BLOCK BOOTSTRAP FOR REALIZED VOLATILITY

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ABSTRACT. Modern procedure of asset return forecasting rely upon asset volatility measurement. Integrated volatility is an intrinsic value of the volatility which can be non-parametrically estimated with high frequency data. Using intra-day observations integrated volatility usually estimated by realized volatility. However, under quite general assumptions this estimator is biased (Barndorff-Nielsen Shephard, 2002a, Meddahi, 2002a). Moreover, the point estimator of integrated volatility do not allow for measuring volatility risks. More accurate measure of integrated volatility is a confidence interval based on inference theory. Asymptotic and bootstrap approaches are two main concepts for confidence interval constructing. It was shown by Goncalves and Meddahi (2009) that application of wild and i.i.d bootstrap for realized volatility may achieve more accurate confidence intervals for integrated volatility than asymptotic theory based on CLT for realized volatility (Barndorff-Nielsen and Shephard, 2002a, 2002b). In this paper we extend the work of Goncalves and Meddahi (2009) suggesting to use the block bootstrap and GARCH residual based bootstrap approaches. Using Monte Carlo simulations technique we show that block bootstrap is more accurate approach on a small frequency data, more robust and valid.

1. INTRODUCTION

Asset return modeling with high frequency data in financial markets become crucial in econometric study since this approach was considered by Andersen and Bollerslev (1997). The advantage of using this type of data is the possibility to apply nonparametric estimation of the volatility process. Using intra-day observation we can derive empirical distribution of daily returns and get information about its moments. More accurately, implementation of quadratic variation theory in asset return distribution modeling allowed to figure out representation of asset return volatility.

In standard continuous-time models return volatility is usually represented by integrated volatility which is actually latent. It was shown that under certain strict assumptions realized volatility is quite accurate and consistent measure of integrated volatility (Andersen, Bollerslev, Diebold, and Labys 2001, henceforth ABDL). Unfortunately, the presence of microstructure effects such as discreteness of prices, bid-ask bounce, irregular trading etc. (ABDL 2000) and the presence of jumps, drifts of asset prices and the leverage effect make realized volatility not exactly perfect estimator in the real world (Meddahi 2002, Barndorff-Nielsen and Shephard 2002a, 2002b).

The significant contribution in theory of realized volatility was made by the introducing asymptotic theorems by Barndorff-Nielsen and Shephard (2002) (henceforth BNS). In particular, authors presented the central limit theorem (CLT) for realized volatility when the frequency of observations is growing. This result allowed for constructing of confidence intervals for integrated volatility under CLT assumptions. Moreover, CLT implies the asymptotic normality of realized volatility as an estimator of integrated volatility. This fact does not contradict to bias of the estimator. With weak assumptions of CLT, asymptotic theory was widely used in constructing confidence intervals for integrated volatility on the real data.

The next step in inference theory for integrated volatility was made by Goncalves and Meddahi (2004) by the proposing bootstrap methods for realized volatility-like measures. The main result was improving the existing asymptotic approximations which was published in 2009 (Goncalves and Meddahi, 2009). They proposed and analyzed two bootstrap methods for realized volatility: an i.i.d. and a wild bootstrap (WB). The i.i.d. bootstrap resampled intra-day returns from the original set of returns. The method was motivated by the model in which volatility is constant and consequently returns during the day are i.i.d. The WB observations are generated by multiplying each original intra-day return by an independent normally distributed random variable. This approach was motivated by Wu (1986). Goncalves and Meddahi (2009) showed the validity of these approaches under stochastic volatility model.

The purpose of this paper is to extend the set of the bootstrap approaches for realized volatility in order to achieve improving the existing results and construct more accurate confidence intervals for integrated volatility when the frequency of observations is small. The motivation to use moderate frequency is the exposition of the real data to microstructure effects. The block bootstrap which was introduced by Hall (Hall, 1995) seems to be an excellent approach for this purpose since it preserves the time structure of the time series. The use of block bootstrap was suggested by Goncalves and Meddahi as a possible extension of their work, however, as far as we know nothing is done. Here we suggest an approach how to construct first-order asymptotic valid block bootstrap. Moreover, we check the validity by the Monte Carlo simulations technique with different frequencies. Considering overlapping and non-overlapping bootstrap separately (Lahiri, 1999) we compare block bootstrap with asymptotic inference, WB and i.i.d. bootstrap. After all, we check these approaches for robustness under the drift of the prices and the leverage effect.

Our results are the following. Under high frequency of the data all approaches are approximately equivalent. Under low frequency asymptotic confidence intervals and i.i.d bootstrap give slightly narrower confidence interval, WB gives slightly wider confidence interval and non-overlapping block bootstrap gives quite precise confidence interval outperforming all of the approaches. Overlapping bootstrap outperforms other approaches at the rather frequent data and it is more robust then others.

We proceed as follows. In Section 2 we introduce the basic concepts of realized and integrated volatility and the main results about them. In Section 3 we introduce block bootstrap. In Section 4 we compare results using Monte Carlo simulations technique and discuss them. In Section 4 we discuss GARCH residual based bootstrap and further research.

2. Basic concepts

In this paper we focus on a single liquid asset, which price S_t has a continuous structure, defined by the following stochastic differential equation:

$$d\log S_t = \mu_t dt + \sigma_t dW_t, \tag{1}$$

where μ_t is the drift term, which has finite variation, σ_t is a volatility process such that $\int_0^t \sigma_u^2 du < \infty$. We denote W_t as a standard Brownian motion at the moment t. σ_t and μ_t are cadlag processes and σ_t is assumed to be independent of W_t. We suppose that μ_t may depend on σ_t and $d\log S_t$. We assume that time t is measured in units of one day. According to the solution of stochastic differential equation (1) we can define one-day continuously compounded return for the price process:

$$r_t = \log S_t - \log S_{t-1} = \int_{t-1}^{t} \mu_u du + \int_{t-1}^{t} \sigma_u dW_u. \tag{2}$$

Integrated volatility is an inherent natural measure of return variability and is defined by

$$IV_t = \int_{t-1}^{t} \sigma_u^2 du.$$

Consider a real number h such that $1/h$ is a positive integer number and define $r_{t-1+ih}^{(h)}$ as the return over the period $[t-1+(i-1)h; t-1+ih]$. The realized volatility $RV_t(h)$ is

$$RV_t(h) = \sum_{i=1}^{1/h} r_{t-1+ih}^{(h)2}.$$

According to quadratic variation theory, the consistency of the realized volatility as an estimator of integrated volatility relies on increasing number of high-frequency observations, or partition diameter $h \to 0$ (BNS, 2002a). However, in real data microstructure effects such as discreteness of prices, bid-ask bounce and irregular trading restrict ultra-high-frequency, since it breaks down semi-martingale properties of returns (Andreou and Ghysels, 2002). According to Andersen et al. (2001b) it is optimal to use intra-day returns not over 30 minutes frequency in order to mitigate microstructure effects. In this paper we abstract from reality and eliminate this factor by considering clear simulated processes. This allows us to consider any sensible frequency. However, in any particular case we consider fixed frequency and therefore we have the measurement of the error term

$$U_t^h = RV_t(h) - IV_t.$$

The main properties of the noise term are well known and firstly was considered by BNS (2002a) and Meddahi (2002a):

(*a*) *The mean of* U_t^h *in general is non-zero when the drift* μ_t *is non-zero.*
(*b*) U_t^h *in general is heteroskedastic.*
(*c*) *Under leverage effect* U_t^h *is correlated with integrated volatility* IV_t*.*

The asymptotic properties of the error term was originally described in CLT for realized volatility (BNS, 2002a). In particular, assuming that drift and volatility processes are jointly independent of $\{W_u, u \geq 0\}$, we have the following asymptotic

$$\sqrt{h^{-1}} \frac{RV_t(h) - IV_t}{\sqrt{2IQ_t}} \xrightarrow{d} \mathcal{N}(0,1), \quad (3)$$

where IQ denotes the integrated quarticity, which is defined by

$$IQ_t = \int_{t-1}^{t} \sigma_u^4 du.$$

Note that this asymptotic result does not require any knowledge about drift and diffusion processes and the asymptotic holds even the fourth moment of the return does not exist. Also, the impact of the leverage effect does not influence the above asymptotic property and non-zero mean of the noise is not in contradiction with the asymptotic result (3).

Using the theory of power variation we can derive realized quarticity, which is a consistent estimator of integrated quarticity under the same assumptions (BNS 2002a, 2004b, 2006) and defined by

$$RQ_t = \frac{1}{h}\frac{1}{3}\sum_{i=1}^{1/h} r_{t-1+ih}^{(h)4}. \tag{4}$$

From privatization of the statistics (3) we can derive feasible CLT and a construct feasible 95% asymptotic confidence interval for integrated volatility

$$CI_t^a = RV_t \pm 1.96\frac{1}{\sqrt{h^{-1}}}\sqrt{2RQ_t}. \tag{5}$$

Goncalves and Meddahi (2004) proposed to use bootstrap for integrated volatility inference. Due to asymptotic refinement property, the bootstrap usually provide a more accurate approximation to the finite-sample distribution of an estimator than its asymptotic distribution. Under assumptions of zero drift $\mu_t = 0$ and no leverage effect Goncalves and Meddahi (2004) derive bootstrap valid statistics for WB and i.i.d bootstrap and prove their first-order asymptotic validity.

The i.i.d. bootstrap was motivated by the model in which volatility is constant and consequently returns during the day are i.i.d. Therefore, this method resample intra-day returns from the original set of returns.

$$r_{t-1+ih}^* \sim i.i.d.\ from\ \{r_{t-1+ih}\}.$$

To construct symmetric 95% percentile-t i.i.d. bootstrap interval for integrated volatility

$$CI_t^{perc-t} = RV_t \pm q_{0.95}^* \frac{1}{\sqrt{h^{-1}}}\sqrt{2RQ_t}$$

we need to take the 95% quantile $q_{0.95}^*$ from the distribution of

$$\left|\frac{\sqrt{h^{-1}}(RV_t^* - RV_t)}{u_t^{(h)*}}\right|,$$

where

$$u_t^{(h)*2} = h^{-1} \sum_{i=1}^{1/h} r_{t-1+ih}^{*4} - \left(\sum_{i=1}^{1/h} r_{t-1+ih}^{*2} \right)^2 .$$

The WB was motivated by the model in which volatility is stochastic. The WB observations are generated by multiplying each original intraday return by an independent normally distributed random variable.

$$r_{t-1+ih}^* = r_{t-1+ih}\eta_i, \quad \eta_i \sim \ i.i.d.\ \mathcal{N}(0,1).$$

To construct symmetric 95% bootstrap percentile-t interval for integrated volatility we need to take the 95% quantile $q_{0.95}^*$ from the distribution of

$$\left| \frac{\sqrt{h^{-1}}(RV_t^* - RV_t)}{\sqrt{2RQ_t^*}} \right| .$$

Goncalves and Meddahi (2009) prove that bootstrap approximation is better than asymptotic ones and demonstrate their robust properties under non-zero drift and leverage effect. Moreover, they demonstrate that i.i.d. bootstrap is useful even in a case of stochastic volatility.

3. Block bootstrap approach

In this section we extend proposed by Goncalves and Meddahi (2009) results on block bootstrap approach for realized volatility. The issue of the extensions is to preserve original returns structure during resampling. The block bootstrap is a popular methods to improve accuracy of bootstrap for time series data (Hall 1995). The main principle relies upon dividing data on several blocks, which can maintain the original structure of initial time series. Blocks may be overlapping and non-overlapping (Lahiri 1999). If the sample size is $T = h^{-1}$ and l is the length of the block then for non-overlapping approach we will divide data on T/l blocks. For overlapping approach we divide initial data on $T - l - 1$ blocks, which block 1 is $\{r_1, r_2, ..., r_l\}$ and block 2 is $\{r_2, r_3, ..., r_{l+1}\}, ...,$ etc. The way to choose the length of blocks was considered by Andrews (2004). Author demonstrated that good asymptotic properties can be achieved if the length of blocks depends on total sample size such that $l(T) \to \infty$ as $T \to \infty$ and $\frac{l(T)}{T} \to 0$. Moreover, the length should be asymptotically equal $T^{1/3}$. We show that in this work good asymptotic properties can be achieved assuming the length of the blocks equal to floor of third root of the sample size. Therefore, we assume that $l = \lfloor T^{1/3} \rfloor$.

We assume the simplified model without drift and leverage effect.

$$d\log S_t = \sigma_t dW_t,$$

Let $\{r^*_{ih,h} : i = 1, ...\frac{1}{h}\}$ be a bootstrap sample from the original set of intra-day returns. Consider for example non-overlapping block bootstrap. Therefore, non-overlapping blocks are

$$\{r_{t-1+ih}\}_{i=0}^{l}, \{r_{t-1+ih}\}_{i=l}^{2l}, ..., \{r_{t-1+ih}\}_{i=1/h-l}^{1/h}.$$

The bootstrap base is formed from randomly drawn blocks with replacement from collection of blocks until a bootstrap sample of length $T = h^{-1}$ is formed.

$$\{r^*_{t-i+ih}\}_{i=0}^{1/h} = \left\{\{r_{t-1+ih}\}_{i=l\zeta_k}^{l\zeta_k+l}\right\}_{k=1}^{T/l},$$

where ζ_k is a random number from $\{0, 1, 2, ..., \frac{T}{l} - 1\}$. Bootstrapped realized volatility is

$$RV^*_t = \sum_{i=1}^{1/h} r^{*2}_{t-1+ih}.$$

The block bootstrap 95% percentile interval is

$$CI^{perc}_t = RV_t \pm Q^*_{0.95},$$

with $Q^*_{0.95}$ the 95% percentile of the distribution of an unstudentized bootstrap version of $\mid \sqrt{h^{-1}}(RV_t - IV_t) \mid$. The block bootstrap 95% percentile-t interval is

$$CI^{perc-t}_t = RV_t \pm q^*_{0.95} \frac{1}{\sqrt{h^{-1}}} \sqrt{2RQ_t}, \tag{6}$$

with $q^*_{0.95}$ the 95% percentile of the distribution of an studentized bootstrap version of $\mid \sqrt{h^{-1}}(RV_t - IV_t) \mid$.

Goncalves and Meddahi (2009) proved that for i.i.d bootstrap and WB

$$\sup_{x\in\mathbb{R}} \left| P^* \left(\frac{\sqrt{h^{-1}}(RV^*_t - RV_t)}{v^{(h)*}_t} \leq x \right) - \Phi(x) \right| \to 0, \tag{7}$$

under conditions similar to CLT as $h \to 0$, where $v^{(h)*2}_t = Var^*(\sqrt{h^{-1}}RV^*_t)$, and $\Phi(x) = P(Z \leq x)$ with $Z \sim \mathcal{N}(0, 1)$. In this paper we assume that this result holds for the block bootstrap.

Corollary 2.1

For the block bootstrap:

(i) $E^*(RV^*_t - RV_t) = 0$ for any h and t.

(ii) Resampled variance of the bootstrapped statistic is

$$v_t^{(h)*2} = Var^*\left(\sqrt{h^{-1}} RV_t^*\right)$$
$$= h^{-1}\left((RV_t^{(1)})^2 + (RV_t^{(2)})^2 + ... + (RV_t^{(T/l)})^2\right) - (RV_t)^2 l,$$

where $RV_t^{(k)}$ denotes realized volatility in block k during the day t:

$$RV_t^{(k)} = \sum_{i=0}^{l} r^2_{t-1+kl+ih}.$$

Note, if the length of the block is equal to one, then this approach will be exactly the same as i.i.d. bootstrap and the first two moments of the bootstrapped statistic will be equal. More formally, if $l = 1$ then

$$v_t^{(h)*2} = Var^*\left(\sqrt{h^{-1}} RV_t^*\right) = h^{-1}\sum_{i=1}^{1/h} r^4_{t-1+ih} - \left(\sum_{i=1}^{1/h} r^2_{t-1+ih}\right)^2$$

and

$$v_t^{(h)*2} \to 3\int_{t-1}^{t} \sigma_u^4 du - \left(\int_{t-1}^{t} \sigma_u^2 du\right)^2 \neq 2\int_{t-1}^{t} \sigma_u^4 du \quad as \quad h \to 0.$$

As we can see, the variance of the bootstrapped statistic in general does not converge to $2IQ_t = 2\int_{t-1}^{t} \sigma_u^4 du$ in probability. Therefore, bootstrap percentile interval based on the quantiles of the distribution of

$$\left|\sqrt{h^{-1}}(RV_t^* - RV_t)\right|$$

is not valid. However, we can define a bootstrap percentile-t interval based on the quantiles of the distribution of

$$\left|\frac{\sqrt{h^{-1}}(RV_t^* - RV_t)}{u_t^{(h)*}}\right|, \tag{8}$$

where

$$u_t^{(h)*} \to v_t^{(h)*}, \quad as \quad h \to 0.$$

If we define

$$u_t^{(h)*} = h^{-1}\left((RV_t^{(1)*})^2 + (RV_t^{(2)*})^2 + ... + (RV_t^{(T/l)*})^2\right) - (RV_t^*)^2 l$$

then bootstrap percentile-t interval based on the quantiles of the distribution of (8) will be valid since we assumed result (7).

Note, that all of the above is correct for overlapping bootstrap approach since the Corollary 2.1 is correct for overlapping one. Proof of the Corollary 2.1 does not use non-overlapping property. Therefore, these two approaches are equal in a first-order asymptotic. However, their properties are not equal since distributions of (8) will have different higher moments. We will consider them separately in the following section.

4. Monte Carlo simulations

In this section we compare asymptotic theory for confidence intervals of integrated volatility, WB and i.i.d bootstrap with proposed above block bootstrap under the finite frequency. In order to achieve clear results we follow the technique used in the paper of Goncalves and Meddahi (2009). Using Monte Carlo simulations proposed by Andersen, Bollerslev and Meddahi (2005) we simulate the following price process

$$d\log S_t = \mu_t dt + \sigma_t \left[\rho_1 dW_{1t} + \rho_2 dW_{2t} + \sqrt{1-\rho_1^2-\rho_2^2}dW_{3t}\right],$$

where W_{1t}, W_{2t} and W_{3t} are three independent standard Brownian motions. In order to achieve diversified presentation of the results we consider four different classes of the volatility process σ_t.

The first one is a $GARCH(1,1)$ diffusion which was implemented by Andersen and Bollerslev (1998):

$$d\sigma_t^2 = 0.035(0.636 - \sigma_t^2)dt + 0.144\sigma_t^2 dW_{1t}$$

The second is log-normal diffusion which was considered by Andersen, Benzoni and Lund (2002):

$$d\log\sigma_t^2 = -0.0136(0.8382 + \log\sigma_t^2)dt + 0.1148dW_{1t}$$

The third is two-factor affine diffusion which was considered by Bollerslev and Zhou (2002):

$$\sigma_t^2 = \sigma_{1,t}^2 + \sigma_{2,t}^2$$
$$d\sigma_{1,t}^2 = 0.5708(0.3257 - \sigma_{1,t}^2)dt + 0.2286\sigma_{1,t}^2 dW_{1t}$$
$$d\sigma_{2,t}^2 = 0.0757(0.1786 - \sigma_{2,t}^2)dt + 0.1096\sigma_{2,t}^2 dW_{2t}$$

The fourth is two-factor diffusion which was implemented by Chernov at al. (2003), and Huang and Tauchen (2003):

$$\sigma_t = exp(-1.2 + 0.04\sigma_{1,t}^2 + 1.5\sigma_{2,t}^2)$$
$$d\sigma_{1,t}^2 = -0.00137\sigma_{1,t}^2 dt + dW_{1t}$$
$$d\sigma_{2,t}^2 = -1.386\sigma_{2,t}^2 dt + (1 + 0.25\sigma_{2,t}^2)dW_{2t}$$

Our baseline model assume $\mu = 0$ and $\rho_1 = \rho_2 = 0$. However, we also check for robust property considering extension of the assumptions which allows for drift and leverage effect. Following to Goncalves and Meddahi (2009) we consider the following parameters. For one-factor diffusion models we assume $\mu = 0.0314$, $\rho_1 = -0.576$ and $\rho_2 = 0$. For two-factor diffusion models we assume $\mu = 0.03$, $\rho_1 = -0.03$ and $\rho_2 = -0.03$. Also we consider two-sided symmetric 95% confidence percentile-t intervals. We use the normal distribution (CLT), the i.i.d. bootstrap (iidB), WB and two types of block bootstrap, non-overlapping (BB1) and overlapping (BB2), to compute critical values. We make 10000 replications for four different sample sizes: $1/\text{h} = 1152$, 288, 48 and 12, corresponding to "1.25-minute", "5-minute", "half-hour", and "2-hour" returns. Bootstrap intervals use 1000 bootstrap replications. Table 1 in Appendix A presents coverage rates for different diffusion models and frequencies. Coverage rate less then 95% imply degree of uncovering, i.e. this model construct too small confidence intervals. Over-coverage property imply too wide confidence intervals. As we can see, under high frequency of the data all approaches are approximately equivalent. Under low frequency asymptotic confidence intervals and i.i.d bootstrap give slightly narrow confidence interval, WB gives slightly wide confidence interval and non-overlapping block bootstrap gives quite precise confidence interval outperforming all of the approaches. Overlapping bootstrap outperforms other approaches at the rather frequent data and it is more robust then another.

5. Further research

Examples of the practical implementation of the bootstrap volatility inference are measuring volatility risks and testing for jumps. Construction one-sided confidence interval for integrated volatility may represent the upper bound for volatility which is an alternative to volatility VaR. This measure may be sensible since volatility became tradable by VIX volatility index options.

Another possible direction for extension of this research is implementation more robust bootstrap approaches such us GARCH bootstrap for realized volatility (Goncalves and Kilian, 2004). This research currently is under consideration. The simplified example of GARCH bootstrap is considered bellow.

Residual based bootstrap under conditional heteroskedasticity. For simplicity we still have restricted model without drift and leverage effect.

$$d \log S_t = \sigma_t dW_t$$

The intrinsic property of the model is that returns are normally distributed with zero-mean. Also we assume that intra-day returns have heteroskedastic structure and follow $\{r_{t-1+ih}\}_{i=0}^{1/h} \sim GARCH(1,1)$ model, which is given by

$$r_{t-1+ih} = \varepsilon_{t-1+ih},$$
$$\varepsilon_{t-1+ih} \sim \psi_{t-1+ih}^{1/2} \mathcal{N}(0,1),$$
$$\psi_{t-1+ih} = \beta_0 + \beta_1 \psi_{t-1+(i-1)h} + \beta_2 (\varepsilon_{t-1+(i-1)h})^2.$$

Using maximum likelihood estimation we can estimate all parameters of the model and derive volatility process $\{\psi_{t-1+ih}\}_{i=0}^{1/h}$. Using volatility process for intra-day returns we define bootstrap base:

$$\{r_{t-1+ih}^*\}_{i=0}^{1/h} \sim \mathcal{N}(0, \psi_{t-1+ih}).$$

From such resampling we can derive the first two moments of the bootstrapped statistic:

Corollary 4.1
For the simplified residual based GARCH bootstrap:
(i) $E^*(RV_t^* - RV_t) = 0$ for any h and t.
(ii) Resampled variance of the bootstrapped statistic is

$$v_t^{(h)*2} = Var^*\left(\sqrt{h^{-1}} RV_t^*\right) = 2h^{-1} \sum_{i=1}^{1/h} \psi_{t-1+ih}^2$$

Resampled variance converge in probability to $6IQ_t$ which is not equal to $2IQ$. Therefore, bootstrap percentile interval based on the quantiles of the distribution of

$$\left|\sqrt{h^{-1}}(RV_t^* - RV_t)\right|$$

is not valid. However, if we make particular correction and take percentile interval based on the quantiles of the distribution of

$$\left|\frac{\sqrt{h^{-1}}(RV_t^* - RV_t)}{\sqrt{3}}\right|$$

it will be valid. A bootstrap percentile-t interval based on the quantiles of the distribution of

$$\left|\frac{\sqrt{h^{-1}}(RV_t^* - RV_t)}{u_t^{(h)*}}\right| \tag{9}$$

is valid, if

$$u_t^{(h)*} \to v_t^{(h)*}.$$

If we define

$$u_t^{(h)*} = \sum_{i=1}^{1/h} \psi_{t-1+ih}^2,$$

then bootstrap percentile-t interval based on the quantiles of the distribution of (9) will be valid.

Note also that in sense of the first two moments of the bootstrapped statistic the bootstrap under conditional heteroskedasticity is asymptotically equal to WB. However, this approach allows for wide extension such as drift persistence, more complex structure of returns, rebootstrapping volatility process on each estimation etc. It may be sensible to use GARCH bootstrap instead of WB in order to preserve more complicated structure of returns. The study, testing and extending this approach is a subject of the future research.

6. Conclusion

In this paper we propose a new method for constructing confidence interval for integrated volatility. It was shown that implementation of the block bootstrap for realized volatility allows to achieve more accurate inference for integrated volatility. Under low frequency non-overlapping block bootstrap outperforms asymptotic theory and described by Goncalves and Meddahi (2004) i.i.d bootstrap and wild bootstrap. Moreover, overlapping bootstrap outperforms other approaches at the rather frequent data and it is more robust then others. Another bootstrap approach, residual based bootstrap, allows for many extensions which are the subject of forthcoming research. Another interesting application of the realized volatility bootstrapping is non-parametric testing of different hypothesis, for instance, test for jumps in returns which were introduced by Andersen, Bollerslev and Diebold (2004). These extension is considered to be a practical application of this research.

Appendix A: Tables and Figures

Table I: Coverage rates of nominal 95% confidence intervals for IV

	No leverage and no drift					Models with leverage and drift				
$1/h$	CLT	iidB	WB	BB1	BB2	CLT	iidB	WB	BB1	BB2
GARCH(1,1) diffusion										
12	85.04	92.8	98.4	95.6	94.6	80.44	90.53	97.95	94.46	93.15
48	92.14	94.91	98.39	96.3	95.3	91.31	93.53	98.04	95.29	94.59
288	94.29	95.17	97.18	95.84	95.80	94.44	95.10	96.50	95.81	95.70
1152	94.51	94.88	95.37	95.33	95.22	94.82	95.03	96.14	95.30	95.47
Log-normal diffusion										
12	84.61	93.38	98.36	96.16	94.48	80.99	94.37	98.77	96.92	95.34
48	91.7	94.88	98.57	95.89	95.3	93.58	96.26	98.97	96.95	96.15
288	94.4	95.31	97.01	96.02	95.43	93.85	95.06	97.23	95.47	95.33
1152	95.15	95.25	96.08	95.44	95.57	94.80	95.03	96.17	95.26	95.37
Two-factor Affine diffusion										
12	83.36	92.02	97.74	95.38	94.02	80.27	90.57	98.45	93.52	92.69
48	90.91	94.73	97.88	95.43	94.69	92.34	94.38	98.13	95.14	94.67
288	94.11	95.22	97.58	96.12	95.78	94.35	95.21	96.67	95.05	95.35
1152	94.65	95.24	96.09	95.65	95.28	95.78	95.45	95.65	95.39	95.22
Two-factor diffusion										
12	79.55	90.63	96.92	94.55	91.96	80.24	90.62	97.20	93.49	92.26
48	90.82	94.46	98.29	95.00	95.12	91.39	94.43	98.14	95.12	94.71
288	94.78	95.17	97.13	95.84	95.56	94.37	95.20	97.03	96.03	95.46
1152	95.10	95.37	95.61	95.16	94.97	95.72	95.00	95.83	95.49	95.38

Appendix B: proofs of results in Sections 2 and 4.

Proof of Corollary 2.1 (i) Due to discreet structure of the bootstrapped realized volatility, we have:

$$E^*\left(RV_t^* - RV_t\right) = \sum_{k=1}^{T/l}\left(\frac{T}{l}\right)^{-1}\left(\sum_{i=0}^{T} r_{t-1+ih}^2\right) - RV_t = \sum_{i=0}^{T} r_{t-1+ih}^2 - RV_t = 0$$

for any h and t.

(ii) Define realized volatility in a block k:

$$RV_t^{(k)} = \sum_{i=0}^{l} r_{t-1+kl+ih}^2.$$

Then

$$v_t^{(h)*2} = Var^*\left(\sqrt{h^{-1}}RV_t^*\right) = h^{-1}Var^*\left(\sum_{i=1}^{T/l} RV_t^{(i)*}\right) = h^{-1}\sum_{i=1}^{T/l} Var^*\left(RV_t^{(i)*}\right)$$

$$= h^{-1}\frac{T}{l}Var^*\left(RV_t^{(i)*}\right). \quad (10)$$

Where

$$Var^*\left(RV_t^{(i)*}\right) = E^*\left(RV_t^{(i)*2}\right) - E^*\left(RV_t^{(i)*}\right)^2$$

$$= \left((RV_t^{(1)})^2 + (RV_t^{(2)})^2 + ... + (RV_t^{(i)})^2\right)\left(\frac{T}{l}\right)^{-1} - (RV_t)^2\left(\frac{T}{l}\right)^{-2}.$$

Substituting this into (10) we receive the desirable result.

Proof of Corollary 4.1 (i) The first moment is follows from zero-mean returns process:

$$E(r_{t-1+hi}^{2*}) = \psi_{t-1+hi} = E(r_{t-1+hi}^2).$$

Summing up all intra-day returns we receive the following:

$$E(RV_t^* - RV_t) = 0.$$

(ii) The second moment is:

$$v_t^{(h)*2} = Var^*\left(\sqrt{h^{-1}}RV_t^*\right) = h^{-1}Var^*\left(\sum_{i=1}^{1/h} r_{t-1+ih}^{2*}\right) = h^{-1}\sum_{i=1}^{1/h} Var^*\left(r_{t-1+ih}^{2*}\right).$$

Due to normality of the return distribution:

$$Var^*\left(r_{t-1+ih}^{2*}\right) = E^*\left(r_{t-1+ih}^{4*}\right) - \left(E^*\left(r_{t-1+ih}^{2*}\right)\right)^2$$

$$= 3\left(Var^*(r_{t-1+ih}^*)\right)^2 - \left(Var^*(r_{t-1+ih}^*)\right)^2 = 2\left(Var^*(r_{t-1+ih}^*)\right)^2$$

$$= 2\left(\psi_{t-1+ih}\right)^2.$$

Substituting this expression into the second moment we receive:

$$v_t^{(h)*2} = 2h^{-1}\sum_{i=1}^{1/h}\psi_{t-1+ih}^2.$$

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