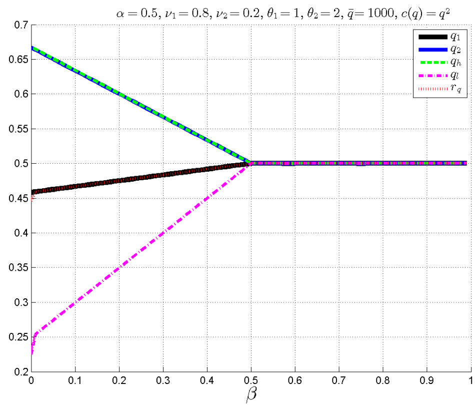
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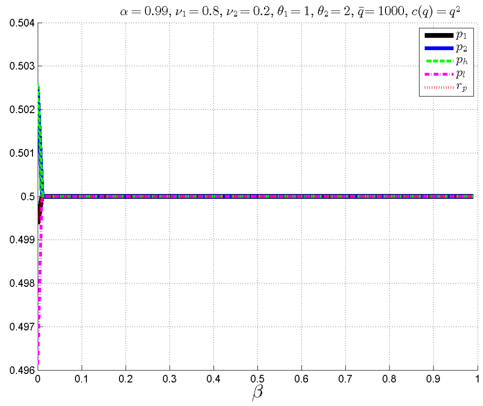
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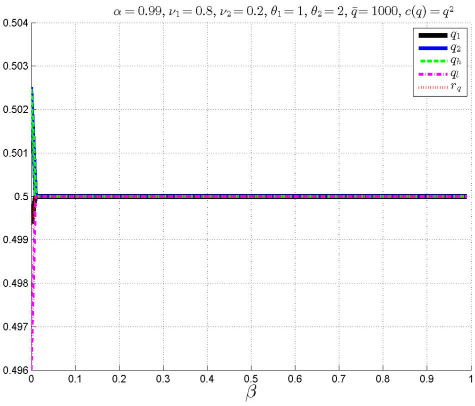
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# B Competition

## B.1 Proof for Claim 3.1

The argument is standard. Assume this is not true. Then either $p > c(q)$ or $p < c(q)$. If $p > c(q)$ and only one firm sells this product, then there is a profitable deviation for any other firm, that is to add this product to its menu. Note that such deviation does not change the whole menu of products and thus it does not change the reference point and the consumers' choices. That leads us to the contradiction. If $p > c(q)$ and at least two firms sell the product, then there is a profitable deviation for each of such firms, that is to decrease price a little bit (say, from $p$ to $p - \varepsilon$, where $\varepsilon > 0$). This deviation might change the reference point but the overall utility increases: $u_i(p - \varepsilon, q; r'_p, r_q) \geqslant u_i(p, q; r_p, r_q) + (1 - \alpha)\varepsilon > u_i(p, q; r_p, r_q)$. This means that consumers will switch from the old product to the new one. Therefore, if $(p, q)$ is sold in equilibrium, then $p \leqslant c(q)$.

If $p < c(q)$, then the profit of a firm that sells this product is negative. In this case, the profitable deviation is just leave the market.

## B.2 Proof for Proposition 3.1

Assume the contrary and from a menu $A$, consumer $i$ chooses $(\tilde{p} = c(\tilde{q}), \tilde{q})$ such that $\tilde{q} \neq q^*$. Then for any $\varepsilon > 0$ there exists $\hat{q}_\varepsilon$ such that $|\hat{q}_\varepsilon - \tilde{q}| = \varepsilon$ and $u_i(c(\hat{q}_\varepsilon), \hat{q}_\varepsilon; r_p, r_q) > u_i(c(\tilde{q}), \tilde{q}; r_p, r_q)$. It is sufficient to show that there exist $\delta > 0$ and $\varepsilon > 0$ such that the profitable deviation for any firm is to offer $(p = c(\hat{q}_\varepsilon) + \delta, q = \hat{q}_\varepsilon)$.

Cases that are possible:

1. $\exists\, \varepsilon > 0$ such that $\min\limits_{(p,q)\in A} p < c(\hat{q}_\varepsilon) < \max\limits_{(p,q)\in A} p$ and $\min\limits_{(p,q)\in A} q < \hat{q}_\varepsilon < \max\limits_{(p,q)\in A} q$,
2. $c(\tilde{q}) = \min\limits_{(p,q)\in A} p$ and $q^* < \tilde{q}$,
3. $c(\tilde{q}) = \max\limits_{(p,q)\in A} p$ and $q^* > \tilde{q}$,
4. $\tilde{q} = \min\limits_{(p,q)\in A} q$ and $q^* < \tilde{q}$,
5. $\tilde{q} = \max\limits_{(p,q)\in A} q$ and $q^* > \tilde{q}$.

In the first case the existence of the profitable deviation is obvious because the reference point does not change. Also note that if $\tilde{q} = \min\limits_{(p,q)\in A} q$, then $c(\tilde{q}) = \min\limits_{(p,q)\in A} p$, and if $c(\tilde{q}) = \max\limits_{(p,q)\in A} p$, then $\tilde{q} = \max\limits_{(p,q)\in A} q$. So, the third and the forth cases are essentially the fifth and the second ones.

Consider the second case. Note that in this case $\tilde{q} > 0$, $\hat{q}_\varepsilon = \tilde{q} - \varepsilon$, and the new reference point is $(r'_p, r'_q)$, where $r_p \geqslant r'_p \geqslant r_p - \frac{1}{2}(c(\tilde{q}) - c(\tilde{q}-\varepsilon))$ and $r_q \geqslant r'_q \geqslant r_q - \frac{1}{2}\varepsilon$. Then

- $|u_j(p,q;r'_p,r'_q) - u_j(p,q;r_p,r_q)| \leqslant \frac{\alpha}{2}(c(\tilde{q}) - c(\tilde{q}-\varepsilon)) + \frac{1}{2}\beta\varepsilon = O(\varepsilon)$ for any $j \in [1,n]$ and any $(p,q) \Rightarrow$ if consumer $j$ strictly prefers $(p,q) \in A$ to $(p',q') \in A$ when she faces menu $A$, then for sufficiently small $\varepsilon$ she will prefers it facing $A \cup (c(\hat{q}_\varepsilon) + \delta, \hat{q}_\varepsilon)$;

- $u_j(c(\hat{q}_\varepsilon), \hat{q}_\varepsilon; r'_p, r'_q) - u_j(c(\tilde{q}), \tilde{q}; r'_p, r'_q) = u_j(c(\hat{q}_\varepsilon), \hat{q}_\varepsilon; r_p, r_q) - u_j(c(\tilde{q}), \tilde{q}; r_p, r_q) + O(\varepsilon) = \theta_j(\hat{q}_\varepsilon - \tilde{q}) + (1-\alpha)(c(\tilde{q}) - c(\hat{q}_\varepsilon)) - \beta|\hat{q}_\varepsilon - r_q| + \beta|\tilde{q} - r_q| + O(\varepsilon) = O(\varepsilon)$ for any $j \in [1,n] \Rightarrow$ if consumer $j$ strictly prefers $(p,q) \in A$ to $(c(\tilde{q}), \tilde{q})$ when she faces menu $A$, then for sufficiently small $\varepsilon$ she will prefers it to $(c(\hat{q}_\varepsilon) + \delta, \hat{q}_\varepsilon)$ facing $A \cup (c(\hat{q}_\varepsilon) + \delta, \hat{q}_\varepsilon)$.

Since a consumer can be indifferent only between products that have price equal to its cost, the only thing to be proved is that consumer $i$ strictly prefers the new product when she faces the new menu, that is[7] $u_i(c(\hat{q}_\varepsilon), \hat{q}_\varepsilon; r'_p, r'_q) > u_i(c(\tilde{q}), \tilde{q}; r'_p, r'_q)$. But this is true because $c(\tilde{q} - \varepsilon) \leqslant r'_p \leqslant r_p$, $c(\tilde{q}) \leqslant r_p$, and therefore for sufficiently small $\varepsilon$ $u_i(c(\tilde{q}-\varepsilon), \tilde{q}-\varepsilon; r'_p, r'_q) - u_i(c(\tilde{q}), \tilde{q}; r'_p, r'_q) = u_i(c(\tilde{q}-\varepsilon), \tilde{q}-\varepsilon; r_p, r_q) - u_i(c(\tilde{q}), \tilde{q}; r_p, r_q) + \alpha(r_p - c(\tilde{q}-\varepsilon)) - \alpha(r_p - c(\tilde{q})) - \alpha(r'_p - c(\tilde{q}-\varepsilon)) + \alpha|r'_p - c(\tilde{q})| + \beta|r_q + \varepsilon - \tilde{q}| - \beta|r_q - \tilde{q}| - \beta|r'_q + \varepsilon - \tilde{q}| + \beta|r'_q - \tilde{q}| \geqslant u_i(c(\tilde{q}-\varepsilon), \tilde{q}-\varepsilon; r_p, r_q) - u_i(c(\tilde{q}), \tilde{q}; r_p, r_q) > 0$ (because if $\tilde{q} \neq r_q$, then $|r_q + \varepsilon - \tilde{q}| - |r_q - \tilde{q}| - |r'_q + \varepsilon - \tilde{q}| + |r'_q - \tilde{q}| = 0$ for small $\varepsilon$, and if $\tilde{q} = r_q$, then $|r_q + \varepsilon - \tilde{q}| - |r_q - \tilde{q}| - |r'_q + \varepsilon - \tilde{q}| + |r'_q - \tilde{q}| = \varepsilon - (r'_q + \varepsilon - \tilde{q}) - (r'_q - \tilde{q}) = 2(r_q - r'_q) \geqslant 0$).

Consider the fifth case. In this case $\tilde{q} < \bar{q}$, $\hat{q}_\varepsilon = \tilde{q} + \varepsilon$, and the new reference point is $(r'_p, r'_q)$, where $r_p + \frac{1}{2}(c(\tilde{q}+\varepsilon) - c(\tilde{q})) \geqslant r'_p \geqslant r_p$ and $r_q + \frac{1}{2}\varepsilon \geqslant r'_q \geqslant r_q$. Then again $|u_j(p,q;r'_p,r'_q) - u_j(p,q;r_p,r_q)| = O(\varepsilon)$ for any $j \in [1,n]$ and any $(p,q)$ and $u_j(c(\hat{q}_\varepsilon), \hat{q}_\varepsilon; r'_p, r'_q) - u_j(c(\tilde{q}), \tilde{q}; r'_p, r'_q) = O(\varepsilon)$ for any $j \in [1,n]$. Since for sufficiently small $\varepsilon > 0$ $u_i(c(\tilde{q}+\varepsilon), \tilde{q}+\varepsilon; r'_p, r'_q) - u_i(c(\tilde{q}), \tilde{q}; r'_p, r'_q) = u_i(c(\tilde{q}+\varepsilon), \tilde{q}+\varepsilon; r_p, r_q) - u_i(c(\tilde{q}), \tilde{q}; r_p, r_q) + \alpha|c(\tilde{q}+\varepsilon) - r_p| - \alpha|c(\tilde{q}) - r_p| - \alpha|c(\tilde{q}+\varepsilon) - r'_p| + \alpha|c(\tilde{q}) - r'_p| + \beta(\tilde{q} + \varepsilon -$

[7]Since $q^* < \tilde{q}$, consumer $i$ cannot be indifferent between two products when she faces menu $A$.

$r_q)-\beta(\tilde{q}-r_q)-\beta(\tilde{q}+\varepsilon-r_q')+\beta|\tilde{q}-r_q'| \geqslant u_i(c(\tilde{q}+\varepsilon),\tilde{q}+\varepsilon;r_p,r_q)-u_i(c(\tilde{q}),\tilde{q};r_p,r_q) > 0$, the claim is proved.

## B.3 Proof for Claim 3.3

**Lemma B.1.** *If a good $(p,q)$ is sold in equilibrium, then $p=c(q)$ and one of the following is true:*

1. *$\exists$ such $i$ that $q \in \left\{(c')^{-1}(\frac{\theta_i-\beta}{1+\alpha}),(c')^{-1}(\frac{\theta_i+\beta}{1+\alpha}),(c')^{-1}(\frac{\theta_i-\beta}{1-\alpha}),(c')^{-1}(\frac{\theta_i+\beta}{1-\alpha}),\bar{q}\right\}$,*
2. *$q=\frac{\bar{q}}{2}$, $\min\limits_{(p,q)\in A} q=0$, and $\max\limits_{(p,q)\in A} q=\bar{q}$,*
3. *$q=r_q$, $c(r_q)=r_p$ and $\exists$ such $i$ that $\theta_i r_q=c(r_q)$,*
4. *$q=c^{-1}(r_p)$, $\min\limits_{(p,q)\in A} q=0$, $\min\limits_{(p,q)\in A} p=0$ and $\exists$ such $i$ that $(c')^{-1}(\theta_i+\beta)\leqslant c^{-1}(r_p)<\min\left\{r_q,(c')^{-1}\left(\frac{\theta_i+\beta}{1-\alpha}\right)\right\}$ and $(\theta_i+\beta)c^{-1}(r_p)=r_p+\beta r_q$,*
5. *$q=c^{-1}(r_p)<r_q$, $\min\limits_{(p,q)\in A} q=0$ and $\exists$ such $i$ that $(\theta_i+\beta)c^{-1}(r_p)=r_p+\beta r_q$ and $\theta_i+\beta=c'(\frac{r_p+\beta r_q}{\theta_i+\beta})$,*
6. *$q=c^{-1}(r_p)$, $c(r_q)<r_p<c(\bar{q})$, $\max\limits_{(p,q)\in A} q=\bar{q}$ and $\exists$ such $i$ that $(\theta_i-\beta)c^{-1}(r_p)+\beta r_q=r_p$ and $\theta_i-\beta=c'(\frac{r_p-\beta r_q}{\theta_i-\beta})$,*
7. *$q=c^{-1}(r_p)$, $c(r_q)<r_p<c(\bar{q})$, $\min\limits_{(p,q)\in A} p=0$, $\max\limits_{(p,q)\in A} q=\bar{q}$ and $\exists$ such $i$ that $(c')^{-1}(\theta_i-\beta)\leqslant c^{-1}(r_p)<(c')^{-1}\left(\frac{\theta_i-\beta}{1-\alpha}\right)$ and $(\theta_i-\beta)c^{-1}(r_p)+\beta r_q=r_p$,*
8. *$q=c^{-1}(r_p)$, $c(r_q)<r_p<c(\bar{q})$, $\min\limits_{(p,q)\in A} p=0$, $(\max\limits_{(p,q)\in A} p, \max\limits_{(p,q)\in A} q)$ is sold and $\exists$ such $i$ that $(c')^{-1}(\theta_i-\beta)\leqslant c^{-1}(r_p)<(c')^{-1}\left(\frac{\theta_i-\beta}{1-\alpha}\right)$ and $(\theta_i-\beta)c^{-1}(r_p)+\beta r_q=r_p$,*
9. *$q=r_q$, $c(r_q)<r_p$, $\min\limits_{(p,q)\in A} p=0$ and $\exists$ such $i$ that $\theta_i r_q=(1-\alpha)c(r_q)+\alpha r_p$ and $\theta_i=(1-\alpha)c'(r_q)$,*
10. *$q=r_q$, $\min\limits_{(p,q)\in A} q=0$, $(\max\limits_{(p,q)\in A} p, \max\limits_{(p,q)\in A} q)$ is sold and $\exists$ such $i$ that $(c')^{-1}\left(\frac{\theta_i-\beta}{1-\alpha}\right)<r_q<\min\left\{c^{-1}(r_p),(c')^{-1}\left(\frac{\theta_i+\beta}{1-\alpha}\right)\right\}$ and $\theta_i r_q=(1-\alpha)c(r_q)+\alpha r_p$,*
11. *$q=r_q$, $\min\limits_{(p,q)\in A} q=0$, $\min\limits_{(p,q)\in A} p=0$ and $\exists$ such $i$ that $(c')^{-1}\left(\frac{\theta_i}{1-\alpha}\right)<r_q<\min\left\{c^{-1}(r_p),(c')^{-1}\left(\frac{\theta_i+\beta}{1-\alpha}\right)\right\}$ and $\theta_i r_q=(1-\alpha)c(r_q)+\alpha r_p$,*

*12.* $q = r_q$, $\max\limits_{(p,q)\in A} q = \bar{q}$, $\min\limits_{(p,q)\in A} p = 0$ *and* $\exists$ *such* $i$ *that* $(c')^{-1}\left(\frac{\theta_i-\beta}{1-\alpha}\right) < r_q < \min\left\{c^{-1}(r_p), (c')^{-1}\left(\frac{\theta_i}{1-\alpha}\right)\right\}$ *and* $\theta_i r_q = (1-\alpha)c(r_q) + \alpha r_p$,

*13.* $q = r_q$, $\min\limits_{(p,q)\in A} p = 0$ *and* $(\max\limits_{(p,q)\in A} p, \max\limits_{(p,q)\in A} q)$ *is sold and* $\exists$ *such* $i$ *that* $(c')^{-1}\left(\frac{\theta_i-\beta}{1-\alpha}\right) < r_q < \min\left\{c^{-1}(r_p), (c')^{-1}\left(\frac{\theta_i}{1-\alpha}\right)\right\}$ *and* $\theta_i r_q = (1-\alpha)c(r_q) + \alpha r_p$.

*Proof.* The proof consists of three steps.

1. Consider a menu $A$ such that if product $(p,q)$ is sold, then $p = c(q^*)$ and $q = q^*$ (according to Claim 3.2, only such menus can appear in equilibrium). Denote $(r_p, r_q)$ the reference point for menu $A$. Note that (1) $c(\min\limits_{(p,q)\in A} q) \leqslant \max\limits_{(p,q)\in A} p$ and (2) $|u_j(p,q;r'_p,r'_q) - u_j(p,q;r_p,r_q)| = O(\varepsilon)$ for all $j$ and $(p,q)$ if $|r'_p - r_p| = O(\varepsilon)$ and $|r'_q - r_q| = O(\varepsilon)$. That means that for small $\varepsilon > 0$ the preferences over the alternatives that are included in $A$ do not change with the change of the reference point. These considerations help to construct some profitable deviations.

   If $\exists i$ such that $(c')^{-1}\left(\frac{\theta_i+\beta}{1+\alpha}\right) \leqslant c^{-1}(r_p) < \min\left\{r_q, (c')^{-1}\left(\frac{\theta_i+\beta}{1-\alpha}\right)\right\}$ and $(\theta_i + \beta)c^{-1}(r_p) > r_p + \beta r_q$ or $\exists i$ such that $\max\left\{r_q, (c')^{-1}\left(\frac{\theta_i-\beta}{1+\alpha}\right)\right\} \leqslant c^{-1}(r_p) < \min\left\{(c')^{-1}\left(\frac{\theta_i-\beta}{1-\alpha}\right), \bar{q}\right\}$ and $(\theta_i - \beta)c^{-1}(r_p) + \beta r_q > r_p \Rightarrow (r_p, c^{-1}(r_p)) \in A$ and consumer $i$ buys it. Then there exists a profitable deviation for any firm, that is to offer two more products $(\max\limits_{(p,q)\in A} p + \varepsilon, \min\limits_{(p,q)\in A} q)$ and $(r_p + \delta, c^{-1}(r_p + \frac{\varepsilon}{2}))$, where $\delta > \frac{\varepsilon}{2}$.

   If $\exists i$ such that $\max\left\{(c')^{-1}\left(\frac{\theta_i-\beta}{1+\alpha}\right), c^{-1}(r_p)\right\} < r_q < (c')^{-1}\left(\frac{\theta_i+\beta}{1+\alpha}\right)$ and $\theta_i r_q + \alpha r_p > (1+\alpha)c(r_q)$ or $\exists i$ such that $(c')^{-1}\left(\frac{\theta_i-\beta}{1-\alpha}\right) < r_q < \min\left\{c^{-1}(r_p), (c')^{-1}\left(\frac{\theta_i+\beta}{1-\alpha}\right)\right\}$ and $\theta_i r_q > (1-\alpha)c(r_q) + \alpha r_p \Rightarrow (c(r_q), r_q) \in A$ and consumer $i$ buys it. Then $\min\limits_{(p,q)\in A} q = 0$ and $\max\limits_{(p,q)\in A} q = \bar{q}$, because otherwise there exists a profitable deviation for any firm, that is to offer two more products: $(c(\tilde{q}), \tilde{q})$, which changes the reference point ($\tilde{q} = \min\limits_{(p,q)\in A} q - \varepsilon$ or $\tilde{q} = \max\limits_{(p,q)\in A} q + \varepsilon$), and $(c(r'_q) + \delta, r'_q)$, where $r'_q = r_q \pm \frac{1}{2}\varepsilon$ is the quality component of the new reference point.

   To sum up,

   (a) if $(r_p, c^{-1}(r_p))$ is sold in equilibrium, then at least one of the following is true:

      i. $\exists\, i : c^{-1}(r_p) = \min\left\{r_q, (c')^{-1}\left(\frac{\theta_i+\beta}{1-\alpha}\right)\right\}$,

ii. $\exists\, i : c^{-1}(r_p) = \min\left\{(c')^{-1}\left(\frac{\theta_i-\beta}{1-\alpha}\right), \bar{q}\right\}$,

iii. the consumer that buys it gets zero utility, which means either $(\theta_i + \beta)c^{-1}(r_p) = r_p + \beta r_q$, or $(\theta_i - \beta)c^{-1}(r_p) + \beta r_q = r_p$.

(b) if $(c(r_q), r_q)$ is sold in equilibrium, then at least one of the following is true:

i. $\exists\, i : r_q = \max\left\{(c')^{-1}\left(\frac{\theta_i-\beta}{1+\alpha}\right), c^{-1}(r_p)\right\}$,

ii. $\exists\, i : r_q = (c')^{-1}\left(\frac{\theta_i+\beta}{1+\alpha}\right)$,

iii. $\exists\, i : r_q = (c')^{-1}\left(\frac{\theta_i-\beta}{1-\alpha}\right)$,

iv. $\exists\, i : r_q = \min\left\{c^{-1}(r_p), (c')^{-1}\left(\frac{\theta_i+\beta}{1-\alpha}\right)\right\}$,

v. $\min\limits_{(p,q)\in A} q = 0$ and $\max\limits_{(p,q)\in A} q = \bar{q}$,

vi. the consumer that buys it gets zero utility, which means either $\theta_i r_q + \alpha r_p = (1+\alpha)c(r_q)$, or $\theta_i r_q = (1-\alpha)c(r_q) + \alpha r_p$.

2. If $c(r_q) = r_p$, then the consumer choice is the following:

(a) $\bar{q} \leqslant (c')^{-1}\left(\frac{\theta_i-\beta}{1+\alpha}\right)$

$\Rightarrow$ consumer $i$ buys $(c(\bar{q}), \bar{q})$ if

$(\theta_i - \beta)\bar{q} + \alpha c(r_q) + \beta r_q \geqslant (1+\alpha)c(\bar{q})$, otherwise she buys nothing;

(b) $r_q \leqslant (c')^{-1}(\frac{\theta_i-\beta}{1+\alpha}) \leqslant \bar{q}$

$\Rightarrow$ consumer $i$ buys $(c((c')^{-1}(\frac{\theta_i-\beta}{1+\alpha})), (c')^{-1}(\frac{\theta_i-\beta}{1+\alpha}))$ if

$(\theta_i - \beta)(c')^{-1}(\frac{\theta_i-\beta}{1+\alpha}) + \alpha c(r_q) + \beta r_q \geqslant (1+\alpha)c((c')^{-1}(\frac{\theta_i-\beta}{1+\alpha}))$, otherwise she buys nothing;

(c) $(c')^{-1}\left(\frac{\theta_i-\beta}{1+\alpha}\right) \leqslant r_q \leqslant (c')^{-1}\left(\frac{\theta_i+\beta}{1-\alpha}\right)$

$\Rightarrow$ consumer $i$ buys $(c(r_q), r_q)$ if

$\theta_i r_q \geqslant c(r_q)$, otherwise she buys nothing;

(d) $(c')^{-1}\left(\frac{\theta_i+\beta}{1-\alpha}\right) \leqslant r_q$

$\Rightarrow$ consumer $i$ buys $(c((c')^{-1}\left(\frac{\theta_i+\beta}{1-\alpha}\right)), (c')^{-1}\left(\frac{\theta_i+\beta}{1-\alpha}\right))$ if

$(\theta_i + \beta)(c')^{-1}\left(\frac{\theta_i+\beta}{1-\alpha}\right) \geqslant (1-\alpha)c((c')^{-1}\left(\frac{\theta_i+\beta}{1-\alpha}\right)) + \alpha c(r_q) + \beta r_q$, otherwise she buys nothing.

Suppose there exists $i$ such that $(c')^{-1}\left(\frac{\theta_i-\beta}{1+\alpha}\right) < r_q < (c')^{-1}\left(\frac{\theta_i+\beta}{1-\alpha}\right)$ and $\theta_i r_q > c(r_q)$. If $\max\limits_{(p,q)\in A} q < \bar{q}$ and $(\max\limits_{(p,q)\in A} p, \max\limits_{(p,q)\in A} q)$ is an "unsold" point (i.e. nobody would buy it), then it can be "shifted" to get the opportunity to profitably deviate: one can offer two more products, $(\tilde{p}, \tilde{q})$, which is "close" to the "unsold" point of maximum price / quality and changes the reference point to $(c(r_q'), r_q')$, and $(c(r_q') + \delta, r_q')$. If $(c')^{-1}\left(\frac{\theta_i - \beta}{1-\alpha}\right) < r_q$, $\max\limits_{(p,q)\in A} q < \bar{q}$ and $(\max\limits_{(p,q)\in A} p, \max\limits_{(p,q)\in A} q)$ is sold (and thus $c(\max\limits_{(p,q)\in A} q) = \max\limits_{(p,q)\in A} p$), then any firm also can get positive profit by offering $(c(\max\limits_{(p,q)\in A} q + \delta) + \eta, \max\limits_{(p,q)\in A} q + \delta)$ and $(c(r_q') + \varepsilon, r_q')$, where new reference point is[8] $(r_p' = \frac{1}{2}(\min\limits_{(p,q)\in A} p + c(\max\limits_{(p,q)\in A} q + \delta) + \eta), r_q' = \frac{1}{2}(\min\limits_{(p,q)\in A} q + \max\limits_{(p,q)\in A} q + \delta))$. If $(c')^{-1}\left(\frac{\theta_i - \beta}{1-\alpha}\right) < r_q$ and $\min\limits_{(p,q)\in A} q > 0$, then a profitable deviation is also possible for any firm: $(\min\limits_{(p,q)\in A} p, \min\limits_{(p,q)\in A} q - \varepsilon)$ would be unsold, which gives the opportunity to offer profitable $(c(r_q') + \delta, r_q')$. If $(c')^{-1}\left(\frac{\theta_i - \beta}{1-\alpha}\right) > r_q$, then by offering $(\max\limits_{(p,q)\in A} p + \delta, \max\limits_{(p,q)\in A} q)$ and $(c(r_q) + \frac{\delta}{2} + \varepsilon, c^{-1}(c(r_q) + \frac{\delta}{2}))$ one can profitably deviate.

3. Note that if consumer $i$ chooses to buy $(c(r_q), r_q)$ or $(r_p, c^{-1}(r_p))$ and gets zero utility, then

   - either $i = 1$, or consumer $i - 1$ does not buy anything $\Rightarrow$
   - either $(\min\limits_{(p,q)\in A} p, \min\limits_{(p,q)\in A} q)$ is not sold, or consumer $i$ chooses to buy it.

   If consumer $i$ chooses to buy $(c(r_q), r_q)$ and $(\min\limits_{(p,q)\in A} p, \min\limits_{(p,q)\in A} q)$ is sold, then $r_q = \min\limits_{(p,q)\in A} q$ and $c(r_q) = \min\limits_{(p,q)\in A} p$. This means that all products in menu $A$ are of the same quality and $c^{-1}(r_p) \geqslant r_q$. If consumer $i$ chooses to buy $(r_p, c^{-1}(r_p))$ and $(\min\limits_{(p,q)\in A} p, \min\limits_{(p,q)\in A} q)$ is sold, then $r_p = \min\limits_{(p,q)\in A} p$ and $c^{-1}(r_p) = \min\limits_{(p,q)\in A} q$. This means that all products in menu $A$ are of the same price and the product of the minimum quality is sold. Therefore, there is only one product in menu $A$.

   Suppose $\exists$ such $i$ that $(c')^{-1}\left(\frac{\theta_i + \beta}{1+\alpha}\right) < c^{-1}(r_p) < \min\left\{r_q, (c')^{-1}\left(\frac{\theta_i + \beta}{1-\alpha}\right)\right\}$ and $(\theta_i + \beta)c^{-1}(r_p) = r_p + \beta r_q$. If $r_p' = r_p + \Delta r_p$ and $r_q' = r_q + \Delta r_q$, then

$$
\begin{array}{rcl}
(\theta_i + \beta)c^{-1}(r_p') & > & r_p' + \beta r_q' \\
& \Updownarrow & \\
r_p' & > & c(\frac{r_p' + \beta r_q'}{\theta_i + \beta}) \\
& \Updownarrow & \\
r_p + \Delta r_p & > & c(\frac{r_p + \beta r_q}{\theta_i + \beta}) + c'(\frac{r_p + \beta r_q}{\theta_i + \beta})\frac{\Delta r_p + \beta \Delta r_q}{\theta_i + \beta} + o(\sqrt{\Delta r_p^2 + \Delta r_q^2}) \\
& \Updownarrow & \\
(\theta_i + \beta - c'(\frac{r_p + \beta r_q}{\theta_i + \beta}))\Delta r_p & > & \beta c'(\frac{r_p + \beta r_q}{\theta_i + \beta})\Delta r_q + o(\sqrt{\Delta r_p^2 + \Delta r_q^2})
\end{array}
$$

[8] $r_q' < c^{-1}(r_p')$ for sufficiently small $\delta > 0$ because $c'(r_q) \leqslant c'(\max\limits_{(p,q)\in A} q)$.

Thus,

- if $\theta_i + \beta > c'(\frac{r_p+\beta r_q}{\theta_i+\beta})$, then there exists a profitable deviation: product $(\max_{(p,q)\in A} p + 2\Delta r_p, \min_{(p,q)\in A} q)$, $\Delta r_p > 0$, changes the reference point and is unsold, product $(r'_p + \varepsilon, c^{-1}(r'_p))$ is profitable;
- if $\theta_i + \beta \leqslant c'(\frac{r_p+\beta r_q}{\theta_i+\beta})$ and $\min_{(p,q)\in A} q > 0$, then the profitable deviation is: product $(\max_{(p,q)\in A} p+2\Delta r_p, \min_{(p,q)\in A} q+2\Delta r_q)$, $\Delta r_p > 0$, $\Delta r_q < 0$, changes the reference point and is unsold, product $(r'_p + \varepsilon, c^{-1}(r'_p))$ is profitable;
- if $\theta_i+\beta < c'(\frac{r_p+\beta r_q}{\theta_i+\beta})$, $\min_{(p,q)\in A} p > 0$ and $(\min_{(p,q)\in A} p, \min_{(p,q)\in A} q)$ is unsold, then a firm can get positive profit by offering $(\min_{(p,q)\in A} p+2\Delta r_p, \min_{(p,q)\in A} q)$, $\Delta r_p < 0$, changes the reference point and is unsold, product $(r'_p + \varepsilon, c^{-1}(r'_p))$.

Therefore[9],

(a) $\min_{(p,q)\in A} q = 0$, $\theta_i + \beta \leqslant c'(\frac{r_p+\beta r_q}{\theta_i+\beta})$,

(b) at least one of the following is true:

  i. $\theta_i + \beta = c'(\frac{r_p+\beta r_q}{\theta_i+\beta})$,

  ii. $\min_{(p,q)\in A} p = 0$.

Suppose $\exists$ such $i$ that $\max\left\{r_q, (c')^{-1}\left(\frac{\theta_i-\beta}{1+\alpha}\right)\right\} < c^{-1}(r_p) < \min\left\{(c')^{-1}\left(\frac{\theta_i-\beta}{1-\alpha}\right), \bar{q}\right\}$ and $(\theta_i - \beta)c^{-1}(r_p) + \beta r_q = r_p$. First note that from the previous discussion $(\min_{(p,q)\in A} p, \min_{(p,q)\in A} q)$ is unsold. If $r'_p = r_p + \Delta r_p$ and $r'_q = r_q + \Delta r_q$, then

$$\begin{array}{rcl}
(\theta_i - \beta)c^{-1}(r'_p) + \beta r'_q & > & r'_p \\
 & \Updownarrow & \\
r'_p & > & c(\frac{r'_p-\beta r'_q}{\theta_i-\beta}) \\
 & \Updownarrow & \\
r_p + \Delta r_p & > & c(\frac{r_p-\beta r_q}{\theta_i-\beta}) + c'(\frac{r_p-\beta r_q}{\theta_i-\beta})\frac{\Delta r_p-\beta\Delta r_q}{\theta_i-\beta} + o(\sqrt{\Delta r_p^2 + \Delta r_q^2}) \\
 & \Updownarrow & \\
(\theta_i - \beta - c'(\frac{r_p-\beta r_q}{\theta_i-\beta}))\Delta r_p & > & -\beta c'(\frac{r_p-\beta r_q}{\theta_i-\beta})\Delta r_q + o(\sqrt{\Delta r_p^2 + \Delta r_q^2})
\end{array}$$

Thus,

[9] I use the fact that if $\min_{(p,q)\in A} q = 0$, then $(\min_{(p,q)\in A} p, \min_{(p,q)\in A} q)$ is unsold.

- if $\theta_i - \beta > c'(\frac{r_p-\beta r_q}{\theta_i-\beta})$, then adding products $(\max\limits_{(p,q)\in A} p + 2\Delta r_p, \min\limits_{(p,q)\in A} q)$, $\Delta r_p > 0$, and $(r'_p + \varepsilon, c^{-1}(r'_p))$ is profitable;
- if $\theta_i - \beta = c'(\frac{r_p-\beta r_q}{\theta_i-\beta})$ and $\max\limits_{(p,q)\in A} q < \bar{q}$, then adding $(\max\limits_{(p,q)\in A} p + 2\Delta r_p, \max\limits_{(p,q)\in A} q + 2\Delta r_q)$, $\Delta r_p > 0$, $\Delta r_q > 0$ and $(r'_p + \varepsilon, c^{-1}(r'_p))$ is profitable deviation ($\max\limits_{(p,q)\in A} p + 2\Delta r_p \geqslant c(\max\limits_{(p,q)\in A} q + 2\Delta r_q)$ if $(\max\limits_{(p,q)\in A} p, \max\limits_{(p,q)\in A} q)$ is sold);
- if $\theta_i - \beta < c'(\frac{r_p-\beta r_q}{\theta_i-\beta})$ and $\min\limits_{(p,q)\in A} p > 0$, then adding $(\min\limits_{(p,q)\in A} p + 2\Delta r_p, \min\limits_{(p,q)\in A} q)$, $\Delta r_p < 0$ and $(r'_p + \varepsilon, c^{-1}(r'_p))$ is profitable deviation;
- if $\theta_i - \beta < c'(\frac{r_p-\beta r_q}{\theta_i-\beta})$, $\max\limits_{(p,q)\in A} q < \bar{q}$ and $(\max\limits_{(p,q)\in A} p, \max\limits_{(p,q)\in A} q)$ is unsold, then adding $(\max\limits_{(p,q)\in A} p + 2\Delta r_p, \max\limits_{(p,q)\in A} q + 2\Delta r_q)$, $\Delta r_p > 0$, $\Delta r_q > 0$ and $(r'_p + \varepsilon, c^{-1}(r'_p))$ is profitable deviation.

Therefore,

(a) $\theta_i - \beta \leqslant c'(\frac{r_p-\beta r_q}{\theta_i-\beta})$,

(b) at least one of the following is true:

  i. $\theta_i - \beta = c'(\frac{r_p-\beta r_q}{\theta_i-\beta})$, $\max\limits_{(p,q)\in A} q = \bar{q}$,

  ii. $\min\limits_{(p,q)\in A} p = 0$, $\max\limits_{(p,q)\in A} q = \bar{q}$,

  iii. $\min\limits_{(p,q)\in A} p = 0$, $(\max\limits_{(p,q)\in A} p, \max\limits_{(p,q)\in A} q)$ is sold.

Suppose $\exists$ such $i$ that $(c')^{-1}\left(\frac{\theta_i-\beta}{1-\alpha}\right) < r_q < \min\left\{c^{-1}(r_p), (c')^{-1}\left(\frac{\theta_i+\beta}{1-\alpha}\right)\right\}$ and $\theta_i r_q = (1-\alpha)c(r_q) + \alpha r_p$. If $(\min\limits_{(p,q)\in A} p, \min\limits_{(p,q)\in A} q)$ is sold, then all products in menu $A$ are of the same quality. If $r'_p = r_p + \Delta r_p$ and $r'_q = r_q + \Delta r_q$, then

$$\begin{aligned} \theta_i r'_q &> (1-\alpha)c(r'_q) + \alpha r'_p \\ &\Updownarrow \\ \theta_i \Delta r_q &> (1-\alpha)c'(r_q)\Delta r_q + \alpha \Delta r_p + o(\Delta r_q) \\ &\Updownarrow \\ (\theta_i - (1-\alpha)c'(r_q))\Delta r_q + o(\Delta r_q) &> \alpha \Delta r_p \end{aligned}$$

Thus,

- if $\min\limits_{(p,q)\in A} p > 0$, $\min\limits_{(p,q)\in A} q > 0$, and $(\min\limits_{(p,q)\in A} p, \min\limits_{(p,q)\in A} q)$ is unsold, then adding products $(\min\limits_{(p,q)\in A} p + 2\Delta r_p, \min\limits_{(p,q)\in A} q + 2\Delta r_q)$, $\Delta r_p < 0$, $\Delta r_q < 0$, and $(c(r_q)' + \varepsilon, r'_q))$ is profitable;

- if $\min_{(p,q)\in A} p > 0$, $\max_{(p,q)\in A} q < \bar{q}$, and $(\min_{(p,q)\in A} p, \min_{(p,q)\in A} q)$ and $(\max_{(p,q)\in A} p, \max_{(p,q)\in A} q)$ are unsold, then adding products $(\min_{(p,q)\in A} p + 2\Delta r_p, \min_{(p,q)\in A} q)$, $(\max_{(p,q)\in A} p, \max_{(p,q)\in A} q + 2\Delta r_q)$,$\Delta r_p < 0$, $\Delta r_q > 0$, and $(c(r_q)' + \varepsilon, r_q'))$ is profitable;
- if $\theta_i > (1-\alpha)c'(r_q)$, $\max_{(p,q)\in A} q < \bar{q}$, and $(\max_{(p,q)\in A} p, \max_{(p,q)\in A} q)$ is unsold, then adding products $(\max_{(p,q)\in A} p, \max_{(p,q)\in A} q + 2\Delta r_q)$, $\Delta r_q > 0$, and $(c(r_q)' + \varepsilon, r_q'))$ is profitable;
- if $\theta_i < (1-\alpha)c'(r_q)$ and $\min_{(p,q)\in A} q > 0$, then adding products $(\min_{(p,q)\in A} p, \min_{(p,q)\in A} q + 2\Delta r_q)$, $\Delta r_q < 0$, and $(c(r_q)' + \varepsilon, r_q'))$ is profitable;
- if $\theta_i = (1-\alpha)c'(r_q)$, $\min_{(p,q)\in A} p > 0$ and $\min_{(p,q)\in A} q > 0$, then adding products $(\min_{(p,q)\in A} p + 2\Delta r_p, \min_{(p,q)\in A} q + 2\Delta r_q)$, $\Delta r_p < 0$, $\Delta r_q < 0$, and $(c(r_q)' + \varepsilon, r_q'))$ is profitable.

Therefore, at least one of the following is true:

(a) $\min_{(p,q)\in A} q = 0$, $\max_{(p,q)\in A} q = \bar{q}$,

(b) $\theta_i = (1-\alpha)c'(r_q)$, $\min_{(p,q)\in A} p = 0$,

(c) $\theta_i > (1-\alpha)c'(r_q)$, $\min_{(p,q)\in A} p = c(\bar{q})$ and all products in menu $A$ are of maximum quality,

(d) $\min_{(p,q)\in A} q = 0$, $(\max_{(p,q)\in A} p, \max_{(p,q)\in A} q)$ is sold,

(e) $\theta_i < (1-\alpha)c'(r_q)$, $\min_{(p,q)\in A} q = 0$, $\min_{(p,q)\in A} p = 0$,

(f) $\theta_i > (1-\alpha)c'(r_q)$, $\max_{(p,q)\in A} q = \bar{q}$, $\min_{(p,q)\in A} p = 0$,

(g) $\theta_i > (1-\alpha)c'(r_q)$, $\min_{(p,q)\in A} p = 0$ and $(\max_{(p,q)\in A} p, \max_{(p,q)\in A} q)$ is sold.

Suppose $\exists$ such $i$ that $\max\left\{(c')^{-1}\left(\frac{\theta_i-\beta}{1+\alpha}\right), c^{-1}(r_p)\right\} < r_q < (c')^{-1}\left(\frac{\theta_i+\beta}{1+\alpha}\right)$ and $\theta_i r_q + \alpha r_p = (1+\alpha)c(r_q)$. From the previous argument, $(\min_{(p,q)\in A} p, \min_{(p,q)\in A} q)$ is not sold. If $r_p' = r_p + \Delta r_p$ and $r_q' = r_q + \Delta r_q$, then

$$\begin{aligned} \theta_i r_q' + \alpha r_p' &> (1+\alpha)c(r_q') \\ &\Updownarrow \\ \theta_i \Delta r_q + \alpha \Delta r_p &> (1+\alpha)c'(r_q)\Delta r_q + o(\Delta r_q) \\ &\Updownarrow \\ (\theta_i - (1+\alpha)c'(r_q))\Delta r_q + o(\Delta r_q) &> -\alpha \Delta r_p \end{aligned}$$

Thus,

- if $\min_{(p,q)\in A} q > 0$, then adding products $(\min_{(p,q)\in A} p, \min_{(p,q)\in A} q + 2\Delta r_q)$, $(\max_{(p,q)\in A} p + 2\Delta r_p, \max_{(p,q)\in A} q)$, $\Delta r_p > 0$, $\Delta r_q < 0$, and $(c(r_q)' + \varepsilon, r_q'))$ is profitable;
- if $\max_{(p,q)\in A} q < \bar{q}$, then adding products $(\max_{(p,q)\in A} p + 2\Delta r_p, \max_{(p,q)\in A} q + 2\Delta r_q)$, $\Delta r_p > 0$, $\Delta r_q > 0$, and $(c(r_q)' + \varepsilon, r_q'))$ is profitable.

Therefore, $\min_{(p,q)\in A} q = 0$ and $\max_{(p,q)\in A} q = \bar{q}$.

□

Assume that in symmetric equilibrium with nonempty menu $A$ all products that offered are sold. Since such products must lie on $p = c(q)$ curve and function $c(q)$ is strictly convex, cost of the quality component of the reference point does not exceed the price component of the reference point, that is $r_q \leqslant c^{-1}(r_p)$. Therefore, if $(p,q) \in A$, then $\exists\, i$ such that

$$q \in \{(c')^{-1}(\frac{\theta_i - \beta}{1+\alpha}), (c')^{-1}(\frac{\theta_i - \beta}{1-\alpha}), (c')^{-1}(\frac{\theta_i + \beta}{1-\alpha}), r_q, c^{-1}(r_p), \bar{q}\}.$$

Denote

- $N_{+-} = \{i \in [1,n] \mid$ consumer $i$ buys $(c((c')^{-1}(\frac{\theta_i+\beta}{1-\alpha})), (c')^{-1}(\frac{\theta_i+\beta}{1-\alpha}))\}$,
- $N_{--} = \{i \in [1,n] \mid$ consumer $i$ buys $(c((c')^{-1}(\frac{\theta_i-\beta}{1-\alpha})), (c')^{-1}(\frac{\theta_i-\beta}{1-\alpha}))\}$,
- $N_{\bar{q}} = \{i \in [1,n] \mid$ consumer $i$ buys $(c(\bar{q}), \bar{q})\}$,
- $N_{-+} = \{i \in [1,n] \mid$ consumer $i$ buys $(c((c')^{-1}(\frac{\theta_i-\beta}{1+\alpha})), (c')^{-1}(\frac{\theta_i-\beta}{1+\alpha}))\} \setminus N_{\bar{q}}$,
- $N_q = \{i \in [1,n] \mid$ consumer $i$ buys $(c(r_q), r_q)\} \setminus (N_{--} \cup N_{+-})$,
- $N_p = \{i \in [1,n] \mid$ consumer $i$ buys $(r_p, c^{-1}(r_p))\} \setminus (N_{--} \cup N_{-+} \cup N_{\bar{q}})$.

It is easy to see that $N_{+-} \neq \emptyset$. Moreover, all consumers $i < i_{min}$ choose not to buy anything, all consumers $i \geqslant i_{min}$ buys something, where $i_{min} = \min\{i \in N_{+-}\}$, and

$$\min_{(p,q)\in A} q = (c')^{-1}(\frac{\theta_{i_{min}} + \beta}{1-\alpha}) \leqslant r_q,$$

$$(\theta_{i_{min}} + \beta)(c')^{-1}(\frac{\theta_{i_{min}} + \beta}{1-\alpha}) \geqslant (1-\alpha)c((c')^{-1}(\frac{\theta_{i_{min}} + \beta}{1-\alpha})) + \alpha r_p + \beta r_q.$$

Furthermore, $\max\limits_{(p,q)\in A} q$ and $\max\limits_{(p,q)\in A} p$ are the quality and the price of the product chosen[10] by the highest type consumer $n$. Therefore, the type $n$ chooses the product which has the price and the quality both above the reference values. Thus, $N_{-+} \cup N_{\bar{q}} \neq \emptyset$ and the highest type chooses $(c(q_n), q_n)$, where

$$q_n = \max_{(p,q)\in A} q = \min \left\{ (c')^{-1}(\frac{\theta_n - \beta}{1+\alpha}), \bar{q} \right\}.$$

What will happen if a firm offers additional product such that it will change the reference point?

For a fixed $i$ such that $(c')^{-1}(\frac{\theta_i+\beta}{1-\alpha}) \leqslant r_q$, as the quality and the price components of the reference point are decreasing to $(c')^{-1}(\frac{\theta_i+\beta}{1-\alpha})$ and to $c((c')^{-1}(\frac{\theta_i+\beta}{1-\alpha}))$ correspondingly, the utility of consumer $i$ from the product $(c((c')^{-1}(\frac{\theta_i+\beta}{1-\alpha})), (c')^{-1}(\frac{\theta_i+\beta}{1-\alpha}))$ is increasing. Moreover, note that if the change of reference point increase the utility of consumer $i_{min}$, then all consumers $i > i_{min}$ choose something to buy in new menu. The maximum $\Delta r_q$ that can be achieved is $\frac{1}{2}(c')^{-1}(\frac{\theta_{i_{min}}+\beta}{1-\alpha})$. The maximum $\Delta r_p$ that can be achieved is $\frac{1}{2}c((c')^{-1}(\frac{\theta_{i_{min}}+\beta}{1-\alpha}))$. Therefore, if the initial situation is equilibrium, then

- for all $i \in N_{+-}$ $(c')^{-1}(\frac{\theta_i+\beta}{1-\alpha}) + \frac{1}{2}(c')^{-1}(\frac{\theta_{i_{min}}+\beta}{1-\alpha}) \leqslant r_q$ and $(c')^{-1}(\frac{\theta_i+\beta}{1-\alpha}) \leqslant c^{-1}(r_p - \frac{1}{2}c((c')^{-1}(\frac{\theta_{i_{min}}+\beta}{1-\alpha})))$,
- if $i_{min} > 1$, then $(\theta_{i_{min}-1} + \beta)(c')^{-1}(\frac{\theta_{i_{min}-1}+\beta}{1-\alpha}) - (1-\alpha)c((c')^{-1}(\frac{\theta_{i_{min}-1}+\beta}{1-\alpha})) + \frac{\alpha}{2}c((c')^{-1}(\frac{\theta_{i_{min}}+\beta}{1-\alpha})) + \frac{\beta}{2}(c')^{-1}(\frac{\theta_{i_{min}}+\beta}{1-\alpha}) \leqslant \alpha r_p + \beta r_q$,
- if $N_{--} \neq \emptyset$, then for all $i \in N_{--}$ $(c')^{-1}(\frac{\theta_i-\beta}{1-\alpha}) \leqslant c^{-1}(r_p - \frac{1}{2}c((c')^{-1}(\frac{\theta_{i_{min}}+\beta}{1-\alpha})))$,
- $N_p = \emptyset$, $N_q = \emptyset$.

If $N_{-+} \neq \emptyset$, then for a fixed $i \in N_{-+}$, as the price component of the reference point is increasing to $c((c')^{-1}(\frac{\theta_i-\beta}{1+\alpha}))$, the utility of consumer $i$ from the chosen product $(c((c')^{-1}(\frac{\theta_i-\beta}{1+\alpha})), (c')^{-1}(\frac{\theta_i-\beta}{1+\alpha}))$ is increasing. In this case, a firm can change the "best" point for consumer $i$ to $(c(\tilde{q}), \tilde{q}) \notin A$, where $\tilde{q} = (c')^{-1}(\frac{\theta_i-\beta}{1+\alpha}) + \varepsilon$ for sufficiently small $\varepsilon > 0$. Therefore, if the initial situation is equilibrium, then $N_{-+} = \emptyset$ and $\max\limits_{(p,q)\in A} = \bar{q}$.

[10] Since menu $A$ is nonempty by assumption, at least one type of consumers is served and that is the highest type. What is more, the highest $|A|$ types are served.

If $r_q = c^{-1}(r_p)$, then menu $A$ consists of only one product $(c(q), q)$, where $q = (c')^{-1}(\frac{\theta_{i_{min}}+\beta}{1-\alpha})$. Moreover, $\frac{3}{2}(c')^{-1}(\frac{\theta_{i_{min}}+\beta}{1-\alpha}) \leqslant r_q = (c')^{-1}(\frac{\theta_{i_{min}}+\beta}{1-\alpha})$, which leads to the contradiction with $(c')^{-1}(\frac{\theta_{i_{min}}+\beta}{1-\alpha}) > 0$. Therefore, $r_q < c^{-1}(r_p)$.

To sum up,

1. $r_q = \frac{1}{2}((c')^{-1}(\frac{\theta_{i_{min}}+\beta}{1-\alpha}) + \bar{q})$, $r_p = \frac{1}{2}(c((c')^{-1}(\frac{\theta_{i_{min}}+\beta}{1-\alpha})) + c(\bar{q}))$;
2. consumer of type $i \in [1, i_{min} - 1]$ buys nothing, consumer of type $i \in [i_{min}, i_{mid} - 1]$ buys $(c((c')^{-1}(\frac{\theta_i+\beta}{1-\alpha})), (c')^{-1}(\frac{\theta_i+\beta}{1-\alpha}))$, consumer of type $i \in [i_{mid}, i_{max} - 1]$ buys $(c((c')^{-1}(\frac{\theta_i-\beta}{1-\alpha})), (c')^{-1}(\frac{\theta_i-\beta}{1-\alpha}))$, and consumer of type $i \in [i_{max}, n]$ buys $(c(\bar{q}), \bar{q})$, where $1 \leqslant i_{min} < i_{mid} \leqslant i_{max} \leqslant n$;
3. $(\theta_{i_{min}} + \frac{\beta}{2})(c')^{-1}(\frac{\theta_{i_{min}}+\beta}{1-\alpha}) \geqslant (1 - \frac{\alpha}{2})c((c')^{-1}(\frac{\theta_{i_{min}}+\beta}{1-\alpha})) + \frac{\alpha c(\bar{q})}{2} + \frac{\beta\bar{q}}{2}$;
4. if $i_{min} > 1$, then $(\theta_{i_{min}-1} + \beta)(c')^{-1}(\frac{\theta_{i_{min}-1}+\beta}{1-\alpha}) - (1 - \alpha)c((c')^{-1}(\frac{\theta_{i_{min}-1}+\beta}{1-\alpha})) \leqslant \frac{\alpha c(\bar{q})}{2} + \frac{\beta\bar{q}}{2}$;
5. $(c')^{-1}(\frac{\theta_{i_{mid}-1}+\beta}{1-\alpha}) \leqslant \min\left\{c^{-1}\left(\frac{c(\bar{q})}{2}\right), \frac{\bar{q}}{2}\right\}$;
6. if $i_{mid} < i_{max}$, then $\frac{1}{2}(c')^{-1}(\frac{\theta_{i_{min}}+\beta}{1-\alpha}) + \frac{\bar{q}}{2} \leqslant (c')^{-1}(\frac{\theta_{i_{mid}}-\beta}{1-\alpha})$ and $(c')^{-1}(\frac{\theta_{i_{max}-1}-\beta}{1-\alpha}) \leqslant c^{-1}\left(\frac{c(\bar{q})}{2}\right)$;
7. $(c')^{-1}(\frac{\theta_{i_{max}}-\beta}{1+\alpha}) \geqslant \bar{q}$.

Note that all potentially profitable deviation were considered. Therefore, these conditions are not only necessary, but also sufficient.

## B.4 Proof for Proposition 3.3

Let $A$ be a menu where $(0, 0) \in A$ and $(c')^{-1}(\frac{\theta_n+\beta}{1-\alpha}) = \min\{r_q, c^{-1}(r_p)\}$. Then for all $i$ consumer $i$ buys either $(c((c')^{-1}(\frac{\theta_i+\beta}{1-\alpha})), (c')^{-1}(\frac{\theta_i+\beta}{1-\alpha}))$, or nothing.

The condition $(\theta_n + \beta)(c')^{-1}(\frac{\theta_n+\beta}{1-\alpha}) - (1 - \alpha)c((c')^{-1}(\frac{\theta_n+\beta}{1-\alpha})) - \alpha r_p - \beta r_q \geqslant 0$ guarantees that the highest type will buy something. To sum up, conditions on $r_p$ and $r_q$:

$$\begin{cases} (c')^{-1}(\frac{\theta_n+\beta}{1-\alpha}) = \min\{r_q, c^{-1}(r_p)\} \\ 2r_q \leqslant \bar{q} \\ (\theta_n + \beta)(c')^{-1}(\frac{\theta_n+\beta}{1-\alpha}) - (1 - \alpha)c((c')^{-1}(\frac{\theta_n+\beta}{1-\alpha})) - \alpha r_p - \beta r_q \geqslant 0 \\ 2\theta_n r_q - 2r_p \leqslant (\theta_n + \beta)(c')^{-1}(\frac{\theta_n+\beta}{1-\alpha}) - (1 - \alpha)c((c')^{-1}(\frac{\theta_n+\beta}{1-\alpha})) \end{cases}$$

The last condition tells that $(2r_p = \max\limits_{(p,q)\in A} p, 2r_q = \max\limits_{(p,q)\in A} q)$ is not sold.

So, the desired equilibrium exists if there exist $r_p$ and $r_q$ such that

$$\left[\begin{array}{l} \begin{cases} (c')^{-1}(\frac{\theta_n+\beta}{1-\alpha}) = r_q \leqslant c^{-1}(r_p) \\ 2r_q \leqslant \bar{q} \\ \theta_n(c')^{-1}(\frac{\theta_n+\beta}{1-\alpha}) - (1-\alpha)c((c')^{-1}(\frac{\theta_n+\beta}{1-\alpha})) \geqslant \alpha r_p \\ (\theta_n-\beta)(c')^{-1}(\frac{\theta_n+\beta}{1-\alpha}) + (1-\alpha)c((c')^{-1}(\frac{\theta_n+\beta}{1-\alpha})) \leqslant 2r_p \end{cases} \\ \begin{cases} (c')^{-1}(\frac{\theta_n+\beta}{1-\alpha}) = c^{-1}(r_p) \leqslant r_q \\ 2r_q \leqslant \bar{q} \\ (\theta_n+\beta)(c')^{-1}(\frac{\theta_n+\beta}{1-\alpha}) - c((c')^{-1}(\frac{\theta_n+\beta}{1-\alpha})) - \beta r_q \geqslant 0 \\ 2\theta_n r_q \leqslant (\theta_n+\beta)(c')^{-1}(\frac{\theta_n+\beta}{1-\alpha}) + (1+\alpha)c((c')^{-1}(\frac{\theta_n+\beta}{1-\alpha})) \end{cases} \end{array}\right.$$

which is equivalent to

$$\left[\begin{array}{l} \begin{cases} c((c')^{-1}(\frac{\theta_n+\beta}{1-\alpha})) \leqslant \theta_n(c')^{-1}(\frac{\theta_n+\beta}{1-\alpha}) \\ 2(c')^{-1}(\frac{\theta_n+\beta}{1-\alpha}) \leqslant \bar{q} \\ (2+\alpha)(1-\alpha)c((c')^{-1}(\frac{\theta_n+\beta}{1-\alpha})) \leqslant (2\theta_n - \alpha\theta_n + \alpha\beta)(c')^{-1}(\frac{\theta_n+\beta}{1-\alpha}) \end{cases} \\ \begin{cases} c((c')^{-1}(\frac{\theta_n+\beta}{1-\alpha})) \leqslant \theta_n(c')^{-1}(\frac{\theta_n+\beta}{1-\alpha}) \\ 2(c')^{-1}(\frac{\theta_n+\beta}{1-\alpha}) \leqslant \bar{q} \\ (\theta_n-\beta)(c')^{-1}(\frac{\theta_n+\beta}{1-\alpha}) \leqslant (1+\alpha)c((c')^{-1}(\frac{\theta_n+\beta}{1-\alpha})) \end{cases} \end{array}\right.$$

Since

$$\frac{\theta_n - \beta}{1+\alpha} < \frac{2\theta_n - \alpha\theta_n + \alpha\beta}{(2+\alpha)(1-\alpha)},$$

these conditions are reduced to

$$c\left((c')^{-1}\left(\frac{\theta_n+\beta}{1-\alpha}\right)\right) \leqslant \theta_n(c')^{-1}\left(\frac{\theta_n+\beta}{1-\alpha}\right) \leqslant \frac{\theta_n\bar{q}}{2}.$$