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Characterization of competitive allocations and the Nash bargaining problem

Working paper # 2000/017

МОСКВА
2000

Sotskov A.I. Characterization of competitive allocations and the Nash bargaining problem. /Working paper # 2000/017.- Moscow, New Economic School, 2000.-34 p.(Engl.)

We discuss here the problem of characterization of competitive allocations in traditional economic equilibrium models. To this end we transfer the bargaining problem to the initial space of alternatives and introduce there the concepts of a Nash agreement point and (for the smooth case) a Nash bargaining point. They look like many-valued rules of choice in a compact convex set with given a status-quo point and «bargaining powers». Competitive allocations are Nash agreement (bargaining) points. We give axiomatic characterizations for these rules and for the Walrasian rule. At the end of the article a simple Nash implementing mechanism for the Walrasian rule is constructed.

Сотсков А.И. Характеризация конкурентных равновесий и задача торга по Нэшу**.**/Препринт # 2000/017.-М.:Российская экономическая школа, 2000.-34 с.(Англ.)

Мы обсуждаем здесь проблему характеризации конкурентных равновесий в традиционных моделях экономического равновесия. С этой целью мы переносим задачу торга (по Нэшу) в исходное пространство альтернатив и вводим концепции Нэшевской точки согласия и, в случае гладких предпочтений, Нэшевского решения торга. Они оформляются как выбор в данном выпуклом компактном множестве с заданными точкой статус-кво и коэффициентами «переговорной силы». Конкурентные распределения являются Нэшевскими точками согласия (решениями торга). Мы даем также аксиоматическую характеризацию этих концепций выбора для Вальрасовского правила. В конце работы мы строим простой механизм для Нэшевской имплементации Вальрасовского правила.

ISBN 5-8211-0104-2

Introduction

In this article we associate the choice of competitive resource allocations in traditional equilibrium models with the relevant Nash bargaining problems. Why do we associate these problems? In both cases the question is to choose a point from a convex compact set (the set of feasible allocations) given a status quo-point (initial endowments) and bargaining powers (production shares). However these similar problems are solved differently: cooperatively in the bargaining problems and non-cooperatively in the market environment; a bargaining solution ignores physical outcomes while the competitive rule ignores utility aspect. The question I would like to discuss here is the following: can we explain the choice of a competitive allocation in terms of a relevant Nash bargaining problem? The positive answer would allow to shed an additional light on the normative properties of competitive allocations. Note that the question is rather opposite to the traditional line of studies concerning the Nash bargaining problem: to support cooperative solutions by equilibria of non-cooperative games (K. Binmore (1987)).

The basic situation is a problem of social choice in a convex set Z given agents' preferences, a status-quo point $\omega \in Z$ and non-negative numbers β_i interpreted as agents' rights or «bargaining powers» in the decision making. As it is known the Nash bargaining solution (NBS) concept suggests to map the initial set of alternatives by means of von Neumann and Morgenstern agents' utilities to the corresponding utility values space and to choose in the image the NBS under given status-quo utility values and «bargaining powers». In the present setting we define a bargaining solution in the initial set of alternatives. To this end given a point $z \in Z$ we define a supporting linear utility profile $\boldsymbol{l}$ at the point z for agents' preference profile $\boldsymbol{P}$ and determine the NBS for the NB problem $\boldsymbol{l}(Z)$ with status-quo point $\boldsymbol{l}(\omega)$ and «bargaining powers» β_i. If $\boldsymbol{l}(z)$ is just the NBS of this problem then z is chosen in Z and we call z a *Nash agreement point (NAP).* In particular, when a continuous choice of supporting linear utilities is possible in Z, we call the corresponding chosen point a *Nash bargaining point (NBP).* We establish existence of NBPs in Z.

We give two equivalent characterizations for NAPs (and NBPs): one interprets them as equilibrium points (EP) with individual prices and another does it in terms of axiomatic requirements. The axiomatic characterization follows the line of studies developed by Polterovich (1973), Sotskov (1987), Nagahisha (1994), Yoshihara (1998). Polterovich's article was the starting point and a sample for our advancements..

Having got all the necessary results in the abstract setting we proceed to applications, in particular, to the characterization of competitive allocations of goods in equilibrium models. The general results imply that competitive allocations of goods are NAPs , and in the case of smooth preferences, NBPs in the set of feasible allocations. That is a competitive allocation z is such a point in set of feasible allocations Z whose image under supporting (to the preferences) linear mapping at z is the NBS. Agents' shares of production perform the role of «bargaining powers» in a model of allocation of goods (model A). In Arrow-Debreu type model (C) this is so in «interior» points z . We consider also a pure exchange model (B) where «bargaining powers» are not given explicitly.

We complete the article by constructing a simple mechanism which implements the Walrasian rule for a pure exchange economy by Nash equilibrium outcomes. This gives one more characterization of competitive allocations.

I□□ Bargaining in the initial space of alternatives

I. The Nash agreement points

We begin with formulating a bargaining problem in the initial space of alternatives. A group of n agents chooses a point from a given set Z. This is a publicly feasible set of alternatives. Besides the agents have individual «consumption» sets $Z_i \supset Z$, $i = 1,\ldots,n$ on which strict preference relations P_i are defined. Given this the agents might choose a Pareto-optimal point but have no ground to prefer one such a point to another one. The problem of choice becomes more definite when there is a status-quo point $\omega \in Z$ to compare with. Sometimes (for example, in pure exchange models) these data already allow to determine unique choice. In many other cases (in particular, in models with production) to narrow the choice we need some additional information in the form of agents' weights, rights, shares, say numbers $\beta_i \geq 0$, $\Sigma\beta_i = 1$ which we call «bargaining powers». The latter are not given sometimes explicitly but for the moment we assume that the both ($\omega \in Z$ and (β_i)) are given. Thus a problem of choice in Z is defined by the data: $S = (Z, \omega, (Z_i, P_i, \beta_i)_{i=1,\ldots,n})$; a class of feasible problems denoted by $\boldsymbol{S}$ is specified below by the conditions A)-D).

A) Z is a convex compact set, $Z \subset Z_i$, $i=1,\ldots,n;$ Z_i are closure of open sets in a finite-dimensional euclidean space L.

Let P_i be agent i's strict preference on Z_i and $\boldsymbol{P}$ be the strict preference profile $(P_i)_{i=1,\ldots,n}$. A linear functional l_i on L is called *supporting* to preference P_i at point z, if $l_i z' > l_i z$ for any $z' \in P_i(z)$. We denote by $\partial P_i(z)$ the set of all supporting linear functionals to preference P_i at point z; $\boldsymbol{l} \in \partial \boldsymbol{P}(z)$ means that $\boldsymbol{l}$ is a supporting profile at z.

Definition 1. A point $z\in Z$ is called a *Nash agreement point* (NAP) in the problem of choice $S = (Z,\omega, (Z_i, P_i, \beta_i)_{i=1,\dots,n})$, if for some supporting profile $\mathbf{l}\in \partial\mathbf{P}(z)$ the point $\mathbf{l}(z-\omega) \in R^n$ is the Nash bargaining solution in the set $\mathbf{l}(Z-\omega)$ with bargaining powers β_i, and disagreement point $0 \in \mathbf{l}(Z-\omega)$.

Under the *Nash bargaining solution* (NBS) for a convex, closed, and bounded above set $M \subset R^n$ with status-quo point $0\in M$ and bargaining powers $\beta_i \geq 0$ we understand the solution of the problem:

$max\, x_1^{\beta_1}\cdots x_n^{\beta_n}$, $x = (x_1,\dots,x_n) \in M\cap R^n_+$, (we set $0^0 = 1$) .

Obviously the NBS is not empty. Below we supply the unique NBS for the NB problems of the form $M=\mathbf{l}(Z-\omega)$. To this end we require the following «resource relatedness» condition B) to hold for the problems $S\in \mathbf{S}$:

B) for any $z\in Z$, $\mathbf{l}\in \partial\mathbf{P}(z)$, $i, j \in \{1,\dots,n\}$ such that $i \neq j$, and $\mathbf{l}(z) \geq \mathbf{l}(\omega)$, there exists $z'\in Z$ such that $l_r(z'-z) \geq 0$ for all $r \neq i$ and $l_j(z'-z)>0$.

The condition B) means that any individually rational with respect to supporting utilities $\mathbf{l}$ state z can be moved in Z so that at the expense of an agent i one can improve the position of any other agent without changing for the worse all other agents' positions.

The «resource relatedness» condition B) implies the following property of the NBSs.

Lemma 1. *For any* $\mathbf{l} \in \partial\mathbf{P}(Z)$ *the NBS* ξ^* *for the set* $M = \mathbf{l}(Z-\omega)$ *is unique and* $\beta_i = 0$ *implies* $\xi_i^* = 0$; *if* $M\cap R^n_+ \neq \{0\}$ *then* $\beta_i > 0$ *implies* $\xi_i^* > 0$.

Proof. First we show that $\beta_i = 0$ implies $\xi_i^* = 0$. Indeed, if $\xi_i^* > 0$ then using condition B) one can increase all ξ_j^* for which $\beta_j > 0$ at the expense of ξ_i^* and thereby to increase the value $x_1^{\beta_1}\cdots x_n^{\beta_n}$. So this is true. If set $M\cap R^n_+ \neq \{0\}$ then it contains a strictly positive point (condition B implies it). It follows that if $\beta_i > 0$ then $\xi_i^* > 0$. Finally, the NBS is unique because the solution of the problem $max\ \Pi_{i:\ \beta_i>0}\ x_i^{\beta_i}$ on the set $Z' = \{(x_i) \in Z:\ x_i = 0$ if $\beta_i = 0\}$ is unique. ·

We come back to the definition of NAPs. Note that if the initial preference profile is linear: ***P* = *l***, then any point $z \in Z$ such that ***l***$(z-\omega)$ is the NBS for the set ***l***$(Z-\omega)$ is a NAP. In other words any preimage point (at the mapping ***l***) of the NBS is a NAP. In Section 3, we consider another definition of a bargaining solution which for a linear preference profile gives only the preimage points of the NBS .

Denote the set of all Nash agreement points in *Z* for a problem *S* by *NAP(S)*; sometimes we write *NAP(**P**)* or *NAP (**l**)*, underlining that the variables in *S* are preference profiles ***P*** or ***l*** . Thus we have a multi-valued rule $NAP : \mathbf{S} \rightarrow 2^Z$ called the *Nash agreement rule.*

2. Equilibrium points

Here we introduce a notion of an equilibrium in a problem $S \in \mathbf{S}$. «Individual prices» $p_i \in L^*$ play the main role in it. Given a bundle of «prices» $\mathbf{p} = (p_1, \ldots, p_n)$ we call a «profit» the following number $\pi(\mathbf{p}) = \max_{z \in Z} \sum_{i=1}^{n} p_i (z - \omega)$. Using «bargaining powers» β_i we can form the «budget sets» $B_i(\mathbf{p}) = \{z \in Z_i \ / \ p_i z \le p_i \omega + \beta_i \pi(\mathbf{p})\}$. Note that a budget set $B_i(\mathbf{p})$ is a subset of Z_i, not *Z*.

Definition 2. A pair $(z^*, \mathbf{p})$ is called an *equilibrium* in a problem $S = (Z, \omega, (Z_i, P_i, \beta_i)_{i=1,\ldots,n})$, if $z^* \in Z$ and

$P_i(z^*) \cap B_i(\mathbf{p}) = \varnothing,\ i = 1, \ldots, n.$

The alternative z^* is called an *equilibrium point (*EP*),* the set of all equilibrium points in a problem S is denoted by $EP(S)$, a multi-valued rule $EP : \mathbf{S} \rightarrow 2^Z$ is called the *equilibrium rule.*

In the sequel the following assumptions C)-D) about agents' preferences in the problems $S \in \mathbf{S}$ are supposed to hold:

C) the sets $P_i(z)$ are non-empty subsets in Z_i for any $z \in Z_i$;

D) if $z' \in P_i(z)$ then $(\alpha z' + (1-\alpha)z) \in P_i(z)$, for all $\alpha \in (0,1]$.

Denote by $\boldsymbol{U}(S)$ a class of preference profiles $\boldsymbol{P} = (P_1,\dots,P_n)$ satisfying the conditions B)-D) in the problem $S = (Z, \omega, (Z_i, \cdot, \beta_i)_{i=1,\dots,n})$; we assume that $\boldsymbol{U}(S)$ contains all preference profiles generated by profiles of linear functionals $\boldsymbol{l} \in \partial \boldsymbol{P}(Z)$.

Using properties of preferences C) and D) one can get the following implications from definition 2.

Lemma 2. *Let $(z^*, \mathbf{p})$ be an equilibrium in a problem $S \in \mathbf{S}$. Then*

i) $p_i z^* = p_i\, \omega + \beta_i\, \pi(\mathbf{p})$ for all i,

ii) $\boldsymbol{p} \in \partial \boldsymbol{P}(z^*)$.

Proof. i). Suppose that $p_i z^* > p_i\, \omega + \beta_i\, \pi(\mathbf{p})$ for some i. Then there is another agent j , for which $p_j z^* < p_j\, \omega + \beta_j\, \pi(\mathbf{p})$. Since $P_j(z^*) \neq \varnothing$ there is a point $z \in P_j(z^*)$. Then for sufficiently small $\alpha > 0$ the point $\alpha z + (1-\alpha) z^*$ belongs to $P_j(z^*)$ and to budget set $B_j(\mathbf{p})$. This contradicts the definition of z^*. So point z^* satisfies all budget equalities.

ii). It follows from the relations $p_i z > p_i z^*$ for any $z \in P_i(z^*)$, $i=1,\dots,n$. ·

Corollary 1. *(Pareto-optimality) If $z^* \in EP(\mathbf{P})$ then there is no $z \in Z$ for which $z \in P_i(z^*)$ and $z \in$ closure $P_j(z^*)$, $j \neq i$.*

Proof. Indeed, the existence of such a point z would contradict the assertions of lemma 2 ii) and equality (1'').

Now we can establish the following theorem.

Theorem 1. *Assume the class of problems* **S** *satisfies the requirements A)-D). Then rule NAP coincides with rule EP on* **S**.

Proof. Assume $z^* \in EP(S)$ in a problem $S = (Z, \omega, (Z_i, P_i, \beta_i)_{i=1,\ldots,n})$. Then there exists a price profile $\mathbf{p} = (p_i)$ such that $(z^*, \mathbf{p})$ is an equilibrium in S. Due to lemma 2 we have the equalities:

$$p_i(z - \omega) = \beta_i \pi(\mathbf{p}),\ i=1,\ldots, n \text{ and so} \quad (1')$$

$$\sum_{i=1}^{n} p_i (z^* - \omega) = \pi(\mathbf{p}) = \max_{z \in Z} \sum_{i=1}^{n} p_i \ (z - \omega). \quad (1'')$$

We consider the set $\mathbf{p}(Z-\omega) \subset R^n$.. Then the number $\pi(\mathbf{p})$ is the maximum of the sum of coordinates of vectors from $\mathbf{p}(Z-\omega)$. This maximum is achieved at the point $\mathbf{p}(z^*-\omega)$ and its coordinates $p_i\ (z^*-\omega)$ are proportional to β_i . But this just means that the point $\mathbf{p}(z^*-\omega)$ is the NBS for the problem $\mathbf{p}(Z-\omega)$ with weights β_i and zero status-quo point. Since due to Lemma 2 ii) vectors p_i are supporting to preferences P_i at z^*, we get the inclusion $z^* \in NAP(S)$.

We prove the converse. Assume $z^* \in NAP(S)$. Then by definition 1 for some supporting profile $\mathbf{l} \in \partial \mathbf{P}\ (z^*)$ the point $\xi^* = \mathbf{l}\ (z^*-\omega) \geq 0$ is the solution of the problem $\max \xi_1^{\beta_1} \cdots \xi_n^{\beta_n}$ when $\xi \in (\mathbf{l}(Z-\omega) - R^n_+) \cap R^n_+$. According to Lemma 1 either 0 is an interior point in $\mathbf{l}(Z-\omega) - R^n_+$ and then $\beta_i = 0 \Rightarrow \xi^*_i = 0$ and $\beta_i > 0 \Rightarrow \xi^*_i > 0$, or 0 is a Pareto-efficient point in A and then all $\xi^*_i = 0$.

We set $\lambda_i = \beta_i / \xi_i^*$ if $\xi^*_i > 0$ and $\lambda_i = 1$ if $\xi^* = 0$. We determine «prices» $p_i = \lambda_i l_i$. Then $\pi(\mathbf{p}) = 1$ and z^* is a solution of the problem

$l_i z \to max$ under constraints:

$p_i\, z \le p_i\, \omega + \beta_i\, \pi(\mathbf{p}),\ i=1,\ldots,n\ ,\ \ z\in Z_i\,.$

Since vectors l_i are supporting functionals to preferences P_i at z^*, point z^* is a solution for the individual maximization problems with preferences P_i. Hence z^* is an EP for the preference profile $\boldsymbol{P}$, that is $z^*\in EP(S)$. ·

According to Corollary 1 any NAP is a Pareto-optimal point in Z.

In order to get rid of choosing appropriate supporting functionals to preferences P_i we consider the case of smooth preferences.

3. Smooth preferences

By a *smoothing out* of a preference P on Z_i we call a mapping which with every point $z\in Z_i$ associates a linear functional $l^z = l(z,\cdot)$ on L, depending continuously on $z\in Z_i$, and such that $l^z\in \partial P(z)$. We call a preference P_i *smooth* if it admits a smoothing out on Z_i. A preference profile $\boldsymbol{P}$ is called smooth if every preference P_i is smooth; $\boldsymbol{l}^z = (l^z_1,\ldots, l^z_n)$ is called the *gradient profile* to preference profile $\boldsymbol{P}$ at z. In this Section, we assume that all preference profiles in the problems $S\in\boldsymbol{S}$ are smooth. The definition 1 now takes the following form.

Definition 1'. A point $z\in Z$ is called a *Nash bargaining point (*NBP*)* in a problem $S = (Z,\omega, (Z_i, P_i, \beta_i)_{i=1,\ldots,n})$, if $\boldsymbol{l}^z(z)$ is the Nash bargaining solution in the set $\boldsymbol{l}^z(Z-\omega)$ with disagreement point 0 and bargaining powers β_i.

Denote by $NBP(S)$ the set of all Nash bargaining points in the problem S. When preferences P_i are generated by linear functionals l_i on Z_i, that is $P_i = l_i$, then $NBP(\boldsymbol{l})$ is the set of points $z\in Z$ for which $\boldsymbol{l}(z)$ is the NBS in the set $\boldsymbol{l}(Z-\omega)$ (or simply the whole preimage in Z at the mapping $\boldsymbol{l}$ of the NBS).

We call agent *i's gradient individual value* at z any functional λl^z_i , $\lambda > 0$. Denote by EP_g the equilibrium rule which uses as individual prices p_i gradient individual values in definition 2. There takes place the double of theorem 1.

Theorem 1'. *$NBP(S) = EP_g(S)$ for any $S \in \mathbf{S}$.*

The proof repeats that of theorem 1.

Do exist NBPs in case of smooth agents' preferences?

Lemma 3. *$NBP(S) \neq \varnothing$ for any $S \in \mathbf{S}$.*

Proof. Let $S = (Z, \omega, (Z_i, P_i, \beta_i)_{i=1,\dots,n})$ be a problem in class $\mathbf{S}$. We define the multi-valued correspondence G from Z to Z , setting

$G(z) = \{ z' \in Z \mid \mathbf{l}^z (z' - \omega)$ is the NBS for the set $\mathbf{l}^z (Z - \omega)\}$.

The set $G(z)$ is non-empty (because set $M^z_+ = \mathbf{l}^z (Z - \omega) \cap R^n_+ \neq \varnothing$ and by lemma 1 the NBS is uniquely defined in it), convex (as a linear preimage of a point), and the correspondence G is closed. Indeed, $\mathbf{l}^z$ is continuous in z, so set M^z_+ changes continuously in Hausdorf metrics in z; the NBS depends continuously on the problem set (because Argmax is a closed correspondence), and at last the preimage correspondence of the NBS (at the mapping $\mathbf{l}^z$) is closed. Thus the correspondence G satisfies the requirements of Kakutani theorem. So there exists a fixed point $z \in Z$ for which vector $\mathbf{l}^z (z)$ is the NBS, that is $z \in NBP(S)$. ▪

Corollary 2. *$EP(S) \neq \varnothing$ for any $S \in \mathbf{S}$.* ▪

In case of smooth preferences the using definition 1' of a NBP seems to be more attractive than definition 1 of a NAP. Obviously $NBP(S) \subseteq NAP(S) = EP(S)$ for any $S \in \mathbf{S}$. The equality holds in particular for smooth preference profiles when all sets $P_i(z)$ belong entirely to $int\ Z_i$. In a more general case the equality may fails. A question arises: when an arbitrary EP z is a NBP (i.e. a preimage point of the NBS for the gradient utilities at z)? This question directly concerns the problem of characterization of equilibrium allocations which we are going to discuss here. We impose a rather strong condition **E** which provides the equality.

E. (Equivalence of outcomes condition): for any $S \in \mathbf{S}$, linear profile $\mathbf{l} \in$ $\mathbf{U}(S)$, and $z, z' \in EP(\mathbf{l})$ there takes place $\mathbf{l}(z)=\mathbf{l}(z')$.

Due to theorem 1 the requirement means that for any linear profile $\mathbf{l} \in \mathbf{U}(S)$ any NAP is a NBP. Later in the applications to equilibrium models we shall discuss when this condition is fulfilled.

Lemma 4. *Assume that the rule EP satisfies the condition* **E**. *Then* $EP(S) \subseteq NBP(S)$ for any $S \in \mathbf{S}$.

Proof. Let $z \in EP(\mathbf{P})$. It is obvious from definition 2 that $z \in EP(\mathbf{l}^z)$, where $\mathbf{l}^z$ is the gradient profile for preference profile $\mathbf{P}$ at z. We take a point z' such that vector $\mathbf{l}^z(z')$ is the NBS for the set $\mathbf{l}^z(Z)$. Due to theorem 1, $z' \in EP(\mathbf{l}^z)$. According to condition **E** $\mathbf{l}^z(z) = \mathbf{l}^z(z')$. Hence by definition 1' $z \in NBP(\mathbf{P})$. ·

We sum up all this in the following theorem.

Theorem 2. *Assume that class* **S** *verifies the conditions A)-D) and agents' preferences are smooth. Then* $NBP(S) = EP_g(S) \subset EP(S)=NAP(S)$ *for any* $S \in \mathbf{S}$. *If rule EP satisfies the condition* **E** *then the equalities hold:* $NAP = EP = NBP = EP_g$.

Proof. The assertion follows from lemma 2', theorem 1, and lemma 4. ·

□heorem 2 characterizes equilibrium points as Nash agreement or under the special conditions as Nash bargaining points. In the following Section we give another characterization.

4. Axiomatic characterization.

Here we give an axiomatic characterization of equilibrium points and thereby due to theorems 1, 2 we get it for NAPs and NBPs. Essentially this will be the Nash's axiomatization transferred to the initial space of alternatives. Note that the NAPs, NBPs, and EPs were defined for a given problem *S*. Now we also take a problem *S* and considering Z, Z_i, ω_i, β_i as fixed we associate the choice for «close» preference profiles from ***U(S)***: that for a given ***P*** and for its gradient or supporting linear profile ***l***. Denote by FM: $\mathbf{S} \to 2^Z$ any multi-valued rule satisfying the following below axioms **F**, **M**. Again we will write *FM(**P**)* or *FM(**l**)* to distinguish the variables in the problem *S* . Let $S\in \mathbf{S}$, ***U***=***U(S)***.

Axiom F **(Fair choice).** Suppose that a linear preference profile $\mathbf{l} \in \mathbf{U}$ and a point $z\in Z$ are such that

$$l_i(z) - l_i(\omega) = \beta_i \cdot \pi(\mathbf{l}) \text{ for all } i=1,...,n, \text{ where } \pi(\mathbf{l}) = \max_{z'\in Z} \sum_{j=1}^{n} l_j(z'-\omega)].$$

Then $z\in FM(\mathbf{l})$.

Axiom M **(Monotonicity).**: Let $z\in Z$, $\mathbf{P} \in \mathbf{U}$, and $\mathbf{l}\in \partial\mathbf{P}(z)$. Then the inclusion $z\in FM(\mathbf{l})$ implies the inclusion $z\in FM(\mathbf{P})$.

Axiom **F** reminds the axiom for the NBS in a symmetrical situation supplemnted by the independence of irrelevant alternatives (IIA). It means that if for a linear preference profile $\mathbf{l} \in \mathbf{U}$ a point $z\in Z$ is such that $\mathbf{l}(z-\omega)$ is the NBS then *z* is chosen by the rule *AX*. Axiom **M** performs the role of the IIA axiom and also makes the choice independent on how the utility scales are calibrated.

Axioms **F** and **M** define a large class of social choice rules, we denote it by ***FM***. We show that class ***FM*** is not empty.

Lemma 5. *The equilibrium rule EP belongs to class* ***FM***.

Proof is almost obvious. Check axiom **F**. Suppose that for some $z \in Z$ and a linear profile $\boldsymbol{l} \in \boldsymbol{U}$ the budget equalities hold:

$l_i(z) - l_i(\omega) = \beta_i \cdot \pi(\mathbf{l})$ for all $i=1,...,n$.

We take the price profile $\boldsymbol{p} = \boldsymbol{l}$. Then obviously $z \in EP(\mathbf{l})$.

Axiom **M** is fulfilled by the definition of equilibrium. ·

The rule *EP* possesses also other useful «natural» properties which we name as axioms and formulate for linear utilities.

Axiom I *If* $z \in EP(\mathbf{l})$ *then* $\mathbf{l}(z) \geq \mathbf{l}(\omega)$.

Axiom P*. If* $z \in EP(\mathbf{l})$ *then there is no* $z' \in Z$ *for which* $\mathbf{l}_i(z') > \mathbf{l}_i(z)$ *and* $\mathbf{l}_j(z') \geq \mathbf{l}_j(z)$ *for* $j \neq i$.

***Axiom N* (Nondiscrimination).** Rule *EP* discriminates no equivalent alternatives for linear preference profiles $\boldsymbol{l} \in \boldsymbol{U}$:

if $z \in EP(\mathbf{l})$, $z' \in Z$ and $\boldsymbol{l}(z) = \boldsymbol{l}(z')$ then $z' \in EP(\mathbf{l})$.

If agents' preferences are smooth then rule EP satisfies the following axiom.

***Axiom L* (Expanding to the linear approximation).**

Let $\boldsymbol{P} \in \boldsymbol{U}$, $z \in EP(\mathbf{P})$ and $\boldsymbol{l}^z$ be the gradient profile of $\boldsymbol{P}$ at z. Then $z \in EP(\mathbf{l}^z)$.

We check only axiom **N** because the other are obviously fulfilled. Let $z \in EP(\mathbf{l})$ for a linear preference profile $\boldsymbol{l} \in \boldsymbol{U}$, $\boldsymbol{p}$ be the equilibrium price profile, and z' be another vector from Z such that $\boldsymbol{l}(z) = \boldsymbol{l}(z')$. Then

$p_i z' \geq p_i \omega + \beta_i \pi(\mathbf{p})$ for any i . Obviously, all these inequalities hold as equalities. It follows then that $z' \in EP(\mathbf{l})$.

One can add axioms **I, P**, **N** to **F**, **M** and to narrow class of rules ***FM*** without changing the result of theorem 3 below. Instead we add axioms **E** (see below)**, L** which together with axioms **F**, **M** implies **I, P**, **N,** (as it will follow from theorem

3) and retain in class ***FM*** only one rule *EP*. We call *EFLM* any rule which satisfies axioms **E**, **F**, **L**, **M**, class of rules *EFLM* is denoted by ***EFLM***.

Axiom E (Equivalence of outcomes). For any linear preference profile $\boldsymbol{l} \in \boldsymbol{U}$ rule *EFLM* chooses only equivalent alternatives:

if $z, z' \in EFML(\mathbf{l})$ then $\mathbf{l}(z) = \mathbf{l}(z')$.

Obviously axioms **M** and **E** act in the opposite directions and could be replaced by one if the sets $P_i(x_i)$ were wholly in $int\ Z_i$:

$z \in EFLM(\mathbf{P})$ *if and only if* $z \in EFLM(\mathbf{l})$.

Axioms **F** and **E** together imply that $EFLM(\mathbf{l})$ is the whole preimage of the NBS, that is $EFLM(\mathbf{l}) = Argmax\ \Pi^n_{i=1}\ l_i(z-\omega)^{\beta_i}$ on Z. Indeed, axiom **F** says that any solution of this maximization problem belongs to $EFLM(\mathbf{l})$, while axiom **E** says that only solutions of this problem enters $EFLM(\mathbf{l})$.

Theorem 3. *Assume that class **S** satisfies requirements A)-D). Then rule EP is the least among the rules FM∈**FM**. Assume that agents' preferences are smooth and rule EP satisfies axiom **E**. Then class **EFLM** consists of one rule {EP} ={NBP} .*

Proof. We check that $EP(\mathbf{P}) \subseteq FM(\mathbf{P})$ for any $\mathbf{P} \in \boldsymbol{U}$, $FM \in \boldsymbol{FM}$. Let $z \in EP(\mathbf{P})$. Then by lemma 2 there exists a price profile $\boldsymbol{p} \in \partial \boldsymbol{P}(z)$ such that

$p_i(z - \omega) = \beta_i \pi(\mathbf{p})$ for all $i = 1,...,n$.

Then according to axiom **F** we get the inclusion: $z \in FM(\mathbf{p})$. Axiom **M** implies then the inclusion: $z \in FM(\mathbf{P})$. The first part of the theorem is proved.

Now we prove the second part. Let $z \in EFLM(\mathbf{P})$ for some rule *EFLM*. By axiom **L** $z \in EFLM(\mathbf{l}^z)$. We take an arbitrary $z' \in EP(\mathbf{l}^z)$. According to Corollary 2 this is possible to do. Since $EP(\mathbf{l}^z) \subseteq EFLM(\mathbf{l}^z)$ (as we proved above, and also because the both verify axioms **E**, **L**) the inclusion $z' \in EFLM(\mathbf{l}^z)$ holds . By axiom

E we get the equality: $\mathbf{l}^z(z)=\mathbf{l}^z(z')$. Axiom **N** which holds for rule EP gives the inclusion $z\in EP(\mathbf{l}^z)$. Using axiom **M** we get: $z\in EP(\mathbf{P})$. Finally by theorem 2 $EP(\mathbf{P})=NBP(\mathbf{P})$. ·

Note that the theorem implies that rules *FM* and *EFLM* are non-empty-valued.

Sometimes a model contains no information about agents' bargaining powers β_i though the choice is quite definite due to a special form of the set *Z* as in the case of pure exchange models. In this case we will replace axiom **F** (retaining the other axioms unchanged) by the individual rationality axiom **I**, or its weaker variant **IR** see Part II, Section 2.

In the next part in applications we take the same individual consumption sets $Z_i\equiv R^K_+$. For this case we mark the following relation between NAPs and NBPs.

Remark. Given $S = (Z,\omega, (R^K_+, P_i, \beta_i)_{i=1,\dots,n})$ any point $z\in NAP(S)$ is a point $z\in NBP(S')$ for $S' = (Z,\omega, (R^K_+, P_i, \beta'_i)_{i=1,\dots,n})$ where β'_i generally differs from β_i if z is located on the boundary of R^K_+. Indeed, if $z\in NAP(S)=E(S)$ then there are individual prices p_i satisfying together with z conditions of definition 2. It follows then that for some gradient individual values l^z_i the following relations hold:

$$l^z_i = p_i - \gamma_i,\ \gamma_i\cdot z = 0,\ \gamma_i \geq 0 .$$

After substituting them into budget constraints $p_i\, z = p_i\, \omega + \beta_i\, \pi(\mathbf{p}),\ i=1,\dots,n,$ we get the new constraints:

$l^z_i\, z = l^z_i\, \omega + \beta'_i\, \pi(\mathbf{l}^z),\ i=1,\dots,n,$ where

$$\beta'_i = (\beta_i\pi(\mathbf{p})+\gamma_i\cdot\omega)/(\pi(\mathbf{p})+ \textstyle\sum_j \gamma_j\cdot\omega),\ i=1,\dots,n . \qquad (2)$$

We discuss the sense of the formula (2) in Part II, where we use it in applications.

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II. Applications. Characterization of competitive allocations.

The essential peculiarity of competitive models is the notion of a resource allocation. The set Z is the set of feasible resource allocations. We assume that agents' preferences P_i depend on their own consumption and indifferent with respect to consumption of other agents. In this case the corresponding coordinates of any supporting linear functional to preference P_i equal to zero and we shall use it. We consider three competitive models and find out that competitive allocations of goods can be interpreted as Nash agreement points or in case of smooth preferences as Nash bargaining points. Besides we give axiomatic characterizations for the models. At last we construct simple Nash implementing mechanisms for the models. In all cases we take consumption sets Z_i for all agents equal to some non-negative orthant. We keep for β_i the role of «bargaining powers» and for α_i the role of shares of production. They can differ at the boundary of the consumption set because of the rents which emerge there and change the utility contribution of agents to the economy. But this is not the case for the first model.

1. A resource allocation model (A)

In model A private and public resources are produced and allocated among n consumers. The consumers have zero personal endowments but are stockholders with shares of production $\alpha_i > 0,\ i=1,...,n$. We take the following notations:

x_i is a vector of the private resources consumed by agent i, $x_i \in R^k{}_+$,

g_i is a vector of the public resources consumed by agent i, $g_i \in R^m{}_+$,

x is a vector of pure outcome of the private resources, $x \in R^k{}_+$,

g is a vector of outcome of the public resources, $g \in R^m{}_+$,

Y is the production set, $Y \subset R^k_+ \times R^m_+$.

The set of feasible allocations Z has the following form:

$$Z = \{ z = (x_i, g_i)_{i=1,\ldots,n} \mid \sum_{i=1}^{n} x_i \le x, g_i \le g, (x,g) \in Y, x_i \ge 0, g_i \ge 0 \}.$$

Here the production set Y is assumed to be convex and compact. It contains 0 and a point $y = (x,g) > 0$. (One can think that in the economy there is a common (state) initial endowment of resources $\omega_0 \ge 0$ and a technology $Y_0 \subseteq R^k \times R^m$ which uses the resources from ω_0 as the input to produce bundles of commodities from the set $Y = \{y \ge 0 : y \in \{\omega_0) + Y_0\})$. Agent i-s strict *p*reference P_i is strictly monotone on his own consumption set $R^k_+ \times R^m_+$, and indifferent with respect to consumption of other agents; P_i satisfies conditions C), D).

A competitive allocation of goods is a point $z^* = (x^*_i, g^*_i)_{i=1,\ldots,n} \in Z$ such that there exist prices of private goods $p \in R^k_+$ and individual prices of public goods $q_i \in R^m_+$ for which z^* is a solution of the problem:

$max\, P_i$ under constraints $px_i + q_i g_i \le \alpha_i \pi(p, q_1, \ldots, q_n)$, $x_i \ge 0$, $g_i \ge 0$, $i=1,\ldots,n$,

where $\pi(p, q_1, \ldots, q_n) = \max_{(x,g) \in Y} (px + \sum q_j g)$.

In order to characterize competitive allocations as a bargaining choice we come back to our notations used in Part I: $L = R^K$, $K=(k+m)n$, $Z \subset L$ is defined above, $\boldsymbol{U}$ is the class of preference profiles on R^K_+ with the properties described above, $\boldsymbol{S} = \{S: S=(Z, 0, (R^{Kn}_+, P_i, \beta_i = \alpha_i)_{i=1,\ldots,n}, \boldsymbol{P} \in \boldsymbol{U}\}$ the class of choice problems in Z. Denote by W the correspondence of competitive allocations $W: \boldsymbol{S} \to 2^Z$ for model A.

Existence of a vector $y > 0$ warrants that the status-quo point $\omega = 0$ will be an interior point in the corresponding Nash bargaining problems; besides, Y is compact and $\alpha_i > 0$. All this yields the result of lemma 1, in particular, the NBS is uniquely determined for any linear profile $\boldsymbol{l} \in \boldsymbol{U}$.

What are the Nash agreement and bargaining points in this model? We show that all they (depending on smoothness of preferences) are identical to competitive allocations of goods. We prove that competitive allocations in model A are equilibrium points in terms of definition 2 (Part 1, Section 2), and then the result follows from the theorem 1.

Proposition 1. *W(S) =NAP (S) for any* $S \in \boldsymbol{S}$. If agents' preferences are smooth then *W(S) =NBP (S) for any* $S \in \boldsymbol{S}$.

Proof. Given $S \in \boldsymbol{S}$, let $z^* = (x_i^*, g_i^*)_{i=1,\dots,n} \in NAP(S)$. By theorem 1 z^* is an equilibrium point from *EP(S)*. The latter means that there exists a profile of individual prices $\boldsymbol{p} = (p_1, \dots, p_n)$ which (together with z^*) satisfy definition 2, in particular :

$$P_i(z^*) \cap B_i(\mathbf{p}) = \varnothing,\ B_i(\mathbf{p}) = \{z \in R^K_+ \,/\, p_i z \le \beta_i\, \pi(\mathbf{p})\}.$$

According to lemma 2 i) $p_i z^* = p_i\, \omega + \beta_i\, \pi(\mathbf{p})$ for all i, and ii) $\boldsymbol{p} \in \partial \boldsymbol{P}(z^*)$. It follows that the summary price vector $p_\Sigma = \sum_{i=1}^{n} p_i$ defines a supporting linear functional to set Z at point z^*. The general form of such a functional p_Σ is the following:

$$p_\Sigma = (p - \delta_i,\ q_i - \gamma_i)_{i=1,\dots,n}, \text{ where } p, \delta_i \in R^k_+,\ q_i, \gamma_i \in R^m_+,\ \delta_i x_i^* = \gamma_i g_i^* = 0, \qquad (3)$$

where $(p, \sum_{i=1}^{n} q_i)$ is a supporting linear functional to the set Y at the corresponding point $y^* = (x^*, g^*) \in Y$. Besides, the individual prices $p_i \in (R^k_+ \times R^m_+)^n$ being a supporting functionals to preferences P_i have zero coordinates corresponding to consumption vectors of agents $j \neq i$. Indeed, suppose that $p_i^r > 0$ for a coordinate r

out of agent i-s consumption space. Let $z \in P_i(z^*)$. Then $z \in P_i(z^* + \lambda e^k)$ for any $\lambda > 0$ since agent i is indifferent with respect to consumption of other agents (e^k is the k-th coordinate vector). From the other side $p_i(z^* + \lambda e^k) > p_i(z)$ for sufficiently big λ that contradicts the definition of a supporting functional. Considering this (3) implies the relations:

$$p_i = (p - \delta_i, q_i - \gamma_i) \in \partial P_i(z^*), \ i=1,\dots,n \tag{4}$$

In formula (4) we consider p_i as vectors from $R^k_+ \times R^m_+$ supporting to preferences P_i at point (x^*_i, g^*_i). Thus the budget equalities take the form: $px^*_i + q_i g^*_i = \beta_i \pi(\mathbf{p})$, $i=1,\dots,n$. Together with relations (4) they mean that the point $z^* = (x^*_i, g^*_i)^n_{i=1} \in Z$ is a solution of the problem:

$max\ P_i$ under constraints $px_i + q_i g_i \le \beta_i\ \pi(p, q_1, \dots, q_n)$, $x_i \ge 0$, $g_i \ge 0$, for every $i=1,\dots,n,$ where

$$\pi(p,q_1,\dots,q_n) = \max_{(x,g)\in Y} (px + \Sigma q_j\, g) = px^* + \Sigma q_j\, g^* = \Sigma(p-\delta_j)x^*_j + \Sigma(q_j - \gamma_j)\, g^*_j = \pi(\mathbf{p}).$$

We have showed that any Nash agreement point $z^* \in Z$ is a competitive equilibrium allocation in model A. It is easy to see that the argument can be reversed. Given an economic equilibrium $(z^*, p, q_1, \dots, q_n)$, and going back we get the optimality conditions (4) of individual maximization problems. Individual prices p_i with z^* satisfy the budget equalities, and so $z^* \in EP(\mathbf{P})$. By theorem 1 then $z^* \in NAP(\mathbf{P})$.

Assume now that agents have smooth preferences in class of problems $S \in \boldsymbol{S}$ and $z^* \in NBP(S)$. Then due to theorem 1' $z^* \in EP_g(S)$. Hence there exists a profile of gradient individual values $p_i = \lambda_i l^z_i$ satisfying definition 2. Further the proof repeats that given above which yields inclusion $z^* \in W(S)$. Conversely, assume that $z^*=(x^*_i\ ,\ g^*_i)\in W(\mathbf{P})$. Then $z^*\in W(\mathbf{l})$ for any profile of gradient individual values $\mathbf{l}^* =(\lambda_i\, l_i^z)$ at z^*. Let a bundle $(z^*, p,\ q_1,...,q_n)$ be an economic equilibrium. From the individual maximization problems we get relations (4) where $p_i = \lambda_i l_i^z = (p - \delta_i\, ,\, q_i - \gamma_i)$ for some $\lambda_i >0$. Substitute individual prices p_i into the budget equalities:

$$p_i z^*= (p - \delta_i)x^*_i+ (q_i - \gamma_i)g^*_i = px^*_i+ q_i g^*_i =\beta_i\, \pi(p,q_1 ,...,q_n)= \beta_i\pi(\mathbf{p}) ,$$

where the last equality was founded above. Thus $z^*\in EP_g(\mathbf{l}^*)$. By theorem 1' $z^*\in NBP(\mathbf{l}^*)$ and so $z^*\in NBP(\mathbf{P})$ ·

Note that the bargaining powers β_i here coincide with the production shares of agents α_i. This is not generally the case when agent *i*-s initial endowment $\omega_i \neq 0$ and the vector of the equilibrium allocation belongs to the boundary of the consumption set.

The axiomatic characterization of rule EP given in theorem 3 is entirely valid here because as it follows from the proof of Proposition 1 rule EP coincides with rule W. Moreover, theorem 3 can be strengthen for model A because condition **E** is fulfilled. It follows from the fact that for a profile of linear utilities $\mathbf{l}$ any NAP in Z is a NBP (with the same bargaining powers β_i , cf. Remark at the end of Part I). Indeed, if $z^*\in NAP(\mathbf{l})$ then there is a profile of supporting functionals $p_i= l_i - \varepsilon_i$, $\varepsilon_i z^*=0$, $\varepsilon_i\geq 0$. Whence $\mathbf{l}\, z^*= \mathbf{p}\, z^*$. So if this point is the NBS then z^* is at the same time a NAP and a NBP. Considering this we can formulate the following variant of theorem 3 .

Proposition 1'. *In model A rule W is the unique rule satisfying axioms* ***E, F, L, M****.*

2. A pure exchange model (B)

We distinguish this model in order to discuss the question about bargaining powers in a simpler situation. In model B set Z has the form: $Z=\{z=(x_i)\mid \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \omega_i,\ x_i \geq 0\}$, where $x_i, \omega_i \in R^k_+$, $\omega_i \neq 0$, $\Sigma\omega_i > 0$; preferences P_i on R^k_+ are smooth and satisfy conditions B), C), D) . Here there is the natural status-quo point ω but nothing is given to take as bargaining powers. When the agents make their choice basing on market prices p the common «profit» $\pi(p)=0$ and since there is nothing to divide the weights β_i are not essential and not determined. Assume now we look at the exchange model as a decision making process where agents use their individual prices $p_i \in \partial P_i(z)$. In this case the common «profit» $\pi(\mathbf{p})$ generally is not zero. Then the bargaining powers β_i are determined depending on individual prices and initial endowments ω_i so that to clean markets .

Proposition 2. *An allocation $z^*=(x^*_i)$ in model B is Walrasian if and only if it is a NAP in Z (NBP in case of smooth agents' preferences* **P***) with disagreement point $\omega=(\omega_1,\dots,\omega_n)$ and bargaining powers β_i determined from the relations:*

$p_i(x_i^*-\omega_i) = \beta_i\, \pi(\mathbf{p})$, (5)

$\beta_i = (p-p_i)\omega_i/\pi(\mathbf{p})$, $i=1,\dots,n$, *where* (6)

$p^s = \max_j p^s_j$, $s=1,\dots,k$. (7)

If $\pi(\mathbf{p})=0$ the weights β_i are indefinite.

The proof of the proposition is given for a more general case in the next Section.

Note that model B as well as model A also verifies the equivalence of outcomes condition, D.Gale (1976).

Consider the distribution of the «profit» $\pi(\mathbf{p})$ (or differently the total value of all bargains in individual values p_i) in equilibrium. Every agent i evaluates the *utility* of his initial endowment ω_i in terms of his (gradient) individual value p_i, that is equal to $p_i\omega_i$. The market price p of every good is maximal among all individual values of the good. If agent i evaluates a good s below the equilibrium market price p^s, he does not consume it and sells the quantity ω_i^s. The difference equal to $(p^s - p_i^s)\omega_i^s$ is the «utility profit» of the society. (This is the rent which the society gets from agent i because of the lower constraint $x^s_i \geq 0$) . The summary «utility profit» of the society $\sum (p - p_i)\omega_i = \pi(\mathbf{p})$ in equilibrium is divided among the agents according to bargaining powers β_i (determined by formula (6)) . If $\pi(\mathbf{p})=0$ (as it happens when the equilibrium allocation is an interior point of the consumption sets and individual values equal to the market prices) β_i are indefinite and inessential. But when $\pi(\mathbf{p})>0$ then β_i is equal to the share of his contribution to the «utility profit» of the society by formula (6).

For finding an equilibrium state agents pick out the lengths of gradients to their preferences at every state $z\in Z$ until they find appropriate $z, (p_i)$ satisfying the relations (5)-(7).

The axiomatic characterization of Walrasian allocations is given by the list of axioms **E**, **I**, **L**, **M**, **N**. The individual rationality axiom **I** can be weaken here (in the resource allocation environment) and takes the form. Let $A: \mathbf{U} \rightarrow 2^Z$ be some rule

Axiom I. For any linear preference profile $\mathbf{l} =(l,...,l)\in \mathbf{U}$ and $z\in A(\mathbf{l})$ there take place the inequalities: $\mathbf{l}(z) \geq \mathbf{l}(\omega)$.

Proposition 2'. *The unique rule which in model B satisfies axioms* **E**, **I**, **L**, **M**, **N** *is rule W*.

The proof follows from the analogous assertion for Arrow-Debreu type model, see Corollary from theorem 4 below in Section 4.

3. An Arrow-Debreu type model (C)

In model C the set Z has the form: $Z=\{z=(x_i)^n_{i=1} \mid \Sigma x_i = \Sigma \omega_i + y,\ y\in Y,\ x_i \in R^k_+\}$ where Y is a convex compact production set in R^k, $0\in Y$; $\Sigma \omega_i > 0$. The consumers have shares $\alpha_i \geq 0$, $\Sigma\alpha_i=1$. Agents's strict preferences P_i on R^k_+ are strictly monotone, and satisfying conditions B), C), D). By ***U*** we denote a class of preference profiles ***P*** on R^k_+ including all those preferences generated by linear functionals (l_i) which satisfy conditions B), C), D). Denote by W: $\boldsymbol{U} \to 2^Z$ the Walrasian rule in this Arrow-Debreu type model . A competitive allocation in model C follows the traditional Arrow-Debreu concept of equilibrium. We keep the notation $\pi(p)$ for the profit when $\boldsymbol{p} = (p,...,p)$.

The next proposition generalizes Proposition 2.

Proposition 3. *An allocation $z^*= (x_i^*)$ in model C is competitive if and only if it is a NAP (NBP when agents' preferences P_i are smooth) with disagreement point $\omega =(\omega_1,..., \omega_n)$ and bargaining powers $\beta_i = \alpha_i$ when all $x_i^*> 0$, and otherwise determined from the general relations:*

$$\boldsymbol{p}\in\partial \boldsymbol{P}\,(z^*) \tag{8}$$

$$p_i(x_i^*- \omega_i) = \beta_i\, \pi(\boldsymbol{p}), \tag{9}$$

$$\beta_i =[\alpha_i\pi(p)+(p-p_i)\omega_i]\,/\,\pi(\boldsymbol{p}),\ i=1,...,n,\ \text{where} \tag{10}$$

$$p^s = \max_j p^s_j,\ s=1,...,k\ . \tag{11}$$

Proof. Sufficiency. Suppose that a point $z^*=(x_i^*)$ is a NAP in Z and a price profile $\boldsymbol{p}\in\partial \boldsymbol{P}\,(z^*)$ satisfy either relations (9)-(11) if $\pi(\boldsymbol{p})>0$ or relation (9) if $\pi(\boldsymbol{p})=0$. (The case $x_i^*> 0,\ i=1,...,n,$ is included here). Then the bundle (x_i^*) gives maximum to $\sum_{i=1}^{n} p_i (x_i - \omega_i)$ on Z equal to $\pi(\mathbf{p})$. It follows that there exist vectors p_0, γ_i such that

$p_i = p_0 - \gamma_i$, $\gamma_i x_i^* = 0$, $p_0, \gamma_i \in R^k_+$, $p_0 y^* = max\, p_0 y$ on Y, $i=1,...,n$, (12)

where $y^* = \sum (x_i^* - \omega_i)$. If agent i consumes good s, i.e. $x_i^{s*} > 0$, then $\gamma_i^s = 0$, whence $p^s_0 = p^s_i = max_j\, p^s_j$. If no agent consumes good t, i.e. $x_i^{t*} = 0$, and $\gamma^t_i > 0$ for $i=1,...,n$, we change p^t_o and γ_i^t setting $p^t_o = p^t = max_j\, p^t_j$, and $\gamma_i^t = 0$ for $i \in I = Arg\, max_j\, p^t_j$. In order to justify this replacement one should only to check that $p_0 y^* = max\, p_0 y$ on Y for new p_0. Indeed, we have from (12): $\sum_{s \notin I} p^s_0 (y^{s*} - y^s) + \sum_{t \in I} p^t_0 (y^{t*} - y^t) \geq 0$, where the first addendum does not change while the second does not decrease because p^t_0 becomes less, and $y^{t*} - y^t = - \sum_i \omega^t_i - y^t \leq 0$ for $t \in I$. So one can take $p_0 = p$ defined in (11). We substitute relations (12) (with $p_0 = p$) and (10) in (9). After cancellations we get:

$p(x_i^* - \omega_i) = \alpha_i \pi(p)$, $i=1,...,n$ (13)

In the case $\pi(\mathbf{p}) = 0$ we also get the budget equalities (13). Indeed, $\pi(\mathbf{p}) = \pi(p) + \sum_j \gamma_j \omega_j = 0$ where every addendum is nonnegative. So we have $\pi(p) = 0$ and $\gamma_i \omega_i = 0$, and hence $p_i(x_i^* - \omega_i) = p(x_i^* - \omega_i) = 0$. Relations (12) together with budget equalities (13) mean that vectors x_i^* are solutions of individual problems at market prices p. Together with inclusion $z^* \in Z$ this means that pair (z^*, p) is a Walrasian equilibrium.

Necessity. Suppose that a pair (z^*, p) compose a competitive equilibrium. Then for some supporting (to $\boldsymbol{P}$ at z^*) linear functionals p_i there take place relations (11), (12), where $p_0 = p$. Substituting $p = p_0$ from (12) to (13), and introducing the notation (10) we obtain (9) if only $\pi(\mathbf{p}) > 0$. If $\pi(\mathbf{p}) = 0$ then noting that $\pi(\mathbf{p}) = \sum p_i (x_i^* - \omega_i)$ and using the same argument as above we get the identities in (9). Hence z^* is an equilibrium point in terms of definition 2. By theorem 1 z^* is a NAP with status-quo ω and bargaining powers β_i determined by (10).

When agents preferences are smooth the same line of proof which use gradient individual values p_i gives the assertion of the proposition that Walrasian allocations are NBPs. ·

Note that here we did not use the equivalence of outcomes condition for linear utilities. Bargaining powers for NAPs and NBPs can differ from each other and the both from shares α_i at the boundary of the orthant R^{kn}_+ . According to general formula (10) β_i is equal to agent *i's* relative summary contribution: to the common production and to the virtual «utility profit» of the society.

4. Axiomatic characterization of competitive allocations in the Arrow-Debreu type model.

The axiomatic characterization given in theorem 3 (Part I) was good in the general setting with fixed bargaining powers β_i . It appeared to be appropriate in model A. For model C (and B) we modify the list of axioms. One cause is the specific form of set *Z*, the other one is that β_i deviate from α_i at the boundary of agents' consumption sets. We will assume here that preferences in class ***U*** are smooth. Denote by $\Sigma: \boldsymbol{U} \to 2^Z$ any non-empty-valued rule satisfying the following below 4 axioms: **I**, **L**, **M**, **N**. Only axiom **I** is new. It replaces axiom **F** and uses the concrete form of the model. For convenience of the reader we give the full list of the axioms.

Axiom I (Individual rationality). For any positive linear functional l on R^k and $z=(x_1,\ldots,x_n)\in \Sigma(\mathbf{l})$, where $\mathbf{l}=(l,\ldots,l)$, the following inequalities hold: $l(x_i - \omega_i)\geq \alpha_i l(y)$ for every $y\in Y$, $i=1,\ldots,n$.

Axiom N (Nondiscrimination). The rule Σ discriminates no equivalent alternatives for linear preference profiles $\boldsymbol{l} \in \boldsymbol{U}$:

if $z\in \Sigma(\boldsymbol{l})$, $z' \in Z$ and $\boldsymbol{l}(z) = \boldsymbol{l}(z')$ then $z' \in \Sigma(\boldsymbol{l})$;

Axiom M (Monotonicity).: Let $\boldsymbol{P} \in \boldsymbol{U}$ and $\boldsymbol{l}\in\partial\boldsymbol{P}(z)$. Then inclusion $z\in\Sigma(\boldsymbol{l})$ implies inclusion $z\in\Sigma(\boldsymbol{P})$.

Axiom L (Expanding to the linear approximation).

Let $\boldsymbol{P}\in \boldsymbol{U}$, $z\in \Sigma(\boldsymbol{P})$ and $\boldsymbol{l}^{z}$ be the gradient profile of $\boldsymbol{P}$ at z. Then $z\in\Sigma(\boldsymbol{l}^{z})$.

Axioms **I**, **L**, **M**, **N** define a class of social choice rules $\boldsymbol{\Sigma}$. We check that the class is not empty.

Lemma* 5.** *The Walrasian rule W satisfies axioms* ***I, L, M, N*.*

Proof. Check axiom **I**. Suppose that $z\in W(\boldsymbol{l})$ for some linear positive profile $\boldsymbol{l} =(l,\ldots,l)$, and p be the equilibrium price. Then the following equalities hold:

$p\,x_i = p\omega_i + \alpha_i\,\pi(p)$, $l = \lambda_i\,p - \gamma_i$, $\gamma_i x_i = 0$ for any i, where $\lambda_i > 0$, $\gamma_i \geq 0$. Here $p>0$ and since $\omega_i \neq 0$, vectors $x_i \neq 0$, whence it follows that vectors γ_i can not be positive. Besides, since $\Sigma\omega_i > 0$ every good s is consumed by an agent i, and so $\gamma^s_i =0$. Given this we can check that all vectors $\gamma_i=0$ and $\lambda_i \equiv\lambda>0$. Indeed, show that $\lambda_i =\lambda_j$. There are some $\gamma^s_i= 0$ and $\gamma^r_j =0$. If $\gamma^s_j =0$ then $\lambda_i=\lambda_j$; if not, then $\lambda_i < \lambda_j$ as it follows from the equality: $l^s= \lambda_i p^s -\gamma^s_i =\lambda_j p^s -\gamma^s_j$. From the other hand the equality $\gamma^r_i =0$ implies again $\lambda_i=\lambda_j$ while $\gamma^r_i >0$ implies from the similar equality that $\lambda_j < \lambda_i$. Thus supposing that some coordinate of a vector γ_i is not zero we get a contradiction. If all $\gamma_i =0$ then $\lambda_i \equiv\lambda>0$.

Now we multiply the budget equalities by λ and get the assertion of axiom 1.

The checking axioms **L**, **M**, **N** does not differ from that made in Section 4, Part I. · Lemma 5 shows that the Walrasian rule is one of the rules $\Sigma\in\boldsymbol{\Sigma}$.

Again in order to narrow class Σ to one element we introduce axiom **E**. We say that rule Σ satisfies axiom **E** if under linear preference profiles $\mathbf{l} \in \mathbf{U}$ rule Σ chooses only equivalent alternatives:

if $z, z' \in \Sigma(\mathbf{l})$ then $\mathbf{l}(z) = \mathbf{l}(z')$.

In order to Walrasian rule W fulfill axiom **E** we impose the AGS-property on production set Y. A production set Y is said to have *the AGS-property* if the correspondence - $Argmax\ py|Y$ possesses this property.

By definition (see Polterovich and Spivak (1983)) a multi-valued correspondence T from R^M to R^M satisfies the AGS-property if for any $p, q \in R^M_+$, $p \leq q$, such that $I(p, q) = \{k \in M / p^k = q^k\} \neq \varnothing$, and for any $d \in T(p)$, $f \in T(q)$ the following inequality holds:

$$\sum_{k \in I(p,q)} (p^k d^k - q^k f^k) \leq 0 .$$

One can show for example that if outcome of every good is determined by the Cobb-Duglas function then Y satisfies the AGS-property.

Now we can prove the theorem analogous to theorem 3 .

Theorem 4. *In model C the Walrasian rule W is the least among the rules* $\Sigma \in \boldsymbol{\Sigma}$. *If the production set Y has the AGS-property then the unique rule* Σ *which satisfies axiom* **E** *is rule W.*

Proof. According to lemma 5 the rule W belongs to class $\boldsymbol{\Sigma}$. We prove that W is the minimal rule in class $\boldsymbol{\Sigma}$. Let $z \in W(\mathbf{P})$, $z = (x_i)$. Let p be the equilibrium prices, then the budget equalities hold: $px_i = p\omega_i + \alpha_i \pi(p)$, $i=1,...,n$. We take the linear preference profile $\mathbf{l} = (p,...,p)$, and a point $z' \in \Sigma(\mathbf{l})$, $z'=(x'_i)$. By axiom **I** (individual rationality of Σ) and the inclusion $z' \in Z$ we get the equalities: $px_i' = p\omega_i + \alpha_i\pi(p)$, $i=1,...,n$. Then by axiom **N** (nondiscrimination) we get the inclusion

$z \in \Sigma(\boldsymbol{l})$. Since $\boldsymbol{l}$ is a supporting linear profile to preference profile $\boldsymbol{P}$ at z, by axiom **M** (monotonicity) $z\in\Sigma(\mathbf{P})$. So $W(\mathbf{P}) \subseteq \Sigma(\mathbf{P})$.

Assume now that production set Y has the AGS-property. Then as it follows from Polterovich and Spivak (1983) rule W satisfies axiom **E**. Suppose that a rule $\Sigma\in\boldsymbol{\Sigma}$ satisfies axiom **E**. We prove the inclusion $W(\mathbf{P})\supseteq \Sigma(\mathbf{P})$ for any preference profile $\boldsymbol{P}\in\boldsymbol{U}$. Let $z \in \Sigma(\mathbf{P})$. Then by axiom **L** we have the inclusion $z \in \Sigma(\boldsymbol{l}^z)$ where $\boldsymbol{l}^z$ is the gradient profile for preference profile $\boldsymbol{P}$ at z. By axiom **E** (equivalence of outcomes) $\boldsymbol{l}^z(\Sigma(\boldsymbol{l}^z)) = \boldsymbol{l}^z(z)$. We proved above the inclusion $W(\boldsymbol{l}^z) \subseteq \Sigma(\boldsymbol{l}^z)$ (at present with axiom **E**). So axiom **N** applied to the rule W gives the inclusion $z\in W(\boldsymbol{l}^z)$. Monotonicity of the rule W (axiom **M**) implies: $z\in W(\mathbf{P})$.

Note also that the strong requirement which is contained in axiom **E** was not used to prove the inclusion $W\subseteq \Sigma$ for any rule $\Sigma\in\boldsymbol{\Sigma}$.

Corollary 3. *For pure exchange models the unique rule* Σ*:* $\boldsymbol{U}\rightarrow 2^Z$ *which satisfies axioms* ***E, I, L, M, N*** *is the Walrasian rule W.*

III. Conclusion

One can see now that decision making based on the Nash bargaining solution for agents' gradient (subgradient) utilities gives the same set of allocations *NBP (S) (NAP(S))* as the Walrasian rule *W(S)* (all applications in Part II were merely examples of different sets *Z)*. We say it more exactly. For model A the Walrasian rule coincides with the NAP-rule or in case of smooth preferences with NBP -rule. For the Arrow-Debreu type model C (in particular B) the set *W(S)* coinsides with *NBP (S) (NAP(S))* where the bargaining powers are equal to the production shares α_i when the allocation $z=(x_i)$ is strictly inside the orthant R^{kn}_+ . In particular when monotone preferences P_i are such that the sets $P_i(x_i)$ are entirely belong to $intR^k_+$

this is so. Thus any interior equilibrium allocation of goods (x_i) is a preimage point of the Nash bargaining solution for the gradient agents' utilities taken at points x_i . At the boundary of the orthant R^{kn}_+ the agents' bargaining powers deviate from α_i and are determined by the formula (10).

The axiomatic characterization exposes the properties of choice of the NBP (NAP)-rule in the initial space of alternatives.

IV. Appendix

A simple mechanism for Nash implementation of Walrasian equilibrium outcomes

We describe here a simple mechanism in the spirit of Maskin whose Nash-equilibrium outcomes are Walrasian equilibrium allocations of goods. The main elements of the construction remind those used by B. Dutta et al. (1995).

We consider a pure exchange economy with k goods and $n \geq 3$ agents. Agent's i strict preference P_i is supposed to be convex and strictly monotone on the open, convex set of admissible exchange bargains $X_i \subset R^k$, $\boldsymbol{P}$ is a preference profile. As usual a positive coordinate of vector $x_i \in X_i$ means that agent i receives the good and a negative one means that he delivers the good. The convexity of the preferences means that if $x' P_i x$ then $(\alpha x'+(1-\alpha)x) P_i x$ for any $\alpha \in (0,1]$. We assume that every set X_i contains a vector $a>0$ and a vector $b = -a/(n-1)$.

Denote by s_i a strategy of agent i. It consists of two parts: $s_i = (x , p)$, where $x \in X=\{(x_1,\ldots,x_n) | \sum x_i = 0, x_i \in X_i\}$ is an allocation of exchange bargains, proposed by agent i, $p \in \Delta$ is the exchange price proposed by agent i, Δ is a unit k-1 -dimensional simplex. Any strategic pair (x, p) is supposed to satisfy the equalities:

$px_j = 0$ for all $j=1,...,n$. So the strategy set of any agent i is a subset $S_i \subset X \times \Delta$ defined above.

Denote by $f\colon S_1 \times \cdots \times S_n \to X$ the mechanism defined by the following rules.

Rule 1. If $s_1 = ... = s_n = (x , p)$ for all $i = 1,...,n$, then $f(s_1,...,s_n) = x$.

Rule 2. If all strategies s_i except s_j are the same: $s_i \equiv (x , p)$ and $s_j = (x', p') \neq (x , p)$ then

$f(s_1,..., s_n) = x'$ if $px'_i \leq 0$ and $f(s_1,..., s_n) = x$ otherwise.

Rule 3. In all other cases the «roulette» mechanism starts functioning, where the winner gets the bargain a and every other agent gets the bargain b.

We define what is the «roulette» mechanism. Let $[np^i_1]$ be the least integer number which is more or equal $n\, p^i_1$ where p^i_1 is the price of the first good, proposed by agent i. Then the winner is determined by the number equal to $\Sigma\, [np^i_1] \ (mod\ n)$.

Denote by $NE(f, \boldsymbol{P})$ the set of Nash equilibrium strategy profiles given mechanism f and preference profile $\boldsymbol{P}$, and let $W(\boldsymbol{P})$ be the set of Walrasian allocations in X under preference profile $\boldsymbol{P}$. The inclusion $x \in W(\boldsymbol{P})$ means that $x \in X$ and there exist prices p such that $px_i=0$, and $P_i(x_i) \cap B_i(p) = \varnothing$ where $B_i(p)=\{x \in X_i,\ px_i \leq 0\}$ is the budget set of agent i.

Proposition 4. $W(\boldsymbol{P}) = f(NE(f, \boldsymbol{P}))$ for any $\boldsymbol{P} \in \boldsymbol{U}$.

Proof. We show the inclusion $\subseteq$. Assume $x \in W(\boldsymbol{P})$ and $p \in \Delta$ be the equilibrium prices. We set $s_i = (x, p)$ for all $i = 1,...,n$. According to rule 1 $f(s_1,...,s_n) = x$. Every agent i can choose another strategy (x', p') and enforce outcome x' if $px'_i \leq 0$. According to rule 2 only such deviations are permissible for individuals. However since the pair (x, p) is equilibrium no such a vector x'_i belongs to the set $P_i(x_i)$. Hence the bundle $(s_i) \in NE(f, \boldsymbol{P}))$ and $x \in f(NE(f, \boldsymbol{P}))$.

Conversely, we show that $f(NE(f, \mathbf{P})) \subseteq W(\mathbf{P})$. Assume $(s_1,...,s_n) \in NE(f, \mathbf{P})$. We check that the outcome $x = f(s_1,...,s_n)$ is determined by the rule 1. Indeed, the «roulette» mechanism has no equilibria, because everybody prefers a to b and can obtain it. So x can't be obtained by the rule 3. Suppose that the outcome x is determined by the rule 2 when only one agent i deviates from the common strategy (p, x). Then any other agent j can activate the «roulette» sending a message (p'', x'') with $p'' \neq p, p'$, which makes him a winner. So if the outcome x is a Nash equilibrium outcome then it is determined by the rule 1. The latter means that all agents propose the same pair (p, x). We show that this is a Walrasian equilibrium. Suppose the converse, i.e. an agent i has a better bargain $x'_i \in P_i(x_i)$ in the budget set $B_i(p)$. Due to monotonicity of the preference one can count that $px'_i = 0$. We take vectors $x'_j \in R^k$, $j \neq i$, such that $x'_i + \sum_{j\neq i} x'_j = 0$, and $px'_j = 0$ (for example, $x'_j = -x'_i/(n-1)$). Then for sufficiently small $\alpha > 0$ the following is true:

a) vector $x^{\alpha}_i = \alpha x'_i + (1-\alpha) x_i$ belongs to X_i and is strictly better than x_i,

b) every vector x^{α}_i belongs to X_i, and all the budget equalities hold.

So agent i can choose the strategy $s'_i = (x', p)$ and enforce x' which he prefers to x. But this contradicts the definition of x as a Nash equilibrium outcome. Thus every x_i is a maximal element in the budget set $B_i(p)$ and $\sum_i x_i = 0$. So $x \in W(\mathbf{P})$. ·

We used open convex sets of admissible bargains X_i. Now we give a condition which allow to justify this requirement. Suppose that the initial allocation of goods $(\omega_i)_{i=1,...,n}$ is fixed, known, and non-manipulated by agents. Agents preferences are defined on R^k_+, their strict preferences are convex and monotone. We impose the following boundary condition on the set of feasible profiles $\mathbf{P} \in \mathbf{U}$: the allocations which give the total endowment of some good to an agent are not Pareto-optimal. Let us give a variant of resource relatedness condition B') under which the requirement holds. We come back to the previous notation: $z=(x_i)$ is a consumption bundle.

B') Let $z \in Z$ be such that $x^r_i = \omega^r$ for an agent i and a good r . Then there is agent j and a good s such that agents i and j have a profitable exchange bargain with goods r and s

$(x_i - \alpha e^r + \beta e^s) P_i x_i$ and $(x_j + \alpha e^r - \beta e^s) P_j x_j$, where

e^r , e^s are unit orths, α ,β are small numbers. To fulfill this condition it suffices that for every good there be at least two agents with high marginal utility of the good when it is not consumed.

Now if the boundary condition is fulfilled then Walrasin allocations z being Pareto-optimal give to each agent $x_i < \omega$. So one can take as a set of admissible bargains for agent i the set $X_i =\{\xi_i \in R^k , \xi_i < \sum_{j\neq i} \omega_j\}$ which is obviously open and convex.

Considering the axiomatic characterization given in theorem 4 and Corollary 3 for the Walrasian rules one can affirm the following.

Corollary. *Mechanism* f *Nash implements the rule of social choice* $\Sigma : \boldsymbol{U} \to 2^Z$, which satisfies the axioms ***E, I, L, M, N***.

The analogous implementation mechanism for the Walrasian rule can be constructed for model C.

The author sincerely thanks V.I. Danilov for a lot of useful remarks and advices.

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