РЭШ
Российская экономическая школа

МАГИСТЕРСКАЯ ДИССЕРТАЦИЯ

MASTER THESIS

Тема: Дизайн рынков контекстной рекламы

Title: Market for Sponsored Links: Mechanism Design Approach

Студент/ Student:

Архангельский Д.И./Arkhangelsky D.I.
(Ф.И.О. студента, выполнившего работу)

Научный руководитель/ Advisor:

Измалков С.Б./Izmalkov S.B.
(ученая степень, звание, место работы, Ф.И.О.)

Оценка/ Grade:

____

Подпись/ Signature:

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Москва 2012

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Market for sponsored links: mechanism design approach

Dmitry Arkhangelsky

Abstract

Sponsored search auctions are the main source of revenue for search engines around the world. In this work I use mechanism design approach to construct efficient market for sponsored links in asymmetric information environment. I explicitly model both sides of the market: users and advertisers. In my model users do not observe true quality of advertised products prior to click and make inferences about this quality. I show that there is an equilibrium in which users believe that higher positions are better than lower ones. This in turn generates a class of advertisers' preferences that contains the preferences considered before in the literature (e.g. Edelman et al. (2007) and Börgers et al. (2008)). I construct a simple ascending auction that allocates sponsored links efficiently given these preferences. I use my auction to analyze the properties of the GSP auction. I show that GSP auction can fail to allocate efficiently if conversion rates are lower at higher positions. I use my model to derive several properties of the optimal mechanism.

1 Introduction

1.1 Motivation

In the last decade sponsored search technologies have been the main source of the revenue of the most search engines around the world. Google, Yahoo around the world, Yandex in Russia, all of these companies earned billions of dollars selling advertising positions. Their business model is quite simple: when users search for some words, e.g., "windows in Moscow" several first results are *sponsored* links that are showed together with *organic* links, i.e., companies have paid to search engine to show these links at the top of the page. Sponsored links are vertically ordered, so there is a top one, a second one, etc. Advantages of such strategy are quite clear: users see only those ads that they might be interested in, while companies can target their potential customers better by using more special word combinations.

In this scheme a search engine can be seen as a mediator between the two parts: users and advertisers (firms, producers, bidders). So, sponsored search might be considered as an example of a *two-sided* market. Pricing in such markets is an issue: in our context one needs to define how to place ads of different advertisers, how often to show them, and how much (and whom) to charge for it. From economic standpoint all these problems are problems of mechanism design with search engine being a designer.

In retrospective, these design problems were addressed in different ways.[1] Today, the most popular mechanism is a Generalized Second Price (GSP) auction. It works roughly in

[1]See short history of sponsored search in Edelman et al. (2007).

the following fashion: at each moment of time each advertiser makes a bid that represents a price that he is ready to pay for a click at the ad, then advertisers are ordered according to their bids, top bidders get higher positions and pay the lowest bid they could have made but still win the same position. It is easy to see that each winning bidder pays the bid of the advertiser that is just below him, in a sense, bidder pays the second highest price for her position, explaining the idea behind the name of the auction format.

To carefully analyze the just described scheme one needs to use the apparatus of the auction theory. In the last 5-10 years much theoretical and empirical work has been done in this direction (see Section 1.2), yet there are still lots of interesting questions. Most authors considered only some fragments of the sponsored search market, e.g., only advertisers. The most cited works (e.g., Edelman et al. (2007),Varian (2007)) postulate some simple form of advertisers' preferences and analyze properties of equilibria in GSP or similar auction format. One important assumption about the preferences is constant *value-per-click*. This effectively means that advertisers believe that users who click at different positions are the same.

One can argue that assumption of constant value per click is quite restrictive. For example, consider the following story. Assume that due to some behavioral reasons top positions receive a lot of "noisy" clicks, i.e., users click on these links but the fraction of those who actually buy something (this fraction is called *conversion rate*) is rather low. This in turn implies that value per click is lower at higher positions. One can expect that GSP auction might not work well in this case since advertisers pay for a click and clicks at lower positions are more valuable comparing to those at top ones. On the other hand top positions receive more clicks, so expected gain from these positions can be higher than from the lower ones.[2]

This simple story shows that advertisers' preferences can be quite different and equilibrium properties of auctions might depend on these preferences. One important thing to notice is the fact that the preferences of advertisers depend on the properties of the users' behavior. In the story I assumed that conversion rates are lower at higher positions and this assumption changed incentives of advertisers. On the other hand, one can assume that conversion rates are higher at higher positions and analyze the game for this case.[3] It is clear that the results depend on this assumption and that both assumptions are in some sense plausible.

The discussion above indicates that in order to understand incentives of advertisers one needs to understand a behavior of users. This behavior can be quite different. For example, users might believe that advertisers at higher positions are in some sense *better* and click more at higher positions. On the other hand, users can believe that advertisers are selected at random, and, consequently, all positions are the same and users then can click at random. Users' beliefs, however, might not be exogenous and can be implied by the fact that search engine places better advertisers at higher places (via some mechanism).

Building on this discussion one can tell the following story: if users believe that advertisers at higher positions are in some sense better then they click more at these positions, consequently advertisers value top positions higher and there is a tough competition for them. Search engine exploits this competition and give top positions to the best advertisers. Consequently, users' beliefs are actually rational.

This story is quite appealing and intuitive but its every step requires clarification. What

[2]Unconditional probability of click is called *click-trough-rate* (CTR). It is usually assumed that higher positions have higher CTR-s.

[3]This is actually the case considered in Börgers et al. (2008).

does *better* means? How users search? What mechanism should search engine use in order to place better advertisers at top positions and does such mechanism exist? What is clear is the fact that in order to answer these questions one needs a framework that includes users, advertiser and a search engine.

One attempt to answer all these questions is presented in Athey and Ellison (2011) and looks like the following: imagine that users want to buy some good and that they value this good in 2$. Producers' price is equal to 1$. All producers can have this good with some exogenous probability (this probability can be interpreted as a conversion rate). Then, better producers are those who have the good with higher probability. Users do not observe this probability, instead they can search for good and stop once they found it. Users believe that better producers are placed at higher positions and search in top-down manner. Advertisers anticipate it and compete for higher positions. Properly defined GSP auction then places advertisers with higher conversion rates at higher positions. Consequently, users' believe are rational.

The just described model is highly stylized but it provides a lot of insights. It rationalizes users' behavior and highlights the role of search engine as an information aggregator. On the other hand, several aspects of the model are misleading and ad hoc in nature. First, while advertisers are different (they have different probabilities of successful trade), their prices and consequently consumer surplus are the same. Next, conversion rates are just exogenously given and depend on positions only in equilibrium. Advertisers' preferences, on the other hand, are quite complex in this story, since they have *interdependent* valuation. This fact makes it difficult to compare the results and insights from the simple reduced form models and this equilibrium model.

In this work I try to address the problems indicated in the previous discussion and construct a simple equilibrium model that includes users, advertisers and a search engine. In contrast with Athey and Ellison (2011) I build a model that endogenizes prices and conversion rates, constructing them from the primitives. I assume that advertisers differ in the quality of their goods. Different advertisers have different prices. I assume that advertisers set their prices at some primary market which has the same characteristics as the on-line market. In equilibrium, producers with higher quality are *better* meaning that they generate a higher consumer surplus for a random user.

Users in my model have idiosyncratic shocks for different advertisers. They do not observe quality of the product and its price, instead they make some inferences about it. In order to simplify the analysis and focus on the main properties of the model, I assume that users can click at most once. In the same time, users' behavior is rational, meaning that they click at the position with the highest expected consumer surplus. I show that there exist an equilibrium in which users rationally believe that *ceteris paribus* their consumer surplus is higher at higher positions. On the other hand, since users have idiosyncratic shocks for advertisers they might actually prefer a lower position to a higher one. [4]

In the proposed equilibrium top positions have higher CTR-s but not necessarily higher conversion rates. Interestingly, in equilibrium my model generates advertisers' preferences that have the same *structural* properties as those considered in Edelman et al. (2007), Varian

[4]This is not true in Athey and Ellison (2011) where all users search in top-down manner starting from the first position.

(2007) and Börgers et al. (2008). As I show, from the mechanism design point of view, the main property of the preferences is *submodularity*: *better* advertisers gain more from increase in position. This fact aligns competition and efficiency and I can use a simple ascending auction similar to GSP in order to allocate positions efficiently.

It is important to note that the proposed ascending auction is different from GSP in one important aspect. In GSP advertisers make bids for a click, while in my ascending auction advertisers make bids for an *impression*. In the literature one can find some discussion about the difference between per click and per impression approach (e.g., see Edelman et al. (2007)). In practice per click pricing seems to be the most reasonable: advertisers pay for the event they observe and click is the best approximation for a purchase. On the other hand, as I discussed above, there might be situations in which higher positions have lower CTR-s and as I show in the paper this fact might lead to inefficient allocation in GSP. Note, that in my model higher conversion rates for higher positions are not guaranteed and consequently GSP might fail to allocate efficiently.[5]

I use structural properties of advertisers' preferences generated by my model in order to address important problem of optimal mechanism design. It appears that this problem has no simple general solution and thus I just derive some properties of the optimal mechanism. It appears that one of the main factors in this problem that was almost ignored before in the literature (e.g., Edelman and Schwarz (2010)) is the relationship between an aggregate CTR of all sponsored search links and a number of this links. I show that if one assumes that total CTR does not depend on the number of positions then it is optimal to sell just one position via second price auction with reserve price. Obviously, this is just an extreme result, since it is reasonable to assume that total CTR increases with the number of positions.[6]

I see my contribution to the literature on position auctions in several things. First, I constructed a new theoretical framework that can be used in further analysis. This framework is quite simple yet general and one can, in principle, incorporate additional structure into it. One important and appealing property of my model is the fact that in equilibrium it generates advertisers' preferences that are similar to those considered in the literature before. Consequently, my model can be seen as a rational and equilibrium justification of the simple reduced form models. Next, I proposed a simple auction format that can be used for efficient allocation. This auction format can be used in theoretical analysis (one can derive equilibrium properties of GSP using this format). Since the proposed auction is quite robust (it works for any submodlar preferences) it actually might be used in practice. My theoretical results also bring new arguments for per impression pricing schemes.

My results on optimal mechanism design can be seen as a contribution to the current discussion of the topic. I show that optimal mechanism problem is more complicated that it was previously thought (Edelman and Schwarz (2010)) and the optimal mechanism actually might be quite sophisticated. Since in reality it is important to have simple form mechanisms it might be important to analyze approximately optimal but robust schemes. Derived properties of optimal mechanism emphasize the most important concepts one need to analyze in order to allocate optimally (virtual valuations and aggregate CTR-s).

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[5]It is insightful to compare this fact with the results and assumptions from Athey and Ellison (2011).

[6]In this context one can see the results of Edelman and Schwarz (2010) as another extreme.

1.2 Literature Review

Starting from 2002 most position auctions are conducted by means of the generalized second price auction (GSP). In this auction each bidder places a sealed bid that represents her payment per click. These bids then are ordered, the K highest bidders are selected, i-th bidder is awarded i-th position; payment of each selected bidder is defined as a minimal bid she has to place to retain a current position, i.e., the $i+1$ bid. Actual auctions are conducted in a similar but distinct fashion: each bidder has an assigned score that depends on her identity and the place won; after submitting, all bids are ordered by their value times their score and k highest bidders are selected from this order. Payment is defined in the same way, but in this case, it's not equal the next highest bid.

In the literature there are several approaches to GSP auctions. The first one is to consider GSP auction as the game with the full information (e.g. Varian (2007),Edelman et al. (2007),Aggarwal et al. (2006) and Börgers et al. (2008)), the second one is to consider it as a game with imperfect information (Gomes and Sweeney (2009)), and the third one is to analyze another auction which has the same equilibrium properties (Edelman et al. (2007), Athey and Ellison (2011)). This auction is called Generalized English auction (GE). In a standard case this auction has an ex post equilibrium that resembles a particular equilibrium of GSP in a complete information game.[7]

The rules of the GE auction are quite simple: the click price is continuously ascending, each bidder indicates her willingness to participate via some device, the price for the j position is equal to the lowest price at which the demand is equal to j (i.e. the price at which the $(N-j)$-th player bowed herself out of the game). In case of one position the auction is the same as the usual English auction. Note, however, that the notion of Generalized English auction is also used in the context of multi-unit auctions (for identical objects), where it has different rules (price rises until demand is equal supply, then each object is sold at this price) and different equilibrium features.[8]

The main result from the firs works on GSP (Edelman et al. (2007); Varian (2007); Aggarwal et al. (2006)) can be formulated in the following way: GSP is not thoughtful mechanism but it has an ex-post equilibrium in which poisitions are allocated efficiently and payments are equal to those in Vickrey-Clark-Grooves mechanism (VCG).[9]

After the first works in the area, second wave of papers can be divided into two parts: the first part somehow changes the initial model, allowing for slightly different designs and preferences (incorporates scores, budget constraints, etc.); the second part tries to bring the sponsored search auctions into more general model, endogenizing users behavior and providing rationale for behavior of advertisers. The important paper that can be related to the first part is Börgers et al. (2008) where authors both allow for more general preferences and ad scores. The results of this paper resemble some of my results. Their analysis, however, is devoted to GSP with full information. They demonstrate the presence of equilibria with very low revenues and less obvious bidders' behavior. Another relevant paper, that uses classical setup but considers design issues is Edelman and Schwarz (2010). In this work

[7] See Varian (2007) and Edelman et al. (2007) for details

[8] The usual references are the works of Ausubel and Crampton on multi-unit auctions (see, e.g., Ausubel and Cramton (2002)).

[9] For the formulation and analysis of VCG mechanism see Krishna (2009).

authors analyze optimal design problems, find out that Generalized English auction with reserve price is optimal and use this fact to justify particular equilibrium of GSP mechanism.

Another methodologically important paper is Gomes and Sweeney (2009). In this paper authors analyze GSP in incomplete information setting. They use the Revenue Equivalence Theorem (Myerson (1981)) to construct an equilibrium in monotone strategies and show that this equilibrium might fail to exist in some cases. These results are challenging from the theoretical point of view, but on the other hand, it seems that it might be not methodologically correct to model GSP as a game of incomplete information. In reality all advertisers have a lot of information about other bids and what is more important it is not very costly for them to experiment with their bids and gain additional information.

Another papers that might be attributed to the second strand of literature are the works of Athey and Ellison (2011); Chen and He (2006) and Gomes (2010). These papers analyze both sides of the market, assuming different things about users' behavior. The most important and relevant paper is Athey and Ellison (2011) where authors explicitly model sophisticated search behavior of users. In their setting in equilibrium users search in the top-down manner. Advertisers in their model differ in exogenously given probability of successful trade. Essentially each advertiser has a fixed conversion rate. Authors also analyze optimal reserve prices and show that in some cases there is no trade-off between efficiency and optimality.

1.3 Outline

The paper proceed in the following way: section 2.1 develops a theoretical framework. Next I analyze users' and producers' behavior in a symmetric setting and then I focus on a mechanism design problem. I describe an efficient mechanism and show that simple ascending auction can be used for its implementation. Next I derive some properties of the auction explaining its connections with the GE auction. In section 3 I consider a general mechanism design probelm and derive several properties of optiml mechanism. I completely characterize optimal mechanism for a simple case when aggregate CTR does not depend on the number of positions. Section 4 concludes.

2 Model

This section develops a general framework for an analysis of sponsored search markets. First, I build a general model and then analyze a particular case with symmetric advertisers. For this case I derive several properties of users' and advertisers' preferences and construct an efficient mechanism.

2.1 Setup

My model consists of three parts: users, advertisers and search engine. In this subsection I describe the structure of users' preferences, pricing decision of advertisers and possible goals of search engine.

2.1.1 Users

There is a continuum of consumers (users) I, where I is a measurable space with total mass equal to one, with i denoting a particular user. Each user is ready to buy at most one unit of some indivisible good. They search for this good and observe K sponsored positions (links), each position being an ad space. This positions are vertically ordered: there is a top position, the second one and so on. In what follows I do not distinguish between an ad and company that placed it.[10]

Each link is associated with some company j from a measurable space J. Quality (in monetary terms) of the good that company j sells might be a priori different, so I denote it by η_j. I assume that each η_j is a random variable with absolutely continuous distribution (measure) that I denote by μ_j. Support of η_j is some open interval in $\mathbb{R}_+$. Obviously, one can represent η_j is the following way:

$$\eta_j = \mathbb{E}_{\mu_j}\{\eta_j\} + \theta_j \tag{1}$$

where the first part is a systematic one, while the second one is a deviation from the average. By construction we have that $\mathbb{E}_{\mu_j}\{\theta_j\} = 0$. I denote a systematic part of the quality of the firm j by $t_j = \mathbb{E}_{\mu_j}\{\eta_j\}$. I assume that all η_j are independent in j although this assumption is not essential.

Each user has preferences over the different goods. Utility (in monetary terms) of user i from a good j is denoted by u_{ij}. I assume that this utility has the following structure:

$$u_{ij} = t_j + \theta_j + \varepsilon_{ij} \tag{2}$$

where ε_{ij} is an idiosyncratic shock that consumer i has for a firm j. Formally ε_{ij} is a measurable function from a product space $J \otimes I$ with some product measure. Note that this shock is a function from $I \otimes J$ not from $I \otimes J \otimes \mathbb{R}_+$. What this effectively means is that a consumer has a shock for a firm not for a particular realization of the product of this firm. This assumption is obviously with loss of generality but it seems to be a reasonable approximation of the true state of the world.[11]

If we control for j then ε_{ij} is a random variable with distribution (image measure) that I denote by λ_j.[12] I assume that $\lambda_j(\cdot)$ is absolutely continuous with respect to Lebesgue measure. For consistency in definition of η_j one should have that $\mathbb{E}_{\lambda_j}\{\varepsilon_{ij}\} = 0$, otherwise η_j cannot represent an average quality. With this restriction we have that $\mathbb{E}_{\lambda_j}\{u_{ij}\} = \eta_j$ so it is correct to think of η_j as an average value of a particular good j.

When a user i observes an ad that company j has placed she does not know her utility level from this product. Instead, I assume that she observes only some part of the utility, namely $\xi_{ij} = t_j + \varepsilon_{ij}$, while another part which is equal to θ_j remains unknown until the user

[10] In reality one company can have many advertising campaigns. In this work I focus on one such campaign, so there is no need to distinguish between ads and firms.

[11] One can relax this assumption and assume that ε depends on the quality. In this case consumers a priori observe only expected shock, and observe the whole ε only after the click. With certain assumptions on the dependence of ε and θ (like affiliation) one can get the same results. This brings not so much into the model but requires some additional notation.

[12] Single index denotes the part that is hold fixed.

actually clicks at the link and visit the site. Now we can represent utility of user i from an ad j in the following way:

$$u_{ij} = \xi_{ij} + \theta_j \tag{3}$$

Equation (3) represented utility levels in a way that is the most convenient for the analysis and for the most part I will be working with this form. The notation might seem a little cumbersome but an actual story behind it is quite simple. Imagine that you search for some good and observe several ads. At that time you obviously cannot tell how much you are ready to pay for these goods because you observe only ads. But at the same time you already have some ideas about these goods, so you have some ex ante utility that in my case is represented by ξ_{ij}. If you click and observe actual characteristics of these goods then you observe a true utility level, in my case – u_{ij}.

Given preferences defined this way consumers search for the best alternative. For the most part of the work I assume that consumers' search is the most simple as possible: they can click only once. This assumption is very restrictive and obviously has nothing to do with the reality. On the other hand, it greatly simplifies analysis and might be some approximation of the reality. If we look at the real data then the number of users who clicked at more than one ad is several times smaller than the number of users who clicked only once. Of course this fact does not justify my assumption since such behavior might easily be endogenous but at least this evidence provides some motivation for the assumption.

2.1.2 Advertisers

As it was noted above, there is a set of firms J, with j as a generic firm. Each firm sells one good with a priori random quality $\theta_j + t_j$ where θ_j is private knowledge of the firm j, while t_j is a common knowledge. Each firm might be characterized then by a tuple (t_j, θ_j).

Each firm has a primary market at which it sells the goods. The fact that this market is primary means that firms set their prices in order to serve this market. Additionally each firm can participate in the Internet market where it can advertise its goods via sponsored links. Note that by construction pricing and advertising decisions are divided here: prices are determined solely at the primary market and do not depend on the Internet campaign and more specifically they do not depend on the mechanism that a search engine uses to sell sponsored links. Such structure might seem a little bit irrational – of course advertising and pricing decisions are interconnected, but, on the other hand, it seems that in reality for most firms on-line market (or on-line customers)constitute only some fraction of an actual demand, so pricing decisions indeed might be considered as independent from advertising ones.

I do not build a detailed model for a primary market in this work, since my interest is in the on-line market. On the other hand, behavior of the firms in the on-line markets depends on their pricing strategies. In order to make things as simple as possible, I assume that a primary market has the most easiest structure possible: each firm behave like a monopolist maximizing its profits; also I assume zero marginal costs. So, if I denote by $D(p, \theta, t)$ a demand for a firm with a type (t, θ) then its problem has the following form:

$$D(p, \theta, t)p \to \max_p \tag{4}$$

This problem requires some comments. First, one needs to describe where the function $D(p, \theta, t)$ comes from. In order to be consistent, I assume that customers at primary markets and on-line markets are the same, meaning that the distribution of their reservation values is the same. Of course, in some cases this is not true and these markets are really different. In this case my pricing assumption might be inadequate. On the other hand, as the first approximation one can think that the customers are the same.

I assume that at the primary market each customer behave in the following way: she searches for goods and she searches in a random way. I assume that all customers solve the following problem: they observe a firm, its good and its price and if their reservation utility is high enough then they buy the good. This behavior might be justified using some more complicated dynamic search model, but that is no my goal here. With this shortcut solution, the demand for each firm has the following structure:

$$D(p, \theta_j, t_j) = \mathbb{P}_{\lambda_j}\{\varepsilon_{ij} + t_j + \theta_j > p + c\} \tag{5}$$

where c is an outside option; I use λ_j as a measure since the distribution of customers at primary and on-line market is the same.

2.1.3 Search engine

Search engine plays a role of a mediator between users and advertisers.[13] What it actually does is that it determines the rules (or designs a mechanism) according to which advertisers are selected and displayed ones a user searches for some keyword.

In practice the usual mechanism is some kind of an auction, typically some variation on Generalized Second Price (GSP) auction. In GSP each advertiser makes a bid, these bids are ordered, advertisers with highest bids are selected and displayed in respecting order. Each advertiser pays the second highest bid (with respect to this order).[14]

In my case I do not focus a priori on GSP, instead I consider general mechanism design problem. In order to solve this problem one should have some criteria to discriminate between different mechanisms. Two typical criteria are efficiency and optimality. As usual, optimality refers to the highest expected revenue for a search engine, while efficiency – to the social optimum.

Note that this case differs from the usual mechanism design problem since modifications in design might easily affect preferences of the participants (advertisers). For example, assume that for some reasons advertisers believe that users click more on higher positions, then advertisers have incentives to compete for higher positions and eventually better advertisers might occupy higher positions and as a consequence higher positions now become more attractive for users. On the other hand, if users a priori believe that advertisers are placed at random and click at all positions in the same way then advertisers have no incentives to compete for higher positions. So, the change in a mechanism affect not only incentives but also preferences of participants on the final outcomes.

[13] In literature on two-sided markets such mediators are usually called *platforms*.

[14] See Edelman et al. (2007),Varian (2007) and Aggarwal et al. (2006).

2.2 Symmetric advertisers

In this section I analyze a particular case which seems to be the first one consider. Formally, I assume that qualities of products defined in 2.1.1 have a particular structure: $\eta_j = t + \theta_j$, where all θ_j are i.i.d. random variables. What this effectively means is that all advertisers are a priori the same.

Also I assume that ε_{ij} has a particular distribution: image measures λ_j are jointly independent and distribution functions that they generate are the same. So ε_{ij} are i.i.d. random variables in j. Hence, one can say that it is the most symmetric case possible: advertisers are symmetric and users treat them as symmetric as well. Nevertheless, this case is interesting since it reflects the main features of the general problem.

For this section I also make the assumption mentioned in 2.1.1 about search model: all users can click only once and have no costs of clicking.

2.2.1 User's behavior

Since we restricted the number of clicks to be at most one and assumed away all search costs, user's problem is relatively simple: decide on which position to click. Remember from Section 2.1.1 that users' utility functions have two parts: observable and unobservable. As we assumed above each firm has the same average quality so what really matters is an unobservable deviation of the real quality from the average one (θ_j) and idiosyncratic shocks. Since in this case θ_j completely describes the firm I will refer to it as its *type*.

From the section 2.1.2 we know that each firm has some pricing function $p(t_j + \theta_j)$. Since there is no variation in t_j, I denote this function simply by $p(\theta_j)$. Consumers do not observe the prices, but have degenerate beliefs about it: they believe that all firms charge prices $p(\theta_j)$. Taking this into account one can say a user should solve the following problem:

$$\text{find } k^\star \in \arg\max_{k \in K} \{\mathbb{E}_k \left\{\max\{u_{ij(k)} - p(\theta_{j(k)}); 0\}\right\} \tag{6}$$

where k is a number of position (so $k = 1$ means that it is the top position) and $j(k)$ stand for index of the firm at k-th position.

Problem (6) requires some explanation. First of all, note that under expectation operator we have maximum function. This reflects the fact that nobody can force a user to buy a product if it is not valuable enough for her, so in the worst case consumer has zero utility level. Next, it is important to explain why expectation operator has an index k. This index reflects the fact that users have different beliefs about firms at different positions. Since the only thing that users do not know about firms is their types this effectively means that users have different beliefs about θ-s. These beliefs comes from the mechanism that search engine uses to place and order advertisers. We have not defined a mechanism yet, but from now on I will be seeking for an equilibrium in which users believe that higher positions are occupied with the firms with higher θ-s. As I show later there is such an equilibrium, but for now this is just an assumption.

Assumption 1: Assume that users believe that $\theta_{j(k)} < \theta_{j(k-1)}$ almost surely.[15]

Assumption 1 is very important for the analysis since it generates a considerable structure on the payoffs. Denote by $f(\varepsilon_{ij}, \theta_{j(k)})$ a payoff of the user if she clicks at position k, that is

$$f(\varepsilon_{ij(k)}, \theta_{j(k)}) = \max\{t + \varepsilon_{ij(k)} + \theta_{j(k)} - p(\theta_{j(k)}); 0\}$$

a priori it is not clear whether $f(\varepsilon_{ij(k)}, \theta_{j(k)})$ is increasing in its second argument. For this to be true I need to assume that $\theta - p(\theta)$ is an increasing function. In the next subsection I show that this is indeed the case, but for now it is just an assumption.

Assumption 2: Assume that $\theta - p(\theta)$ is an increasing function of θ.

With the second assumption at hand, I denote an expected payoff of a user if she clicks at k-th position by $f(\varepsilon, k)$:

$$f(\varepsilon, k) = \mathbb{E}_k\{f(\varepsilon, \theta)\} \tag{7}$$

The following Proposition is quite simple but important.

Proposition 2.2.1. *In the symmetric model under Assumption 1 and 2 expected payoff of a user is decreasing in k and increasing in ε*

Proof. By Assumption 1 distribution of $\theta_{(k)}$ FOSD distribution of $\theta_{(k+1)}$ and by Assumption 2 $f(\varepsilon, \theta)$ is increasing in θ, consequently (by properties of the domination) $f(\varepsilon, k)$ is increasing in k. The fact that it increases in ε is trivial. □

Proposition 2.2.1 effectively tells us that users tend to prefer higher positions to lower ones. But it describes the behavior of just one user. What we need is an aggregate demand for product j. Obviously, it can be expressed in the following form:

$$D(p(\theta) - \theta, k) = \mathbb{P}\{k = \arg\max_l\{f(\varepsilon(l), l)\}; t + \varepsilon > p(\theta) - \theta\} \tag{8}$$

The first condition tells us that a user has selected the k-th position and the second one – that she is ready to buy the good. In order to derive some features of this expected demand, I need to introduce the following notation. Denote by $g(\varepsilon, k, l)$ the infimum value of ε_l such that $f(\varepsilon_k, k) = f(\varepsilon_l, l)$, that is $g(\varepsilon, k, l) = \inf_{\varepsilon_l}\{f(\varepsilon_l, l) > f(\varepsilon, k)\}$. This infimum exists due to monotonicity of functions $f(\varepsilon, k)$ in ε. In words $g(\varepsilon, k, l)$ is the value of ε_l consumer should have to be indifferent between position k with ε and l with ε_l. With this notation at hand we are ready to state the following Proposition.

Proposition 2.2.2. *In the symmetric model under Assumption 1 and 2 expected demand is decreasing in k and $D_1(p(\theta) - \theta, k)$ is increasing in k.*

[15] There is a trick here, since in order to specify these beliefs from the basics we need to assume that users actually know what is the number of competitors in the mechanism. Formally, we might be tempted to say that $\theta_j(k)$ is the k-highest statistics but in this case the number of competitors becomes important. In reality users do not know this number, so we need to specify their beliefs about competition. As it really has no effect on anything that follows I do not do this. It is not hard but technically demanding and as far as number of competitors is not near infinity this really doesn't matter.

Proof. First, we can express expected demand in the following way:

$$D(p(\theta)-\theta,k)=\mathbb{E}_{\lambda_{j(k)}}\left\{\mathbb{P}_{\lambda_{-j(k)}}\{\varepsilon_l\le g(\varepsilon,k,l)\text{ for each } l\neq k\mid\varepsilon\}\mathbb{I}\{t+\varepsilon>p(\theta)-\theta\}\right\} \tag{9}$$

First, it is important to note that $g(\varepsilon,k,l)$ is decreasing in k due to Proposition 2.2.1. It follows from the fact that $f(\varepsilon,k)$ is decreasing in k and increasing in ε and hence $\inf_{\varepsilon_l}\{f(\varepsilon_l,l)>f(\varepsilon,k)\}$ is weakly decreasing in k. Consequently $\mathbb{P}_{\lambda_{-j(k)}}\left\{\{\varepsilon_l\le g(\varepsilon,k)\}_{l\neq j(k)}\mid\varepsilon\right\}$ is decreasing in k and $\mathbb{I}\{t+\varepsilon>p(\theta)-\theta\}$ does not depend on k, thus $D(p(\theta)-\theta,k)$ is decreasing in k.

The fact that $D_1(p(\theta)-\theta,k)$ is increasing is quite obvious:

$$D_1(p(\theta)-\theta,k)=-\mathbb{P}_{\lambda_{-j(k)}}\left\{\{\varepsilon_l\le g(\varepsilon,k)\}_{l\neq j(k)}\mid\varepsilon=p(\theta)-\theta-t\right\}d\lambda_j(p(\theta)-\theta-t)$$

where $d\lambda_j(\cdot)$ is a density with respect to Lebesgue measure. The first multiplier in this expression is decreasing in k, consequently the whole expression in increasing. The logic is very simple: at higher positions you have uniformly more customers, consequently increase in price leads to larger decrease in demand. □

This Proposition says that higher positions are unambiguously better than lower ones. What we can conclude is that in symmetric model with simple search model, expected demand has anticipated features. In the next subsection we use these qualities to establish some properties of producer's problem.

2.2.2 Producer's behavior

In this subsection I establish some properties of producer's behavior. As I described in section 2.1.2 producers select their prices at the primary market. Since their pricing behavior is important for the justification of Assumption 2 I need to explore it in some detail. I denote by $F_\varepsilon(\cdot)$ a cumulative distribution function of ε (since they are i.i.d. this function is the same for all j). Then the following assumption provides all the necessary properties of $p(\theta)$.

Proposition 2.2.3. *Assume that $1-F_\varepsilon(\cdot)$ is log-concave. Then $p(\theta)$ is increasing in θ and $\theta-p(\theta)$ is increasing in θ.*

Proof. First, note that $D(p,\theta,t)=1-F_\varepsilon(p-t-\theta-c)$ from (5). Then the price elasticity of the demand is equal to $e(p,\theta)=-p\left(\frac{f_\varepsilon(p-t-\theta-c)}{1-F_\varepsilon(p-t-\theta-c)}\right)$ (density exists due to absolute continuity). By log-concavity $\frac{f_\varepsilon(p-t-\theta-c)}{1-F_\varepsilon(p-t-\theta-c)}$ is increasing in p and decreasing in θ. The first fact implies that $p(\theta)$ is well defined, the next one – that it is increasing in θ.

To see why $\theta-p(\theta)$ is increasing consider the following reasoning: assume that p is optimal price for some level of θ, then increase both p and θ by a small amount equal to δ. From the first order conditions we know that $e(p,\theta)=-1$; after the increase we have that

$$e(p+\delta,\theta+\delta)=-(p+\delta)\left(\frac{f_\varepsilon(p-t-\theta-c)}{1-F_\varepsilon(p-t-\theta-c)}\right)=e(p,\theta)-\delta\left(\frac{f_\varepsilon(p-t-\theta-c)}{1-F_\varepsilon(p-t-\theta-c)}\right)<-1$$

consequently, optimal p for $\theta+\delta$ is less than $p+\delta$ which is equivalent to the fact that $\theta-p(\theta)$ is increasing. □

It is easy to see that the most important fact that I used in the proof is the log-concavity. I cannot justify this assumption by some empirical evidence. On the other hand it is quite common in auction theory and it seems to be not so restrictive, meaning that the class of distributions that satisfies this property is quite large. Note that Proposition 2.2.3 justifies Assumption 2, so now it is logical to restate it in the following way:

Assumption 2_a: Assume that $1 - F_\varepsilon(\cdot)$ is log-concave.

With these properties of pricing strategies I am ready to establish some properties of the valuations of advertisers, that are given by the following equation:

$$\nu(\theta, k) = D(p(\theta) - \theta, k)p(\theta) \tag{10}$$

where $D(p(\theta) - \theta, k)$ was defined in (8). Properties of the valuations are important for determining the incentive of advertisers. The following Proposition describes some important features of the valuations.

Proposition 2.2.4. *In the symmetric model, under Assumption 1 and 2_a, $\nu(\theta, k)$ is decreasing in k, increasing in θ and submodular in (k, θ).*

Proof. The first two properties are obvious: the first one comes from Proposition 2.2.2, the second one – from Proposition 2.2.3 and definition of $D(p - \theta, k)$. The only thing that I actually need to prove is submodularity. First, note that by construction $D(p(\theta) - \theta, k)$ is differentiable in the first argument, consequently we have the following expression:

$$\nu_\theta(\theta, k) = p'(\theta)D(p(\theta) - \theta, k) + pD_1(p(\theta) - \theta, k)(p'(\theta) - 1)$$

To prove submodularity I need to show that $\nu_\theta(\theta, k)$ is decreasing in k. But this is actually straightforward, since by Proposition 2.2.2 $D(p(\theta)-\theta, k)$ is decreasing in k and $D_1(p(\theta)-\theta, k)$ is increasing in k, by Proposition 2.2.3 $p'(\theta)$ is positive and $p'(\theta)-1$ is negative. Consequently the whole expression is decreasing in k and valuations are submodular. □

Proposition 2.2.4 is very important. As I will show below, submodularity is just enough to make mechanism design problem relatively easy and intuitive. Note that submodularity is a consequence of two basics: properties of the pricing function and properties of the expected demand. One important property that I haven't mentioned yet but which is rather important is the fact that $\nu_\theta(\theta, k)$ does not depend on the types of other firms. This fact is a consequence of my simple search model: the only thing that user knows when she buys a good is the type of the one particular ad. If we consider another search model then this fact might not be true.

In the next subsection I consider the mechanism design problem. It will be more convenient to introduce some additional notation that is common in sponsored search literature. More specifically, I denote by α_k – a click-through rate (CTR) of position k. Formally it is equal to the following:

$$\alpha_k = \mathbb{P}\{k = \arg\max_l\{f(\varepsilon(l), l)\}\} = \mathbb{E}_{\lambda_{j(k)}}\left\{\mathbb{P}_{\lambda_{-j(k)}}\{\varepsilon_l \leq g(\varepsilon, k, l) \text{ for each } l \neq k\}\right\} \tag{11}$$

In words, CTR of k-th position is just a probability of click at this position. Observe that in my model α_k is decreasing in k – the usual property of the CTR-s. In most papers on sponsored search this property (along with almost any other) is just assumed with the reference that this is true in the real world. In the symmetric case I generated it (conditional on Assumption 1 and 2_a) as an equilibrium effect.

With this notation I can rewrite the valuations of advertisers in the following way:

$$\nu(\theta, k) = \alpha_k \pi(\theta, k) \tag{12}$$

where $\pi(\theta, k) = \frac{\nu(\theta,k)}{\alpha_k}$. Function $\pi(\theta, k)$ is usually called a value-per-click. In the usual case $\pi(\theta, k)$ is assumed to be independent from k (see Edelman et al. (2007),Varian (2007) etc). Hence, any click at any position has the same expected value. In my case this is not true and this comes from the fact that different users click at different positions.

Note that α_k does not depend on θ. This is only logical because at the time of clicking user cannot know θ. On the other hand, α_k implicitly depends on j through t_j which is observable. But in this symmetric case all t_j are equal, hence I can omit them. Another interesting fact is that α_k doesn't depend on θ_{-j} – the types of others. One can say that there are no competition effects. This is an obvious consequence of my simple search model.

All these facts are pretty obvious but at the same time they are important. In practice search engines are trying to forecast CTR-s and then it is important to understand what structural properties this object has. My simple model indicates that position effects are always there, while other effects, e.g. idiosyncratic effects and competition effects, come from observable heterogeneity of advertisers (different t_j) and from more sophisticated search models (dependence on θ_{-j}).

In practice functions $\pi(\theta, k)$ also divided into two parts:

$$\pi(\theta, k) = \gamma_k(\theta) p(\theta) \tag{13}$$

where $\gamma_k(\theta)$ is called conversion rate (CR) at k-th position. Conversion rate is a probability of buying the good if user has already clicked at the position. So formally, it is equal to the following:

$$\gamma_k(\theta) = \frac{\mathbb{E}_{\lambda_{j(k)}} \left\{ \mathbb{P}_{\lambda_{-j(k)}} \{\varepsilon_l \leq g(\varepsilon, k, l) \text{ for each } l \neq k \mid \varepsilon\} \mathbb{I}\{t + \varepsilon > p(\theta) - \theta\} \right\}}{\mathbb{E}_{\lambda_{j(k)}} \left\{ \mathbb{P}_{\lambda_{-j(k)}} \{\varepsilon_l \leq g(\varepsilon, k, l) \text{ for each } l \neq k\} \right\}} = \frac{D(p(\theta) - \theta, k)}{\alpha_k}$$

Conversion rates are very important for search engines because they indicate how well their advertising schemes work. Unfortunately (for search engines) these rates are not primary observed and their forecasting is a practically relevant task.

With all this new notation I can express the valuations of advertisers in the following form:

$$\nu(\theta, k) = \alpha_k \gamma_k(\theta) p(\theta) \tag{14}$$

This way of presenting the valuations is quite common but might not be the most convenient one. In the next subsection I will be using both this form and the usual one.

2.2.3 Different number of positions

Analysis in the previous section was developed under two implicit assumptions. First, I assumed that the total number of positions is equal to some K and this K is constant, second, I assumed that consumers observe only sponsored links and nothing else. In this subsection I derive some properties of valuations for different K-s and relax the second assumption.

One can expect that variation in K has two effects on users and consequently on advertisers. First, decreasing the number of positions should have positive effect on CTR-s (α-s) due to the fact that users have less options to select from. On the other hand, it can have negative effect since the number of shown positions affects user's beliefs. In what follows I assume away the second effect.

Assumption 3: Users' beliefs about $\theta_j(k)$ do not depend on the number of shown positions.

Assumption 3 is restrictive and simplifying but one can justify it in some way with the following logic. In reality users do not know what kind of mechanism is used by the search engine. They just believe that advertisers at higher positions are better on average. If users have the same beliefs about competition when they observe different number of positions then Assumption 3 is only logical. Actually, empirical analysis of this fact is quite interesting and challenging.

Note that under Assumption 3 expected payoff function $f(\varepsilon, k)$ defined in 7 does not depend on K. Consequently, functions $g(\varepsilon, k, l)$ are also the same. Expected demand, on the other hand, depends on K this time:

$$D(p(\theta) - \theta, k, K) = \mathbb{E}_{\lambda_{j(k)}} \left\{ \mathbb{P}_{\lambda_{-j(k)}} \{ \{ \varepsilon_l \leq g(\varepsilon, k, l) \}_{l \neq k, l \in K} \mid \varepsilon \} \mathbb{I}\{ t + \varepsilon > p(\theta) - \theta \} \right\} \quad (15)$$

where I (with a slight abuse of notation) use K for a set of positions. The following Proposition is obvious but nevertheless is important.

Proposition 2.2.5. *In symmetric model and under Assumption 2_a and 3 $D(p(\theta) - \theta, k, K)$ is decreasing in K, $D_1(p(\theta) - \theta, k, K)$ is increasing in K and $\alpha(k, K)$ is decreasing in K.*

Proof. The fact that $D(p(\theta) - \theta, k, K)$ are decreasing is obvious – increase in K is equivalent to additional restriction in definition of expected demand in (15) and CTR in (11).The fact that $D_1(p(\theta) - \theta, k, K)$ is increasing in K is of similar nature (see Propositions 2.2.2). $\square$

The most important part of the Proposition is the fact that CTR-s are decreasing in K. This fact is expected but it makes analysis of mechanisms rather complex. Notice that while all CTR-s are decreasing in K, their sum remains the same and is equal to $\sum_{k \in K} \alpha(k, K) = 1$. This fact is a consequence of my modeling assumption. I assumed that users observe only sponsored positions and have no search costs. With this assumption obviously all users eventually click at some position.

Constant total CTR is not so important when the number of positions does not change. When this is the case we can always normalize total CTR to unity, considering only users who clicked at some position. When the number of positions is changing one cannot do this. One can introduce dependence of the total CTR on K in the model in different ways.

One obvious solution is to introduce search costs. It seems that this is not the best way to approach the problem. Search costs should be taken into consideration if we assume more complicated search model. On the other hand, there is a way to introduce dependence on K in simple and intuitive way without any search costs. One just needs to introduce organic links into the model.

One can introduce organic links in the same fashion as I did with sponsored links. The only difference is the fact that these links are free. For consumers, however, there is no difference. I assume that consumers have some beliefs about quality of the organic search. In this case organic links are just equal to additional restriction in definition of expected demand. On the other hand, $\sum_{k \in K} \alpha(k, K)$ is depends on K now. It is obvious that total CTR of sponsored positions is increasing in K, while CTR of each ad is decreasing.

2.3 Efficient mechanism

In this subsection I consider the problem of the search engine that needs to design a proper mechanism to discriminate between advertisers. Here I focus on the efficiency: I consider the mechanism that maximizes the utility of all participants (both users and advertisers). Usually users are not modeled explicitly so efficient mechanism is the one that generates the greatest total revenue for advertisers (I call such mechanisms pseudo efficient). Here I model the users, so their interests should be considered as well. In this section I assume that total number of positions is exogenously given.[16]

In my model users have quasilinear preferences, zero search costs, and advertisers have zero production costs. Consequently the only thing that matters is the total consumer surplus of all users (everything else is just a transfer between users and advertisers).

The theoretical framework of most papers on sponsored search markets differs from my approach.[17] Most authors consider reduced form preferences of advertisers. In terms of the Section 2.2.2 these preferences can be described as the following:

$$\nu(\theta, k) = \alpha_k \pi(\theta)$$

where $\pi(\cdot)$ is does not depend on k and α_k are decreasing in k. Some authors (e.g. Börgers et al. (2008)) consider another case where $\pi(\cdot)$ is increasing in k. All these preferences are just particular examples of the valuations that come from my model.

The results of the most papers that analyze GSP and GE auctions can be briefly summarized in the following way: GSP auction with complete information has a pseudo efficient equilibrium and GE auction has equilibrium that is outcome equivalent to the aforementioned GSP equilibrium (and consequently is pseudo-efficient).

In order to find efficient mechanism I use revelation principle and consider (at first) direct mechanisms. In the direct mechanisms each participant (advertiser) just tells his type to mechanism designer (search engine). Since I am working with the symmetric case the only difference between firms are θ-s, so each firm j reports its θ_j (not necessarily truthfully).

[16]One can introduce different number of positions, but results here are trivial: the more links – the better. In order to have some non-trivial connection between the number of positions and efficiency one needs to introduce some adverse effects of advertising or consider more complicated search models.

[17]Notable exceptions are Gomes and Sweeney (2009), Athey and Ellison (2011), Chen and He (2006).

Efficient mechanism should select and place K advertisers in such a way that it will maximize total consumer surplus.

In order to find the efficient mechanism, I should describe the set of possible mechanisms. Here I work only with the set of mechanisms $\mathcal{S}$ that I call *simple order mechanisms.* This is a class of mechanisms that receives a vector $\times_{j\in J}(\theta_j)$ maps it into an order on $\{\theta_j\}_{j\in J}$ (so the highest θ receive the highest number, the second one – the second and so on) and then work only with this order. Simple order mechanisms do not depend on anything but the order of θ-s. For example, one such mechanism selects advertisers with K highest θ-s and give the first position to the highest one, the second to the next and so on. Another example selects K highest θ-s and places them at random.

One can easily imagine a mechanism that does not belong to $\mathcal{S}$. For example any auction that has reserve price does not belong to $\mathcal{S}$, because the fact that you receive the good depends not only on the fact that you have the highest valuation but also on the level of this valuation. A more sophisticated example is the following one: assume that search engine has K slots for ads and divides it into the two groups: N special and $K - N$ usual. The mechanism works in the following way: it selects all firms with θ_j higher than some level $\theta^\star$, orders them and places the first N of them at special positions, while next $N - K$ highest occupy the usual positions; if the number of ads with θ_j higher than $\theta^\star$ is smaller than N then not all special positions are occupied.

The just described mechanism is actually the one that is used in practice. For example, Yandex has (roughly) two groups of positions: "special placement" and "guarantee". These groups differ in the placement at the search query page: "special offereing" is just below the search bar and "guarantee" is to the right of the organic search links. In order to get into special placement your bid should be high enough. Other engines, e.g., Google, have similar rules.

Here I focus on mechanisms that belong to $\mathcal{S}$ primary for technical reasons. Set of this mechanisms is quite simple and it is easy to find the best one. More complicated mechanisms might be more efficient for the reasons that I describe below, but they are much harder to analyze.

The following Proposition describes one particular efficient mechanism from $\mathcal{S}$.

Proposition 2.3.1. *The mechanism that selects firms with K highest reported θ-s and places them into descending order is efficient in $\mathcal{S}$*

Proof. First, note that it is efficient to select firms with highest θ-s. This comes from the fact that distribution of ε is the same for all j and the fact that $\theta - p(\theta)$ is increasing in θ, consequently there is no average gain from selecting firms with lower θ: total measure of buyers shrinks and their utility decreases.

Next it is optimal to place firms in descending order (a firm with highest θ occupies the first place and so on). To understand this fact, note that such ordering of the ads maximizes the amount of information that users have when they are making decisions. Formally, such ordering provides the most finest information partition and any decision rules that are measurable with respect to any other partitions are measurable with respect to this one. Obviously, ascending order is not unique – any other non-random permutation has the same properties. With better information users can make better decisions, consequently their ex ante utility is the highest possible. □

The proof requires some comments. The first part is obviously true for any set of mechanisms not just $\mathcal{S}$. The second one is more complicated. If there are no restrictions on the number of positions then on can easily construct the mechanism that is better than any simple one. It works in the following way: you select K highest ads, place them in descending order and if the first ad is good enough you show it on specific position that reflects this fact. On the other hand, such construction is not absolutely correct because search engine shows K positions but has $K+1$ slots. One can construct then the mechanism with "guarantee" and "special offering" described above. a priori it is not clear whether this mechanism is better from informational point of view but it actually might be. On the other hand, there is loss in efficiency since in some cases not all slots are occupied.

In practice, complicated mechanism with different sets of positions are justified by another reasons: they generate more revenue. As the previous discussion indicates it is not a priori clear whether there is a loss in efficiency in this case.[18] On the other hand, Proposition 2.3.1 tells us that in $\mathcal{S}$ the efficiency problem has easy and intuitive solution.

Since we found how efficient mechanism should work what we need now is to find some incentive compatible mechanism that implements efficient outcome. One obvious candidate is a GSP auction since it works in some particular cases. But it turns out that GSP has some restrictions that makes it efficient only in specific cases. Instead, consider the following mechanism: each advertiser makes a bid, these bids are ordered, K highest are selected, k-th advertiser pays the bid of $(k+1)$-th advertiser. Scheme is the same as in GSP, the only difference is that advertiser pays for show not for click. This might appear as a tiny difference, but it is actually matters. To implement the mechanism I use a simple ascending auction (Ascending auction). It works just in the same way as GE auction but with price per position not per click. The following Proposition describes an equilibrium behavior in Ascending auction.

Proposition 2.3.2. *Assume that number of participants in the auction is $N > K$, then under Assumptions 1 and 2_a Ascending auction has an efficient equilibrium.*

Proof. Consider the following strategy for player j:

$$\beta(p, k, \theta_j) = \begin{cases} 1, & \text{if } \nu(k, \theta_j) - p > \nu(k+1, \theta_j) - p_{k+1}; \\ 0, & \text{otherwise}; \end{cases} \tag{16}$$

The strategy prescribes to stay active ("1") if current price p for the position k is such that surplus from this position is greater than the current outside option: surplus from getting the position $k+1$ at the price p_{k+1} (already determined), and otherwise go out of the auction (receiving $k+1$ position and determining the price for k position). Assuming that ties happen with zero probability, such strategies uniquely determine all the prices and winners.

To show that this strategy is indeed an equilibrium assume that everybody except the player j is following this strategy. The only thing that player j might change is the price at which she exits. Assume that current prices and places are such that $\beta(p, k, \theta_j) = 1$, then it is clearly not optimal to exit, as the surplus of getting current position is larger than the value of the current outside option. So, the only potential deviation is to wait and stay in the auction

[18] See Athey and Ellison (2011) for the discussion of the trade-off between optimality and efficiency.

when $\beta(p, k, \theta_j) = 0$. This, however, implies two things: (1) $\nu(k, \theta_j) - \nu(k+1, \theta_j) \leq p - p_{k+1}$ and (2) $\nu(k, \theta_l) - \nu(k+1, \theta_l) > p - p_{k+1}$ for all $l \neq j$ that are currently active. These two facts imply that $\nu(k, \theta_j) - \nu(k+1, \theta_j) < \nu(k, \theta_l) - \nu(k+1, \theta_l)$ and by submodularity of $\nu(k, \theta)$ this implies that $\theta_j < \theta_l$ for all $l \neq j$. Hence, bidder j has the least type among all currently active bidders. This implies that in order to outbid any bidder for any higher position bidder j should foregone some surplus (because, as everybody else are following their equilibrium strategies, the prices for all higher positions will be such that it would be better for a bidder to be outside of the game). Thus the K highest types win the auction and get K positions, respectively. By Proposition 2.3.1 this outcome is efficient. □

This proposition is important for two reasons: it describes the efficient mechanism and justifies Assumption 1. Consequently I can state the following Corollary.

Corollary. Under assumption 2_a there is a $\mathcal{S}$-efficient mechanism that in equilibrium generates users' beliefs from Assumption 1 and advertisers' preferences from (10).

Proposition 2.3.2 and the Corollary effectively tells us that the whole story was consistent: users have consistent beliefs and advertisers given these beliefs act as anticipated. On the other hand, there is some value in Proposition 2.3.2 as it is, since it generalizes all the results on the GSP auction for more general class of preferences – any submodular preferences.

The mechanism proposed above is quite good from theoretical point of view but in reality its implementation has some problems. Obviously, advertisers are risk averse (at least to some extent) and for them it is better to pay per click. So, it is only logical to ask when GE auction can be used instead of Ascending auction. The following proposition describes the sufficient conditions.

Proposition 2.3.3. *Assume that $\gamma_k(\theta)$ is decreasing in k, then GE auction has an efficient equilibrium.*

Proof. The argument is quite straightforward. If average prices in Ascending auction increase then GE auction has the same outcome and the same payments. From Proposition 2.3.2 we can derive the following equalities for prices:

$$p_k = p_{k+1} + \alpha_k \gamma_k(\theta_{k+1}) p(\theta_{k+1}) - \alpha_{k+1} \gamma_{k+1}(\theta_{k+1}) p(\theta_{k+1}) \tag{17}$$

This equation and the fact that $\gamma_k(\theta)$ is increasing in k implies the following:

$$\frac{\alpha_k \gamma_k(\theta_{k+1}) p(\theta_{k+1}) - \alpha_{k+1} \gamma_{k+1}(\theta_{k+1}) p(\theta_{k+1})}{\alpha_k - \alpha_{k+1}} > \gamma_{k+1}(\theta_{k+1}) p(\theta_{k+1}) \tag{18}$$

Now, for the last position we have the following expression:

$$\frac{p_K}{\alpha_K} < \frac{\alpha_K \gamma_k(\theta_k) p(\theta_k)}{\alpha_K} = \gamma_K(\theta_K) p(\theta_K) \tag{19}$$

From (19) and (25) one can infer that average price is lower than marginal increase in price for every position starting from K, consequently average prices increases. This fact implies that equilibrium outcome in Ascending auction is the same as in GE auction. □

Proposition 2.3.3 is quite important since it provides sufficient conditions for AE auction to be efficient. Note that for efficiency we need conversion rates be higher at higher positions the fact that one can test empirically. My model does not place a priori restrictions on the dependence of $\gamma_k(\theta)$ on k. It seems that in reality conversion rates on higher positions are higher but this is just a speculation.

Proposition 2.3.3 can be reversed in the following way: if average prices in Ascending auction decrease (for some positions), than there is no efficient equilibrium (in undominated strategies) in Ascending auction. To see why it is true, consider any efficient equilibrium of GE auction. This equilibrium naturally corresponds to efficient equilibrium of Ascending auction. So, the point is, that the only efficient equilibrium in Ascending auction is the one considered in Proposition 2.3.2. The following Proposition states this formally.

Proposition 2.3.4. *Assume that conditions of Proposition 2.3.2 hold. Then there is a unique ex-post equilibrium in undominated strategies in Ascending auction.*[19]

Proof. Consider any efficient equilibrium of Ascending auction. Without loss of generality assume that there are only $K+1$ bidders for K position. Consider the last position. While it is traded each bidder should choose a price at which she exits the auction (thus defining the price for the last position). Clearly, all exit prices that are less than $\nu(K, \theta_j)$ are dominated for bidder j: if she drops out before $p_j = \nu(K, \theta_j)$ she does not win anything but actually might have won something. So, the exit price for K-th position is not less than in ex-post equilibrium in Proposition 2.3.2. Now, proceed by induction: assume that p_{k+1} is not less than $p^\star_{k+1}$ – price in the ex-post equilibrium. It is clear then, that p_k (for player j) should be greater than or equal to $\nu(k, \theta_j) - \nu(k+1, \theta_i) + p_{k+1} \geq \nu(k, \theta_j) - \nu(k+1, \theta_j) + p^\star_{k+1} = p^\star_k$. The logic is the same as before: exit at earlier prices is dominated. Hence, we get that all prices at this equilibrium should be not lower than in the ex-post equilibrium. By Revenue Equivalence they should be equal to the prices in ex-post equilibrium, otherwise proposed equilibrium generates strictly higher revenue but allocates in the same way (we do not consider differences on the subsets of types with zero measure).[20] □

Corollary. If average prices in Ascending auction decreases than there is no efficient equilibrium in GE auction.

2.4 Discussion

In this section I have built a model of a market of sponsored links. In my model advertisers have endogenous preferences over different positions generated by users' endogenous beliefs. The basic idea behind this whole construction is relatively simple: users just believe that those advertisers who made their way to higher positions are somehow better and in equilibrium this is indeed the fact.

The main contribution of this section can be divided into two parts. The first part is developing a consistent framework for analysis of both advertisers and users. Existing models differ greatly from the model presented here. In most of them users' behavior is completely

[19] Up to zero-measure subsets of types.

[20] See, e.g., Williams (1999) for the statement of the Revenue Equivalence that I am using here.

ignored and preferences of the advertisers are just given. The most developed framework is presented in the article by Athey and Ellison (2011) but they focus on different things. Their main contribution is a construction of the model with complicated consumer search and analysis of this model, but their assumptions about producers' behavior are very stylized. Here I considered the simplest search model possible but instead endogenized producers' preferences. I see this as an important development since it provides the rationale for existing reduced form models of sponsored search auctions. Also, as I model explicitly both parts of the market my framework can be used for a welfare analysis.

The second contribution is a development of Ascending auction and deriving its properties. These results generalize the works of Edelman et al. (2007) and Varian (2007) in several aspects. First of all, here I consider more general setting and show that all results in those papers hinges on two assumptions: submodularity of the valuation and constant values per click. While submodularity is indeed very important and guarantees simple efficient allocations, constant values are not so important, meaning that one can use similar format and allocate efficiently. Proposition 2.3.3 shows that GE auction works well when CR-s are increasing but otherwise might fail. In this case the failure is drastic as there is no efficient equilibrium in GE auction.

Idea to use submodularity in context of sponsored search auctions is not brand new. Independently, Börgers et al. (2008) proposed the similar conditions. In the paper authors assumed submodularity of $\nu(k, \theta)$ and increasing CR-s, although, they used different notation. One of the results of these papers corresponds to my Propositions 2.3.2 and 2.3.3 for GE auctions. In their work Börgers et al. (2008) considered GSP with complete information and showed that it has the same type of equilibrium as the usual model. In their paper Börgers et al. (2008) underline the fact that increasing values per click assumption is quite restrictive but did not do anything about it. My results can be seen as (independent) generalization of these works: first of all, I considered more general case and, secondly, I show that the role of increasing values is quite crucial, next, I proposed a useful and intuitive auction format.

One can generalize my model in several ways. Firstly, I have described only symmetric case, while in reality it is clear that most auctions are asymmetric. Secondly, one can introduce more complicated search into the model. This will generate several new effects. On the other hand analysis of such models might be quite complicated and probably one should assume particular functional forms in order to get some non-trivial results.

3 Optimal mechanism

3.1 Motivation

In this section I derive some properties of the efficient mechanism. This question already received some attention in the work of Edelman and Schwarz (2010) where authors show that GE auction with reserve price is optimal in the class of all mechanisms which allocate one position to one bidder. Their analysis is, however, severely limited. The main problem with optimal auctions in context of sponsored links is the fact that changes in design affect preferences of advertisers. The most important effect comes from the following fact: different mechanisms generate different number of positions in equiibrium. For example, GE auction with reserve price might result in different number of positions, depending on the types of participants. On the other hand, different number of positions in equilibrium generate different preferences of the firms as I noted in Section 2.2.3.

Unfortunately complete equilibrium analysis of optimal mechanism is very complicated and requires very special properties of distributions, etc. Instead I proceed in the following fashion: I take some properties of advertisers' valuations derived in previous sections and then derive some properties of optimal mechanism. In some places I assume purely ad hoc properties of valuations that cannot be generated by the previous model.

I see the results of this section as a preliminary study of optimal mechanism. In what follows I highlight several essential features that are the most important for the analysis. I believe that careful analysis of these features and their incorporations into the framework can be interesting and challenging area of further research.

3.2 Properties of optimal mechanism

In this section I build on the results of Section 2.2 and assume that advertisers' valuations have the following properties: $\nu(\theta, k, K)$ is submodular in (θ, k), increasing in θ and decreasing in k. I assume that θ_i is distributed according to the cumulative distribution function $G(\cdot)$ with support $[a, b]$. I also assume that maximal number of positions is equal to some exogenous $\bar{K}$.

I begin with a usual notation. I consider direct mechanisms, so I ask bidders for their types. Define by $\hat{\theta} = (\hat{\theta}_1, \ldots, \hat{\theta}_n)$ the vector of reported types. Define by $Q_i(\hat{\theta}, k, K)$ probability that player i get position k while the total number of positions is equal to K, given vector of reported types. Define by $M_i(\hat{\theta})$ the payment of firm i given vector of all reports. Denote by $q_i(\hat{\theta}_i, k, K) = \mathbb{E}_{\theta_{-i}} \left\{ Q_i(\theta_{-i}, \hat{\theta}_i, k, K) \right\}$ – expected probability that advertiser i get the position k and the total number will be K, given that everybody reports his types truthfully and agent i reports $\hat{\theta}_i$. Denote by $m_i(\hat{\theta}_i) = \mathbb{E}_{\theta_{-i}} \left\{ M_i(\theta_{-i}, \hat{\theta}_i) \right\}$ – expected payment of agent i given his report. Now, denote by $\bar{\nu}_i(\theta_i, \hat{\theta}_i) = \mathbb{E}_{q(\hat{\theta}_i, \cdot, \cdot)} \left\{ \nu(\theta_i, k, K) \right\}$ – expected gain of the firm i from the mechanism. Then expected payoff of agent i looks like the following:

$$v_i(\theta_i, \hat{\theta}_i) = \bar{\nu}(\theta_i, \hat{\theta}_i) - m_i(\hat{\theta}_i) \tag{20}$$

With utility given by equation (20), **IC** constraints can be expressed in the following way: (1) $\bar{\nu}_i(\theta_i, \hat{\theta}_i)$ is supermodular and (2)

$$v_i(\theta_i, K) = v_i(\theta_i, \theta_i, K) = \max_{\hat{\theta}_i} \{ v_i(\theta_i, \hat{\theta}_i) \} = v_i(0) + \int_a^{\theta_i} \bar{\nu}_1(\tau, \tau)\, d\tau$$

where $\bar{\nu}_1(\cdot)$ is a partial derivative with respect to the first argument. Conditions (1) and (2) are well known in mechanism design literature: the first condition asks for some sort of monotonicity, while the second one comes from the envelope theorem[21] Using the second

[21] For careful discussion of these conditions see Milgrom (2004), Chapters 3 and 4.

condition I can express expected payment in the following form (assuming that expected gain of the firm of the type a is equal to zero):

$$m_i(\theta_i) = m_i(a) + \bar{\nu}(\theta_i, \theta_i) - \int_a^{\theta_i} \bar{\nu}_1(\tau, \tau)\, d\tau \tag{21}$$

From (21) one can construct the following representation of the expected revenue (all $m_i(a)$ are set to zero):

$$ER = \mathbb{E}_\theta \left\{ \sum_{K=0}^{\bar{K}} Q(\theta, K) \sum_{i=1}^{N} \sum_{k=1}^{K} Q_i(\theta, k \mid K) \left\{ \nu(\theta_i, k, K) - \upsilon_1(\theta_i, k, K) \frac{1 - G(\theta_i)}{g(\theta_i)} \right\} \right\} \tag{22}$$

where $Q(\theta, K)$ is probability that the total number of positions will be equal to K given reports while $Q_i(\theta, k \mid K)$ is a conditional probability that the firm i get position k given reports, when the total number of positions is equal to K. Obvious restrictions on these probabilities are the following:

$$\sum_{K=0}^{\bar{K}} Q(\theta, K) = 1 \tag{23}$$

$$\sum_{k=1}^{K} Q_i(\theta, k \mid K) \leq 1 \tag{24}$$

$$\sum_{i=1}^{N} \sum_{k=1}^{K} Q_i(\theta, k \mid K) = K \tag{25}$$

Optimal mechanism design problem then looks in the following way:

$$\begin{cases} (22) \to \max \\ \text{s.t.: (23)-(25) and } \textbf{IC} \text{ constraints} \end{cases} \tag{26}$$

Problem (26) appears to be rather complicated. In order to simplify analysis I make several purely technical assumptions.

Assumption 4: Assume that function $r(\theta_i, k, K) = \nu(\theta_i, k, K) - \nu_1(\theta_i, k, K) \frac{1-G(\theta_i)}{g(\theta_i)}$ is submodular in (θ_i, k) and is increasing in θ.

Assumption 4 plays the same role as regularity condition in the usual mechanism design problem. With this assumption at hand we can have the following first part of the characterization of the optimal mechanism.

Proposition 3.2.1. *Under Assumption 4 optimal deterministic mechanism has the following property: $Q_i(\theta, k \mid K) = \mathbb{I}\{\theta_i \neq \theta^k\}$, where θ^k is the k-th highest among all θ-s.*

Proof. Result is the direct consequence of the submodularity of $r(\theta_i, k, K)$. □

Proposition 3.2.1 is the first step for characterization of optimal mechanism. In a sense it says that given that the number of positions has been already decided it is optimal to give these positions with respect to the order of the types. This fact is intuitive and anticipated. It corresponds to the regular case in the usual optimal mechanism.

If we assume that CTR-s do not depend on the total number of positions, then Proposition 3.2.1 almost characterizes the optimal mechanism. The only thing one should control for is the fact that $r(\theta_i, k, K)$ are positive. This can be done with optimally chosen reserve price. This is actually the result that one can find in Edelman and Schwarz (2010) (actually it is a slight generalization). However, in reality if we change the number of positions CTR-s also change.

Unfortunately I cannot solve Problem (26)in general and even under severe assumptions. The main obstacle is the fact that optimal $K(\theta)$ is quite a complicated function. My next result highlights new properties of the optimal mechanism under new technical assumption.

Assumption 5: Assume that $\nu(\theta, k, K) = \alpha(k, K)\pi(\theta, k)$.

Assumption 5 says that the only difference that we have from different number of positions comes from difference in CTR-s, while values per click are not affected. This is not true in my model, since values per click actually depend on K. However, in practice this might be quite a reasonable assumption. Given this new condition I can derive another property of optimal auction.

Proposition 3.2.2. *Under Assumptions 4 and 5 optimal auction has the following property: if* $\pi(\theta^k, k) - \pi_1(\theta^k, k)\frac{1-G(\theta^k)}{g(\theta^k)} < 0$ *then* $K^\star < k$.

Proof. First note that $r(\theta_i, k, K) = \alpha(k, K)\left(\pi(\theta, k) - \pi_1(\theta, k)\frac{1-G(\theta)}{g(\theta}\right)$ and the second multiplier does not depend on K. From Proposition 3.2.1 we know that positions should be allocated with respect to order on θ-s. Moreover, since $r(\theta, k, K)$ is increasing in θ and is submodular in (θ, k) due to Assumption 4 we know that $r(\theta^k, k, K)$ is decreasing in k. From Proposition 2.2.3 we know that $\alpha(k, K)$ is decreasing in K. Consequently, if for some k we have that $\pi(\theta^k, k) - \pi_1(\theta^k, k)\frac{1-G(\theta^k)}{g(\theta^k)} < 0$ then we have the same inequality for all (θ^j, j), $j > k$ and consequently it is optimal to have less than k positions since decrease in the number of positions increases α-s of those who receive positions and by constructions this has positive effect on the total revenue. □

Proposition 3.2.2 does not completely characterize optimal auction. It just says that it is not optimal to give positions to firms with negative $r(\theta, k, K)$ since under Assumption 5 sign of $r(\theta, k, K)$ does not depend on K. Without further assumptions I cannot go beyond Propositions 3.2.1 and 3.2.2. However, under one additional assumption optimal mechanism can be characterized.

Assumption 6: Assume that $\pi(\theta, k) - \pi_1(\theta, k)\frac{1-G(\theta)}{g(\theta)}$ is submodular in (θ, k). Further assume that $\sum_{k=1}^{K} \alpha(k, K) = c$, where c does not depend on K.

Proposition 3.2.3. *Under Assumptions 3-6 it is optimal to sell at most one price and optimal mechanism can be implemented via second price auction with reserve price.*

Proof. By Proposition 3.2.1 we know that it is optimal to allocate positions with respect to other on θ-s. From Assumption 6 one can infer that $\pi(\theta^k, k) - \pi_1(\theta^k, k)\frac{1-G(\theta^k)}{g(\theta^k)}$ is decreasing in k (consequence of submodularity and monotonicity). If we normalize all CTR-s by c, we get that the sum $R(K) = \sum_{k=1}^{K} \frac{\alpha(k,K)}{c}\left(\pi(\theta^k, k) - \pi_1(\theta^k, k)\frac{1-G(\theta^k)}{g(\theta^k)}\right)$ can be interpreted as expectation of $\pi(\theta^k, k) - \pi_1(\theta^k, k)\frac{1-G(\theta^k)}{g(\theta^k)}$ under measure $\frac{\alpha(k,K)}{c}$. By Proposition 2.2.3 the measure $\{\alpha(k, K)\}_{k=1}^{K}$ is FOSD (with respect to reversed order on k) by the measure $\{\alpha(k, K-1)\}_{k=1}^{K-1}$, and since $\pi(\theta^k, k) - \pi_1(\theta^k, k)\frac{1-G(\theta^k)}{g(\theta^k)}$ is decreasing in k we get that $R(K)$ is decreasing in K. Consequently it is optimal to sell only one position and this is equivalent to selling one good. Optimal mechanism in this case can be implemented via second price auction with reserve price (Myerson (1981)). □

Proposition 3.2.3 is rather striking since it tells us that there is no gain in selling more positions than just one. Obviously this result is completely driven by Assumption 6. Note that the first part of the Assumption is satisfied in the case when $\pi(\theta, k)$ does not depend on k – the case that was analyzed before in the literature. On the other hand, the second part of the Assumption is the most important. Note that it is generated by my symmetric model, where $\sum_{k=1}^{K} \alpha(k, K) = 1$ for all K. This is obviously driven completely by construction of the model.

I see Proposition 3.2.3 as an interesting result that indicates the property of the model that is crucial for optimality – dependence of $\sum_{k=1}^{K} \alpha(k, K)$ on K. One can introduce this dependence in the model via different mechanisms. One obvious solution is introduction of search costs. It seems that this is not the best way to approach the problem. Search costs should be taken into consideration if we assume more complicated search model. On the other hand, there is a way to introduce dependence on K in simple and intuitive way without any search costs. One just need to introduce organic links into the model. With organic search links we can generate two effects that are observed in practice: CTR-s decrease in K but total CTR of the sponsored links is increasing.

Results of this section can be considered as some preliminary results on the optimal design problem. It appears that previous analysis from Edelman and Schwarz (2010) is incomplete and results from this Section indicates the main features one should focus on in order to find optimal mechanism in general case. One important lesson one can draw from my analysis is the fact that optimal mechanism is quite complicated and has little to do with conventional auction formats.

4 Conclusion

In this paper I developed a new framework for analysis of markets of sponsored links. This framework incorporates both advertisers and users. In equilibrium this model generates preferences of advertisers that are similar to those considered by other authors. Also I introduced simple Ascending auction that generates efficient allocation in equilibrium. Using this new format I derived some new properties of GE auctions. I also considered problem of optimal mechanism design and found several properties of optimal mechanism.

My analysis indicates several possible aspects of further research. It seems that one can introduce more complicated search model a la Athey and Ellison (2011) into my framework. Another possible generalization is introduction of asymmetry beyond advertisers. This will generate several additional complications but it seems that in principle it can be done.

Another possible line of research considers the problem of optimal design. As I show in the text this problem is not so easy that was previously believed and its complicatedness comes from purely equilibrium effects. It is important to emphasize that previous reduced form models a la Edelman et al. (2007) cannot deal with optimal design problem since in order to analyze it one needs to model users' behavior.

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