РОССИЙСКАЯ ЭКОНОМИЧЕСКАЯ ШКОЛА

NEW ECONOMIC SCHOOL

A. Vasin, Yu. Sosina, D. Stepanov

Endogenous formation of the coalitional structure in a heterogeneous population

Working Paper # WP2007/71

This paper is prepared in the framework of the 2005-2006 NES research project "Analysis and Design of Electricity Markets". We would like to thank Fuad Aleskerov and Viktor Polterovich for useful comments on our presentation at the NES research conference.

Moscow
2006

A. Vasin, U. Sosina, D. Stepanov. Endogenous formation of the coalitional structure in a heterogeneous population. / Working Paper # WP2007/72. – Moscow, New Economic School, 2007. –21 p. (Engl.)

We consider a model of population self-organization. Individuals distributed in some parameter form coalitions that provide them with some good. A strategy of each coalition corresponds to some point in the same space. It is determined according to a certain rule depending on set of participants. The payoff function of any individual increases in the number of participants of his coalition, and decreases in the distance between the coalition strategy and his individual parameter. A strategy of each individual is a choice of the coalition.

In contrast to the known papers by Weber, Savvateev et al. we consider the game without side payments. We study existence, uniqueness and computation problems for Nash equilibria as well as coalitional equilibria. We find out their properties, in particular the number of coalitions at the equilibrium. We discuss the results in context of political parties' formation.

Key words: Coalition, stability, Nash Equilibrium, Weak Coalitional Equilibrium

А.А. Васин, Ю.В. Сосина, Д.С. Степанов. Модель эндогенного формирования коалиционной структуры в неоднородной популяции. / Препринт # WP2007/72. - М.: Российская Экономическая Школа, 2007. – 21 с. (Англ.)

Рассматривается модель самоорганизации населения. Граждане, распределенные по некоторому параметру, формируют коалиции, обеспечивающие их определенными благами. Стратегия каждой коалиции представляет собой точку в том же пространстве параметров и определяется согласно некоторому правилу в зависимости от состава участников. Функция выигрыша индивидуума возрастает с числом участников коалиции, в которую он входит, и убывает по расстоянию между стратегией коалиции и его индивидуальным параметром. Стратегией индивидуума является выбор коалиции.

В отличие от известных работ Вебера, Савватеева и др. рассматривается игра без побочных платежей. Исследуются вопросы существования и вычисления равновесий Нэша, а также коалиционных равновесий. Выясняются свойства этих решений, в частности, количество коалиций в равновесии в зависимости от параметров модели. В качестве приложения рассматривается формирование политических партий.

Ключевые слова: Коалиция, устойчивость, равновесие Нэша, слабое коалиционное равновесие

ISBN

1. Introduction

The present paper aims to study endogenous formation of coalitional structures in framework of the non-cooperative game theory. We assume that each individual of the population is characterized by some parameter (for instance, her location or bliss point). The continuous distribution over this parameter describes the whole population. We propose the following simple model of coalitions' formation. There is a large finite set of labels: "coalition 1", "coalition 2",…, "coalition M". Each individual (player) chooses one of these labels and becomes a member of the corresponding coalition, or decides to abstain and stay alone.

A given strategy profile determines the set of non-empty coalitions, the size and the strategy of each coalition from this set. We assume that the strategy is a point in the same parameter space. This point is determined depending on the distribution of coalition members' parameters according to a certain rule (for instance, a median or mean rule). For each player, her payoff depends on two values: it increases in the size of the coalition including the player, and decreases in the distance between the individual parameter and the coalition strategy.

For this game, we study Nash and coalitional equilibria and characterize corresponding coalitional structures. Coalition formation in practice is a complicated dynamical process, and we assume that some equilibrium realizes as its outcome.

There are two main streams in the literature related to endogenous formation of coalitional structures. One considers formation of jurisdictions (municipalities or regions) (Alesina, Spolaore (1997,2003), Weber, Le Breton (2002), Haimanko, Le Breton, Weber (2002a,b)) by individuals located on some line or plain. They form coalitions in order to provide for themselves public goods (a school, a library, a hospital,…). Each coalition builds a center including all these institutes. Its strategy is a location of the center.

The literature considers several rules that determine the coalition strategy depending on its members' parameters: (a) median rule, (b) Rowlsian rule, (c) mean rule.

The payoff function of each individual includes two negative terms: the fixed cost of building the center is divided by the number of individuals of the coalition, and the travel cost is proportional to the distance between the locations of the individual and the center. The model assumes the good to be necessary for each individual.

The authors consider this model as a cooperative game with side payments and study the core of the game.

Savvateev (2003, 2005), Bogomolnaia, Le Breton at al (2005) consider Nash and coalitional equilibria for similar games without side payments with a small number of players. They provide some results on existence, uniqueness and computation of equilibria. However, these results cast poor light upon the properties of the equilibria in large populations. Another stream of the literature relates to endogenous formation of political parties.

Caplin, Nalebuff (1997), Ortuno-Ortin, Roemer (2000), Gomberg, Marhuenda, Ortuno-Ortin (2000, 2005) consider continuous distribution of players bliss points in the political space. Important difference with the present paper is that the number of parties is fixed and the payoff function of each individual does not depend on the size of his party. Meanwhile, this term of the utility seems to be practically important. Besides that, such settings do not permit to determine the number of the parties at the Nash equilibrium and the coalitional equilibrium structures. That setting seems to better describe distribution of voters over existing parties rather than their formation.

Thus, the following characters distinguish the present paper from the existing literature in this field: continuous distribution of individuals in the space, no side payments, the individual payoff dependent on the coalition size and the distance between the individual bliss point (or location) and the coalition strategy, non-cooperative solutions.

Our main results are as follows. Section 2 considers an n-dimensional Euclidian parameter space with a uniform distribution of individuals. There exist different types of Nash equilibrium (NE) structures, and we focus on the structures corresponding to the uniform rectangular grids. If any coalition corresponds to the rectangular parallelepiped with the edges parallel to the axis, only such grids determine NE coalitional structure. For these structures we consider several concepts of coalitional stability. The structure is *stable with respect to a split* if there exists no new coalition that is a proper subset of some coalition in the structure and provides greater payoffs to all its members. The structure is *stable with respect to a local unification* if there is no new coalition that is a union of several neighbor coalitions and provides greater payoffs to all its members. We obtain necessary and sufficient conditions of stability with respect to some types of unions and splits. We show that existence of non-trivial stable structures crucially depends on relation between the space dimension *n* and the degree *k* of the main term in the payoff function Taylor expenditure in the distance between the individual bliss point and the coalition strategy. If $k \le n$ then the only possible stable structure is atomic (nobody joins any coalition) or the global union (everybody joins one coalition). The first variant takes place if the coefficient before the main term (*the non-conformity coefficient*) is larger than some threshold, and the second variant occurs if the coefficient is less than this threshold. For $k > n$ we determine the interval for the non-conformity coefficient where the non-trivial stable structures exist.

Section 3 provides more complete results for the one-dimensional parameter space – interval $(0,1)$. We show that for any regular NE (with different strategies of different coalitions) the coalitional structure is a partition of the space into intervals corresponding to different coalitions, or including abstainers. Besides that, there might be irregular NE including two coalitions with equal sizes and strategies. Any such NE is unstable in some sense. In particular, individuals of the two coalitions are interested in their merger.

Typically there exist many regular NE with different numbers of coalition. We call NE a *weak coalitional equilibrium* (WCE) if there is no new coalition that provides greater payoffs to all its members. We determine WCE for several types of payoff functions and distributions. We assume that the payoff linearly increases in the coalition size and either linearly or quadratically decreases in the distance between the coalition strategy and the individual bliss point. For linear payoff function the WCE is typically unique and corresponds to some trivial structure: if the non-conformity coefficient is less than 2 than this is a global union, and if the coefficient is more than 2 than this is an atomic structure. For a quadratic payoff function we limit our study with the case of the uniform distribution. We show that for any non–conformity coefficient below some threshold the only WCE is the global union. Above this threshold the number of WCE, the minimum and the average number of coalitions in the WCE increase in the non-conformity coefficient.

In conclusion, we consider applications of the theoretical results to formation of political parties and, more generally, development of the civil society in transition countries. Proceeding from our results, we discuss some reasons for different political structures and different number of political parties in the modern world.

2. Models with n-dimensional parameter space

2.1. Formal model

Consider a population of individuals distributed in parameter space X (for instance, this might be a geographic space or a space of political parties) according to their preferences. Let A denote the set of individuals, $x^a \in X$ is a bliss point (or location) of a player $a \in A$. The whole population is characterized by a distribution function $F(x)$ with the density $f(x)$. For any $a \in A$ the set of strategies $S^a = \{0,1,\ldots,m\} \equiv I^0$ is the set of "labels" (e.g., communists, socialists, LDP and so on, 0 means "abstainer"). If individual a sets $s^a = i \in I$, $I = \{1,\ldots,m\}$, then she joins coalition i, if she sets $s^a = 0$ she stays alone. Below we consider such outcomes where each coalition is characterized by the integrable density function $f_i(x) = \delta_i(x) f(x)$, $i \in I^0$, where $\delta_i(x)$ is the share of players using strategy i among the players with bliss point x.

For a given strategy profile, let $\bar{I} \subseteq I$ denote the set of coalitions with positive sizes. Without loss of generality, let $\bar{I} = \{1,2,...,m\}$, $m \le M$. For any coalition $i \in \bar{I}$

$X_i \overset{def}{=} \{x \in X \mid f_i(x) > 0\}$ is a support of its density function. Each coalition is characterized by its size (or its share in the whole population) $V_i = \int_0^1 f_i(x)dx$ and its strategy p_i. The strategy p_i is a point in the set X. In general, the density function $f_i(x)$ determines this strategy. Below we consider several particular rules for such determination.

A coalitional structure is a partition of the population in coalitions and the set of abstinents. Formally we denote a coalitional structure as $P = \{\delta_i, p_i, i \in \bar{I}\}$, where δ_i is a density of coalition i and p_i is its strategy.

Now, let us determine the payoff functions. If a player with a bliss point x chooses coalition i with size V_i and strategy p_i then her payoff is $U(x, V_i, p_i) = R(V_i) - \alpha L(\rho(p_i, x))$, where $\rho(\cdot)$ is a metrics on X, $R(\cdot)$ and $L(\cdot)$ are increasing functions, $\alpha > 0$ is a nonconformity coefficient. For a player with strategy 0 (who stays alone), the payoff is $U_0 = U(x, 0, x) = R(0) - \alpha L(0)$.

Below we examine the following variants of the coalition strategy determination.

a) The median rule: $X \subseteq \mathrm{E}^1$, coalition strategy p_i is such that $\int_{x \le p_i} f_i(x)dx = \int_{x > p_i} f_i(x)dx$.

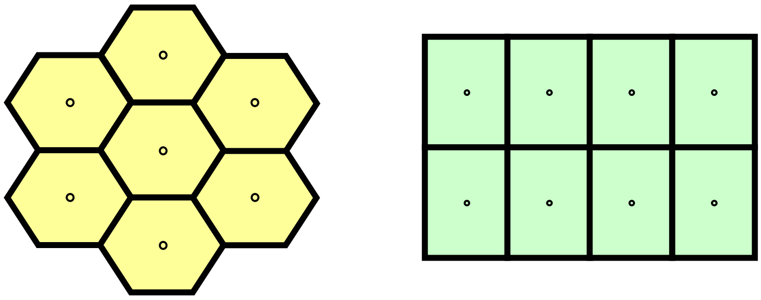
b) The Rowlsian rule: p_i realizes $\max_{p \in X} \inf_{x \in X_i} U(x, V_i, p)$. If $X \subseteq \mathrm{E}^1$ then $p_i = (\inf X_i + \sup X_i)/2$.

c) The mean rule: $p_i = \int_X x f_i(x)dx \Big/ V_i$, i.e. p_i is the mass center of the corresponding coalition.

Nash equilibrium (NE) is such coalitional structure where each individual joins a coalition that maximizes her payoff: $\forall i \quad \delta_i(x) > 0 \;\Rightarrow\; i \in \underset{j \in I}{Argmax}\, U(x, V_j, p_j)$. Note that the atomic structure $\{\delta_0(x) \equiv 1\}$ is NE.

Our first task is to examine possible NE structures. Figure 1 provides some examples of NE structures for the uniform distribution of players on $X \subset \mathrm{E}^2$. Note that any rectangular uniform grid in E^n determines NE structure (for any space X that consists of the cells in this structure). Besides that, there exist many other types of NE structures.

Figure 1



2.2. Properties of NE structures

This section examines general properties and describes some particular types of NE in the game. Let $\pi(V,P)$ denote a coalition with strategy $P \in X$ and size $V \in E_+$. Below we sometimes omit these parameters.

Theorem 1. If the term $L(\cdot)$ in the utility function is a convex function then for any Nash equilibrium structure $P=\{f_i, p_i, i \in \overline{I}\}$, for any coalitions $i, j \in \overline{I}$ with strategies p_i, p_j and sizes V_i, V_j the following relations hold: if $p_i = p_j$ then $V_i = V_j$; if $p_i \neq p_j$ then there exists $(n-1)$-dimensional hyper-surface meeting equation

$$L(\rho(x, p_i)) - L(\rho(x, p_j)) = (R(V_i) - R(V_j))/\alpha .$$

and separating individual bliss points of the coalition i members from the bliss points of the coalition j members.

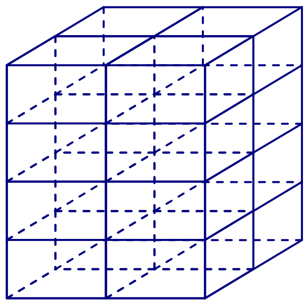
Now consider the payoff function $U(x,V,p) = V - \alpha \|p - x\|^k$. Then the boundary between two coalitions meets equation

$$\|p_i - x\|^k - \|p_j - x\|^k = (V_i - V_j)/\alpha . \tag{1}$$

Equation (1) determines a hyperplane if and only if $k = 2$ (in this case V_1 and V_2 might be different) or $V_1 = V_2$.

Now consider a coalitional structure where each coalition corresponds to some rectangular parallelepiped (see Figure 2). In order to make the exposition short and clear, we identify a coalition and its geometric image. Let $\bar{a} = (a_1,...,a_n)$ denote the lengths of the sides. Then the coalition size is $V = \prod_{i=1}^{n} a_i$, and the diagonal length is $D = \sqrt{\sum_{i=1}^{n} a_i^2}$.

Figure 2



Proposition 1. The coalitional structure corresponding to a uniform rectangular grid is a NE for any sufficiently small non-conformity coefficient, $\alpha \le R(V)/L(D/2)$. Vice versa, if NE structure determines a partition of the space X into rectangular parallelepipeds, $k \ne 2$, $n \ge 2$, then this structure corresponds to a uniform rectangular grid, in particular all coalitions are equal to each other and each corner player lies on the boundaries of 2^n coalitions.

2.3. Stability

The study above shows that, for a given space X, there typically exist many regular NE structures. On the other hand, proceeding from its definition, NE is stable only with respect to individual deviations. In this section we shall find out what NE are stable with respect to deviations of coalitions. Below we consider *weak coalitional equilibria* (WCE).

Coalitional structure is a *weak coalitional equilibrium* (WCE) if no new coalition exists such that it provides greater payoffs to all its members.

We focus on stability analysis of rectangular parallelepiped coalitional structures under a homogeneous distribution of agents in the parameter space. (So we assume the corresponding form of the space X). Our approach to the stability investigation is as follows.

For any x, let $i(x)$ be a coalition containing x in the coalitional structure, $V(x)$ and $p(x)$ - size and program of this coalition. Consider a new coalition j with size V_j and program p_j.

Denote $\Delta U_j(x) = U(x, V(x), p(x)) - U(x, V_j, p_j)$ the increment of agent x payoff due to joining the new coalition j. Then the original structure is not stable with respect to formation of new coalition j if and only if $\min_{x \in X_j} \Delta U_j(x) > 0$. Thus, for a coalitional structure under investigation, we should consider possible deviating coalitions and check this condition. Below we consider deviations of coalitions corresponding to rectangular parallelepipeds with the sides parallel to the axes in the original structure. In this case, some corner agent $x \in X_j$ usually realizes the value $\min_{x \in X_j} \Delta U_j(x)$.

First we establish the conditions of stability with respect to unification of several coalitions from the rectangular homogeneous structure. We consider following types of unification:

I) the union of 2^n neighbor coalitions;

II) the union of r^n neighbor coalitions;

III) the union of 2 neighbor coalitions.

Under the uniform distribution of agents, the strategy of the union is determined as the average of the included coalitions' strategies:

$$\text{if } X_j = \bigcup_{s=1}^{S} X_s \text{ then } p_j = \frac{1}{S}\sum_{s=1}^{S} p_s .$$

Thus, if 2^n neighbor coalitions with a common corner agent form a new similar coalition j then the corner agents that are most distant from the center of X_j realize the value $\min_{x\in X_j} \Delta U_j(x)$. This property implies the following necessary condition for WCE.

L e m m a 1 . The rectangular homogeneous coalitional structure is stable with respect to the union of 2^n neighbor coalitions with a common corner agent if and only if

$$\alpha \geq \frac{(2^n - 1)}{(2^k - 1)} \cdot \frac{V}{(D/2)^k} . \tag{2}$$

N o t e . A similar stability condition for the union of r^n coalitions (where $r = 2,...,\bar{r}$, the upper bound $\bar{r}$ is determined by the size of X) to the similar coalition is: $\alpha \geq \frac{(r^n - 1)}{(r^k - 1)} \cdot \frac{V}{(D/2)^k}$.

L e m m a 2 . The rectangular homogeneous coalitional structure is stable with respect to the union of two neighbor coalitions if and only if

$$\alpha \geq \frac{2^k}{(1+3\gamma)^{k/2} - 1} \cdot \frac{V}{(D/2)^k} \text{, where } \gamma \equiv \left(\frac{a_{\min}}{D}\right)^2 ,\ a_{\min} \equiv \min_{1\leq l\leq n} a_l . \tag{3}$$

What condition - (2) or (3) - is stronger, depends on relations among $a_1,\ldots,a_n$. In particular, if $a_{min} \to 0$ then the right-hand side of (3) tends to infinity and determines the minimal value of α that provides stability with respect to unification. On the other hand, if $a_1 = \ldots = a_n$ then the right-hand side of (2) exceeds the right-hand side of (3). Compare

$$\frac{(2^n - 1)}{(2^k - 1)}\left(2^k \frac{V}{D^k}\right) \vee 2^k \frac{V}{\left(D^2 + 3a_{min}^2\right)^{k/2} - D^k} \text{, or equivalently } \frac{2^n - 1}{2^k - 1} \vee \frac{1}{(1+3\gamma)^{k/2} - 1} .$$

Note that $\gamma \in \left(0, \frac{1}{n}\right]$ and reaches the maximum under $a_1 = \ldots = a_n$. Thus it suffices to check if $(2^n - 1)\left(\left(1 + 3\frac{1}{n}\right)^{k/2} - 1\right) \geq 2^k - 1$ for any k. Note that the both sides are equal for $n = 1$, and the left-hand side increases in n for $n \geq 2$. The inequality also holds for $k = 1$ and $n \geq 2$.

In general the both conditions (2) and (3) are necessary for stability with respect to unification types I, II.

Now we find out under what conditions some proper subset of the coalition in the NE structure can profitably separate. As above, we consider the coalitional structure corresponding to a uniform rectangular grid, so the original coalition corresponds to the rectangular parallelepiped.

Let i denote the original coalition, and j with $X_j \subset X_i$ be the separating coalition. The following conditions are necessary for any profitable split.

L e m m a 3 . If the split is profitable for any $x \in X_j$ then 1) $p_i \notin X_j$; 2) $V_j < V_i/2$.

Below we derive propositions on stability with respect to 3 types of the split and thus obtain necessary conditions of WCE for the structure under consideration:

I) the split of similar coalition with the sides $b_l = \lambda a_l$, $l = 1,...,n$, $\lambda > 0$;

II) the split of coalition with the sides $b_l = a_l$ for any $l = \{1,...,n\} \setminus l'$, and $b_{l'} = \lambda a_{l'}$;

III) the split of coalition with the sides $b_l \leq a_l/2$, $l = 1,...,n$.

Consider the case where new coalition j is similar to the original coalition i (i.e. X_j is similar to X_i). Then j is a rectangular parallelepiped with the sides $b_l = \lambda a_l$, $l = 1,...,n$, $\lambda > 0$. Proceeding from Lemma 3 it suffices to examine $\lambda \in \left(0, \frac{1}{2}\right)$, otherwise $p_i \in X_j$. Note that the corner agent of coalition j that is the closest to the strategy p_i realizes $\min_{x \in X_j} \Delta U_j(x)$. Thus we obtain the following lemma.

L e m m a 4 . The coalition i is stable with respect to the split of a similar coalition if and only if

$$\alpha \leq \frac{V}{(D/2)^k}. \tag{4}$$

Now consider another type of split. Let the sides of X_j for coalition j meet condition $b_l = a_l$ for any $l = \{1,...,n\} \setminus l'$, and $b_{l'} = \lambda a_{l'}$ where $\lambda \in \left(0, \frac{1}{2}\right)$ (otherwise $p_i \in X_j$).

L e m m a 5 . The coalition i is stable with respect to the split type II if and only if

$$\alpha \leq \frac{V}{(a_{max}/2)^k}, \text{ where } a_{\max} = \max_{1 \leq l \leq n} a_l. \tag{5}$$

The inequality (4) implies the condition (5). Moreover, this inequality is equivalent to the non-negative payoff requirement for a corner agent of the coalition i, and this is a necessary and sufficient condition for the coalition structure to be a NE. Thus any NE structure generated by the uniform rectangular grid is stable with respect to the considered types of split.

Now consider a rectangular coalition j with the sides that meet condition $b_l \le a_l/2$, $l = 1,...,n$. The next lemma establishes the stability condition for this case.

L e m m a 6 . The coalition i is stable with respect to the split type III if and only if

$$\alpha \le \frac{V}{(D/2)^k}.$$

Summarizing abovementioned results on stability of the homogenous rectangular structure, we obtain the following stability condition with respect to all considered types of unification and split.

P r o p o s i t i o n 2 . A homogenous rectangular coalitional structure is stable with respect to the mentioned types of unification and split if and only if

$$\alpha \le \frac{V}{(D/2)^k} \cdot \left[\max\left\{ \max_{r=2,...,\bar{r}} \frac{r^n - 1}{r^k - 1}, \frac{1}{(1+3\gamma)^{k/2} - 1} \right\}, 1 \right] \equiv A(\bar{a}). \tag{6}$$

Let us find out, what conditions on the parameters of the model provide non-emptiness of this set. Note that inequality $\max\left\{ \max\limits_{r=2,...,\bar{r}} \frac{r^n - 1}{r^k - 1}, \frac{1}{(1+3\gamma)^{k/2} - 1} \right\} \le 1$ is equivalent to system $\begin{cases} 4 \le (1+3\gamma)^k \\ n \le k \end{cases}$. Since the value in brackets is greater than 1, for sufficiently large k there exists the non-empty interval for α such that the structure is stable with respect to the mentioned deviations:

$$\forall \gamma \in (0, 1/n] \;\; \exists \underline{k} \ge n : \forall k \ge \underline{k} \Rightarrow A(\bar{a}) \ne \phi$$

P r o p o s i t i o n 3 .

a) For $k < n$, non-trivial stable structures do not exist: either the only stable structure is $\delta_0(x) \equiv 1$ (atomic structure), or, under sufficiently small α, everybody joins one coalition $\delta_1(x) \equiv 1$.

b) For $k = n$ the result is similar: a non-trivial stable structure does not exist for any $\alpha \ne A(\bar{a})$.

c) For $k > n$, stable structures with respect to the mentioned deviations exist whenever (6) holds.

3. Models for one-dimensional space

3.1. A coalition formation game

Let $X = [0,1]$ (for instance, X is a space of political programs, 0 corresponds to the extreme left, 1 – to the extreme right program). Let A denote the set of individuals. For any $a \in A$ the set of strategies $S^a = \{0,1,...,M\} \equiv I^0$. If individual a sets $s^a = i \in I$, $I = \{1,...,M\}$, then she joins coalition i, if she sets $s^a = 0$ she stays alone. Below we consider such outcomes where each coalition is characterized by the piece-wise continuous density function $f_i(x) = \delta_i(x) \cdot f(x), i \in I^0$, where $\delta_i(x)$ is the share of players using strategy i among the players with bliss-point x. For any such function, there exists the value $V_i = \int_0^1 f_i(x)dx$ that we call a coalition size.

For a given strategy profile, let $\bar{I} \subseteq I$ denote the set of coalitions with positive sizes. Without loss of generality, let $\bar{I} = \{1,2,...,m\}$, $m \leq M$. For any coalition $i \in \bar{I}$ $X_i \overset{def}{=} \{x \in X \mid f_i(x) > 0\}$ is a support of its density function. Each coalition is characterized by its size $V_i = \int_0^1 f_i(x)dx$ and its strategy p_i.

The strategy p_i is a point in the set X. In general, the density function $f_i(x)$ determines this strategy. Below we consider several particular rules for such determination.

The payoff function of a player with bliss-point x and strategy $i \in \bar{I}$ is

$$U(x,V,p) = R(V) - \alpha L(|p - x|), \tag{7}$$

where α is a positive non-conformity coefficient, $R(\cdot)$ and $L(\cdot)$ are monotonously increasing functions on $[0,1]$, $R(\cdot)$ is piece-wise continuous, and $L(\cdot)$ is continuous.

For a player a with strategy $s^a = 0$ (who stays alone), the payoff is $U_0 = U(x,0,x) = R(0) - \alpha L(0)$.

Consider the following variants of determination of coalition strategy.

V a r i a n t 1 . The coalition strategy is a median of its players' distribution over bliss-points: $\int_0^{p_i} f_i(x)dx = \int_{p_i}^1 f_i(x)dx$.

If the median is not unique then any median point may be taken as a coalition strategy.

Such rule is employed by the majority of the papers on endogenous formation of political parties (see Savvateev (2003,2005), Bogomolnaia, LeBreton, Savvateev, Weber (2005)). The following two propositions justify this choice.

P r o p o s i t i o n 4 (common knowledge). The median is a Condorcet winner in the competition with any alternative strategy.

P r o p o s i t i o n 5 . Assume that function $L(\cdot)$ in the payoff (7) is linear in the distance between x and p. Then the median maximizes the total payoff to all individuals in the coalition.

V a r i a n t 2 . Let $\overline{X}_i = \left[C_i^l, C_i^r\right]$ denote the minimal segment including the support of $f_i(x)$. The coalition strategy is determined as a median of this segment, that is $p_i = \left(C_i^r + C_i^l\right)/2$. Such choice maximizes the minimal payoff for individuals in the coalition: $p_i = \arg\max_{p \in X} \inf_{x \in X_i} U(x, V_i, p)$. If the coalition is not stable under such rule then it is not stable under any other rule.

V a r i a n t 3 . The strategy is determined as a mean of individual bliss points for coalition members

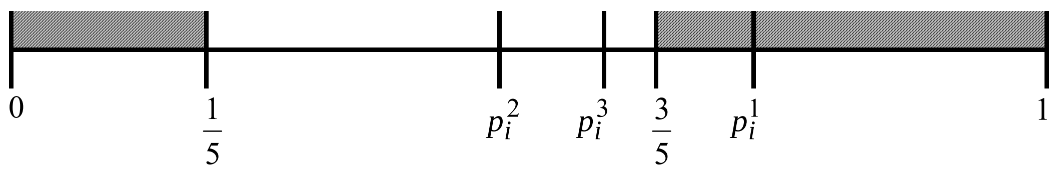
$$p_i = \int_0^1 x \frac{f_i(x)}{r_i} dx. \tag{8}$$

P r o p o s i t i o n 6 . Let function $L(\cdot)$ in the payoff function $U(x, V_i, p)$ be quadratic in the distance $|x - p|$. Then the mean value (8) maximizes the total payoff to all individuals in the coalition.

N o t e . In the case of connected support of X^i and the homogeneity distribution of individuals in X, all the given rules determine the same coalition strategy. Otherwise the results may essentially differ.

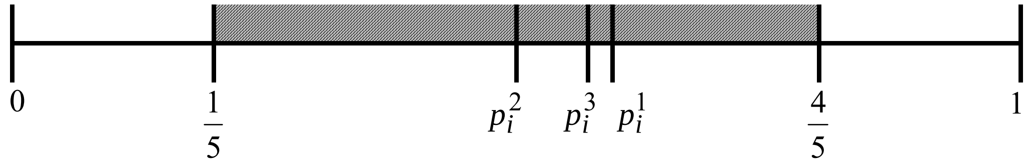
E x a m p l e 1 . Assume that members of coalition are homogenously distributed in the set $X_i = [0, 1/5] \cup [3/5, 1]$. Figure 3 shows the coalition strategies for each variant of stability determination. For the variant 1 $p_i^1 = 21/30$, for the variant 2 $p_i^2 = 15/30$, for variant 3 $p_i^3 = 17/30$.

F i g u r e 3 .



E x a m p l e 2 . Let $f_i(x)$ increase on the set $X_i = [1/5, 4/5]$: $f_i(x) = 2x, \forall x \in X_i$. Figure 4 shows the coalition strategies for each variant of stability determination. For the variant 1 $p_i^1 \approx 0{,}58$, for the variant 2 $p_i^2 = 0{,}50$, for variant 3 $p_i^3 = 0{,}56$.

F i g u r e 4 .



3.2. Nash equilibria

Let each player join the coalition that maximizes his gain (or stays alone if this is the most profitable strategy). Formally this individual rationality means that the strategy profile is a *Nash equilibrium* (NE) (see Nash (1951), and Maynard Smith (1982) for population games).

In context of endogenous formation of coalitions, it is not obvious if an individual can change her strategy (for instance, leave one party and join another one) without permission from other players. So, for the games with infinite set of players Bogomolnaia, Jackson (2002) introduce the weaker concept of *individual stability*.

A strategy profile is *individually stable* if no player can gain by such deviation from strategy i to strategy j, $i, j \in I^0$, that any other player employing j (that is, choosing coalition j) does not lose under this deviation.

For the population game under consideration, any reasonable deviation of one player (who joins coalition j with positive size) does not change the gains of other players in this coalition. So Nash equilibrium and individual stability concepts coincide for this game.

For any subset $Y \subseteq X$ let $\overline{Y}$ denote the minimal segment including Y, and $\widetilde{Y}$ - the set of interior points of $\overline{Y}$. The theorem below specifies the structure of Nash equilibrium and facilitates their determination.

T h e o r e m 2 . For any NE structure $P = \{f_i, p_i, i \in \bar{I}\}$

$U(x, V_i, p_i) \geq U_0$ for any $i \in \bar{I}$, $x \in \overline{X}_i$ and $U(x, V_i, p_i) > U_0$ for any $x \in \widetilde{X}_i$.

Thus, a coalition member with the most distant bliss point gets the payoff not less then the payoff to an abstinent, and any other coalition member get the grater payoff.

If the term $L(\cdot)$ in the utility function is a convex function on $[0,1]$ then for any NE structure $P = \{f_i, p_i, i \in \bar{I}\}$, for any coalitions $i, j \in \bar{I}$ with strategies p_i, p_j and sizes V_i, V_j the following relations hold:

if $p_i \neq p_j$ then $\forall i, j \in \bar{I}$ $\widetilde{X}_i \cap \widetilde{X}_j = \phi$,

if $p_i = p_j$ then $V_i = V_j$.

The theorem implies that coalitions with the same strategy should be of the same size at any NE. If the strategies differ then there exists a separating point such that the bliss points for members of one coalition lie to the left, and the bliss points for the other coalition lie to the right of this point.

N o t e . This theorem holds for any variant of coalition strategy determination if a single player cannot change a strategy of any coalition with a positive size.

Regular Nash equilibrium strategies.

Coalition structure $P=\{f_i, p_i, i\in \bar{I}\}$ is called *regular* if:

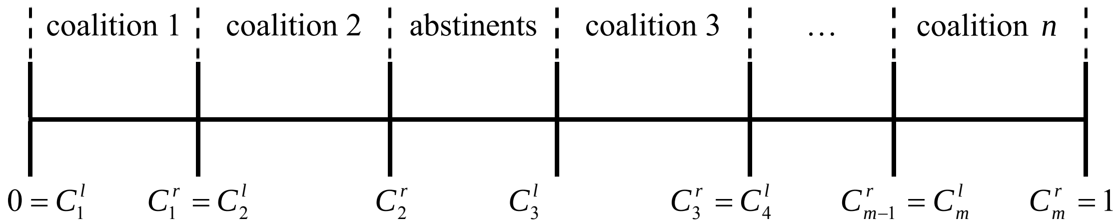
1) for any coalition $i\in \bar{I}$ the support of the set X_i of its members' bliss points is a connected set;

2) for any $x\in \tilde{X}_i$ $f_i(x)=f(x)$, that is, if x is an interior point of X_i then all individuals with this bliss point join coalition i.

Let C_i^l, C_i^r denote respectively the infinum and the supremum of the set $\bar{X}_i, i\in I$. The players with bliss-points C_i^l or C_i^r are called *boarder agents* of coalition. Coalitions $i,j\in \bar{I}$ in a regular structure are n*eighbor* if $C_i^r=C_j^l$. Below we show that at any NE the boarder agents may belong to any of the neighbor coalitions since it does not influence the payoffs.

Without loss of generality, let coalitions in a regular structure be enumerated from the left to the right: $\forall i,j\in \bar{I}\quad i<j\quad if\ C_i^r\le C_j^l$. Figure 5 shows a typical regular coalitional structure.

F i g u r e 5 .



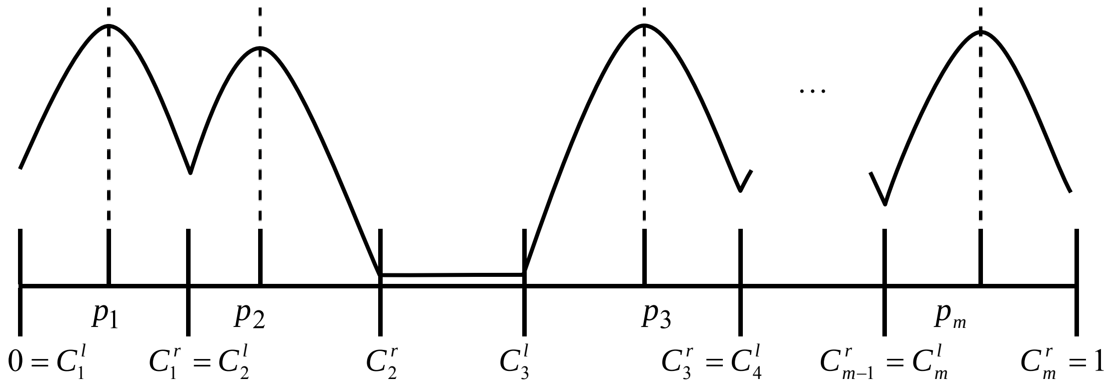
P r o p o s i t i o n 7 . NE coalitional structure $P=\{f_i, p_i, i\in \bar{I}\}$ is regular if and only if $p_i\ne p_j$ for any different coalitions $i,j\in \bar{I}$ in this structure.

P r o p o s i t i o n 8 (Irrelevance condition for boarder agents). Let $L(\cdot)$ be convex on $[0,1]$. Then a regular coalitional structure $P=\{f_i, p_i, i\in \bar{I}\}$ is NE if and only if the following conditions hold for the boarder agents:

$$\begin{aligned}
&U(C_i^r,V_i,p_i)=U(C_{i+1}^l,V_{i+1},p_{i+1})\ge U_0,\ \forall i=1,\dots,m-1,\\
&\textit{in particular, if } C_i^r\ne C_{i+1}^l\textit{, then } U(C_i^r,V_i,p_i)=U_0;\\
&U(C_1^l,V_1,p_1)\ge U_0\textit{, and if } C_1^l\ne 0\textit{ then } U(C_1^l,V_1,p_1)=U_0;\\
&U(C_m^r,V_m,p_m)\ge U_0\textit{, and if } C_m^r\ne 1\textit{ then } U(C_m^r,V_m,p_m)=U_0.
\end{aligned} \tag{9}$$

Figure 6 shows a typical structure of agents payoffs at the regular NE.

F i g u r e 6.



3.3. Strong equilibria and weak coalitional equilibria

Under general assumptions on the distribution of agents and the payoff function, there exist many different NE structures. For instance, any partition in m equal coalitions is such structure under the homogeneous distribution and the linear payoff with nonconformity coefficient $\alpha < 2$.

This section aims to find out what NE structures are stable with respect to deviations of coalitions of players. The conventional concept of coalitional stability is *a strong equilibrium*. According to Aumann (1961) strategy profile is a strong equilibrium if there is no coalition such that all its members can increase their payoffs by setting other strategies under fixed strategies of the rest individuals. This concept permits to change the strategies irrespective of the payoffs to those who do not change them. However, in context of political parties formation, such assumption seems to be unreasonable: usually a grope of individuals can not join some party if its members would lose because of the program change. So below we consider the weaker concept of the coalitional stability.

Obviously any strong equilibrium is a WCE.

T h e o r e m 3 (on the structure of WCE). Let $L(\cdot)$ be convex on X. Then any WCE is a regular NE.

Below we also employ two particular stability concepts.

Regular coalition structure $P = \{f_i, p_i, i \in \bar{I}\}$ is stable *with respect to a split* if there is no new coalition that is a connected subset of some coalition in the $P = \{f_i, p_i, i \in \bar{I}\}$, and providing greater payoffs to all its members.

L e m m a 7. Regular coalition structure $P = \{f_i, p_i, i \in \bar{I}\}$ is stable with respect to a split if and only if $\forall i \in \bar{I} \cup \{0\}$, $\forall [C', C''] \subseteq X_i$ $\exists x \in [C', C'']$: $U(x, V', p') \leq U(x, V_i, p_i)$, where V', p' − the size and the program of the coalition, including all individuals with bliss-points from the set $[C',C'']$.

Regular coalition structure $P=\{f_i, p_i, i\in\bar{I}\}$ is *stable with respect to a local unit* if there is no new coalition consisting of several neighbor coalitions in the $P=\{f_i, p_i, i\in\bar{I}\}$, and providing greater payoffs to all its members.

L e m m a 8 . Regular coalition structure $P=\{f_i, p_i, i\in\bar{I}\}$ is stable with respect to a local unit if and only if $\forall i, i+1\in\bar{I}$: $C_i^r = C_{i+1}^l$ either $\exists x\in X_i$: $U(x,V',p')\le U(x,V_i,p_i)$ or $\exists x\in X_{i+1}$: $U(x,V',p')\le U(x,V_{i+1},p_{i+1})$, where V', p' − the size and the program of the coalition, including all individuals with bliss-points from the set $X_i \cup X_{i+1}$.

Regular coalition structure $P=\{f_i, p_i, i\in\bar{I}\}$ is *locally stable* if it is stable with respect to a split and local unit.

Obviously, any WCE structure is locally stable. Below we show that, under certain assumptions, any profile with the locally stable coalitional structure is a WCE.

3.4. A model with the homogeneous distribution of players

This section studies coalition formation under the following assumptions:

- Distribution of players over bliss-points is homogeneous: $f(x)\equiv 1$, $\forall x\in[0,1]$;
- Individual payoff linearly depends on the coalitional size: $R(V)=V$.

We consider two variants of the payoff dependence on the distance between the individual bliss-point and coalitional strategy:

- the linear dependence $L(\Delta)=\Delta$;
- the quadratic dependence $L(\Delta)=\Delta^2$.

Below we describe WCE structures for these variants. Proceeding from Theorem 3, we consider only regular coalitional structures and assume that the coalitions are enumerated from the left to he right: $\forall i,j\in\bar{I}$ $i<j$ if $C_i^r \le C_j^l$.

The following two theorems characterize regular NE structures for each variant.

T h e o r e m 4 (regular NE for the linear payoff). Let $U(x,V,p)=V-\alpha|p-x|$.

1) If $\alpha>2$ then the only regular NE is $\{\delta_0(x)\equiv 1\}$ (the atomic structure).

2) If $\alpha<2$ then the set of regular coalition structures is coincides with the set of partitions of X into m coalitions of the same size $1/m$, $m=1,2,...$.

3) For $\alpha=2$, any regular structure (i.e., any partition of X) is NE.

T h e o r e m 5 (regular NE for the quadratic payoff). Let $U(x,V,p)=V-\alpha(p-x)^2$.

1) Partition of X into m coalitions of the same size $1/m$ is NE structure if and only if $\alpha/m \le 4$.

2) Partition of X into m_1 coalitions with size V_1 and m_2 coalitions with size V_2 such that $m_1V_1 + m_2V_2 = 1$ (the coalitions may be located in any order) is NE structure if and only if $\alpha = 4/(V_1+V_2)$.

No other regular NE structure exists in this case.

Theorems 4 and 5 show that the structure of NE essentially depends on the utility function term related to the distance between the agent bliss point and the coalition strategy.

Now let us describe locally stable coalitional structures.

T h e o r e m 6 (locally stable structures for the linear payoff). Let $U(x,V,p)=V-\alpha|p-x|$.

1) Regular NE structures is stable with respect to the local split if and only if $\alpha \le 2$. Thus all the structures specified in Theorem 4, p.p. 2, 3, are stable with respect to the local split under this condition.

2) Regular NE structures including $m \ge 2$ is stable with respect to the local unit if and only if $\alpha = 2$.

T h e o r e m 7 (locally stable structures for the quadratic payoff). Let $U(x,V,p)=V-\alpha(p-x)^2$.

1) Regular NE structures is stable with respect to the local split if and only if $\alpha \le 4m$.

2) Regular NE structures including $m \ge 2$ is stable with respect to the local unit if and only if $\alpha \ge \frac{4}{3}m$.

T h e o r e m 8 (WCE structures for the linear payoff). Let $U(x,V,p)=V-\alpha|p-x|$.

1) The atomic structure $\{\delta_0(x)\equiv 1\}$ is a WCE if and only if $\alpha \ge 2$.

2) The global union $\{\delta_1(x)\equiv 1\}$ is a WCE if and only if $\alpha \le 2$.

3) WCE with the nontrivial structure ($m \ge 2$ or $m=1$ and $X = X_1 \cup X_0$) exist if and only if $\alpha = 2$.

T h e o r e m 9 (WCE structures for the quadratic payoff). Let $U(x,V,p)=V-\alpha(p-x)^2$.

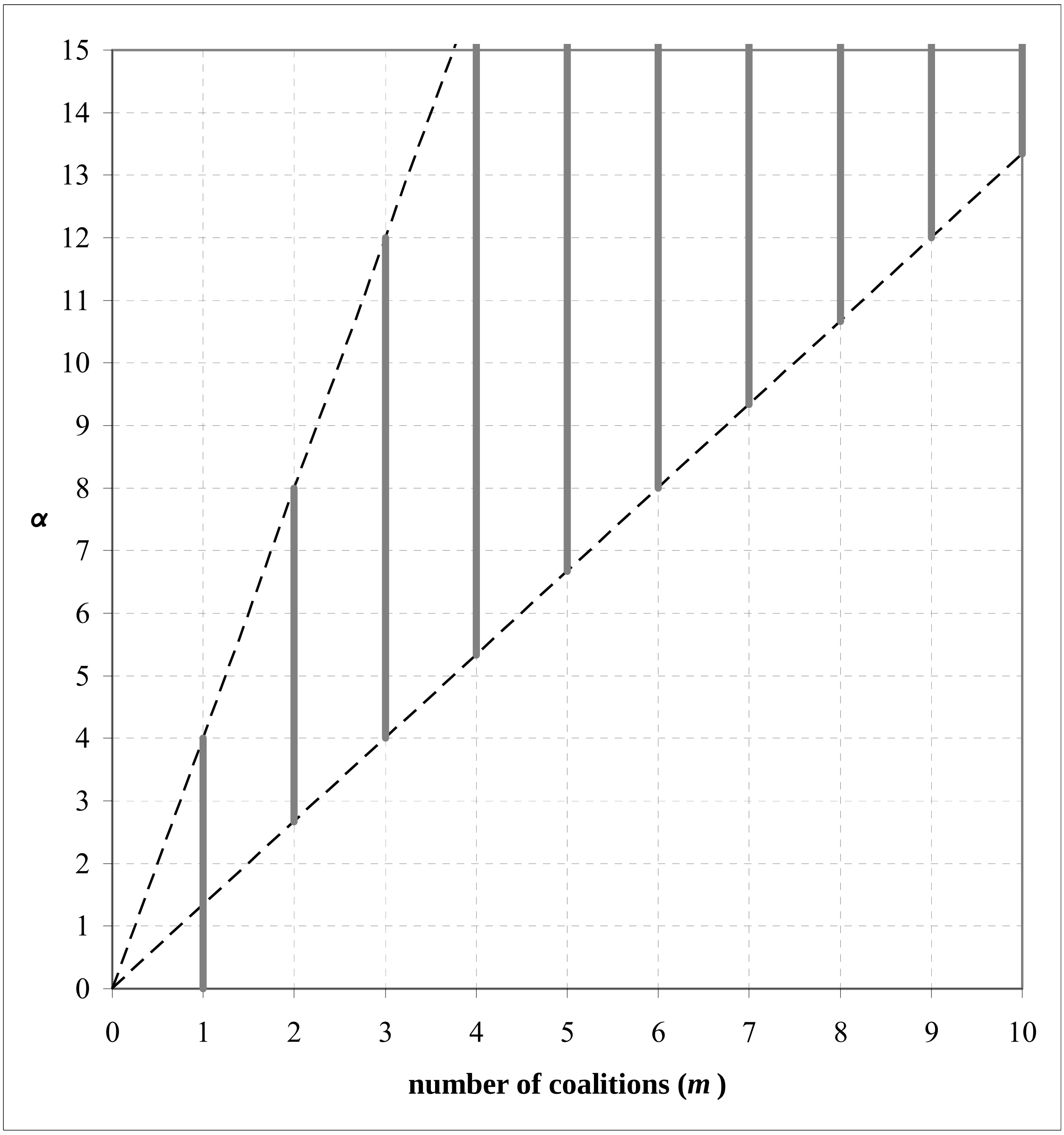
1) The atomic structure $\{\delta_0(x)\equiv 1\}$ is not a WCE under any $\alpha > 0$.

2) The global union $\{\delta_1(x)\equiv 1\}$ is a WCE if and only if $\alpha \le 4$.

3) The NE structure with $m \geq 2$ coalitions is a WCE if and only if $\frac{4}{3}m \leq \alpha \leq 4m$.

Thus, for the quadratic payoff function, the set of WCE structures coincides with the set of locally stable structures. For any $\alpha \geq 8/3$, the WCE is not unique, and the number of such structures increases with α. Figure 7 shows how WCE structures depend on α.

Figure 7.



Conclusion

Self-organization of individuals plays an important role in formation of political parties and other voluntary unions of citizens in the modern society. Of course, other forces also take part in this process: the state services, as well as private centers possessing financial and informational resources, aim to form the political structure according to their own interests. Professional politicians, who often consider political structures in concern with their power and welfare, also make an essential impact on the process. Nevertheless, voluntary unification of individuals proceeding from their interests is a crucial factor of the process in many cases. In the present paper we constructed and studied a mathematical model of such unification.

In our analysis we assumed that the result of the process should be a political structure stable with respect to individual and coalitional deviations. We showed that the properties of the stable coalitional structures essentially depend on the parameters of the individual utility functions: the non-conformity coefficient and the sensitivity to the distance between the coalition strategy and the individual bliss point. In particular, we distinguished the class of utility functions (including linear functions) such that only trivial stable structures (the atomic structure where individuals abstain from coalition formation, and the global union where all individuals join one coalition) may exist for the functions from this class. Under variation of the environment, the society with such utility functions may suffer a sharp transition from one stable structure to the other. We also determined another class of the functions (with the grater sensitivity) that permit non-trivial stable structures, and examined how the number and the characters of stable structures depend on the non-conformity coefficient.

Proceeding from these results, we put forward the following conjecture: the variety of political structures and transition processes in different countries may concern with the differences between the utility functions of the population in these countries. The grater sensitivity to the distance between the coalition strategy and the individual bliss point and the grater non-conformity coefficient usually imply the grater number of coalitions (in particular, political parties) in the stable structure and the smoother transition under the change of the environment.

Of course, there exist other factors that can essentially influence the political structure of the country. (The recent paper by Polterovich et al, 2007, distinguish the resource abundance). Nevertheless we suppose that the conjecture on the role of the utility functions is worth the careful empirical and theoretical analysis.

Literature

1. Alesina A., Spolaore E. (1997). On the Number and Size of Nations //Quarterly Journal of Economics. №112: 1027-1056.
2. Alesina A., Spolaore E. (2003). The size of Nations. MIT Press, Cambridge, MA.
3. Aumann R.J. (1961). The Core of a Cooperative Game Without Side Payments. //Transactions of American Mathematical Society. №98: 535-552.
4. Bogomolnaia A., LeBreton M., Savvateev A., Weber S. (2005). Stability of Jurisdiction Structures under the Equal Share and Median Rules. CORE Discussion Paper. №32.
5. Bogomolnaia A., Jackson M. (2002). The Stability of Hedonic Coalition Structures// Games and Economic Behavior. №38: 201-230.
6. Caplin A., Nalebuff B. (1997). Competition among Institutions //Journal of Economic Theory. №72: 306-342.
7. Gomberg A.M., Marhuenda F., Ortuno-Ortin I. (2000). Equilibrium in a Model of Endogenous Political Party Formation // Proceedings for 5th International Meeting of the Society for Social choice and Welfare, Alicante.
8. Gomberg A.M., Marhuenda F., Ortuno-Ortin I. (2005). Endogenous Platforms: The Case of Many Parties. Alicante University Press.
9. Haimanko O., Le Breton M., Weber S. (2002a). On Efficiency and Sustainability in a Collective Decision Problem with Heterogeneous Agents. Mimeo.
10. Haimanko O., Le Breton M., Weber S. (2002b). Transfers in a Polarized Country: Bridging the Gap between Efficiency and Stability //CORE Discussion Paper. №18.
11. Maynard Smith J. (1982). Evolution and the Theory of Games, Cambridge, Cambridge University Press.
12. Nash J. (1951). Non-cooperative games //Annals of Mathematics. №54: 286-295.
13. Ortuno-Ortin I., Roemer J.E. (2000). Endogenous Party Formation and the Effect of Income Distribution on Policy //University of Alicante. IVIE working paper.
14. Polterovich V., Popov V., Tonis A. (2006). Resourse Abundance, Political Corruption and Instability of Democracy// Contributed paper for the XX NES Research Conference.
15. Savvateev A. (2003). (Non)existence of Secession-Proof Partitions. Moscow, New Economic School. Mimeo.
16. Savvateev A. (2005). Achieving Stability in Heterogeneous Societies: multi-jurisdictional structures, and redistribution policies. Moscow: EERC.
17. Sosina Y. (2004). "Endogenous Forming of Political Structures and Investigation of Their Stability" (in Russian). Moscow, Higher School of Economics. Preprint WP7/20.