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«Time-consistent government policies in the Sidrausky's model with foreign currency»

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I consider a model of a small open economy of Sidrausky type with foreign currency, which imperfectly substitutes the domestic currency. The foreign currency performs the essential role for the results, theoretical as well as practical. In the theoretical part I prove the stationarity of optimal consistent in time policies and then derive analogies of the Phelps rule. The rule states that at the optimal stationary regime the marginal budget revenues from a unit of each kind of receipts should be equal.

In the practical part I calibrate the stationary models on the Russian data, then calculate equilibrium trajectories for different values of initial external debt and growth rate. In particular, I evaluate welfare welfare losses from «unjustified» dollarization of the Russian economy as 2.5 - 3%.

Сотсков А.И. Оптимальные политики правительства в модели Сидравского с иностранной валютой. Препринт # 2003/034. - М.: Российская экономическая школа 2003. - 25 стр. (Англ.)

Я рассматриваю модель малой открытой экономики типа Сидравского. Особенность модели в том, что она содержит иностранную валюту, которая несовершенно замещает внутреннюю. Присутствие иностранной валюты с фиксированным темпом инфляции, а также с фиксированным мировым реальным процентом позволяет доказать стационарность оптимальной консистентной во времени политики и вывести аналоги правила Фелпса. Последнее утверждает, что на стационарном режиме маргинальный вклад в бюджет с единицы каждого источника поступлений одинаков.

В практической части я откалибровал модель по Российским данным и вычислял равновесные траектории для различных значений внешнего долга и темпа роста. В частности, получается оценка потерь благосостояния от избыточной долларизации экономики в 2.5 - 3 %.

About the contents of work and main outcomes

The purpose of work stated in the title, consists in characterizing optimal consistent in time government's policies in the framework of Sidrausky model. In the theoretical part we justify the stationarity of consistent policies and derive analogies of the Phelps relation, which characterize optimal policies. In the practical part we calibrate on the Russian data the stationary models, calculate and characterize the equilibrium trajectories for different values of external debt and growth rate. Let's tell about it more in detail.

The major part of the work is theoretical. It is considered a model of small open economy with domestic and foreign currencies. The latter substitutes the first in transactions and as a carrier of value. It is supposed that the rate of inflation of the good in foreign currency and the real interest rate in the external world are constant, and under perfect foresight the PPP и UIP conditions are fulfilled. The domestic government chooses a policy, i.e. the income tax rate, the rate of nominal monetary growth, real government expenditure, and also issues bonds for the internal and external market. The government seeks to determine the policy, which maximizes the intertemporal welfare function of the representative individual subject to equilibrium constraints. The additional requirement is the consistency in time of the optimal policy. We show that in the model with two currencies the optimal consistent policy is necessarily stationary, and can be found from a static system of equations. In case of one domestic currency the consistency requirement fails. Thus in the conditions of constant rate of inflation and real interest rate the foreign currency is like an anchor that does not allow disorganizing the finance system due to consistency requirement. Considering further the static optimization problem we derive the analogies of the Phelps relation. The basic papers in this part were E. Phelps (1973), S. Turnovsky and W. Brock (1980), S. Turnovsky (1987). My contribution in the topic here is the consistency of the Phelps problem in a model with foreign currency, justifying the Phelps relation.

In the above the variables jumped to the stationary states at $t=0$ so the model did not contain capital. By the next step we consider a model of stationary growth with capital, which linearly enters the production function. This model can be reduced to one without capital, and we make the similar work with it. As a result we get an analogy of the Phelps relation for the model of stationary growth.

The theory is added by the calibration of stationary equilibrium models on the Russian data and analysis of equilibrium solutions. Evaluating the parameters of the models we based on some fundamental proportions, which have turned out in Russian economy to the present time. I mean such as the share of labor in outcome, the shares in GDP of real government expenditure, internal and external debt, of consumption, of investment, and also the tax rate, the real money balances to consumption ratio, and the value of dollarization. Dollarization and its aspects represent a big special topic. Here we touch it because the Russian economy is very much dollarized. The average dollar to ruble balances ratio is equal approximately 6:5 that seems unjustified when the inflation rate less 20% per year. So we say here about cost of dollarization and the numerical results show the losses. The general idea of the calculations consisted in the following. Keeping the real proportions we observe the changes of equilibrium rates of inflation, inflation tax, and cost of inflation and dollarization for different values of debt and rate of growth.

In all calculations the rate of inflation of the good in dollar is equal zero, and the real world interest rate equals to 0.05. In the static model we consider external indebtedness at the level 20%-35% GDP, and in the growth model at the rate of growth 3.5%-4.5% and under 35% GDP of external indebtedness. The last numbers are close to the actual data for Russian economy. In order to evaluate the cost of dollarization, we give the parallel calculations for models with one and two currencies. The general conclusion is that in the model with dollars the consumer has larger losses of welfare (about 2.5-3%) and higher inflation rate than without dollars at least for the rate of inflation less than 20% per year. This conclusion does not contradict to the recent investigation by A. Friedman and A. Verbetsky (2001), where they say about negative effect of dollarization at not high inflation.

I calculated equilibrium trajectories but did not succeeded in calculation of optimal ones because of technical problems.

1. Consistency in time of optimal policies and the Phelps problem

1.1. Introduction. In 1973 E. Phelps derived a simple rule, which related the optimal (in Ramsey sense) rate of income tax τ and the nominal interest (inflation rate) *n*. In general form, it could be stated as follows:

$$[\partial(\tau z + nm)/d\tau] : z = [\partial(\tau z + nm)/dn]: m = \text{const},$$

where z is non-interest income of the representative consumer, m – real money balances, nm – inflation tax (by Phelps' definition); so τz + nm is the total budget revenue which (as it was assumed) must not change with policies. The rule states that the marginal revenues from a unit of each kind of receipts should be equal. Thus, the Phelps' result allowed (if not stated) the existence of positive inflation tax, as opposed to Friedman rule, which demanded for a zero inflation tax. The main simplification, allowed by Phelps, was that the framework was essentially static. Clearly, there was a need to derive the Phelps' result in a dynamic setting. However the subsequent authors working with dynamic models did not reproduce the Phelps result. Moreover C. Turnovsky and W. Brock (1980) proved time inconsistency of the Phelps problem in dynamic setting. In spite of these facts the Phelps' rule looks very natural and «almost obvious». It would be interesting to give some conditions and still obtain the Phelps' rule (or its analogy) from a dynamic setting. In Section 1 we consider a small open economy without capital (like Turnovsky (1987)) where a foreign currency with a given constant rate of inflation and imperfect substitution of domestic currency additionally enters. We show that if the rate of time preference is equal to the real world interest rate and government controls the initial values of real money balances and bonds through an open market operation with domestic residents then the optimal consistent policy is stationary as well as the corresponding perfect foresight equilibrium. They can be found from a static system of equations. After this we are able to get the Phelps relation.

Such a claim with only domestic currency and without some kind of commitments would be untrue. (In particular, in Sotskov (2003) government starting at arbitrary moment *t>0* takes the current real wealth of the consumer as the initial data). Since the state coordinates must jump from the initial to the stationary values we did not include capital in the model. In Section 2 we consider a growth model with capital linearly entering the production function. Such a model can be reduced to one without capital and can be elaborated in the similar way as the previous one. In particular, we get an analogy of the Phelps relation for the model of growth.

Some generalizations of the Phelps rule have been obtained due to dynamic setting and refuse from invariability of government revenue: «distorted Friedman rule» by S. Turnovsky (1987,2000), see also A. Draizen (1979), C. Chamley (1985). A. Sotskov (2001) considered a model with utility function of the form $U(c+h(l)+v(m))$ and obtained the Phelps' rule for dynamic optimal trajectories (not necessarily stationary).

1.2 Structure of economy. We describe the basic model on which the optimality and consistency questions are studied. This is a model of a small open economy in perfect foresight equilibrium as it was given in S. Turnovsky (1987) where a foreign currency additionally enters which imperfectly substitutes the domestic currency. The economy is supposed to consist of three sectors: consumers, firms, and domestic government (fiscal-monetary authority). The consumers and firms are assumed to be identical, so we speak about a representative consumer and a firm. The firm produces a single good whose foreign price Q is fixed on the world market. Since there is only a single traded good the PPP property holds: $P=QE$, P is domestic price of the good and E is exchange rate, i.e. units of domestic currency per unit of foreign one. Domestic residents may hold three assets: domestic currency, foreign currency, and traded bonds denominated in foreign currency issued by the domestic government. The representative consumer solves the following optimization problem:

$$max \sum_{t=0}^{\infty} (1+\rho)^{-t} U(C_t, M_t/P_t, F_t/Q_t, l_t, G_t)$$

subject to:

$$M_{t+1} + F_{t+1}E_t + B_{t+1}E_t = M_t + F_tE_t + (1-\tau_t)(w_t l_t + \Pi_t) + E_t(1+n^F_t)B_t - P_tC_t,$$

and initial conditions: $M_0>0, F_0>0, B_0>0, P_0, E_0$.

where C_t - private real consumption, M_t, F_t - nominal money balances in domestic and foreign currencies, G_t - real government expenditure, P_t - domestic price level, l_t - labor, w_t - nominal wage rate, Π_t - nominal profit, τ_t - income tax rate, E_t - exchange rate (ruble/ \$), ρ - rate of time preference; B_{t+1} - nominal stock of one-period traded bonds; denominated in foreign currency, bought in period t, n^F_t - foreign nominal interest rate. We assume that the cost of change of the currencies is equal zero. Also for simplicity, we assume that interest income is untaxed. In determining the optimal plan for C_t, M_t, l_t, B_t, F_t, the consumer takes parameters τ_t, G_t, Π_t, w_t, , n^F_t, P_t, Q_t, E_t, ρ as given for all $t=0,1,...$

Denote $\overline{M}=M/P$ и $\overline{F}=F/Q$. One-period utility function $U(C, \overline{M}, \overline{F}, l, G)$ is assumed to be concave and twice differentiable in its five arguments, with positive marginal utility in $C, \overline{M}, \overline{F}$, G and negative one in l. Since the consumer produces demand in real terms, it is convenient to set his problem in real term also::

$$max \sum_{t=0}^{\infty} (1+\rho)^{-t} U(C_t, \overline{M}_t, \overline{F}_t, l_t, G_t) \qquad (1)$$ subject to:

$(1+\pi_{t+1})\,\overline{M}_{t+1} + (1+q_{t+1})(\overline{F}_{t+1} + \overline{B}_{t+1}) = \overline{M}_t + \overline{F}_t + (1-\tau_t)(\overline{w}_t l_t + \overline{\Pi}_t) + (1+n^F_t)\,\overline{B}_t - C_t$ (2)

and initial conditions: $\overline{M}_0 = M_0/P_0$, $\overline{F}_0 = F_0/Q_0$ и $\overline{B}_0 = B_0/Q_0$.

Here $\pi_{t+1} = (P_{t+1} - P_t)/P_t$ - rate of inflation of good in domestic currency, q_{t+1} - rate of inflation of good in foreign currency, $\overline{w} = w_t/P_t$, $\overline{\Pi}_t = \Pi_t/P_t$, $\overline{B}_t = B_t/Q_t$. By definition $1+n_t = (1+r_t)(1+\pi_t)$, or approximately $n_t = r_t+\pi_t$, where r_t - is real world interest rate, analogously $n^F_t = q_t+r_t$. In real terms we have got a new regulator π_t instead of P. Since the initial prices P_0 и Q_0 are given, and values M_0, B_0 и F_0 are in «his hands», then the real values $\overline{M}_0, \overline{B}_0, \overline{F}_0$ are determined. The values $\overline{B}_0$ and $\overline{F}_0$ are predetermined by past accumulation, while $\overline{M}_0$ is determined by the equilibrium price P_0, and consequently is endogenous. Sum $\overline{M}_0 + \overline{B}_0 + \overline{F}_0$ is the real financial wealth of the consumer at the moment $t=0$.

The second sector is production. Firms for producing output hire labor and maximize real profit $\overline{\Pi}_t = f(l_t) - \overline{w}_t l_t$, where $\overline{w}_t$ is real wage rate, $f(l)$ is the production function, assumed to possess the usual property of positive but diminishing marginal product of labor.

The third sector is government. It is assumed to control the income tax rate τ_t, the rate of money emission $\theta_{t+1} = (M_{t+1} - M_t)/M_t$, government expenditures G_t, and issues debt. The stock of the government 's debt denote by A_t, $\overline{A}_t = A_t/Q_t$, is real debt. The government problem consist in keeping the state budget:

$$\tau_t f(l_t) + \overline{M}_t\,\theta_{t+1} + q(\overline{F}_{t+1} + \overline{B}_{t+1} - \overline{F}_t - \overline{B}_t) - G_t = (1+r)\,\overline{A}_t - \overline{A}_{t+1}.$$

The resource balance of the economy can be presented as accumulation law of the foreign assets. Denote by $\overline{D}_t = \overline{A}_t - \overline{B}_t$ the external government debt, and by $\overline{R}_t = \overline{F}_t - \overline{D}_t$ the «disposable» foreign assets. Then the resource balance takes the form:

$$\overline{R}_{t+1} - \overline{R}_t = r_t \overline{R}_t + f(l_t) - C_t - G_t - (r_t + q_t)\,\overline{F}_t. \quad (3)$$

The mentioned relation $P = EQ$ implies after differentiation that $\pi = q + e$, where e is rate of exchange depreciation. Adding to the both parts real interest r we have got another so called UIP relation: $n = n^F + e$. In the sequence we set q and r fixed and constant in time. Thus the nominal interest rate n и the rate of depreciation e are determined by the inflation rate: $n = \pi + r$, $e = \pi - q$.

1.3 Perfect foresight equilibrium. A bundle of policy instruments (τ_t, θ_t, A_t, G_t) for all t from 0 till ∞, subject to balance constraint (i.e. three of four are independent) is called *policy*. It is assumed that a declared government policy is implemented further in life. From the other side the consumer possesses the perfect foresight w.r.t. the parameters of equilibrium corresponding to the declared policy. Given a policy (τ_t, θ_t, A_t, G_t) parameters (π_t, n_t, e_t, $\overline{\Pi}_t$, $\overline{w}_t$) are called *perfect foresight equilibrium* if the planned demand and supply functions (C_t, $\overline{M}_t$, $\overline{F}_t$, $\overline{B}_t$, l_t), $t=0,1,...,$ solve the consumer (1),(2) and producer problems and satisfy the resource balance (3) at all points of time.

We now proceed to develop the conditions for perfect foresight equilibrium. We begin with the consumer. Let λ_t be discounted Lagrange multiplier. Then the equilibrium conditions for consumer are given by

$$U_C(t) = \lambda_t,$$

$$U_M(t) = \lambda_t (r + \pi_t),$$

$$U_F(t) = \lambda_t (r + q),$$

$$U_l(t) = -\lambda_t (1-\tau_t)\, \overline{w}_t ,$$

$$\lambda_t = \lambda_{t-1} (1+\rho)/(1+r).$$

One should add to this the transversality conditions at infinity:

$$lim\ \lambda_t \overline{M}_t (1+\rho)^{-t} = 0,\ lim\ \lambda_t \overline{B}_t (1+\rho)^{-t} = 0,\ lim\ \lambda_t \overline{F}_t (1+\rho)^{-t} = 0 \text{ при } t \to \infty . \qquad \text{(T)}$$

The equilibrium conditions for the representative firm are the usual marginal product condition $\overline{w}_t = f'(l_t)$ and the definition of profit: $\Pi_t = f(l_t) - \overline{w}_t l_t$.

The government choice of a rate of printing money $\theta_{t+1} = (M_{t+1} - M_t)/M_t$ gives in real terms the equality: $(1+\pi_{t+1}) \overline{M}_{t+1} = \overline{M}_t (1+\theta_{t+1})$. Now the perfect foresight equilibrium can be specified as follows:

$$U_C(t) = \lambda_t , \tag{4a}$$

$$U_M(t) = \lambda_t (r + \pi_t), \tag{4b}$$

$$U_F(t) = \lambda_t (r + q), \tag{4c}$$

$$U_l(t) = -\lambda_t (1-\tau_t) f'(l_t), \tag{4d}$$

$$\lambda_t = \lambda_{t-1} (1+\rho)/(1+r). \tag{4e}$$

$$(1+\pi_{t+1}) \overline{M}_{t+1} = \overline{M}_t (1+\theta_{t+1}) \tag{4f}$$

$$\overline{M}_t \theta_{t+1} + (1+q)(\overline{F}_{t+1} + \overline{B}_{t+1}) + C_t = \overline{F}_t + (1-\tau_t) f(l) + (1+q+r) \overline{B}_t \tag{4g}$$

$$\overline{R}_{t+1} - \overline{R}_t = r_t \overline{R}_t + f(l_t) - C_t - G_t - (r + q) \overline{F}_t . \tag{4h}$$

where $\overline{R}_t = \overline{F}_t + \overline{B}_t - \overline{A}_t$, initial conditions $\overline{M}_0, \overline{F}_0, \overline{B}_0$, and transvesality conditions (T). Here $\overline{M}_0 = M_0 / P_0$, where equilibrium price $P_0 = Q_0 E_0$, so one can say that $\overline{M}_0$ is determined by the equilibrium rate of change E_0 at $t=0$ (which, note it, does not enter the equilibrium conditions).

We have got a dynamic system of 8 equations (4a)-(4h) with 7 variables C_t, $\overline{M}_t$, $\overline{F}_t$, l_t, $\overline{B}_t$, λ_t, π_t and one of policy instruments τ_t, θ_t, A_t, G_t, which solves at given initial data $\overline{B}_0$, $\overline{F}_0$. The choice of a policy instrument for supporting equilibrium S. Turnovsky calls *accommodation*. We shall choose in calculations (Section 3) the rate of money emission θ.

Obviously, a stationary solution is possible only if $r \equiv \rho$ and $\theta = \pi$. If we exclude the regimes where U_c and U_F converge to 0 or ∞ then r can not be strictly more or less than ρ.

1.4 Optimal government policies. We shall call the bundle of policy instruments (τ_t, θ_t, A_t, G_t) a *general policy*. Let the government seeks to determine a general policy, which maximizes the welfare function of the representative consumer (1) subject to the equilibrium constraints (4). The problem has the form:

$$max \sum_{t=0}^{\infty} (1+\rho)^{-t}\, U(-\overline{R}_{t+1} + (1+r)\overline{R}_t + f(l_t) - G_t - (r+q)\overline{F}_t ;\ \overline{M}_t ;\ \overline{F}_t ;\ G_t ;\ l_t) \tag{5}$$

subject to:

$$U_C\,(t) = \lambda_t\,, \tag{5a}$$

$$U'_l\,(t) = -\,\lambda_t\,(1\text{-}\tau_t) f'(l_t)\,, \tag{5b}$$

$$U'_M\,(t) = \lambda_t(r + \pi_t), \tag{5c}$$

$$U_F\,(t) = \lambda_t\,(r + q)\,, \tag{5d}$$

$$\tau_t f(l_t) + \overline{M}_t\,\theta_{t+1} + q(\overline{F}_{t+1} + \overline{B}_{t+1} - \overline{F}_t - \overline{B}_t) - G_t\; = (1+r)\,\overline{A}_t - \overline{A}_{t+1}\,, \tag{5e}$$

$$\overline{M}_{t+1} = \overline{M}_t\,(1+\theta_{t+1})/(1+\pi_{t+1}), \tag{5f}$$

$$\lambda_t = \lambda_{t-1}\,(1+\rho)/(1+r). \tag{5g}$$

The intertemporal welfare function (5) is maximized on policy parameters τ_t, θ_t, A_t, G_t, and variables of the model l, $\overline{M}_t$, $\overline{F}_t$, $\overline{B}_t$, π_t, λ_t. The solution of the problem is to satisfy also the transversality conditions (T). The initial values of $\overline{A}_0, \overline{B}_0, \overline{F}_0$ are given, (we shall make more precise below), values of $\overline{M}_0$, λ_0 are free.

We have the following Lagrangian expression:

$L_t = (1+\rho)^{-t}U(-\overline{R}_{t+1}+(1+r)\,\overline{R}_t + f(l_t) - G_t - (r+q)\,\overline{F}_t\,;\,\overline{M}_t\,;\;\overline{F}_t\,;\,G_t;\,l_t)+\nu_1(1+\rho)^{-t}[\lambda_t - U_c] - \nu_2(1+\rho)^{-t}[(1-\tau_t)f'(l_t)\lambda_t + U_l] + \nu_3(1+\rho)^{-t}[(r+\pi_t)\lambda_t - U_M] + \nu_4(1+\rho)^{-t}[\lambda_t\,(r+q) - U_F] - s_1(1+\rho)^{-t}[\tau_t f(l_t) + \overline{M}_t\,\theta_{t+1} + q(\overline{F}_{t+1} + \overline{B}_{t+1}) - q(\overline{F}_t + \overline{B}_t) - G_t - (1+r)\,\overline{A}_t + \overline{A}_{t+1}] + s_2(1+\rho)^{-t}\,[\,\overline{M}_t\,(1+\theta_{t+1}) - (1+\pi_{t+1})\,\overline{M}_{t+1}] + s_3(1+\rho)^{-t}[(1+\rho)\lambda_t - (1+r)\,\lambda_{t+1}]$.

The first order condition w.r.t. variables τ_t, G_t, θ_t, $\overline{A}_t$, l, $\overline{M}_t$, $\overline{B}_t$, π_t, λ_t, $\overline{F}_t$ have the form (index of moment t is omitted):

$$\nu_2 f'(l)\lambda_t - s_1 f(l) = 0, \tag{6a}$$

$$-U_C + U_G + s_1 = 0, \tag{6b}$$

$$s_1 - s_2 = 0, \tag{6c}$$

$$(1+r)[-U_C+\nu_1 U_{CC}+\nu_2\,U_{lC}+\nu_3\,U_{MC}+\nu_4 U_{FC}+\,s_1] - (1+\rho)[-U_C(t\text{-}1)+ \tag{6d}$$
$$\nu_1(t\text{-}1)U_{CC}(t\text{-}1)+\,\nu_2\,(t\text{-}1)U_{lC}\,(t\text{-}1) + \nu_3(t\text{-}1)U_{MC}\,(t\text{-}1) + \nu_4(t\text{-}1)U_{FC}\,(t\text{-}1)+\,s_1(t\text{-}1)] = 0\,;$$

$$[U_C f_l + U_l] - \nu_1\,[U_{CC} f_l + U_{Cl}] - \nu_2\,[U_{lC} f_l + U_{ll} + f_{ll} U_C(1\text{-}\tau_t)] - \tag{6e}$$
$$\nu_3\,[U_{MC} f_l + U_{Ml}] - \nu_4\,[U_{FC} f_l + U_{Fl}] - s_1\tau_t f_l = 0,$$

$$U_M - \nu_1 U_{CM} - \nu_2\,U_{lM} - \nu_3 U_{MM} - \nu_4 U_{FM} - s_1\theta_{t+1} + s_2(1+\theta_{t+1}) - s_2(t\text{-}1)(1+\rho)(1+\pi_t) = 0, \tag{6f}$$

$$(1+r)[-U_C+\nu_1 U_{CC}+\nu_2\,U_{lC}+\nu_3\,U_{MC}+\nu_4\,U_{FC} - s_1\,q] - (1+\rho)[-U_C(t\text{-}1)+ \tag{6g}$$
$$\nu_1(t\text{-}1)U_{CC}(t\text{-}1)+\,\nu_2\,(t\text{-}1)U_{lC}\,(t\text{-}1) + \nu_3(t\text{-}1)U_{MC}\,(t\text{-}1)+\,\nu_4\,(t\text{-}1)U_{FC}(t\text{-}1) - s_1(t\text{-}1)q] = 0\,;$$

$$\nu_3\lambda_t\; - s_2(t\text{-}1)(1+\rho)\,\overline{M}_t = 0, \tag{6h}$$

$$\nu_1 - \nu_2\,(1\text{-}\tau_t)f_l + \nu_3\,(r+\pi_t) + \nu_4(r+q) + s_3(t)(1+\rho) - s_3(t\text{-}1)(1+\rho)(1+r) = 0, \tag{6i}$$

$$(1+r)[-U_C+\nu_1 U_{CC}+\nu_2\,U_{lC}+\nu_3\,U_{MC}+\nu_4\,U_{FC} - s_1\,q] - (1+\rho)[-U_C(t\text{-}1)+\nu_1(t\text{-}1)U_{CC}(t\text{-}1) \tag{6j}$$
$$+\nu_2\,(t\text{-}1)U_{lC}\,(t\text{-}1) + \nu_3(t\text{-}1)U_{MC}\,(t\text{-}1)+\,\nu_4\,(t\text{-}1)U_{FC}(t\text{-}1) - s_1(t\text{-}1)q] + U_C(r+q) - U_F = 0\,;$$

Taking into account (5a) and (5d), we see that equation (6j) is identical to (6g) (that is optimization on $\overline{F}_t$ gives no additional information comparing with $\overline{B}_t$). At the ends of trajectories the transversality conditions hold. At infinity: $s_1\,\overline{B}_t\,(1+\rho)^{-t}\rightarrow 0$, $s_1\,\overline{F}_t\,(1+\rho)^{-t}\rightarrow 0$, $s_1\,\overline{A}_t\,(1+\rho)^{-t}\rightarrow 0$, $s_2(t)\,\overline{M}_t\,(1+\rho)^{-t}\rightarrow 0$,

$s_3(t)\lambda(t)(1+\rho)^{-t}\to 0$ when $t\to\infty$. At the left end: $s_2(0)=0,\ s_3(0)=0$. Besides, the transversality conditions from consumer problem:

$lim\ \lambda_t \overline{M}_t\,(1+\rho)^{-t} = 0,\ lim\ \lambda_t \overline{B}_t\,(1+\rho)^{-t} = 0,\ lim\ \lambda_t \overline{F}_t\,(1+\rho)^{-t} = 0$ при $t\to\infty$. (T)

These are the necessary conditions for a *general policy* ($\tau_t,\ \theta_t,\ A_t,\ G_t$) to be optimal. The further aims the property of the consistency in time of optimal policy.

1.5 Consistency problem. If the optimization problem starting at arbitrary moment $t>0$ takes as initial conditions the values of the state coordinates determined by past accumulation then the optimal policy of moment $t=0$ is consistent in time. This is the dynamic programming principle. In terms of our problem suppose that

1) rate of time preference is equal to the real interest rate: $\rho = r$;

2) government, starting at any moment $t\geq 0$ commits to take the accumulated from the past values $\overline{A}_t$, $\overline{B}_t$, $\overline{F}_t$, $\overline{M}_t$ as given initial conditions, that is does not allow jumps in exchange rate $E(t)$ and the stock of debt liabilities $\overline{A}_t$. (In case of continuous time we would require continuous change of state coordinates). One can formulate

Proposition 1. *Let the conditions 1) and 2) hold. Then an optimal policy and the corresponding equilibrium trajectory with* ***given initial conditions*** $\overline{A}_0, \overline{B}_0, \overline{F}_0, \overline{M}_0$ *are consistent in time.*

If condition 1) fulfills, (5g) implies $\lambda_t = const$, and constraint (5g) disappear together with multiplier s_3. Since the other state variables obey condition 2), the solution is consistent. ·

In the sequel we assume that condition 1) holds. Condition 2) is relaxed in the following way.

2') The government, starting at $t=0$ commits to take as initial condition the accumulated from the past real financial wealth of the consumer $\overline{W}_0$, i.e. it has a constraint on initial conditions: $\overline{M}_0 + \overline{F}_0 + \overline{B}_0 = \overline{W}_0$. The government, starting at $\tau > 0$ takes $\overline{A}_\tau$, $\overline{B}_\tau$, $\overline{F}_\tau$ as given from the past, and $\overline{M}_\tau$ free.

The sense of difference between 2) и 2') is that the first government, which optimizes policy needs reorganize the initial conditions of the consumer (with the help of an operation at the open market) so that to engage the stationary equilibrium positions. The commitment in 2') aims to do this without jump in exchange rate E_0 or P_0. (We discuss it below.) The next governments will stay on the stationary way without commitments. Let us consider the consequences of conditions 1) and 2').

At any $t>0$ we get the transversality condition $s_2(t)=0$; then (6c) implies $s_1(t)\equiv 0$; besides, $\lambda_t=\lambda=const$, and $s_3=0$. Equations (6d), (6g), (6j) become identical and give the relation:

$$-U_C+\nu_1 U_{CC}+\nu_2 U_{lC}+\nu_3 U_{MC}+\nu_4 U_{FC} = \mu,$$

where μ is some constant. System (6a)-(6j) reduces to the equations:

$$U_C = U_G, \tag{7a}$$

$$(U_C - \nu_1 U_{CC} - \nu_4 U_{FC}) f_l + (U_l - \nu_1 U_{Cl} - \nu_4 U_{Fl}) = 0, \tag{7b}$$

$$U_M - \nu_1 U_{CM} - \nu_4 U_{FM} = 0, \tag{7c}$$

$$\nu_1 + \nu_4 (r + q) = 0, \tag{7d}$$

$$U_C = \lambda, \tag{7e}$$

$$U_l = - \lambda (1-\tau) f_l, \tag{7f}$$

$$U_M = \lambda(r + \pi), \tag{7g}$$

$$U_F = \lambda (r + q), \tag{7h}$$

$$U_C - \nu_1 U_{CC} - \nu_4 U_{FC} = \mu. \tag{7i}$$

This is a static system of 9 equations which may be solved for 9 variables: ν_1, ν_4, τ, π, C, $\overline{M}$, $\overline{F}$, l и G in terms of constants λ and μ. So the optimal solution is stationary, in particular, optimal policy τ, $\theta = \pi$, G is stationary.

The equilibrium solution has to satisfy also two balances: (4g) and (4h) which take now the form:

$$(1 + q)(\overline{B}_{t+1} - \overline{B}_t) = r \overline{B}_t + [(1-\tau)f(l) - C - \pi \overline{M} - q \overline{F}],$$

$$(\overline{B}_{t+1} - \overline{A}_{t+1}) - (1+r)(\overline{B}_t - \overline{A}_t) = f(l) - C - G - q \overline{F} = 0.$$

We set new initial conditions: $\overline{B} = \overline{W}_0 - \overline{F} - \overline{M}$ и $\overline{A} = \overline{A}_0 - \overline{B}_0 + \overline{B}$. Developing the balance relation from $t=0$, and applying the transversality conditions for $\overline{B}_t$ and $\overline{A}_t$ at infinity we get still two static equations:

$$r(\overline{B} + \overline{F} + \overline{M}) + (1-\tau)f(l) - C - (\theta + r) \overline{M} - (q+r) \overline{F} = 0, \tag{7j}$$

$$f(l) - C - G - q \overline{F} - r(\overline{A} - \overline{B}) = 0. \tag{7k}$$

Constants λ and μ may be chosen so that (7j),(7k) hold. Thus the optimal stationary equilibrium is determined.

In order to get the stationary equilibrium all initial values $\overline{A}_0, \overline{B}_0, \overline{F}_0, \overline{M}_0$ must jump to the equilibrium values $\overline{A}, \overline{B}, \overline{F}$ and $\overline{M}$ at $t=0$. (It is the single jump on the trajectory). The optimization problem of moment $t=0$ with constraint $\overline{F} + \overline{B} + \overline{M} = \overline{W}_0$ has the same first order conditions (6), as well as transversality condition at the left end: $s_1(0) = s_2(0)$ which agrees with (6c). Since $\overline{A} - \overline{B} = \overline{A}_0 - \overline{B}_0$, and $\overline{B} + \overline{F} + \overline{M} = \overline{B}_0 + \overline{F}_0 + \overline{M}_0$ balances (7j), (7k) will be the same as with the original initial conditions. Thus we get that the jump at t=0 and the consistency requirement keep within the necessary conditions of optimization problem of moment $t=0$ which give the stationary solution.

Now we say how the jumps can be realized in practice. The desired jump of the debt from $\overline{A}_0$ to $\overline{A}$ can be achieved by an «instantaneous» open-market purchase or sale of government debt from or to public. Domestic residents buy or sell such a quantity of bonds for foreign currency and also using domestic currency and exchange rate E_0. Doing so a consumer comes to optimal values $\overline{F}$, $\overline{M}$ and $\overline{B}$. Hence the following relations hold: $B-B_0 = A-A_0$ and $F-F_0 + (M-M_0)/E_0 = B_0 - B$. This operation leave invariable the equalities: $\overline{F} + \overline{B} + \overline{M} = \overline{W}_0$ and $\overline{A} - \overline{B} = \overline{A}_0 - \overline{B}_0$, that is real financial wealth of the representative consumer $\overline{W}_0$ and original external debt $\overline{A}_0 - \overline{B}_0$ do not change. We emphasize that rate of exchange E_0, and hence price P_0 do not jump at $t=0$, but the nominal money balances and nominal value of stock of bonds B_0

jump from M_0, B_0 to M and B. They stay invariable after while nominal money M and price P increase with the rate θ.

Another way of coming to the stationary trajectory may be in jumping of the rate of exchange (and hence price) at $t=0$, see S.Turnovsky (1987). Then bonds exchange only on foreign currency. One can formulate the result.

Proposition 2. Let $\rho = r$ *and condition 2') holds. Then the optimal consistent in time policy and the corresponding equilibrium trajectory are necessarily stationary. In particular, the initial real financial wealth of the representative consumer and external debt of the government stay invariable. The jump of initial conditions to the stationary ones is provided by an operation of the government at the open market of bonds.*

Remark. Without foreign currency an optimal policy of horizon $[0,\infty]$ would be inconsistent. Really, in this case system (7a)-(7i) gives $U_M = 0$ (7c), $\pi = \theta = -r$ (7j), $\tau = 0$ (7b),(7f). Together (if they are possible separately) this would bring to negative income of the state budget:

$$\tau f(l) + \theta \overline{M} = G + r\overline{A}\,.$$

S. Turnovsky and W. Brock obtained the result in (1980) ·

If in equilibrium $U_M > U_F > 0$ (i.e. domestic currency has higher marginal transactional utility than foreign one (that would be natural) then (7j),(7h) imply that $\pi > q > -r$, that is the Friedman regime is impossible. A more precise assertion about optimal rate of inflation is contained in the Phelps relation.

1.6 The Phelps problem. E.Phelps (1973) obtained a natural characterization of optimal policy for a stationary model. We are going to get the analogous result for our model. To this end we solve equations (5a)-(5d) and (7j) for C_t, l_t, $\overline{M}_t$, $\overline{F}_t$, λ_t in terms of τ, $\theta = \pi$, G. Since policy is constant the variables are also constant. We consider the problem of choosing optimal constant policy τ, θ, G, subject to the government budget constraint. Given a policy τ, θ, G denote by $V(\tau, \theta, G)$ maximal gain of the consumer. The problem is:

$$\max V(\tau, \theta, G)$$

$$\tau f(l(\tau, \theta, G)) + \theta \overline{M}(\tau, \theta, G) - G - r\overline{A}(\tau, \theta, G) = 0.$$

The first order conditions are:

$$V_\tau = -\eta\,[\tau f + \theta \overline{M} - G - r\overline{A}\,]'_\tau\ ,$$

$$V_n = -\eta\,[\tau f + \theta \overline{M} - G - r\overline{A}\,]'_\theta\ ,$$

$$V_g = -\eta\,[\tau f + \theta \overline{M} - G - r\overline{A}\,]'_G,$$

where η is a scalar Lagrange multiplier. We write derivatives V_τ, V_θ и V_G taking into account (5a)-(5d):

$$V_\tau = U_C \cdot [C_\tau + (\theta+r)\,\overline{M}_\tau + (q+r)\,\overline{F}_\tau - (1-\tau) f'(l) l_\tau\,]\,,$$

$$V_\theta = U_C \cdot [C_\theta + (\theta+r)\,\overline{M}_\theta + (q+r)\,\overline{F}_\theta - (1-\tau) f'(l) l_\theta\,],$$

$$V_G = U_C \cdot [C_G + (\theta+r)\,\overline{M}_G + (q+r)\,\overline{F}_G - (1-\tau) f'(l) l_G\,] + U_G\,.$$

Now we differentiate on τ, θ and G the budget constraint of the consumer (7j):

$$r\overline{W}_0 + (1\text{-}\tau)f(l) - C - (\theta+r)\,\overline{M} - (q+r)\,\overline{F} = 0.$$

We get the relations:

$$f(l) + [C_\tau + (\theta+r)\,\overline{M}_\tau + (q+r)\,\overline{F}_\tau - (1\text{-}\tau)\,f'(l)\,l_\tau] = 0\,,$$

$$\overline{M} + [C_\theta + (\theta+r)\,\overline{M}_\theta + (q+r)\,\overline{F}_\theta - (1\text{-}\tau)\,f'(l)\,l_\theta] = 0\,,$$

$$[C_G + (\theta+r)\,\overline{M}_G + (q+r)\,\overline{F}_G - (1\text{-}\tau)\,f'(l)l_G] = 0.$$

Substituting the brackets from the previous expressions we get:

$$V_\tau = -\,U_c\;f(l)\,,$$

$$V_n = -\,U_c\;\overline{M}\,,$$

$$V_G = U_G\,.$$

Coming back to the optimality conditions we obtain finally:

$$[\tau f + \theta\,\overline{M} - G - r\,\overline{A}\,]'_\tau \,:\, f(l) = [\tau f + \theta\,\overline{M} - G - r\,\overline{A}\,]'_\theta \,:\, \overline{M} = \tag{8}$$

$$-\,[\tau f + \theta\,\overline{M} - G - r\,\overline{A}\,]'_G \,:\, (U_G / U_c) = U_c : \eta.$$

The Phelps requirement of invariability of the sum-of-income-effect is replaced here on the requirement of the government budget without deficit at variations of policy. More exactly this means that the difference between the tax income and the government expenditure including the payments on external debt must be zero at variations of policy τ, θ, g around their optimal values. The first and the last equalities in (8) represent a variant of the Phelps relation. The second emerges due to government expenditure as a policy.

Proposition 3. *If conditions 1), 2') are satisfied, then the optimal stationary equilibrium is characterized by the relation (8).*

In this Section we have considered a model without capital. All variables allowed in principle jumps to the stationary values. In the next Section we consider a model of stationary growth with capital for which one can get the analogous results.

2. The Phelps relation in a model of stationary growth

We assume now that the model admits endogenous growth of technical progress type. We consider the dependence of stationary growth regimes on constant government policies and derive an analogy of the Phelps relation.

2.1 Description of the model. The model is essentially the same. The representative consumer solves the problem:

$$max \sum_{t=0}^{\infty} (1+\rho)^{-t}\, U(C_t\,, M_t/P_t\,, F_t\,/Q_t,, G_t, l_t)$$

subject to

$$M_{t+1} + F_{t+1}E_t + B_{t+1}E_t = M_t + F_tE_t + (1\text{-}\tau_t)(W_t\,l_t + \Pi_t + S_t\,K_t) + E_t(1+n^F)B_t - P_tC_t - P_t\,I_{t+1},$$

$$K_{t+1} = K_t(1\text{-}\delta) + I_{t+1}\,,$$

and initial conditions $K_0 > 0, M_0 > 0, F_0 > 0, B_0 > 0, P_0$ and E_0.

We use the same notations as before, and: K - capital, S_t - nominal interest rate on capital, I_t – capital investment, δ - rate of depreciation,.

In order for the properties of the model were compatible with growth we assume that one-period utility function $U(C, M/P, F/Q, G; l)$ is homogenous of $(1\text{-}\sigma)$ – degree in the first four arguments ($C, M/P, F/Q, G$), convex and twice continuously differentiable in all arguments.

For production process we imposed Romer's (1986) AK-function. See also Turnovsky (1995, ch. 13), or Turnovsky (2000). The productive sector of the economy is presented by N identical firms, each endowed by K units of capital and the maximal labor supply is limited by one unit of time. The accumulated capital $k = NK$ is also a measure of the stock of knowledge in the economy; the knowledge is assumed to be non-rival good, having the productive value. Since, in this competitive setting, every firm is small, it takes k as given. It is convenient to assume that the productive capacity function is of the Cobb-Douglas form, i.e., the potential output is $y = F(K, k)= K^{\varepsilon} k^{1\text{-}\varepsilon} = N^{1\text{-}\varepsilon}K$. Thus, we come to the Romer AK-model, where A= $N^{1\text{-}\varepsilon}$. Since we allow for elastic labor supply, l , the real output of each firm is given by $f(l)y$. Hence, the production function is represented by: $F(K, l) = AKf(l)$.

Now denoting by $\gamma_{t+1} = I_{t+1}/K_t$ - δ, we come to the following basic relation

$$y_{t+1} = y_t\,(1+ \gamma_{t+1})\ ;$$

γ_{t+1} is the growth rate of potential output in the period $t+1$. The own capital of the firm enters the production function as K^{ε}. The economy capital is denoted for a while as K_1. The firm maximizes the net profits in each period, choosing the optimal values of K_t, l_t:

$$max\ [p_t\, AK_t^{\varepsilon}\ K_{1t}^{1-\varepsilon}\ f(l_t) - W_tl_t - S_tK_t] = \Pi_t \text{ with } p_t\, , A\, K_{1t}^{1-\varepsilon}\, , W_t\, , S_t \text{ given.}$$

(Note that if we set $f(l) = l^{1\text{-}\varepsilon}$ the production function would be of Cobb-Douglas form on the own capital and labor).

Government is the same as in the previous Section, the same policy (τ_t, θ_t, G_t, A_t) and external debt A_t - B_t.

Because of homogeneity, we can rewrite the consumer's problem in terms of variables relative to the full output:

$$max \sum_{t=0}^{\infty} I_t{\cdot}U(c_t\, , m_t\, ,m_t^*, g_t\, ; l_t) \tag{9}$$

subject to:

$$(1+\mu_{t+1})m_{t+1} + m_{t+1}^*(1+q)(1+ \gamma_{t+1})+ b_{t+1}(1+q)(1+ \gamma_{t+1})= m_t+ m_t^* + \tag{9a}$$
$$(1\text{-}\tau_t)(w_tl_t+ profit + s_t\, N^{\varepsilon\text{-}1}) + (1+q)(1+r)b_t - c_t\, \text{-}i(\gamma_{t+1}),$$

Notations are:

$\beta_t = \prod_{s=1}^{t} (1+\gamma_s)^{1\text{-}\sigma}/(1+\rho)^t$, $1+\mu_t = (1+\gamma_t)(1+\pi_t)$, where $\pi_{t+1}=(P_{t+1} - P_t)/P_t$, $m_t = M_t/(P_ty_t)$, $m_t^* =F_t\,/(P_ty_t)$, $b_t = B_t\,/(P_ty_t)$, $w_t = W_t\,/(P_ty_t)$, $profit = \Pi_t/P_ty_t$, $s_t=S_t\,/P_t$, $c_t = C_t\,/y_t$, $g_t = G_t\,/y_t$, $i(\gamma_{t+1}) = (\gamma_{t+1} +\delta)N^{\varepsilon\ \text{-}1}$. . As before we assume rate of inflation in foreign currency q and real interest r given and constant. Unknown variables are (c_t, m_t, l_t, b_t, γ_t), values π_t , $profit_t$, τ_t, w_t , s_t, g_t рассматриваются потребителем как заданные.

Profit maximization determines the demand for labor and capital by firms, given interest rate s_t and real wage w_t :

$$w_t = f'(l_t), \quad s_t = \varepsilon N^{1-\varepsilon} f(l_t).$$

The choice of rate of money emission by the government $\theta_{t+1} = (M_{t+1} - M_t)/M_t$ gives in relative real terms the relation:

$$m_{t+1} = m_t (1+\theta_{t+1})/(1+\mu_{t+1}).$$

The first order condition for the consumer take the form:

$$U_c'(c, m, m^*, g; l) = \lambda, \tag{10a}$$

$$U'_m(c, m, m^*, g; l) = \lambda(\pi + r), \tag{10b}$$

$$U'_{m^*}(c, m, m^*, g; l) = \lambda(q + r), \tag{10c}$$

$$U'_l(c, m, m^*, g; l) = -\lambda(1-\tau)f'(l), \tag{10d}$$

where λ_t is discounted (on β_t) Lagrangian multiplier in problem (9), (9a).

2.2 Stationary solution for the consumer problem In Section 1 we proved stationarity of the optimal policy in the perfect foresight equilibrium model. We shall base on this result here counting that $\tau_t \equiv const, \; g_t \equiv const, \quad \theta_t \equiv const$. Let $c(\lambda,\tau,\pi,g)$, $m(\lambda,\tau,\pi,g)$, $m^*(\lambda,\tau,\pi,g)$, $l(\lambda,\tau,\pi,g)$ - implicit differentiable functions solving system (10a)-(10d) for c, m, m^*, l in terms of constants λ,τ,π,g. We can get the following facts:

Equality $\lambda_t = const$ gives by force of the first order conditions (in dynamic) the relation: $1+r = (1+\rho)(1+\gamma_t)^\sigma$, or approximately:

$$r = \rho + \sigma\gamma,$$

i.e. optimal rate of growth γ can be determined immediately given parameters r, ρ, σ and so becomes itself as a parameter. (cf. R. Lucas (2000), where real interest was introduced so by definition given rate of growth γ).

Equalities $\theta_t \equiv const$, $m_t \equiv const$ and $m_{t+1} = m_t (1+\theta_{t+1})/(1+\mu_{t+1})$ give:

$$\theta = \mu = const.$$

Given θ and γ one can calculate rate of inflation π from equality $1+\theta = (1+\gamma)(1+\pi)$, it is equal approximately:

$$\pi = \theta - \gamma.$$

Discount β_t in problem (9), (9a) at stationary growth is equal:

$$\beta = (1+\gamma)^{1-\sigma}/(1+\rho), \text{ or approximately:}$$

$$\beta = (1+\gamma)/(1+r).$$

In a more habitual form the relation between r, rate of time preference β and γ has the form: $(1/\beta) - 1 = r - \gamma$. Such a relation is necessary for existence of stationary growth. (Before we had $\gamma = 0$, and $r = \rho$) .

We show now that b_t remains constant when policy τ, g, θ, rate of inflation π, rate of growth γ and variables c, m, m^*, l, λ stay constant. We substitute the expressions c, m, m^*, l through τ, $\pi = \theta - \gamma$, g to budget equality of the consumer (9a) taking into account optimality of production. Denote:

$$\Delta b = (1-\tau)f(l) - c - \theta\, m - (q+\gamma)m^* - i(\gamma).$$

Then we can rewrite it as:

$$\Delta b = (1+q)(1+\gamma)b_{t+1} - (1+q)(1+r)b_t \,.$$

Here Δb is constant in t. We consider the equation:

$$b_{t+1} = (1+r)b_t/(1+\gamma) + \Delta b\,/(1+q)(1+\gamma).$$

The transversality condition on b_t, considering $\beta = (1+\gamma)/(1+r)$ (####), takes the form:

$$\lambda b_t\,(1+\gamma)^t/(1+r)^t \rightarrow\ 0 \text{ при } t \rightarrow \infty.$$

Here $(1+\gamma)/(1+r)<1$ (see below). As to $m_t = const$, the transversality condition holds. Developing the equation from $t=0$, and applying the transversality condition we get: $\Delta b = b_0\,(\gamma - r)$. The point $b_t \equiv b_0$ is the unique stationary point of the equation.

Now we can rewrite the budget equality of the consumer in the form:

$$(b_0 + m + m^*)\,(r - \gamma) + (1-\tau)f(l)\ -\ (\pi + r)\,m - (q+r)m^* - c - i(\gamma)\ = 0.$$

Here c, m, m^*, l are functions of λ, τ, π, g. The last relation allow to exclude λ and to get functions: $c(\tau, \theta, g)$, $m(\tau, \theta, g)$, $m^*(\tau, \theta, g)$, $l(\tau, \theta, g)$ (further we shall write θ instead of π, remembering that $\pi = \theta - \gamma$ and γ is a parameter).

Above we used inequality $r - \gamma > 0$. Really this is true. The maximization problem (9), which is equivalent to the original consumer problem has a solution if discount $(1+\gamma)^{1-\sigma}/(1+\rho)$ less than 1. This implies: $(1+\gamma)/(1+r) < 1$ whence $\gamma < r$.

2.3 Stationary equilibrium growth. We assume the government declares and then pursue a constant policy τ, g, θ. A stationary perfect foresight equilibrium growth is a bundle scalars π, *profit*, w, s such that the constant in time functions c, m, m^*, l, b, a are solutions of the consumer problem (9), producer problem, and the resource balance is fulfilled in the system. The latter (in relative terms) has the form:

$$c + i(\gamma) + g + (q+\gamma)m^* - f(l) = (1+q)(1+\gamma)d_{t+1} - (1+q)(1+r)d_t\,,$$

where $d_t = a_t - b_t$ is relative external government debt.

Denote the left side of the balance by Δd, thus $\Delta d = (1+q)(1+\gamma)d_{t+1} - (1+q)(1+r)d_t$ is relative payment on the external debt in period t which in fact constant in time. We require that the no Ponzi game condition hold:

$$d_t(1+\gamma)^t/(1+r)^t \rightarrow 0 \text{ when } t \rightarrow \infty\,.$$

In the same way as we did before with the internal debt one can get the final relation: $\Delta d = -\,d_0\,(r - \gamma)$ and the unique stationary point $d_t \equiv d_0$. So we can write the stationary resource balance in the form:

$$c + i(\gamma) + g + (q+\gamma)m^* + d_0\,(r - \gamma) = f(l).$$

If initial external debt $d_0 = a_0 - b_0$ is positive (the country is a debtor), and $\gamma < r$, we have $d_0\,(r - \gamma) > 0$.

The government problem is not constrained by choosing a policy (τ, g, θ) and a. As before initial values b_0, m_0, m^*_0 and a_0 accumulated from the past generally do not coincide with equilibrium values b, m, m^* and a. So the same intervention at the open market is needed in order to reorganize the initial conditions. This results in relations: $b + m + m^* = b_0 + m_0 + m_0^*$ and $a - b = a_0 - b_0 = d_0$. The reasons given in Section 1 are entirely suitable here.

We summarize the all obtained relations in the system of equations the equilibrium stationary growth corresponding to a given constant policy τ, g, θ must satisfy :

$$U_c'\,(c\,,\,m\,,\,g\,;\,l) = \lambda\,, \tag{11a}$$

$$U'_m\,(c\,,\,m\,,\,g\,;\,l) = \lambda\,(\theta + r - \gamma), \tag{11b}$$

$$U'_{m^*}\,(c\,,\,m\,,\,m^*,\,g\,;\,l) = \lambda(q + r), \tag{11c}$$

$$U'_l\,(c\,,\,m\,,\,g\,;\,l) = -\,\lambda\,(1-\tau)f'(l), \tag{11d}$$

$$(b+m+m^*)\,(r-\gamma) + (1-\tau)f(l) - (\theta+r)\,m - (q+r)m^* - c - i(\gamma) = 0. \tag{11e}$$

$$c + i(\gamma) + g + (q+\gamma)m^* - f(l) + d_0\,(r - \gamma) = 0. \tag{11f}$$

2.3 Optimal policy. The Phelps relation. System (11a)-(11f) is static and we assume that it has a unique solution. The maximizing consumer utility on constraints (11a)-(11f) for constant policies gives the necessary conditions for optimal stationary (and so consistent) growth. This work is quite analogous to that in Section 1. We drop it. Instead we give an analogy of the Phelps relation

Let again $V(\tau,\theta, g)$ be the maximal value of the consumer utility function in the problem (9) under a policy τ,θ, g. The government problem is:

$$\max V(\tau,\theta, g))$$

subject to:

$$\tau f(l(\tau, \theta, g)) + \theta\, m(\tau, \theta, g) - g - a(\tau, \theta, g)(r - \gamma) = 0.$$

The result can be get quite in the same as in Section 1. So we simply give it:

$$[\tau f + \theta\, m - g - (r - \gamma)a]'_\tau \,:\, f(l) = [\tau f + \theta\, m - G - (r - \gamma)a]'_\theta \,:\, m = \tag{12}$$

$$-\,[\tau f + \theta\, m - g - (r - \gamma)a]'_g \,:\, (U_g / U_c) = U_c : \eta.$$

Relation (12) is direct generalization of the relation (8) for the growth model.

We note that here given initial capital K_0 is not an obstacle for stationary processes. The optimality conditions and the Phelps relation (12) are given in relative terms. Initial capital K_0 determines initial potential output, which will grow after with the rate γ.

3. Numerical calculations[1]

3.1 Evaluations of parameters. We study two stationary models: static, given by equations (4a)-(4h) and of stationary growth given by (11a)-(11f). The parameters of production, consumer preferences, and government's policy are evaluated by average values taken from the following Russian sources: Goscomstat RF, Russian statistical annual, and Short-time economic parameters, and also Central Bank of RF, Bulletin of bank statistics after 1999 year.

Production. In the static model output is produced with one factor - labor l, and production function has the form: $Y=l^\zeta$. We set $\zeta = 0.5$ which is equal to labor share in GDP in Russian national accounts and corresponds to the case when capital stock is fixed that corresponds to constant returns to scale case and in this case labor can be thought of a composite of all factors. In the growth model we also set the standard for Russian economy rate of depreciation of capital $\delta = 0.1$.

Government policy. We set share of government spending to GDP g = 32% which reflects the share of expenditure of consolidated Government in Russia.; we set constant b which is the internal debt-to-GDP ratio to 0.09 and constant d which is the external debt-to-GDP ratio to 0.35 ($130 billions from 350 is even 0.37), i.e. 0.44 is equal to total Federal Government debt to GDP ratio for Russia. We set rate of associated tax (i.e. federal tax and subjects of Federation taxes) τ =32%.

Preferences. One period utility function is taken of standard form (widely used beginning from Kydland and Prescott (1982)):

$$U = [(c^{1-\nu} s^{\nu} l^{-\kappa})^{1-\sigma} - 1]/(1-\sigma).$$

Here share of consumption c in GDP changes within 0.66 - 0.685 in static model and including investment in growth model; labor l, $0<l<1$, approximately equal 1/3 (8 h. from 24), s is money services in transactions reflecting imperfect substitution of currencies, are given by the standard formula:

$$s = [(1-\xi)m^{-\eta} + \xi\, m^{*\,-\eta}]^{-1/\eta}.$$

Parameters of utility function, which are subject to evaluating:

ν is share of money services in utility function, $0< \nu <1$,

ξ is relative efficiency of foreign currency in production of money services, $0<\xi <1$,

κ is parameter of elasticity of labor supply, $\kappa>0$,

η governs the elasticity of substitution between domestic and foreign money w.r.t. domestic and foreign interest rates ratio. It is equal $1/(1+\eta)$ and ordinarily is taken between 2 and 3, see Busman and Alderman (1992). Like Friedman and Verbetsky (2001) we took η = -0.5, so elasticity is equal 2,

σ is risk aversion parameter, $1/\sigma$ is elasticity of intertemporal substitution which is not revealed from static model but determines the rate of growth by formula: $r = \rho + \sigma\gamma$, where the world real interest assumed to be fixed at 5%,

ρ is subjective discount factor, in the growth model we set ρ =0.03 to be compatible with formula $r = \rho + \sigma\gamma$, where we change parameter σ between 0.4 and 0.8 to consider rate of growth between 3.5% and 4.5%, this are just actual growth rates for Russian economy.

We set the rate of inflation of good in foreign currency q =0. Using the data on stock of foreign currency in circulation in Russia and ruble money supply we calculated the ratio of domestic and foreign money which turned out to be equal 6/5, i.e. dollarization surpass 1/2. Assuming that domestic interest rate n= 15-20%, and foreign interest rate n^*=5%, and using the first order conditions (3.a) and (3.b) (see below), one can get as a guide the parameter of relative efficiency of foreign currency ξ from relation:

$$(1-\xi)(m^*/m)^{1+\eta}/\xi = n/n^*.$$

After substitution of indicated values we find $\xi \approx$ 0.21-0.25. However, when the rate of inflation decreases to zero ξ increases till 0.65.

Approximate evaluation for share of money services ν one can get from the first order conditions, using the evaluations of share of consumption in GDP c =0.65-0.7 (this is together with investment), and the real money balances (M_2) to consumption ratio m/c equal to 1/8. These proportions give us approximate

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[1] I also thank prof. K. Sossounov (NES) for help in calibrating the models..

values for m=0.08 and $m^*=6m/5$= 0.1. Substituting them to the first order condition (3.a) we find that ν changes between 0.018 and 0.035.

Elasticity of labor supply κ is evaluated now from the necessary condition (3.c) after substitution there the evaluated parameters ζ, ν and variables c= 0.7, l = 1/3 and τ =0.32. We get κ =0.28.

4.2 Analysis of equilibrium trajectories depending on parameters ζ and ν. For the static system we calculated equilibrium solutions on the following system including the first order conditions and balance relations:

$$\nu(1-\xi)/(1-\nu) = [1 - \xi + \xi(m^*/m)^{-\eta}](\pi + r)m/c \tag{3.a}$$

$$\nu\xi /(1-\nu) = [(1-\xi)(m^*/m)^{\eta} + \xi](q + r)m^*/c \tag{3.b}$$

$$(1-\nu)(1-\tau)\zeta\, l^{\zeta} = \kappa\, c \tag{3.c}$$

$$rb_0 + (1-\tau) - c - \pi m - qm^* = 0 \tag{3.d}$$

$$1 - c - g - qm^* - rd_0 = 0 \tag{3.e}$$

The last two balances are divided by output, so all real values are shares of unit. The influence of parameters ξ and ν on equilibrium values of variables, inflation tax and losses of welfare from inflation are given below in tables 1 and 1'.

Таблица 1 (τ =0.32, g = 0.32, d_0 = 0.35)

ξ	ν	m	m*	θ%	inflation tax %	Welfare and losses of inflation %	
0,2	0,039	0,086	0,123	20.7	1.8	0.612	4.2
0,2	0,04	0,094	0,122	19.1	1.8	0.613	4.2
0,2	0,041	0,101	0,122	17.1	1.8	0.613	4.0
0,17	0,039	0,0887	0,093	20,4	1.8	0.612	4.6
0,18	0,039	0,0877	0,104	20.5	1.8	0.613	4.4
0,19	0,039	0,087	0,113	20.6	1.8	0.613	4.3

Таблица 1’ (τ =0.32, g = 0.32, d_0 = 0.3)

ξ	ν	m	m *	θ%	inflation tax%	Welfare and losses of inflation %	
0,17	0,039	0,129	0,071	11.6	1.5	0.622	3.5
0,18	0,039	0,127	0,080	11.8	1.5	0.622	3.4
0,19	0,039	0,125	0,089	12	1.5	0.622	3.4
0,20	0,039	0,123	0,098	12.2	1.5	0.622	3.2
0,21	0,039	0,122	0,108	12.3	1.5	0.622	3.2
0,2	0,04	0,132	0,098	11.4	1.5	0.622	3.2
0,2	0,041	0,14	0.098	10.7	1.5	0.622	3.2
0,2	0,042	0,148	0,098	10.1	1.5	0.622	3.2

One can see from the tables 1 and 1’ that given ν and increasing ξ the share of equilibrium stock of dollars m^* in GDP increases; and from the other hand given ξ and increasing ν the equilibrium share of real domestic money balances increases. When share of external debt decreases from 0.35 to 0.3 very sensitively decrease inflation, inflation tax, losses from inflation. We would get decreasing inflation on 8-9% and increasing welfare on 1.4% (losses from inflation decrease on 0.9% GDP).

The losses from inflation seem to be higher than their ordinary values. I calculated them comparing percent changing of welfare function (i.e. literally) , not increment of consumption as it is done ordinarily. I did so also calculating losses from dollarization.

4.2 Losses from dollarization. I am going to compare two static models: one with foreign currency for which $m^*/m \approx 6/5$ as it takes place now in Russia, and another without foreign currency. The latter means that efficiency of dollar ξ is set equal zero (while ν remains the same) and equation (3b) is excluded. Since I do not study dynamic here the results seems to be more relevant with actual level of dollarization. I’ll compare the models assuming that external debt taken under 5% runs sequence 0.35, 0.3, 0.25 and 0.2 of GDP. (In fact, the Russian debt was taken under bigger interest rate and I would take it into account as the cost borrowing in the future)

Let us consider solutions of system (3a)-(3e) for these values of debt. If we fix evaluated in 4.1 values of parameters ξ and ν the sharp changes in solutions of the system will testify about necessary changes in the finance system, in particular in parameters ξ and ν. So I corrected the parameters so that the real proportions (as they are now) remained approximately invariable: c≈ 0.665, $m^*/m \approx 6/5$, $m/c \approx 1/8$. The calculations of the corresponding stationary equilibria are given in tables 2 and 2'.

From comparison of two tables one can see that when debt is equal 0.35 GDP rate of inflation in the model with foreign currency is about 10% higher and welfare is 2.6% ((0.628-0.612)/0.612=2.6%) lower than without foreign currency. The similar picture takes place for lower debts 0.3,.0.25, 0.2. One could propose the following reasons for this losses. We keep equilibria corresponding to high level of dollarization (6:5). Dollar substitutes ruble in transactions until its efficiency $\xi > 0$. Assume that because of institutional or other causes the efficiency became close to zero. All money services are fulfilled by ruble. Its real quantity increases (from 0.08 till 0.16). Without printing new money it can be only due to lowering prices. A new equilibrium is set with lower rate of inflation corresponding the increased real money balances. Since ruble is essentially more efficient in transactions (it economizes more resources) than dollar utility value will become bigger than before. The calculations were made at values τ =32% and g=0,32. Labor services are found independently from equation (3c). I did not give them because, partly absolute value of output does not influence the conclusions; besides, until income tax is fixed, dependence of labor on parameters ξ and ν is not very much informative.

Table 2. Influence of external debt in static model with foreign currency.

ξ	ν	d_0	m	m^*	θ %	c	Inflation tax %	Welfare and losses of inflation %	
0.18	0.039	0.35	0.083	0.104	20	0.662	1.8	0.613	4.4
0.2	0.034	0.3	0.083	0.101	18	0.665	1.5	0.620	3.5
0.23	0.03	0.25	0.085	0.102	14	0.668	1.2	0.629	2.4
0.25	0.027	0.2	0.083	0.102	12	0.67	1.0	0.634	1.8

Tables 2' . Influence of external debt in static model without foreign currency.

ν	d_0	m	θ %	c	Inflation tax %	Welfare and losses of inflation %	
0.039	0.35	0.17	11	0.662	1.8	0.628	4.2
0.034	0.3	0.16	8.9	0.665.	1.54	0.635	3.3
0.03	0.25	0.17	6.9	0.668	1.12	0.641	2.5
0.027	0.2	0.17	5.8	0.67	1.0	0.646	1.9

4.3 Influence of rate of growth. We consider the model of stationary growth. The corresponding system of equations has the form:

$$\nu(1-\xi)/(1-\nu) = [1 - \xi + \xi(m^*/m)^{-\eta}](\theta + r - \gamma)m/c \qquad (3.a')$$

$$\nu\xi/(1-\nu) = [(1-\xi)(m^*/m)^{\eta} + \xi](q + r)m^*/c \qquad (3.b')$$

$$(1-\nu)(1-\tau)\zeta\, l^{\zeta} = \kappa\, c \qquad (3.c')$$

$$(r - \gamma)b_0 + (1-\tau) - c - \theta\, m - (q+\gamma)m^* - i(\gamma) = 0 \qquad (3.d')$$

$$1 - c - g - i(\gamma) - (q+\gamma)m^* - (r-\gamma)d_0 = 0 \qquad (3.e')$$

Here also balances are divided by output. We consider solutions of the system for sequence of γ = 3.5%, 4% and 4.5% and fixed values τ =32%, g=0,32, d_0 = 0.35. Again we compare the results of calculations for the models with and without foreign currency. As above we shall correct parameters ξ and ν so that the real proportions approximately corresponded the actual values. Now the essential part of general consumption consists of investment (mainly because of depreciation of capital) share of which in GDP in Russia is about 0.3. Proceeding from this we evaluate (after dividing on output) share of investment as follows: $i(\gamma) = (\gamma + \delta)/N^{1-\varepsilon}l^{\zeta} = 2(\gamma + \delta)$.

The calculations are given in tables 3 and 3'.

Таблица 3. Influence of growth rate γ% in model with foreign currency

ξ	ν	γ%	*m*	*m**	θ%	*c*	inflation tax%	Welfare
0.39	0.0301	3.5	0.0825	0.104	7.27	0.401	0.6	0.384
0.44	0.0272	4	0.084	0.103	5.9	0.392	0.5	0.377
0.65	0.019	4.5	0.084	0.1	2.3	0.383	0.2	0.373

Таблица 3'. Influence of growth rate γ% in model without foreign currency and with corrections of parameter ν.

ξ	ν	γ%	M	m *	θ%	C	inflation tax%	Welfare and losses of inflation %
0	0.02	3.5	0.15	0	4.0	0.405	0.6	0.397 2.4
0	0.0164	4	0.16	0	3.9	0.396	0.5	0.390 2.2
0	0.0072	4.5	0.16	0	1.23	0.388	0.2	0.386 0.7

Comparison of tables 3 and 3' shows the similar picture as tables 2, 2', i.e. values of welfare in the model with dollars are lower than without them. Correcting share of money services (in Table 3') taking into account the increased role of ruble we find that losses of welfare will be about 2.5 –3%. The latter is the effect of «forced» high dollarization.

In a recent and more specific investigation A. Friedman and A. Verbetsky (2001) note that the effect of dollarization is negative when rate of inflation is lower, and positive under high rates of inflation.

Obviously the authors consider rate of inflation as high if it is $\geq 2\%$ per month when the economy get the non-efficient part of Laffer curve. In our case (if we keep the actual proportions between real variables) rate of inflation hot higher 20% per year. So one can say that our conjecture about losses of dollarization does not contradict their conclusion.

4. Conclusion

The peculiarity of the work is foreign currency in hands of the representative consumer. This performed the role in both parts: theoretical and practical. In theoretical part we proved with help of foreign currency the stationarity of optimal consistent policies and could derive analogies of the Phelps relation. Without foreign currency this could be made only with special governments commitments. In practical part presence or absence of foreign currency was the central point for comparison of stationary equilibria. As the static and growth models shown the «unjustified» high dollarization brings welfare losses (given actual real proportions) about 2.5-2.8% for static as well as growth models.

Two parameters were of special interest: efficiency of dollar in production of money services ξ and share of money services in utility function ν (elasticity of currency substitution $1/(1+\eta)$ was taken invariably equal 2, that is approximately typical for countries with abnormally high inflation). The calculations results on static model showed that given real proportions in Russian economy $\xi \approx 0.2$ and $\nu \approx 0.039$. In A.Friedman and Verbetsky (2001) we find (for period after crisis 1998) ξ (at their paper λ) between 0.20 and 0.29 and ν (at their paper γ) = 0.05. The growth model shows values of ξ between 0.39 and 0.65 when we increase rate of growth from 3.5% till 4.5%. This is rather typical for period before crisis at the mentioned paper since they have ξ between 0.38 and 0.57. Thus we note that if rate of growth increases while level of dollarization remains invariable (6/5) the efficiency of dollar ξ necessarily increases whereas share of money services decreases. Evidently this witnesses that actually the level of dollarization has to decrease. This high level probably was justified at high inflation at the beginning of reforms. However, crisis of 1998 brought to distrust of public to the government, which supports high dollarization. It is pertinent note else that real Russian interest payments on debt are higher than in the model. I hope that making more precise the model in this direction will help to get a clearer picture.

All calculations were made at fixed income tax τ =0.32. One can see from static equilibrium equations that increasing tax brings to equivalent decreasing consumption (as it happened when investment emerged), and accordingly to decreasing money demand.

Appendix

The proof of stationarity of consistent policies in continuous time looks more compact and clear. I give it here.

The government problem is:

$$max \int_0^\infty U(c, m, m.^*, l, g)e^{-rt}dt$$

$$U_c = \lambda,$$

$$U_l =- (1-\tau)f'\lambda,$$

$$U_m = (\pi+r)\, \lambda,$$

$$U_{m^*} = (q+r)\, \lambda,$$

$$\dot{m} =(\theta -\pi)m,$$

$$\dot{v} = rv + [\, c + (q+r)m^* + g - f(l)],$$

$$\dot{z} = rz + [(1-\tau)f(l) - c - \theta m - (q+r)m^*],$$

where $z = m^* + b$ is foreign assets, $v = a - z$ - «pure» external debt. Denote by $\nu_1 e^{-rt}$,..., $\nu_4\, e^{-rt}$ discounted Lagrange multiplier to the first four constraints, and $s_1\, e^{-rt}$,..., $s_3\, e^{-rt}$ to the last three constraints. The first order conditions are:

(c): $U_c - \nu_1 U_{cc} - \nu_2 U_{cl} - \nu_3 U_{mc} - \nu_4 U_{m^*c} - s_2 + s_3 = 0,$

(l): $U_l - \nu_1 U_{cl} - \nu_2\, [U_{ll} + f''(1-\tau)\lambda] - \nu_3 U_{ml} - \nu_4 U_{m^*l} + s_2 f' - s_3(1-\tau)f' = 0,$

(m): $U_m - \nu_1 U_{cm} - \nu_2 U_{lm} - \nu_3 U_{mm} - \nu_4 U_{m^*m} - s_1(\theta -\pi) + s_3\theta = \dot{s}_1 - rs_1,$

(m^*): $U_{m^*} - \nu_1 U_{cm^*} - \nu_2 U_{lm^*} - \nu_3 U_{mm^*} - \nu_4 U_{m^*m^*} - s_2(q+r) + s_3(q+r) =0,$

(g): $U_g - \nu_1 U_{cg} - \nu_2 U_{lg} - \nu_3 U_{mg} - \nu_4 U_{m^*g} - s_2 = 0,$

(λ): $\nu_1 - \nu_2(1-\tau)f' + \nu_3(\pi + r) + \nu_4(q+r) = 0,$

(π): $\nu_3\, \lambda + s_1\, m = 0,$

(τ): $\nu_2\, f'\lambda + s_3\, f =0,$

(θ): $-s_1 + s_3 = 0,$

(z): $\dot{s}_3 =0,$

(v): $\dot{s}_2 = 0.$

Since $m(0)$ is free end, and for the consistent in time solution it has to be free at every moment of time then $s_1 \equiv 0$. It follows from (θ) $s_3 \equiv 0,$ from (τ) $\nu_2 =0$ (since generally $\lambda \neq 0$), from (π) $\nu_3 =0,$ from (v) $s_2 \equiv const$. We get system:

$$U_c - \nu_1 U_{cc} - \nu_4 U_{m^*c} - s_2 = 0,$$

$$U_l - \nu_1 U_{cl} - \nu_4 U_{m^*l} + s_2 f' = 0,$$

$$U_m - \nu_1 U_{cm} - \nu_4 U_{m^*m} = 0,$$

$$U_{m^*} - \nu_1 U_{cm^*} - \nu_4 U_{m^*m^*} - s_2(q+r) =0,$$

$$U_g - \nu_1 U_{cg} - \nu_4 U_{m^*g} - s_2 = 0,$$

$$\nu_1 + \nu_4(q+r) = 0.$$

(The system differs from analogous system (7) in the basic text because in (7) argument c in utility function is replaced by its expression from resource balance). We add first order constraints:

$$U_l = -(1-\tau)f'\lambda,$$

$$U_m = (\pi+r)\lambda,$$

$$U_{m*} = (q+r)\lambda .$$

We have got a static system of 9 equations which can be solved for $c, m, m^*, l, g, \tau, \pi, \nu_1, \nu_4$ in terms of constant s_2. Thus policy and variables from this list are constant. In particular, we get $\theta = \pi = const$. Constancy of a and и b is obtained in standard way with help of transversality conditions at infinity. Thus the system is filled up one more two static equations:

$$r(a_0 - b_0) + c + qm^* + g - f(l),$$

$$rb_0 + (1-\tau)f(l) - c - \theta m - qm^* = 0,$$

where a_0 and b_0 are new initial values corresponding to stationary value m^*: In the basic text we discussed how they emerge.

In S. Turnovsky (1987) the similar proof was developed for a model without foreign currency and fixed τ. Maximization on τ would bring to incompatible system. Really, without foreign currency $\nu_4 = 0$, $\nu_1 = 0$, and so $U_m = 0$, i.e. $\pi = \theta = -r$, and $\tau = 0$. Together this gives generally incompatible government balance:

$$ra_0 + c + g = \tau f(l) + \theta m.$$

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