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THE WAR OF ATTRITION WITH EXPECTED CHANGES OF FUTURE TERMS

Working paper # BSP/99/016

This paper presents a Master's Thesis completed at NES in 1999.

The paper was written within the research program "Transforming Government in Economies in Transition" (GET) sponsored by the Ford Foundation (Grant No.950-1503), project "Institutional Changes and Traps in Transition Economies".

This paper was prepared under the supervision of Prof. L. Polishchuk and Prof. V. Polterovich. I would like to thank them for their helpful guidance.

МОСКВА

1999

Ковтуненко Б.И. Война на выживание с ожидаемыми изменениями во внешних обстоятельствах в будущем./ Препринт # BSP/99/016.- М.: Российская Экономическая Школа, 1999. - 57с. (Англ.)

В данной работе рассматривается война на выживание с двумя игроками и неполной информацией. В отличие от общепринятой версии войны на выживание с бесконечным горизонтом мы ввели внешнее ограничение на длительность «войны». В работе показано, что данное ограничение приводит к появлению «мертвой зоны» непосредственно перед моментом ограничения. В «мертвой зоне» игра заканчивается с нулевой вероятностью и, следовательно, если игра дошла до «мертвой зоны», она продлится до самого конца, и этот факт является общим знанием для обоих игроков. В работе также показано, что в некоторых случаях введение внешнего ограничения на продолжительность «войны» может привести не только к увеличению, но и к уменьшению ожидаемого общественного благосостояния.

Kovtunenko B.I. The War Of Attrition With Expected Changes Of Future Terms. / Working Paper #BSP/99/016. –Moscow, New Economic School, 1999, -57p. (Engl.)

This paper considers a two-player incomplete information war of attrition. In contrast to the conventional infinite horizon war of attrition setting, we introduce an external constraint on the duration of the “war”. It is shown that this constraint results in the occurrence of the “dead zone” right before the moment of the constraint. In the “dead zone” the “war” ends with probability zero and, thus, once the game evolved into the “dead zone”, it is common knowledge for the players that the “war” will last till the very end. It is also shown that in some cases the introduction of the external constraint on the duration of the “war” can not only raise but also lower the expected social welfare.

ISBN 5-8211-0054-2

Contents

1. Introduction

The war of attrition was introduced in theoretical biology by Maynard Smith (1974) to explain animals' fights for prey. This approach was also used in industrial organization for the case of a natural monopoly (on this point, Tirole (1993, pp.311-314)). Two animals fighting for prey may resemble two firms fighting for control of an increasing returns industry. Fighting is costly to the animals; at the very least, they forgo the opportunity of other activities and become exhausted. Similarly, duopoly competition may be costly because it generates negative profits. In both cases, the object of the fight is to induce the rival to give up. The winning animal keeps the prey; the winning firm obtains monopoly power. The loser is left wishing it had never entered the fight. In a war of attrition, each player waits and suffers for a while. If at some point in time his rival has not yet quit, a player gives up. The same framework, with minor changes, can be used for the case of two firms engaged in a patent race (see Tirole (1993, pp.394-399)). In the conventional versions of the stationary war of attrition with two identical players, there is a unique symmetric Nash equilibrium, which is stationary, involves mixed strategies, and has a property that at each date the players are indifferent between stopping at time t and waiting a bit longer, until $t+\varepsilon$, to see if the opponent stops first (see Fudenberg and Tirole (1991, pp.119-121))[1].

The cases described above are the examples of complete information wars of attrition. In this paper, we will mainly concentrate on the incomplete information war of attrition first introduced in the theoretical biology literature by Bishop, Cannings, and Maynard Smith (1978), and extended by Kreps and Wilson (1982). In the incomplete information war of attrition players' types are private information and independently drawn from the same distribution function. The analysis of such games is performed within the concept of Bayesian Nash equilibrium. Alesina and Drazen (1991) used the incomplete information war of attrition approach to explain the delays in providing socially beneficial reforms[2,3]. Their model of delayed stabilization considers an economy in which the government is running a deficit due to failure of interest groups to agree on a deficit reduction program. In the absence of a consensus, only highly distortionary taxes can be used to finance government expenditures, and the revenue from those taxes is insufficient to fully cover expenditures. The budget deficit prior to stabilization is financed by a growing external debt. A fiscal reform program replaces highly distortionary taxes with less distortionary taxes large enough to cover government expenditures and close the deficit. Taxes after stabilization, however, must be distributed unequally across different groups in the economy. As a result, each group would like the burden of higher taxes placed elsewhere and refuses to agree to bearing a large fraction of the taxes in the hope that some other group will concede and accept (or no longer block) a fiscal reform placing a high burden on them. As any group can obstruct program it dislikes, fiscal reform requires consensus. Groups in the economy differ from one another in the welfare loss they suffer from the distortions associated with the prestabilization methods of government finance. Since each particular group does not know the welfare losses of the other groups, characterized by their types, at the beginning of the "game" there is a positive option value from waiting for any group with a loss lower than the maximum possible loss. Only when a group realizes that it can only do worse by waiting any

[1] For a more general analysis of complete information wars of attrition in continuous time see Hendricks et al. (1988).
[2] For a good survey of recent approaches in explaining the delays in reforms see Drazen (1996).
[3] The incomplete information war of attrition has also been used to describe industrial competition. The classic reference is Fudenberg and Tirole (1986). In contrast to the classic war of attrition for a natural monopoly industry, the authors assume that with (arbitrarily small) probability the market may accommodate two existing firms. This assumption results in a unique perfect Bayesian equilibrium.

more, will they concede and accept a reform with unfavorable distributional implications.

Drazen and Grilli (1993), following Alesina and Drazen(1991), considered an economy in which the government budget deficit is fully covered by the inflation tax. Stabilization raises taxes and eliminates the budget deficit. There are two individuals in the economy, representing two different groups, and taxes are divided half-half before stabilization and fall entirely on one individual after stabilization. The individuals differ in the group-specific costs of inflation. As in Alesina and Drazen (1991), the distributional conflict between groups leads to a delay in reaching an agreement. The general argument of the paper is that highly distortionary finance may improve the expected social welfare if the government must finance some portion of its expenditures in a distortionary way. Higher inflation, by raising the cost of living in the economy prior to stabilization, will shorten the delay in reaching agreement. There is thus a trade-off, with a higher inflation lowering welfare until an agreement is reached, but inducing earlier time of agreement on use of nondistortionary financing. There should therefore be a positive but finite level of inflation that maximizes expected utility. More precisely, the war of attrition analysis may be seen as a formalization of the view that policies which reduce (but do not eliminate) either inflation or the costs associated with inflation may be counterproductive, since they make it more difficult to gain agreement on undertaking painful policy steps to eliminate inflation.

All the models described above explicitly assume that the war can, in principle, last forever. For simplicity, no changes in external circumstances following the original shock were considered. As a result, there is a positive probability for the game to last any finite amount of time. This assumption might sound a bit artificial in some cases. More generally, during a war of attrition, a change in the environment may lead to a change in agents' behavior and the end of the "war". In this paper we argue that even (or especially) when this change is expected, the war of attrition is crucial in the delay of stabilization until the external change. For example, a country financing its budget deficit by borrowing from abroad, that is a country with a rising debt to GNP ratio as in Alesina and Drazen (1991), usually find much more difficult to borrow more when the debt/GNP ratio exceeds some critical value, and this fact is common knowledge. We argue that in some cases it is a fact of closing the credit line from abroad that might induce a reform.

Even without the external constraint on the amount of debt, there is a feasibility issue when the debt and, therefore, interest payments are so high that the loser of the war of attrition is unable to bear the cost of a stabilization. As argued in Alesina and Drazen (1991), as the value of initial output to debt ratio increases, the fraction of the distribution of groups whose behavior is not affected by this constraint rises. However, a problem of delayed reforms is often a problem of poor countries, whose initial output to foreign debt ratio in many cases can not be considered high enough to neglect the issue of constraints on the duration of the delay in reforming the economy.

Elections might also be considered as an example of expected future changes in the external circumstances. Even in developing countries with not very long-lasting democratic traditions, the time of the nearest elections is often known with a great degree of confidence long before the moment of the elections. Thus, this knowledge might affect the behavior of different groups with conflicting distributional objectives since these groups often represent different political parties. As the moment of elections approaches, the assumption that those groups behave as if they were unaware about the probable change in the external circumstances in the future becomes less and less justified.

In this paper, we present a simple two players war of attrition model with external constraint on the duration of the "war". It is common knowledge for the players that due to some changes in the environment in the future[4] the game can not last more than some finite amount of time. If no one concedes till that time, the coin is flipped to determine the loser. We investigate the changes in the equilibrium outcome of the game as compared with the infinite horizon version of the model. It will be shown that, because of the constraint on the duration of the game, the "dead zone" occurs right before the moment of the constraint. In the "dead zone" the game ends with probability zero. Thus, once the game evolved into the "dead zone", the "war" will last till the very end with probability one and this fact is common knowledge. It will be also shown that the expected duration of the finite horizon game is always lower than the expected duration of the game with an infinite horizon. However, our analysis will yield an unexpected result: in some (not extremely specific) cases the relative size of the "dead zone" does not approach zero as the time of the constraint rises.

The occurrence of the "dead zone" raises an important question: whether the expected social welfare of the finite horizon war of attrition always exceeds the expected social welfare of the infinite horizon "war". The introduction of the external constraint on the duration of the "war", T_M , has two opposite effects on the expected social welfare. First of all, agents who conceded within the "dead zone" (before T_M) in the infinite horizon game now wait until T_M, which obviously decreases the expected social welfare. On the other hand, agents who conceded later than T_M in the infinite horizon "war" concede earlier in the finite horizon

[4] These expected changes are of the types considered above, but we do not explicitly model them here.

game, which increases the expected social welfare. Therefore, despite the fact that the expected duration of the "war" is always lower in the finite horizon model, the sign of the change in the expected social welfare might be ambiguous. In this paper we will show that this sign is indeed ambiguous. It means that in some cases the introduction of the external constraint on the duration of the "war" can not only increase the expected social welfare but also decrease it. Moreover, for some region of T_M a decrease in the external constraint on the duration of the game is associated with a decrease in the expected social welfare. Therefore, policies which reduce the duration of the war of attrition but do not eliminate it completely might be counterproductive[5]. The striking result is that the decrease in the expected social welfare appears not for high but for relatively low values of the external constraint T_M.

The only work of a similar theme of which we are aware is that of Cannings and Whittaker (1995). They considered an infinite population of identical individuals who have fixed finite time available for war of attrition type contests; individuals start a new trial as soon as their old one has finished. Thus choosing to play for a long period of time in any particular trial will increase an individual's chance of winning that trial, but will tend to decrease the number of trials it is possible for that individual to play before the population runs out of time, and the contestant must balance these two factors. The paper presents the analytical analysis together with computer simulations of the evolutionary stable strategies[6] of the model. Despite some similarities, our approach is quite different. Cannings and Whittaker (1995) considered an infinite set of identical individuals, therefore their model is a symmetric information war of attrition. They also used discrete

[5] This argument is similar to that of Drazen and Grilli (1993).
[6] Following Selten (1980), a strategy r is said to be evolutionary stable if (i) r is a best response to itself and (ii) for any alternative best response r' to r, r is a better response to r' than r' to itself.

strategy space. We will consider an incomplete information war of attrition with two agents and infinite strategy space and focus on the symmetric Bayesian Nash equilibrium.

The paper is organized as follows. Section 2 presents a simple infinite horizon war of attrition model and investigates basic features of the equilibrium outcome of this model which will be used in the following sections. Some generalizations are also considered. Section 3 introduces the external constraint on the duration of the game considered in Section 2 and investigates the changes in the symmetric equilibrium outcome induced by this constraint. Section 4 examines the change in the expected social welfare after the introduction of the external constraint on the duration of the "war". The final section briefly summarizes the results and suggests extensions.

2. The Infinite Horizon Model

Consider an incomplete information infinite horizon war of attrition with two risk-neutral players. Player *i* chooses his strategy T_i (the time of concession if the rival has not conceded before) from $[0, +\infty]$. Both players choose their strategies simultaneously at the beginning of the game. The player who chooses the longer time wins. The payoffs are

$$u_i(T_i, T_j; \theta_i) = \begin{cases} b - \theta_i T_i & \text{if } T_i < T_j \\ B - \theta_i T_j & \text{if } T_j < T_i \\ \frac{1}{2}(b+B) - \theta_i T_i & \text{if } T_i = T_j \end{cases} \tag{2.1}$$

Player *i*'s type, θ_i, characterizes the utility loss the player suffers from the "war"[7], is private information, and takes values in $[\underline{\theta}, \bar{\theta}]$, $0 \le \underline{\theta} < \bar{\theta}$, with cumulative distribution $F(\theta_i)$ and strictly positive on $(\underline{\theta}, \bar{\theta})$ density $f(\theta_i)$. Types are independent between the players. The payoff received by the winner, *B*, is greater than the payoff received by the loser, *b*, that is $\Delta B \equiv B - b > 0$. When both players choose the same time of concession, either they share the total payoff (as in (2.1)) or the coin is flipped to determine the loser[8]. Since we suppose that players are risk-neutral, they are indifferent between these two alternatives.

Let us look for a pure-strategy Bayesian Nash equilibrium $(T_1(\theta), T_2(\theta))$ of this game. For each θ_i, $T_i(\theta_i)$ must satisfy

$$T_i(\theta_i) = \arg\max_{T_i} \left\{ (b - \theta_i T_i) \Pr\left[T_j(\theta_j) > T_i\right] + \left(\frac{b+B}{2} - \theta_i T_i \right) \Pr\left[T_j(\theta_j) = T_i\right] \right. \\ \left. + \int_{\{\theta_j | T_j(\theta_j) < T_i\}} (B - \theta_i T_j(\theta_j)) f(\theta_j) d\theta_j \right\} \quad (2.2)$$

Let us consider some properties of an equilibrium profile of this game.

Lemma 1: *Equilibrium strategies must be nonincreasing.*

(See Appendix for a proof).

It is worth noticing that monotonicity property of equilibrium strategies holds not only for this particular game but also in a rather general case. Namely, consider a war of attrition with two players of the types specified above but with a more general payoff for a player if the game ends at moment T: $u_i(T;\theta_i) = H(x, \theta_i, T)$, where *H*(.) is a strictly increasing function of *x* and strictly

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[7] Following Fudenberg and Tirole (1986) θ_i might be interpreted not only as resources devoted to the "war" but also as opportunity costs (the benefits from forgone activity). Opportunity costs are particularly likely to be private information.

[8] We assume that there is no dissipation of the total surplus when both players concede at the same time.

decreasing in *T*, and $x = \{b, \bar{b}, B\}$ depending on whether the player *i* loses, ties, or wins the game, $b \le \bar{b} \le B$. Then Lemma 1 can be generalized as follows

Theorem 1: *If* $H(x, \theta_i, T)$ *is a function specified above and* $\frac{\partial H}{\partial \theta_i}(x, \theta_i, T)$ *is nonincreasing (nondecreasing) in T and x and strictly decreasing (increasing) in at least one of T and x for any feasible* (x, θ_i, T)*, then equilibrium strategies are nonincreasing (nondecreasing).*

(See Appendix for a proof).

The reason for this monotonicity property is rather clear intuitively. If the cost per unit of time of the "war", $-\frac{\partial H}{\partial T}$, is increasing with the type of the agent, and a marginal gain from winning the game, $\frac{\partial H}{\partial x}$, is decreasing with the type, that is for any time of the end of the game a player with higher type incurs higher costs than a lower type player and obtains lower gain from winning the game as compared with losing it, then a player with a higher type will always choose lower time of concession.

The intuition that the time of concession is strictly decreasing with the type is also justified. To obtain this result we should eliminate the equilibrium where one player waits forever and the other one concedes immediately. For our simple model, we have the following lemma.

Lemma 1': *If T=0 is not a mass point of one of equilibrium strategies, then equilibrium strategies are strictly decreasing.*

(See Appendix for a proof).

Again, this result appears to be rather general.

Theorem 1': *If all the conditions of the first (second) part of Theorem 1 are satisfied,* $H(x,\theta,T)$ *is continuous in T for any* (x,θ)*, and T=0 is not a mass point of one of equilibrium strategies, then equilibrium strategies are strictly decreasing (increasing)*[9].

(See Appendix for a proof).

The intuition is straightforward: If the strategy is not strictly decreasing (increasing), there must be a mass point T in the distribution of a player's concession time. Thus, the other player will never set his strategy closely below that mass point since he can be better off setting it just above T. But in that case playing T is no more optimal for the first player since it can lower his strategy not affecting the probability of winning the game but reducing the expected cost.

Similar intuition underlines the argument that strategies must be continuous.

Theorem 2: *If equilibrium strategies are monotonous, they must be continuous.*

(See Appendix for a proof).

Theorems 1, 1', and 2 allow us to investigate general properties of the war of attrition considering rather simple model (2.1). For example, if we introduce a discount r in model (2.1), we obtain

$$H(x,\theta_i,T) = xe^{-rT} - \theta_i \int_o^T e^{-rt}dt = xe^{-rT} - \frac{\theta_i}{r}\left(1-e^{-rT}\right) \qquad (2.3)$$

Since $\frac{\partial H}{\partial \theta_i} = \frac{1}{r}\left(e^{-rT}-1\right)$ is strictly decreasing in T and does not depend on x, all the conditions of Theorem 1 hold. Therefore, the presence of a discount does not change the equilibrium behavior of the agents but makes calculations more cumbersome. The same is true for the model considered in Alesina and Drazen (1991). In our notations, the model takes the form

[9] It is possible for an equilibrium strategy to have a mass point at $T = +\infty$. In that case, the strategy is

$$H(x,\theta_i,T)=\bar{b}e^{-\gamma rT}\left[-\left(\frac{1}{2}+\theta_i\right)(e^{\gamma rT}-1)-\alpha+x(2\alpha-1)\right] \qquad (2.4)$$

where $\bar{b}$ is the present discounted value of future tax payments before and after stabilization, $(1-\gamma)$ is a fraction of government expenditures covered by issuing debt until the date of stabilization (a fraction γ is covered by distortionary taxation), $\alpha > \frac{1}{2}$ is a share of tax burden levied on the loser after stabilization, *r* is world interest rate, which is equal to the discount rate, and *x*=0 for a loser and *x*=1 for a winner (see Alesina and Drazen (1991)). Again, all the conditions of Theorem 1 hold. Therefore, most of the results of this paper might be applied to the model of Alesina and Drazen (1991).

Let us return to our simple model (2.1) and denote by $G_i(T)$ the distributions of the players optimal times of concession (they are, of course, endogenous and will be derived below) and by $g_i(T)$ the associated density functions (thus, we implicitly assume that equilibrium strategies are differentiable). The expected payoff of player *i* as a function of his strategy T_i is

$$U_i(T_i,\theta_i)=\int_0^{T_i}(B-\theta_i t)g_j(t)dt+[1-G_j(T_i)](b-\theta_i T_i) \qquad (2.5)$$

We did not include in (2.5) the case of $T_i = T_j$ as we consider an equilibrium in strictly decreasing strategies (that is, without a mass point at *T*=0 for any agent) and, therefore, for a finite density function $f(\theta)$ the probability of a tie is zero.

Maximizing (2.5) with respect to T_i we obtain

$$\left[\frac{g_j(T_i)}{1-G_j(T_i)}\right]\Delta B=\theta_i \qquad (2.6)$$

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monotonous until at some $\tilde{\theta}$, $T(\tilde{\theta})=+\infty$.

If there exists a solution to the first order conditions (2.6), it also satisfies the second order conditions. Indeed, from (2.5)-(2.6) we have

$$\frac{\partial U_i}{\partial T_i}(T_i(\theta_i'),\theta_i) = (\theta_i' - \theta_i)(1 - G_j(T_i(\theta_i'))) \tag{2.7}$$

Thus, according to Lemma 1,

$$sign\frac{\partial U_i}{\partial T_i}(T_i(\theta_i'),\theta_i) = sign(\theta_i' - \theta_i) = sign(T_i(\theta_i) - T_i(\theta_i')) \tag{2.8}$$

Therefore, we have proved the following corollary:

Corollary 1: *A solution to (2.6) constitutes Bayesian Nash equilibrium of our game.*

The right-hand side of (2.6) is the cost of waiting another instant to concede. The left-hand side is the expected gain from waiting another instant to concede, which is the product of the conditional probability that one's opponent concedes (the hazard rate in brackets) multiplied by the gain if the other player concedes. The concession occurs when the (player-specific) cost of waiting just equals the expected benefit from waiting.

We now want to find a symmetric Bayesian Nash equilibrium in which each player's concession behavior is described by the same function $T(\theta)$. Then (2.6) takes the form

$$\left[\frac{g(T)}{1-G(T)}\right]\Delta B = \theta \tag{2.9}$$

Since $T(\theta)$ is strictly decreasing in θ (see Lemma 1'), $G(T(\theta)) = 1 - F(\theta)$ and $g(T) \equiv G'(T) = -\frac{f(\theta)}{T'(\theta)}$. Substituting this into (2.7), we finally obtain the equation that implicitly defines a symmetric equilibrium $T(\theta)$

$$\left[-\frac{f(\theta)}{F(\theta)}\frac{1}{T'(\theta)}\right]\Delta B = \theta \tag{2.10}$$

To derive the initial boundary condition, note first that, for any value of $\theta \in [\underline{\theta}, \overline{\theta}]$ and for any time *T*, the gain from having the opponent to concede is positive. Therefore, as long as there exists ε>0 such that $f(\theta) > 0$ for $\theta \in (\overline{\theta} - \varepsilon, \overline{\theta})$, players with $\theta < \overline{\theta}$ will not concede immediately as there is a positive probability to have an opponent with higher θ. This in turn implies that a player with $\theta = \overline{\theta}$ (i.e., a player that knows he has the highest possible cost of waiting) will find it optimal to choose

$$T(\overline{\theta}) = 0 \tag{2.11}$$

It can be also proved that equilibrium strategies in (2.10)-(2.11) constitute a perfect Bayesian equilibrium, which means that once a player chooses his concession time at the beginning of the game, he will never want to change it during the game.

Lemma 2: *Bayesian Nash equilibrium strategies from* (2.6) *are perfect Bayesian equilibrium strategies.*[10]

(See Appendix for a proof).

Lemma 2 is useful in understanding the evolution of the war of attrition from viewpoint of one side. If the types of the agents were common knowledge, an agent with the higher type would concede immediately[11]. However, players do not know the types of their opponents and form expectations about those types. Consider a

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[10] Again, this result is very general. Since each player knows that his decision to concede is only relevant if the other player has not conceded before, the game is strategically equivalent to a static game in which players simultaneously choose concession times. Thus, if there exist Bayesian Nash equilibrium strategies, they also constitute a perfect Bayesian equilibrium.

[11] When agents have an infinite horizon, stationary wars of attrition with complete information have an infinite number of subgame perfect equilibria (see Hendricks et al. (1988)). In particular, there is a subgame perfect equilibrium in which anyone concedes immediately. However, if we assume instead that agents have a finite horizon, the game has a unique subgame perfect equilibrium in which the agent with the higher type concedes immediately and the other one waits. This remains true even when the time horizon goes to infinity. If we treat the infinite horizon game as the limit of the set of finite horizon games, this outcome stands out even in infinite horizon games (on this issue, though in a slightly different setting, see Bilodeau and Slivinski (1996)).

player with $\theta < \overline{\theta}$. At time 0, there is some probability that his opponent has $\theta = \overline{\theta}$ and will concede immediately. If no one concedes at time 0, both sides know that their opponent is not type $\overline{\theta}$. A the "next" instant the "next-highest" type concedes and so on, so as time elapses each side learns that his opponent does not has a cost above a certain level. When the conditional probability of an opponents concession in the next instant (based on what the player has learnt about his highest possible cost) is such that (2.6) holds, it is time to "throw in the towel".

We derived (2.6) and thus (2.10) under the assumption that equilibrium strategies are differentiable. However, in the symmetric equilibrium case it can be proved directly.

Lemma 3: *Symmetric equilibrium strategies are differentiable.*

(See Appendix for a proof).

From Lemma 3, the first-order condition (2.10) and the initial boundary condition (2.11) characterize the equilibrium uniquely. Therefore, we have proved the following corollary:

Corollary 2:[12] *The unique symmetric perfect Bayesian equilibrium of the game is defined by*

$$T(\theta) = \Delta B \int_{\theta}^{\overline{\theta}} \frac{f(x)}{F(x)} \frac{dx}{x} \tag{2.12}$$

Given concession times as a function of θ, the expected duration of the game is then the expected minimum *T*, the expectation taken over $F(\theta)$. Following Alesina and Drazen (1991), the expected value of minimum *T* in a two-player case is

$$T^E = 2\int_{\underline{\theta}}^{\overline{\theta}} T(x) F(x) f(x) dx \tag{2.13}$$

[12] Similar results can also be found in Bliss and Nalebuff (1984), Bulow and Klemperer (1998), and elsewhere.

Therefore, both the equilibrium strategy and the expected duration of the game are directly proportional to the difference between the payoffs of the winner and of the loser ΔB.

As long as all participants in the process initially believe that someone else may have a higher θ, the game does not stop immediately. The cumulative distribution of duration times *T* is therefore 1 minus the probability that every player has θ lower than the value consistent with the game's end at *T*. With two players it is

$$S(T) = 1 - [F(\theta(T))]^2 \qquad (2.14)$$

where $\theta(T)$ is defined by $T(\theta) = T$.

Consider the influence of an increase in the costs associated with the “war” on the equilibrium strategy. For this purpose, let us denote $\Delta\theta \equiv \overline{\theta} - \underline{\theta}$. An increase in the costs may be of two different types. First, it may be a multiplicative shift in θ that is an increase in $\underline{\theta}$ for unchanged $\Delta\theta$. Second, it may be an increase in $\Delta\theta$ for unchanged $\underline{\theta}$. Finally, it may be a combination of these two increases.

Lemma 4: *An increase in the costs associated with the “war”, for an unchanging distribution of* θ *, lowers the equilibrium strategies and the expected duration of the game.*[13]

(See Appendix for a proof).

Finally, consider the behavior of the equilibrium strategy $T(\theta)$ as θ approaches $\underline{\theta}$.

Lemma 5: *If* $f(\theta)$ *is continuous at* $\underline{\theta}$ *and* $f(\underline{\theta}) > 0$*, then* $T(\theta) \to +\infty$ *as* $\theta \to \underline{\theta}$ *.*[14]

[13] Unchanging distribution of θ means that the probability for an agent to have a type $\theta \in [\theta_1, \theta_2]$, where the interval $[\theta_1, \theta_2]$ constitutes the share $\alpha = \dfrac{\theta_2 - \theta_1}{\overline{\theta} - \underline{\theta}}$ of the whole interval, does not change for any constant α when $[\underline{\theta}, \overline{\theta}]$ changes.

(See Appendix for a proof).

Therefore, in the symmetric equilibrium there is a positive probability for the game to last any finite amount of time. Thus, even without an explicit constraint on the duration of the game the issue of feasibility cannot be avoided once initial endowments of players are limited.

To illustrate the results of this section, consider a uniform distribution $f(\theta)=\frac{1}{\Delta\theta}$ over $[\underline{\theta},\overline{\theta}]$. The symmetric equilibrium strategy in that case is

$$T(\theta)=\Delta B\left(\frac{1}{\theta}-\frac{1}{\overline{\theta}}\right) \qquad \text{for } \underline{\theta}=0$$
$$T(\theta)=\frac{\Delta B}{\underline{\theta}}\ln\left[\frac{\theta}{\overline{\theta}}\frac{\Delta\theta}{\theta-\underline{\theta}}\right] \qquad \text{for } \underline{\theta}>0 \tag{2.15}$$

The expected duration of the game is

$$T_{\infty}^{E}=\frac{\Delta B}{\overline{\theta}} \qquad \text{for } \underline{\theta}=0$$
$$T_{\infty}^{E}=\frac{\Delta B}{\Delta\theta}\left\{1-\frac{\underline{\theta}}{\Delta\theta}\ln\frac{\overline{\theta}}{\underline{\theta}}\right\} \qquad \text{for } \underline{\theta}>0 \tag{2.16}$$

3. The Finite Horizon Version of the Model

Let us consider a finite horizon version of the model (2.1). Players are allowed to play the war of attrition within a finite time, that is player i chooses his strategy T_i (the time of concession if the rival has not conceded before) from $[0,T_M]$, where $0<T_M<+\infty$. If no one has considered before T_M, a tie-breaking rule is used to determine the winner (each agent having a probability 1/2 to win).

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[14] The result of Lemma 5 also holds if $f(\theta)$ is n times $(n\geq 1)$ differentiable at $\underline{\theta}$, $f^{(k)}(\underline{\theta})=0$ for $0\leq k\leq n-1$, and $f^{(n)}(\underline{\theta})>0$.

To get an intuition of the change in players' equilibrium behavior in that case consider the symmetric equilibrium strategy $T(\theta)$ in the infinite horizon version of the model (of course, it is not an equilibrium strategy for this game, but as we will show, it does make sense)[15]. The constraint implies that now the distribution of concession times will have a mass point at T_M, with concession occurring at that moment with probability 1 if it has not occurred before. It is clear that an agent who conceded after T_M without the constraint will concede at T_M now. Let us denote $\theta_M : T(\theta_M) = T_M$. The existence of a mass point at T_M implies that players with costs close to but above θ_M (i.e., players that would have conceded before T_M under strategy $T(\theta)$ if there were no mass point at T_M) will now find it preferable to wait until T_M to end the "war" under a tie-breaking rule. Define $\tilde{\theta} > \theta_M$ as the cost when a player is indifferent between being the loser at $\tilde{T} = T(\tilde{\theta})$ and waiting until T_M to be the winner with probability 1/2. Thus, in the symmetric equilibrium there will be a "dead zone" of times $T : \tilde{T} < T < T_M$ when the game ends with probability zero. That means that if no one has conceded before or at $\tilde{T}$, both players know for sure that the game will last till T_M. Being perfectly rational, they will wait until T_M to take their chances in a tie-breaking contest. Note that waiting from $\tilde{T}$ to T_M with probability 1 is highly inefficient from the social point of view and the best outcome (conditional on reaching $\tilde{T}$) is to make the tie breaking contest immediately, but due to lack of coordination the probability for the players to concede at the same time is zero for any $T < T_M$. As a result, the "dead zone" occurs (any war of attrition is a consequence of the lack of

[15] Here we closely follow the logic from Alesina and Drazen (1991) but focus our attention on the "dead zone" occurrence and investigate the properties of this zone in a formal way.

coordination problem, but, in our opinion, this result is one of the most impressive consequences).

Let us now present a more formal proof of the equilibrium properties considered above. First of all, notice that the proof of Lemma 1 is valid for this model since it does not use the fact that players' strategies are nonlimited. Therefore, equilibrium strategies are still nonincreasing.

Lemma 6: *If* $T(\overline{\theta})=0$*, then symmetric Bayesian Nash equilibrium strategies cannot be continuous.*

(See Appendix for a proof).

The condition $T(\overline{\theta})=0$ is significant since in this model the player with the highest possible cost does not always concede immediately. Making his decision, he compares his expected payoff from conceding at $T=0: U(0)=b$ (we will show below that there cannot be mass points other than T_M) with the expected payoff from waiting till T_M: $U(T_M)=\frac{b+B}{2}-\overline{\theta}T_M$. He will choose $T(\overline{\theta})=0$ only if $U(0)>U(T_M)$, that is

$$\overline{\theta}T_M>\frac{\Delta B}{2} \tag{3.1}$$

Otherwise players of all the types will wait until T_M and, therefore, the game ends at T_M with probability 1 (this is the case when $\widetilde{T}=0$).

The discontinuity point of the symmetric equilibrium strategy is T_M and the reason for that is clear: by conceding at T_M a player cannot lose, the worst outcome he can get is a tie, while for any concession time below T_M a player can lose the game with positive probability. Since the payoff from a tie (without costs of waiting) is strictly greater than the payoff from losing the game, this gap must be compensated by the difference in the costs of waiting and, therefore, a symmetric equilibrium strategy cannot be continuous at T_M.

Lemma 7: *In a symmetric equilibrium, the distribution of concession times cannot have a mass point other than* T_M.

(See Appendix for a proof).

From this Lemma, it follows that a symmetric equilibrium strategy is strictly decreasing for any $\theta : T(\theta) < T_M$.

Lemma 8: *In a symmetric equilibrium strategy, there cannot be a discontinuity point other than* T_M.

(See Appendix for a proof).

Therefore, if the condition (3.1) holds, a symmetric equilibrium strategy of the finite horizon war of attrition must be found in the form

$$T(\theta) = \begin{cases} \text{continuous strictly decreasing } T(\theta) & \text{for } \theta > \tilde{\theta} \\ T_M & \text{for } \theta < \tilde{\theta} \end{cases} \tag{3.2}$$

and $\tilde{\theta}$ having a property that the agent with $\tilde{\theta}$ is indifferent between being a loser at $\tilde{T} = T(\tilde{\theta})$ and being a winner at T_M with probability 1/2

$$b - \tilde{\theta} T(\tilde{\theta}) = \frac{b+B}{2} - \tilde{\theta} T_M \tag{3.3}$$

It can be proved that $T(\theta)$ is differentiable for $\theta > \tilde{\theta}$[16]. Therefore, denoting by $G(T)$ the distribution of one of the players optimal time of concession and by $g(T)$ the associated density function for $T < \tilde{T}$, the expected payoff for the player θ from playing T is

[16] The proof is the same as the proof of Lemma 3.

$$U(T,\theta)=\begin{cases}\int_0^T (B-\theta t)g(t)dt+[1-G(T)](b-\theta T) & \text{for } T<\tilde{T}\\ \int_0^{\tilde{T}} (B-\theta t)g(t)dt+[1-G(\tilde{T})]\left(\frac{b+B}{2}-\theta T_M\right) & \text{for } T=T_M\end{cases} \quad (3.4)$$

Maximizing $EU(T,\theta)$ for $T<\tilde{T}$ leads to (2.9) and, therefore, (2.10). Thus, we are ready to make the following statement:

Corollary 3: *The symmetric perfect Bayesian equilibrium strategy of the finite horizon model is unique. It has the same functional form* (2.12) *as in the infinite horizon model for* $\theta>\tilde{\theta}$ *and equals* T_M *for* $\theta<\tilde{\theta}$, *where* $\tilde{\theta}$ *is determined according to* (3.3).[17]

Therefore, the change in the symmetric equilibrium strategy in the finite horizon model as compared with the infinite horizon one is fully determined by $\tilde{\theta}$ (or $\tilde{T}=T(\tilde{\theta})$).

Let us investigate the behavior of $\tilde{T}$ when T_M changes. From (3.1) it follows that when $T_M \le \frac{\Delta B}{2\bar{\theta}}$, then $\tilde{T}=0$ and the game ends at $T\in[0,T_M)$ with probability zero. Consider T_M large enough so that (3.1) holds. In that case, from (3.3), $\tilde{T}=T_M-\frac{\Delta B}{2\tilde{\theta}}>0$.

Lemma 9: $\tilde{\theta}$ *is decreasing in* T_M *and, if* $f(\theta)$ *is continuous at* $\underline{\theta}$ *and* $f(\underline{\theta})>0$, $\tilde{\theta}\to\underline{\theta}$ *when* $T_M\to+\infty$.[18]

(See Appendix for a proof).

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[17] Corollary 3 justifies the intuition considered at the beginning of this section.

[18] As in Lemma 5, $\tilde{\theta}\to\underline{\theta}$ as $T_M\to+\infty$ if $f(\theta)$ is n times $(n\ge 1)$ differentiable at $\underline{\theta}$, $f^{(k)}(\underline{\theta})=0$ for $0\le k\le n-1$, and $f^{(n)}(\underline{\theta})>0$.

Thus, both the beginning $\widetilde{T}$ and the size $T_M - \widetilde{T}$ of the "dead zone" are increasing in T_M, $\widetilde{T}$ approaches $+\infty$, and $T_M - \widetilde{T}$ approaches $\frac{\Delta B}{2\underline{\theta}}$ for $\underline{\theta} > 0$ and $+\infty$ for $\underline{\theta} = 0$ as $T_M \to +\infty$. The existence of the "dead zone" in the equilibrium outcome of the game can be neglected for T_M high enough only if its relative size $\frac{T_M - \widetilde{T}}{T_M}$ approaches zero as T_M rises to infinity, but this is not always the case.

Lemma 10: *For $f(\theta)$ continuous at $\underline{\theta}$ and $f(\underline{\theta}) > 0$, $\frac{T_M - \widetilde{T}}{T_M}$ is decreasing in T_M high enough. If $\underline{\theta} > 0$, then $\frac{T_M - \widetilde{T}}{T_M} \to 0$ when $T_M \to +\infty$. If $\underline{\theta} = 0$, then $\frac{T_M - \widetilde{T}}{T_M} \to \frac{1}{3}$ when $T_M \to +\infty$*[19].

(See Appendix for a proof).

Therefore, even if the probability for a player not to bear the costs from living in the "war" is zero, once there is a positive probability to face a player with the costs below any positive value, the relative size of the "dead zone" will be above some positive level for whatever high T_M. Moreover, if $\underline{\theta} > 0$, $\frac{T_M - \widetilde{T}}{T_M} \approx \frac{\Delta B}{2\underline{\theta} T_M}$ for T_M high enough, thus the relative size of the "dead zone" becomes low only for $T_M > \frac{1}{\underline{\theta}}$, which might be very high for $\underline{\theta}$ small enough. In addition, it can be easily proved that a multiplicative downward shift of the

[19] The result of Lemma 10 also holds if $f(\theta)$ is n times $(n \geq 1)$ differentiable at $\underline{\theta}$, $f^{(k)}(\underline{\theta}) = 0$ for $0 \leq k \leq n-1$, and $f^{(n)}(\underline{\theta}) > 0$. If $\underline{\theta} = 0$, then $\frac{T_M - \widetilde{T}}{T_M} \to \frac{1}{2n+3}$ as $T_M \to +\infty$.

distribution $f(\theta)$ increases the relative size of the "dead zone" for any fixed high enough T_M (see Lemma 4). Thus, for small $\underline{\theta}$, and especially for $\underline{\theta}=0$, the fact that the game cannot last forever (arising whether from a change in the external circumstances or from the feasibility problem) should be treated with some caution.

The expected duration of the finite horizon game is (see (2.13) and (3.2))

$$T^E(T_M)=2\int_{\tilde{\theta}}^{\bar{\theta}}T(x)F(x)f(x)dx+T_M\left[F(\tilde{\theta})\right]^2 \qquad (3.5)$$

As $T_M \to +\infty$, the first part of the sum in the right-hand side of (3.5) approaches the expected duration of the infinite horizon game (2.13) (denote it T_∞^E). The question is whether the second part of the sum approaches zero.

Lemma 11: *If $f(\theta)$ is continuous at $\underline{\theta}$ and $f(\underline{\theta})>0$, then $T^E(T_M)$ is strictly increasing for all $T_M \geq 0$ and $T^E(T_M) \to T_\infty^E$ when $T_M \to +\infty$.*[20]

(See Appendix for a proof).

Thus, the presence of the "dead zone", even with strictly positive relative size, does not affect the behavior of the expected duration of the game in the limit $T_M \to +\infty$.

For example, in the case of a uniform distribution over $[0,\bar{\theta}]$ (see (2.15)-(2.16)) we have

$$T^E(T_M)=\begin{cases} T_M & \text{for } \alpha \leq \frac{1}{2} \\ T_\infty^E\left[\left(1-\frac{3}{2(1+\alpha)}\right)^2+\alpha\left(\frac{3}{2(1+\alpha)}\right)^2\right] & \text{for } \alpha > \frac{1}{2} \end{cases} \qquad (3.6)$$

[20] Again, the result of Lemma 11 is also true if $f(\theta)$ is n times $(n \geq 1)$ differentiable at $\underline{\theta}$, $f^{(k)}(\underline{\theta})=0$ for $0 \leq k \leq n-1$, and $f^{(n)}(\underline{\theta})>0$.

where $\alpha = \dfrac{\overline{\theta} T_M}{\Delta B}$. Thus, we can see that $T^E(T_M)$ is strictly increasing in T_M for any $\alpha \geq 0$ and $T^E(T_M) \to T_\infty^E$ as $\alpha \to \infty$.

Discounting

The introduction of discounting in the model does not change the equilibrium outcome much. All the results about the general behavior of the symmetric equilibrium strategies for both infinite and finite horizon models studied in Sections 2 and 3 are also valid in this case. However, the behavior of the "dead zone" changes slightly.

In the case when players' payoffs take the form $H(x,\theta,T)$ (see Section 2) and symmetric equilibrium strategies of the infinite horizon game are differentiable, the first order condition (2.10) takes the from

$$\left[-\frac{f(\theta)}{F(\theta)}\frac{1}{T'(\theta)}\right]\{H(B,\theta,T(\theta)) - H(b,\theta,T(\theta))\} = -\frac{\partial H}{\partial T}(b,\theta,T(\theta)) \qquad (3.7)$$

When all the conditions of Theorem 1' are satisfied, a solution to (3.7) also satisfies the second order condition and, therefore, constitutes a symmetric Bayesian Nash equilibrium of the game.

For the model with discounting (2.3) it can be easily proved that symmetric equilibrium strategies are differentiable and constitute Perfect Bayesian equilibria of the infinite horizon game. Therefore, the unique perfect Bayesian equilibrium in this case, according to (3.7), is defined by

$$T(\theta) = \Delta B \int_{\theta}^{\overline{\theta}} \frac{f(x)}{F(x)} \frac{dx}{x + br} \qquad (3.8)$$

For the finite horizon model, as in the case without discounting, the only difference in the symmetric equilibrium strategy as compared with (3.8) is the presence of the "dead zone". Similar to (3.1), an agent with the highest possible cost of the "war" will concede immediately only if

$$b > \left(\frac{b+B}{2} + \frac{\overline{\theta}}{r}\right)e^{-rT_M} - \frac{\overline{\theta}}{r} \quad \Leftrightarrow \quad T_M > \frac{1}{r}\ln\left(1 + \frac{r\Delta B}{2(rb+\overline{\theta})}\right) \tag{3.9}$$

As before, the "dead zone" is determined by type $\widetilde{\theta}$ having the property that an agent with this type is indifferent between conceding at $T\left(\widetilde{\theta}\right)$ and waiting until T_M:

$$\left(b + \frac{\widetilde{\theta}}{r}\right)e^{-rT(\widetilde{\theta})} = \left(\frac{b+B}{2} + \frac{\widetilde{\theta}}{r}\right)e^{-rT_M} \quad \Leftrightarrow \quad T_M - T\left(\widetilde{\theta}\right) = \frac{1}{r}\ln\left(1 + \frac{r\Delta B}{2(rb+\widetilde{\theta})}\right) \tag{3.10}$$

For the model with discounting the case when the cost of waiting another instant to concede for the lowest type player $\underline{\theta}$, $-\frac{\partial H}{\partial T}\left(b,\underline{\theta},T\right)$, is equal to zero, corresponds to $\underline{\theta} = -br$ $(b < 0)$. For $\underline{\theta} \geq -br$, as before, $\widetilde{\theta}$ is decreasing in T_M and approaches $\underline{\theta}$ when $T_M \to \infty$. Therefore, from (3.10), the absolute side of the "dead zone" is increasing in T_M and bounded from above for $\underline{\theta} > -br$ and approaches infinity for $\underline{\theta} = -br$. However, in contrast with Lemma 10, the relative size of the "dead zone" approaches zero in both cases:

$$\frac{T_M - T\left(\widetilde{\theta}\right)}{T_M} \approx -\frac{1}{\ln(\widetilde{\theta} - \underline{\theta})}\frac{\underline{\theta} + rb}{r\Delta B}\ln\left[1 + \frac{r\Delta B}{2(\underline{\theta} + rb)}\right] \quad \text{for } \underline{\theta} > -rb$$

$$\frac{T_M - T\left(\widetilde{\theta}\right)}{T_M} \approx -\frac{(\widetilde{\theta} - \underline{\theta})\ln(\widetilde{\theta} - \underline{\theta})}{r\Delta B} \quad \text{for } \underline{\theta} = -rb \tag{3.11}$$

Nevertheless, when $r \to 0$ the rate of convergence to zero of the relative size of the "dead zone" is unaffected for $\underline{\theta} > -br$ but becomes slower and slower for $\underline{\theta} = -br$ (in the limit we have the result of Lemma 10).

4. Expected Social Welfare

The expected social welfare of the war of attrition equals the expected total surplus minus the expected costs of the "war". For our setting (2.1) we have

$$ESW = b + B - E[(\theta_1 + \theta_2)T(\theta_1, \theta_2)] \quad (4.1)$$

where $T(\theta_1, \theta_2)$ is the time of the end of the "war" when players' types are θ_1 and θ_2 and expectations $E[.]$ are taken over $[\underline{\theta}_1, \overline{\theta}_1] \times [\underline{\theta}_2, \overline{\theta}_2]$.

Since the expected total surplus is constant for our model without discounting, the expected total costs of the "war" completely determine the expected social welfare. Let us try to investigate the changes in the expected social welfare when the external constraint on the duration of the "war" is introduced into the model.

A disappointing result is that in our case the appropriate version of the Revenue Equivalence Theorem[21] is reduced to the trivial statement:

Lemma 12: (The Revenue Equivalence Theorem) *Consider a war of attrition setting with two risk-neutral players, in which player i has privately known costs of the "war" θ_i independently drawn from interval $[\underline{\theta}_i, \overline{\theta}_i]$, $0 \le \underline{\theta}_i < \overline{\theta}_i$, with cumulative distribution $F_i(\theta_i)$ and strictly positive on $(\underline{\theta}_i, \overline{\theta}_i)$ density $f_i(\theta_i)$. Suppose that given pair of Bayesian Nash equilibria of two different wars of attrition are such that:* (i) *For each possible realization of (θ_1, θ_2) the time of the end of the "war", $T(\theta_1, \theta_2)$, is the same in both equilibria; and* (ii) *For any i=1,2 player i has the same expected utility level in the two wars of attrition when his cost of the "war" is at its highest possible level. Then these equilibria of the two wars of attrition generate the same expected social welfare.*

(See Appendix for a proof).

Since the function of the end of the "war", $T(\theta_1, \theta_2)$, changes after the introduction of the external constraint on the duration of the "war" (see Section 3), the Revenue Equivalence Theorem loses its predictive power in our case.

Let us return to the symmetric war of attrition setting (2.1) and consider the symmetric perfect Bayesian equilibria for the infinite and finite horizon wars of attrition studied in Sections 2 and 3 correspondingly. The expected total costs of the "war" for the infinite horizon game are

$$\begin{aligned} ETC_\infty &= \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} (\theta_1 + \theta_2) T(\max\{\theta_1, \theta_2\}) f(\theta_1) f(\theta_2) d\theta_1 d\theta_2 \\ &= 2 \iint_{\theta_1 \geq \theta_2} (\theta_1 + \theta_2) T(\max\{\theta_1, \theta_2\}) f(\theta_1) f(\theta_2) d\theta_1 d\theta_2 \\ &= 2 \int_{\underline{\theta}}^{\overline{\theta}} \left(\theta_1 + E\left[\theta_2 \middle| \theta_2 \leq \theta_1\right]\right) T(\theta_1) F(\theta_1) f(\theta_1) d\theta_1 \end{aligned} \qquad (4.2)$$

where $T(\theta)$ is symmetric equilibrium strategy (2.12).

Now consider the finite horizon model with the external constraint on the duration of the game T_M and let $\tilde{\theta}$ be the solution of (3.3) for $T_M \geq \frac{\Delta B}{2\overline{\theta}}$ and $\tilde{\theta} = \overline{\theta}$ otherwise. Notice that the outcome of the "war" does not change as compared with the infinite horizon model when one of the players has costs above $\tilde{\theta}$. Thus, the difference in the expected social welfare between the infinite and the finite horizon "wars" is associated with players' types within $[\underline{\theta}, \tilde{\theta}] \times [\underline{\theta}, \tilde{\theta}]$. The expected costs for these agents in the infinite and finite horizon models are

$$\begin{aligned} ETC_\infty(\tilde{\theta}) &= 2 \int_{\underline{\theta}}^{\tilde{\theta}} \left(\theta + E\left[\theta' \,\middle|\, \theta' \leq \theta\right]\right) T(\theta) F(\theta) f(\theta) d\theta \\ ETC_{T_M}(\tilde{\theta}) &= 2 T_{\mathrm{M}} \left[F(\tilde{\theta})\right]^2 E\left[\theta \middle| \theta \leq \tilde{\theta}\right] \end{aligned} \qquad (4.3)$$

---

[21] See Myerson (1981) and Riley and Samuelson (1981) for the earliest statements of the Revenue

correspondingly. The difference in the expected total costs between the models is

$$ETC_{\infty} - ETC_{T_M} = ETC_{\infty}(\widetilde{\theta}) - ETC_{T_M}(\widetilde{\theta}) \tag{4.4}$$

The introduction of the external constraint on the duration of the "war", T_M, has two opposite effects on the expected total costs. First of all, agents with types within $[\theta_M, \widetilde{\theta}]$, where θ_M is defined as $T(\theta_M) = T_M$, who conceded before T_M in the infinite horizon game now wait until T_M, which obviously increases the total expected costs of the "war". On the other hand, agents with types below θ_M concede earlier in the finite horizon model, which decreases the expected total costs. Therefore, despite the fact that the expected duration of the game is always lower in the finite horizon model (see Lemma 11), the sign of the change in the expected social welfare might be ambiguous.

In this paper we argue that this sign is indeed ambiguous. It means that in some cases the introduction of the external constraint T_M on the duration of the "war" will increase the expected social welfare, but in other cases it will actually decrease it. In order to prove this argument we will consider two examples. In the first one the expected social welfare of the finite horizon model is always higher than in the model with the infinite horizon. In the second example, however, for some values of T_M the expected social welfare of the finite horizon model is lower than in the infinite horizon one.

Equivalence Theorem.

Example 1

Consider a uniform distribution $f(\theta)=\frac{1}{\Delta\theta}$ over $[\underline{\theta},\overline{\theta}]$. The symmetric equilibrium strategy for this case is given by (2.15). For $T_M \geq \frac{\Delta B}{2\overline{\theta}}$ we have $\underline{\theta}<\tilde{\theta}\leq\overline{\theta}$ (see Section 3) and, according to (3.3) and (4.3),

$$
\begin{aligned}
\frac{1}{\Delta B}ETC_{\infty}(\tilde{\theta}) &= \left[F(\tilde{\theta})\right]^2\left(\frac{\underline{\theta}+\tilde{\theta}}{\underline{\theta}}\ln\left(\frac{\tilde{\theta}}{\underline{\theta}F(\tilde{\theta})}\right)+\frac{1}{2}\right)+\frac{\underline{\theta}}{\Delta\theta}\left(F(\tilde{\theta})-\frac{\underline{\theta}}{\Delta\theta}\ln\frac{\tilde{\theta}}{\underline{\theta}}\right) \\
\frac{1}{\Delta B}ETC_{T_M}(\tilde{\theta}) &= \left[F(\tilde{\theta})\right]^2\left(\frac{\underline{\theta}+\tilde{\theta}}{\underline{\theta}}\ln\left(\frac{\tilde{\theta}}{\underline{\theta}F(\tilde{\theta})}\right)+\frac{1}{2}+\frac{\underline{\theta}}{2\tilde{\theta}}\right)
\end{aligned}
\tag{4.5}
$$

where $F(\tilde{\theta})=\frac{\tilde{\theta}-\underline{\theta}}{\Delta\theta}$.

Therefore, according to (4.5),

$$
ESW_{T_M}-ESW_{\infty}=\Delta B\frac{\underline{\theta}}{\Delta\theta^2}\left(\frac{\tilde{\theta}^2-\underline{\theta}^2}{2\tilde{\theta}}-\underline{\theta}\ln\frac{\tilde{\theta}}{\underline{\theta}}\right) \tag{4.6}
$$

First of all, notice that the change in the expected social welfare is proportional to $\underline{\theta}$. Therefore, if $\underline{\theta}$ =0, $ESW_{T_M}-ESW_{\infty}=0$ for $T_M \geq \frac{\Delta B}{2\overline{\theta}}$. It means that for $\underline{\theta}$=0 in the symmetric equilibrium of the finite horizon game the agents change their strategies in such a way that the expected social welfare remains unchanged in spite of the reduced expected duration of the "war" (see Figure 1a).

For the case $\underline{\theta}$>0, $ESW_{T_M}-ESW_{\infty}\to 0$ as $\tilde{\theta}\to\underline{\theta}$ (that corresponds to $T_M\to\infty$ according to Lemma 9) and $\frac{d}{d\tilde{\theta}}\left(ESW_{T_M}-ESW_{\infty}\right)=\Delta B\frac{\underline{\theta}}{2\Delta\theta^2}\left(1-\frac{\underline{\theta}}{\tilde{\theta}}\right)^2>0$ for any $\tilde{\theta}>\underline{\theta}$. Therefore, $ESW_{T_M}-ESW_{\infty}>0$ for any $T_M \geq \frac{\Delta B}{2\overline{\theta}}$ (see Figure 1b).

For $T_M < \frac{\Delta B}{2\overline{\theta}}$ agents of all the types will wait until T_M in the finite horizon model, that means $\tilde{\theta} = \overline{\theta}$. Therefore, we have

$$\begin{aligned} ESW_\infty &= b + B - \frac{\Delta B}{2} - \underline{\theta} T_\infty^E \\ ESW_{T_M} &= b + B - (\underline{\theta} + \overline{\theta}) T_M \end{aligned} \tag{4.7}$$

Thus, $ESW_{T_M} - ESW_\infty > 0$ for $T_M < \frac{\Delta B}{2\overline{\theta}}$ in both cases (see Figures 1a and 1b).

Summing up this example, for the case of a uniform distribution over $[\underline{\theta}, \overline{\theta}]$ the introduction of the external constraint on the duration of the "war" can only raise the expected social welfare, though in the case of $\underline{\theta}$=0 the expected social welfare increases only for low enough T_M (see Figures 1a and 1b).

Example 2

Our second example is a little bit tricky and the reasons for that are as follows. As can be seen from Example 1, a uniform distribution does not allow us to obtain the result that the expected social welfare can fall due to the introduction of the external constraint on the duration of the "war". For this to be true, we should "overload" the right-hand side of the distribution of agents' types that means to give more weight (in terms of probability) to the higher types than to the lower ones. But even for a uniform distribution the calculations are quite cumbersome, and they become much more cumbersome for an increasing density function (even in the case of "stairs" made from uniform distributions).

In order to avoid unnecessary calculations and to make the result more tractable we will use the following trick. Consider the following probability density function

$$
f(\theta) = \begin{cases} \dfrac{1}{a}\left(\pi - \dfrac{\varepsilon}{2}\right) & \text{for } \theta \in (\underline{\theta}, \underline{\theta} + a) \\ \dfrac{\varepsilon}{\Delta\theta - a - b} & \text{for } \theta \in (\underline{\theta} + a, \overline{\theta} - b) \\ \dfrac{1}{b}\left(1 - \pi - \dfrac{\varepsilon}{2}\right) & \text{for } \theta \in (\overline{\theta} - b, \overline{\theta}) \end{cases} \tag{4.8}
$$

where a, b, and ε are strictly positive and small enough and $0<\pi<1$ (see Figure 2). For this distribution, all the conditions of Sections 2 and 3 are satisfied. Thus, we can find the symmetric equilibrium strategies and calculate the expected social welfare for both infinite and finite horizon models. After that we can take a limit $\varepsilon \to 0$, $a \to 0$, and $b \to 0$. In that case the distribution (4.8) approaches to the degenerate distribution: $\theta = \underline{\theta}$ with probability π and $\theta = \overline{\theta}$ with probability $1-\pi$. Though the symmetric equilibrium strategy $T(\theta)$ does not, strictly speaking, have a limit when $(a,b,\varepsilon)\to 0$,[22] the expected social welfare is continuous in $(a,b,\varepsilon)>0$. Therefore, we can (after some simple transformations) change the order of the limit and the integral in (4.3) and take the limit within the integral before we take the integral itself. As will be seen below, it will reduce the calculations significantly without affecting the result much.

First of all, notice that most of the results of Sections 2 and 3 are valid for the case when $f(\theta)=0$ for some interval $[\theta',\theta'']\subset(\underline{\theta},\overline{\theta})$ (in that case the symmetric equilibrium strategy will be constant on $[\theta',\theta'']$ and, therefore, will not be strictly

[22] For such a degenerate distribution function, agents play mixed strategies in the symmetric perfect Bayesian equilibrium. Thus, Example 2 can also be considered as an example of the procedure for calculating such an equilibrium using the idea that any degenerate probability distribution can be considered as a limit of differentiable probability distributions with strictly positive densities.

decreasing). Thus, we can set $\varepsilon \to 0$ even before we calculate $T(\theta)$ without loss of generality.[23]

Consider $\theta \in [\bar{\theta} - b, \bar{\theta}]$. Then $f(\theta) = \frac{1-\pi}{b}$, $F(\theta) = \pi + (1-\pi)\left(1 - \frac{\bar{\theta} - \theta}{b}\right)$.

According to (2.12) for the symmetric equilibrium strategy of the infinite horizon model we have

$$T(\theta) = \frac{\Delta B}{\bar{\theta} - \frac{b}{1-\pi}}\left[-\ln F(\theta) + \ln\left(1 - \frac{b}{\bar{\theta}}F(\theta)\right)\right] \to -\frac{\Delta B}{\bar{\theta}}\ln F(\theta) \text{ as } b \to 0 \quad (4.9)$$

For $\theta \in (\underline{\theta}, \underline{\theta} + a]$ we have $f(\theta) = \frac{\pi}{a}$, $F(\theta) = \pi\frac{\theta - \underline{\theta}}{a}$ and, according to (2.12),

$$T(\theta) = \frac{\Delta B}{\underline{\theta}}\left[-\ln y(\theta) - \ln\frac{\underline{\theta} + a}{\underline{\theta} + ay(\theta)}\right] + T(\bar{\theta} - b) \to -\Delta B\left[\frac{\ln y(\theta)}{\underline{\theta}} + \frac{\ln \pi}{\bar{\theta}}\right] \quad (4.10)$$

where $y(\theta) = \frac{F(\theta)}{\pi} \in (0,1]$ and we set $a \to 0$ and $b \to 0$ in the limit. Thus, the symmetric equilibrium strategy of the infinite horizon game for small enough a and b has a form presented in Figure 3.

We also have

$$E[\theta' \mid \theta' \le \theta] = \underline{\theta} + \frac{a}{2}y(\theta) \to \underline{\theta} \qquad \text{for } \theta \in (\underline{\theta}, \underline{\theta} + a]$$

$$E[\theta' \mid \theta' \le \theta] = \frac{1}{F(\theta)}\left[\pi\left(\underline{\theta} + \frac{a}{2}\right) + (F(\theta) - \pi)\left(\bar{\theta} - \frac{b}{2}(1 - \pi + F(\theta))\right)\right] \to \bar{\theta} - \pi\frac{\Delta\theta}{F(\theta)} \quad (4.11)$$

$$\text{for } \theta \in [\bar{\theta} - \text{b}, \bar{\theta}]$$

Therefore, the integral in (4.3) is in fact taken over $F(\theta) \in (0, F(\tilde{\theta}))$ and the upper limit of the integral, $F(\tilde{\theta})$, is continuous in $(a,b)>0$. Thus, in order to

[23] It is possible to strictly follow the logic of the previous paragraph and set $\varepsilon \to 0$ only within the integral.

calculate the limit of the integral (4.3) we can use the following procedure: first, we take the limits $(a,b)\to 0$ within the integral; after that take the integral itself. As a result, we can obtain

$$ETC_{\infty}(\tilde{\theta}) \to \Delta B \pi^2 \tilde{y}^2 \left(-2\ln\tilde{y} + 1 - 2\frac{\underline{\theta}}{\overline{\theta}}\ln\pi \right) \tag{4.12}$$

for $\tilde{\theta} \in (\underline{\theta}, \underline{\theta}+a]$, where $\tilde{y} = \dfrac{\tilde{\theta}-\underline{\theta}}{a}$;

$$ETC_{\infty}(\tilde{\theta}) \to \Delta B \left(-2\tilde{z}\ln\tilde{z}\left(\tilde{z} - \pi\frac{\Delta\theta}{\overline{\theta}}\right) + \tilde{z}^2 - 2\pi(\tilde{z}-\pi)\frac{\Delta\theta}{\overline{\theta}} \right) \tag{4.13}$$

for $\tilde{\theta} \in [\overline{\theta}-b, \overline{\theta}]$, where $\tilde{z} = F(\tilde{\theta})$.

Now we can interpret our results in terms of the symmetric perfect Bayesian equilibrium of the infinite horizon war of attrition with a degenerate distribution of players' types: $\theta = \underline{\theta}$ with probability π and $\theta = \overline{\theta}$ with probability 1-π. In this equilibrium an agent with $\theta = \underline{\theta}$ plays a mixed strategy: 'Concede at $T^{(1)}(y) = -\Delta B\left[\dfrac{\ln y}{\underline{\theta}} + \dfrac{\ln\pi}{\overline{\theta}}\right]$ if the rival has not conceded before', where y is uniformly distributed over [0,1]. An agent with $\theta = \overline{\theta}$ plays a mixed strategy: 'Concede at $T^{(2)}(z) = -\dfrac{\Delta B}{\overline{\theta}}\ln z$ if the rival has not conceded before', where z is uniformly distributed over [π,1]. (4.11) presents the expected type of the rival if no one has conceded before a certain moment of time. If this time is less than $T^{(2)}(\pi)$, a player is still not sure about the type of his rival and forms expectations about it according to Bayesian rule. Once the time becomes more than $T^{(2)}(\pi)$, the player knows for sure that his rival has type $\underline{\theta}$. (4.12) and (4.13) present the expected total costs for the "war" between two agents on condition that they will not

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The result will be the same, but the calculations will be more cumbersome.

concede before $T^{(1)}(\tilde{y})$ and $T^{(2)}(\tilde{z})$ correspondingly. From (4.13) we can also obtain the expected social welfare of this game

$$ESW_{\infty} = b + B - \Delta B\left(1 - 2\pi(1-\pi)\frac{\Delta\theta}{\overline{\theta}}\right) \qquad (4.14)$$

Now let us consider the finite horizon version of this "war" and use the following procedure: first, we work with distribution (4.8) (with ε=0 without loss of generality); once the symmetric equilibrium strategy is substituted into the expected total costs (4.3), we take the limit $(a,b)\to 0$.

Let $T_M \le \frac{\Delta B}{2\overline{\theta}}$. Then agents of any type will wait until T_M, that means $\tilde{\theta} = \overline{\theta}$ and, according to (4.3),

$$ETC_{T_M} = 2\overline{\theta}T_M\left(1 - \pi\frac{\Delta\theta}{\overline{\theta}}\right) \qquad (4.15)$$

Comparing (4.14) and (4.15), we can see that for $\pi < \frac{1}{2}$, $ETC_{T_M} > ETC_{\infty}$ for $\tilde{T}_M^{\min} < T_M \le \frac{\Delta B}{2\overline{\theta}}$ (see Figure 4) and, therefore, the expected social welfare of the finite horizon game is lower than in the case of the infinite horizon. The largest difference in the expected social welfare, $ETC_{T_M}\left(T_M = \frac{\Delta B}{2\overline{\theta}}\right) - ETC_{\infty}$, has its maximum value $\frac{\Delta B}{8}\frac{\Delta\theta}{\overline{\theta}}$ for $\pi = \frac{1}{4}$.

Now consider $\frac{\Delta B}{2\overline{\theta}} < T_M \le T(\overline{\theta} - b) + \frac{\Delta B}{2(\overline{\theta} - b)}$, which corresponds to $\frac{\Delta B}{2\overline{\theta}} < T_M \le \frac{\Delta B}{\overline{\theta}}\left(\frac{1}{2} - \ln\pi\right)$ in the limit $b\to 0$. In that case all the agents of the lower types, $\theta \in [\underline{\theta}, \underline{\theta} + a]$, will wait until T_M, but some agents of the higher types, namely those with types higher than $\widetilde{\theta}$: $\overline{\theta} - b \le \widetilde{\theta} < \overline{\theta}$, will concede according to the infinite horizon strategy. In the limit $b \to 0$ for the degenerate distribution function it corresponds to agent $\overline{\theta}$ playing the mixed strategy: 'Wait until T_M' with probability $\frac{\widetilde{z} - \pi}{1 - \pi}$ or 'concede at $T^{(2)}(z)$ if the rival has not conceded before' with probability density function $(1 - \widetilde{z})z$, where z is uniformly distributed over $[\pi, 1]$ and $\widetilde{z} = \lim_{b \to 0} F(\widetilde{\theta}) = \exp\left(\frac{1}{2} - \frac{\overline{\theta} T_M}{\Delta B}\right) \in [\pi, 1]$. Then, according to (4.3) and (4.11), in the limit we have

$$ETC_{T_M}(\widetilde{\theta}) = \widetilde{z}\left(\widetilde{z} - \pi \frac{\Delta\theta}{\overline{\theta}}\right)(1 - 2\ln \widetilde{z}) \tag{4.16}$$

Therefore, from (4.13) and (4.16) we can obtain that in the limit $(a,b,\varepsilon) \to 0$

$$ESW_{T_M} - ESW_{\infty} = ETC_{\infty}(\widetilde{\theta}) - ETC_{T_M}(\widetilde{\theta}) = \Delta B \pi \frac{\Delta\theta}{\overline{\theta}}(2\pi - \widetilde{z}) \tag{4.17}$$

Form (4.17) we can see that for $\pi < \frac{1}{2}$, $ESW_{T_M} < ESW_{\infty}$ for $\frac{\Delta B}{2\overline{\theta}} < T_M < \widetilde{T}_M^{\max}$ (see Figure 4).

For

$$T_M \in \left[T(\overline{\theta} - b) + \frac{\Delta B}{2(\overline{\theta} - b)}, T(\overline{\theta} - b) + \frac{\Delta B}{2(\underline{\theta} + a)}\right] \to \left[\frac{\Delta B}{\overline{\theta}}\left(\frac{1}{2} - \ln \pi\right), \Delta B\left(\frac{1}{2\underline{\theta}} - \frac{\ln \pi}{\overline{\theta}}\right)\right]$$

all the lower type agents will wait until T_M and all the higher type agents will follow the infinite horizon strategy. Formally, we have $\widetilde{\theta} \in [\underline{\theta} + a, \overline{\theta} - b]$ and, therefore, $F(\widetilde{\theta}) = \pi$ and $E[\theta' \mid \theta' \le \widetilde{\theta}] = \underline{\theta} + \frac{a}{2} \to \underline{\theta}$. Thus, according to (4.3),

$$ETC_{T_M}(\widetilde{\theta}) = 2T_M \pi^2 \underline{\theta} \tag{4.18}$$

Therefore, the expected social welfare of the finite horizon game is higher than the expected social welfare of the infinite horizon one for that region of T_M (see Figure 4).

Now consider the last region for the external constraint: $T_M > T(\overline{\theta} - b) + \frac{\Delta B}{2(\underline{\theta} + a)}$ that corresponds to $T_M > \Delta B\left(\frac{1}{2\underline{\theta}} - \frac{\ln \pi}{\overline{\theta}}\right)$ in the limit. Now some of the higher type agents, namely those with type above $\widetilde{\theta}$: $\widetilde{\theta} \in (\underline{\theta}, \underline{\theta} + a)$, concede as if they were playing the infinite horizon game. Therefore, according to (4.9),

$$T_M = \Delta B\left[\frac{1}{\underline{\theta}}\left(\frac{1}{2} - \ln \widetilde{y}\right) - \frac{\ln \pi}{\overline{\theta}}\right] \tag{4.19}$$

From (4.19) and (4.11) we finally obtain

$$ETC_{T_M}(\widetilde{\theta}) = \Delta B \pi^2 \widetilde{y}^2 \left(-2\ln \widetilde{y} + 1 - 2\frac{\underline{\theta}}{\overline{\theta}} \ln \pi\right) \tag{4.20}$$

Thus, comparing (4.12) and (4.20) we can see that $ESW_{T_M} = ESW_\infty$ for this region of T_M (see Figure 4). This result is a consequence of $a \to 0$. In the case of a (nondegenerate) uniform distribution with $a>0$, according to Example 1, we have $ESW_{T_M} > ESW_\infty$ and the difference in the expected social welfare is proportional to a (from (4.6) for the case $a = \Delta\theta << \underline{\theta}$ we can obtain $ESW_{T_M} - ESW_\infty \approx \frac{\Delta B}{6} \widetilde{y}^3 \frac{a}{\underline{\theta}}$).

Summing up this example, for the case of a degenerate distribution (and for distribution (4.8) with a, b, and ε low enough) the introduction of the external constraint on the duration of the "war" can not only raise but also lower the expected social welfare for some small (around $\frac{\Delta B}{2\overline{\theta}}$) values of T_M (see Figure 4).

And this effect occurs when the probability share of the higher type agents exceeds that of the lower type agents, that is just in the case when it is more probable to face an agent with higher costs of the “war” than with low costs.

Discounting

The introduction of discounting into the model does not change the general logic of this section much. Now not only the total costs of the “war” but also the total surplus depend on the time of the end of the game. The earlier the “war” ends the more the total surplus is. Since the expected duration of the final horizon war of attrition is always lower than that of the infinite horizon game (see Section 3), the expected total surplus always rises after the external constraint on the duration of the “war” is introduced. However, the effect of this constraint on the expected total costs is still ambiguous. Moreover, since the cost of the “war” is discounted, the society gains less (in terms of present value) from the reduction of the total costs due to the introduction of the external constraint on the duration of the game. Therefore, the sign of the change in the expected social welfare might still be ambiguous.

Since players’ utilities are continuous in the discount factor, the sign of the change in the expected social welfare is still ambiguous for low enough discounting. Higher discounting, according to (3.8), lowers the symmetric equilibrium strategies of the players and, therefore, the “war” becomes effectively “shorter”. At the present state of our research we do not know for sure whether the extremely heavy discounting can eliminate the possibility of a welfare deteriorating constraint on the duration of the “war”, though this effect seems to us rather improbable.

5. Summary and extensions

In the incomplete information war of attrition, the players' uncertainty about the type of their rivals leads to delays in the end of the game. In this paper we showed that in the case when the symmetric equilibrium strategy of the infinite horizon war of attrition is nonlimited, the introduction of an expected change in the external circumstances to the model leads to the occurrence of the “dead zone” right before the moment of the change. In the “dead zone” the game ends with probability zero and, therefore, once no one has conceded before the beginning of this zone, both players know for sure that the game will continue until the date of the change in the external circumstances. The type of the change in the circumstances considered in this paper, namely the end of the game under a tie-breaking rule, is, of course, the simplest one. Nevertheless, the “dead zone” result can be generalized for other types of external changes. For example, for a country financing the fiscal deficit by borrowing abroad, as in Alesina and Drazen (1991), there might be the case that, as the debt to output ratio reaches a certain value, there is a positive probability for the foreign credit line to be closed. Another possible extension which might be relevant to the present situation in Russia is that a country with a full financing of the fiscal deficit by the inflation tax (as in Drazen and Grilli (1993)) might have a positive probability for a foreign loan on favorable terms to be granted at a certain moment of time. In all the cases, the fact that this change is expected by all the agents might lead to the “dead zone” occurrence. If we consider elections as an example of expected future changes in the external circumstances, the occurrence of the “dead zone” might be considered as a formal proof of the commonly observed fact that reforms are almost never implemented right before the elections. Our result suggests that if the reform has not been implemented before a certain moment of time, it will not be implemented until the time of the elections.

In this paper it is also shown that in the case when there is a strictly positive probability for an agent to face an opponent with the costs below any positive value, the fact that the “war” can not last forever due to some external constraint leads to the following unexpected result: for the model without discounting the relative size of the "dead zone" stays above some strictly positive level for whatever high value of the external constraint. This result, however, does not hold for the model with a strictly positive discount factor. Nevertheless, when the discount factor is small, the relative size of the “dead zone” will be significant even for high enough values of the constraint on the duration of the game[24].

We also showed in this paper that the introduction of the external constraint on the duration of the war of attrition could not only increase but also decrease the expected social welfare. Namely, the decrease in the expected social welfare might occur when the right-hand side of the distribution of agents’ types is “overloaded” (that means that the probability for an agent to have a cost of the “war” from the higher cost region is always more than the same probability for the lower cost region). A striking result is that the decrease in the expected social welfare occurs for relatively low rather than for relatively high values of the constraint on the duration of the game. It means that when a third party (for example, the government in the case of the “war” for a monopoly profit or an international monetary organization in the case of a country with a fiscal deficit) tries to interfere into the war of attrition with a generous goal to increase the social welfare, the effect of such an interference might be quite the opposite. Moreover, this perversity result might occur just in the case when it is much more probable to face an agent with high costs of the "war" than with low costs. Another possible application of our result might be the case of pre-term elections, which is relevant to the present Russian situation. Consider a country with a "weak" government, that means that the government is unable to undertake painful policy steps to introduce necessary reforms into the economy. A usually suggested "cure" in such a case is to hold pre-term elections as soon as possible in order to elect a sufficiently "strong" government. Our analysis suggests that such pre-term elections might, in fact, lower the expected social welfare and this effect might appear just in the case when most of the population suffers significantly from living in an unstabilized economy.

A possible extension of our model, which might be interesting to consider, is the model in which the properties of Theorem 1 do not hold and, consequently, equilibrium strategies can be nonmonotonous. For example, for the case $H(x,\theta,T) = x\theta - cT$,[25] all the properties of Theorem 1 hold and the symmetric equilibrium strategy is increasing in the type θ and nonlimited for the case of the infinite horizon war. However, the introduction of the discount into this model, $H(x,\theta,T) = x\theta e^{-rT} - \frac{c}{r}\left(1 - e^{-rT}\right)$, might change the agents' behavior significantly. Now a player with higher type θ not only gains a higher prize for winning the game but also has higher costs from waiting another instant to concede $-\frac{\partial H}{\partial T}(b,\theta,T) = (rb\theta + c)e^{-rT}$. Choosing his equilibrium strategy, a player should balance these two factors. In this case, the monotonicity property of equilibrium

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[24] If r is the discount factor, then the "scale" of the constraint on the duration of the "war" has an order of $1/r$.

[25] This is the classic case studied in most of the auctions and war of attrition literature (see, for example, Myerson (1981), Bishop, Cannings, and Maynard Smith (1978), and Bulow and Klemperer (1998)).

strategies might be lost[26]. If the equilibrium strategies in the infinite horizon model appear to be limited, the players will have a finite planning horizon and the expected future changes in the external circumstances at the time beyond this horizon might not influence the agents' equilibrium behavior.

Finally, it is worth noticing that the case when the time of the change in the external circumstances is known for sure by the agents is very extreme. In most of the real life situations, the moment of the external change T_M is not known for sure and economic agents form expectations about it. The random nature of T_M might become very important as the time horizon of the agents rises (it occurs either when T_M rises with unchanged costs distribution or when T_M does not change but the costs of the “war”, including a discount rate as a special case, rise). Thus, the war of attrition model with a random constraint on its duration might deserve certain consideration.

[26] It might appear if $rb > \Delta B$, that is if the difference between the winner's and the loser's payoffs is not very high or the discounting is very heavy.

Appendix

Proof of Lemma 1: Equilibrium requires that type θ_i' prefers $T_i' \equiv T_i(\theta_i')$ to $T_i'' \equiv T_i(\theta_i'')$ and that type θ_i'' prefers T_i'' to T_i'. Thus, according to (2.2), we have

$$\begin{aligned}&(b-\theta_i'T_i')\Pr[T_j(\theta_j)>T_i']+\left(\frac{b+B}{2}-\theta_i'T_i'\right)\Pr[T_j(\theta_j)=T_i'] \\ &+\int_{\{\theta_j|T_j(\theta_j)<T_i'\}}(B-\theta_i'T_j(\theta_j))f(\theta_j)d\theta_j \geq (b-\theta_i'T_i'')\Pr[T_j(\theta_j)>T_i''] \\ &+\left(\frac{b+B}{2}-\theta_i'T_i''\right)\Pr[T_j(\theta_j)=T_i'']+\int_{\{\theta_j|T_j(\theta_j)<T_i''\}}(B-\theta_i'T_j(\theta_j))f(\theta_j)d\theta_j\end{aligned} \tag{A1}$$

and

$$\begin{aligned}&(b-\theta_i''T_i'')\Pr[T_j(\theta_j)>T_i'']+\left(\frac{b+B}{2}-\theta_i''T_i''\right)\Pr[T_j(\theta_j)=T_i''] \\ &+\int_{\{\theta_j|T_j(\theta_j)<T_i''\}}(B-\theta_i''T_j(\theta_j))f(\theta_j)d\theta_j \geq (b-\theta_i''T_i')\Pr[T_j(\theta_j)>T_i'] \\ &+\left(\frac{b+B}{2}-\theta_i''T_i'\right)\Pr[T_j(\theta_j)=T_i']+\int_{\{\theta_j|T_j(\theta_j)<T_i'\}}(B-\theta_i''T_j(\theta_j))f(\theta_j)d\theta_j\end{aligned} \tag{A2}$$

Summing (A1) and (A2), we obtain

$$\begin{aligned}&(\theta_i''-\theta_i')\Big\{T_i'\Pr[T_j(\theta_j)\geq T_i']-T_i''\Pr[T_j(\theta_j)\geq T_i''] \\ &+\int_{\{\theta_j|T_j(\theta_j)<T_i'\}}-\int_{\{\theta_j|T_j(\theta_j)<T_i''\}}T_j(\theta_j)f(\theta_j)d\theta_j\Big\}\geq 0\end{aligned} \tag{A3}$$

Suppose that for $\theta_i''>\theta_i'$ we have $T_i''>T_i'$. Then (A3) takes the form

$$(\theta_i''-\theta_i')\left\{(T_i'-T_i'')\Pr[T_j(\theta_j)\geq T_i'']+\int_{\{\theta_j|T_i'\leq T_j(\theta_j)<T_i''\}}(T_i'-T_j(\theta_j))f(\theta_j)d\theta_j\right\}\leq 0 \tag{A4}$$

And (A4) equals to zero only if $\Pr[T_j(\theta_j)\geq T_i']=0$. In that case, player θ_i'' does not change his payoff by choosing any $T_i''\geq T_i'$, but he might increase his expected payoff by choosing other T_i''. For this case without loss of generality we can set $T_i''\leq T_i'$. Therefore, in equilibrium if $\theta_i''>\theta_i'$, $T_i(\theta_i'')\leq T_i(\theta_i')$. □.

Proof of Theorem 1: In the full analogy with deriving (A3) we can obtain

$$\begin{aligned}&\Delta H(b,T_i')\Pr[T_j(\theta_j)>T_i']-\Delta H(b,T_i'')\Pr[T_j(\theta_j)>T_i'']\\&+\Delta H(\bar{b},T_i')\Pr[T_j(\theta_j)=T_i']-\Delta H(\bar{b},T_i'')\Pr[T_j(\theta_j)=T_i'']\\&+\int_{\{\theta_j|T_j(\theta_j)<T_i'\}}-\int_{\{\theta_j|T_j(\theta_j)<T_i''\}}\Delta H(B,T_j(\theta_j))f(\theta_j)d\theta_j\geq 0\end{aligned}\tag{A5}$$

where $\Delta H(x,T)\equiv H(x,\theta_i',T)-H(x,\theta_i'',T)$.

Consider $\theta_i''>\theta_i'$. Then $\Delta H(x,T)=-\int_{\theta_i'}^{\theta_i''}\frac{\partial H}{\partial\theta_i}(x,\theta_i,T)d\theta_i$ is a nondecreasing function of *x* and *T* (we consider the first part of the theorem). Suppose $T_i''>T_i'$. Then (A5) takes the form

$$\begin{aligned}&\{\Delta H(b,T_i')-\Delta H(b,T_i'')\}\Pr[T_j(\theta_j)>T_i'']+\{\Delta H(b,T_i')-\Delta H(\bar{b},T_i'')\}\Pr[T_j(\theta_j)=T_i'']\\&+\Big\{\Delta H(b,T_i')\Pr[T_i'<T_j(\theta_j)<T_i'']+\Delta H(\bar{b},T_i')\Pr[T_j(\theta_j)=T_i']\\&-\int_{\{\theta_j|T_i'\leq T_j(\theta_j)<T_i''\}}\Delta H(B,T_j(\theta_j))f(\theta_j)d\theta_j\Big\}\leq 0\end{aligned}\tag{A6}$$

since all the differences in {} in (A6) are nonpositive. According to the theorem conditions, the second and the third differences are strictly negative. Therefore, (A6) equals to zero only if $\{\Delta H(b,T_i')-\Delta H(b,T_i'')\}\Pr[T_j(\theta_j)>T_i'']=0$ and $\Pr[T_i'\leq T_j(\theta_j)\leq T_i'']=0$. In that case, since player θ_i'' is indifferent among choosing any $T_i''\geq T_i'$, without loss of generality we can set $T_i''\leq T_i'$ (see the proof of Lemma 1). Therefore, in an equilibrium if $\theta_i''>\theta_i'$, $T_i(\theta_i'')\leq T_i(\theta_i')$.

The proof of the second part of the theorem is the same as above. □.

Proof of Lemma 1': According to Lemma 1, if equilibrium strategies were not strictly decreasing, there would be a mass point in the distribution of concession times. Let *T*>0 be a mass point of agent *j*'s equilibrium strategy, that is $\Pr[T_j(\theta_j)=T]\equiv P>0$. In this case, player *i* would assign probability 0 to the interval (T-ε ,T) as he does better playing just above T. Indeed, for $\delta > 0, \varepsilon > 0$ we have he following difference in expected payoffs for player i

$$U_i(T+\delta,\theta_i)-U_i(T-\varepsilon,\theta_i)=-\theta_i(\delta+\varepsilon)\Pr[T_j(\theta_j)\geq T+\delta]+\frac{\Delta B}{2}\Pr[T_j(\theta_j)=T+\delta]$$
$$+\theta_i(T-\varepsilon)\Pr[T-\varepsilon\leq T_j(\theta_j)<T+\delta]-b\Pr[T-\varepsilon<T_j(\theta_j)<T+\delta] \quad \text{(A}$$
$$-\frac{b+B}{2}\Pr[T_j(\theta_j)=T-\varepsilon]+\int_{\{\theta_j|T-\varepsilon\leq T_j(\theta_j)<T+\delta\}}(B-\theta_i T_j(\theta_j))f(\theta_j)d\theta_j$$

7)

Since a distribution can not have two infinitely close mass points, $\lim_{\delta\to 0}\Pr[T_j(\theta_j)=T+\delta]=\lim_{\varepsilon\to 0}\Pr[T_j(\theta_j)=T-\varepsilon]=0$. Therefore

$\lim_{\delta\to 0}U_i(T+\delta,\theta_i)-\lim_{\varepsilon\to 0}U_i(T-\varepsilon,\theta_i)=P\Delta B>0$ for any θ_i.

Therefore, there exists $\varepsilon > 0$ such that player i assigns probability 0 to the interval (T-ε ,T). Thus, the types of player j that play T would be better off playing T-ε , because it would not reduce the probability of winning and would lead to reduced cost. Therefore, T can not be a mass point of agent j's equilibrium strategy. □.

Proof of Theorem 1': As in Lemma 1' we can obtain for a mass point T>0 of agent j's equilibrium strategy and for $\delta > 0, \varepsilon > 0$:

$\lim_{\delta\to 0}U_i(T+\delta,\theta_i)-\lim_{\varepsilon\to 0}U_i(T-\varepsilon,\theta_i)=P(H(B,\theta_i,T)-H(b,\theta_i,T))>0$ for any θ_i,

since $H(x,\theta_i,T)$ is strictly increasing in x. The rest of the proof closely follows that of Lemma 1'. □.

Proof of Theorem 2: If agent j's equilibrium strategy were discontinuous, then there would be $T'\geq 0$ and $T''>T'$ such that $\Pr[T'<T_j(\theta_j)<T'']=0$ while $T_j(\tilde{\theta}_j)=T''+\varepsilon$ for some small $\varepsilon\geq 0$ for some $\tilde{\theta}_j$. In this case, player i strictly

prefers $T_i = T'$ to any $T_i \in (T', T'')$, as the probability of winning is the same and the expected cost is reduced. But then the quitting "at or just beyond" T'' is not optimal for player *j* with type $\widetilde{\theta}_j$. □.

Proof of Lemma 2: Consider agent *i* with the time of concession T_i chosen at the beginning of the game according to (2.6). At any moment during the game either one of the players has conceded and the game is over or the "war" is still going on. Consider $\widetilde{T} < T_i$ and suppose that no one has conceded yet. Player *i* at time $\widetilde{T}$ updates his beliefs about the time of concession of his rival according to the Bayesian rule: $g_j\left(T_i | \widetilde{T}\right) = \dfrac{g_j(T_i)}{1 - G_j\left(\widetilde{T}\right)}$ for $T_i \geq \widetilde{T}$. Thus, his expected payoff from time $\widetilde{T}$ on from choosing T_i is

$$U_i\left(T_i | \widetilde{T}\right) = \int_{\widetilde{T}}^{T_i} (B - \theta_i t) \frac{g_j(t)}{1 - G_j\left(\widetilde{T}\right)} dt + \frac{\left[1 - G_j(T_i)\right]}{1 - G_j\left(\widetilde{T}\right)} (b - \theta_i T_i) \qquad \text{(A8)}$$

which leads to (2.6) after maximizing with respect to T_i. □.

Proof of Lemma 3: According to Lemma 1', the expected utility of the player θ in a symmetric equilibrium is

$$U(\theta) = b + \Delta B \Pr[\theta' > \theta] - \theta E\left[T\left(\max\{\theta, \theta'\}\right)\right] \qquad \text{(A9)}$$

where $T(.)$ is a symmetric equilibrium strategy and expectations $E[.]$ are taken over θ'.

Now notice that since in equilibrium no type of agent can gain by following any other type's concession rule,

$$U(\theta^a) \geq U(\theta^b) + (\theta^b - \theta^a) E\left[T\left(\max\{\theta^b, \theta'\}\right)\right] \qquad \text{(A10)}$$

So $U(\theta)$ is differentiable

$$\frac{dU}{d\theta} = -E[T(\max\{\theta,\theta'\})] = -\left(T(\theta)F(\theta) + \int_{\theta}^{\bar{\theta}} T(\theta') f(\theta') d\theta' \right) \qquad \text{(A11)}$$

On the other hand, from (A9),

$$U(\theta) = b + \Delta B[1 - F(\theta)] - \theta \int_{\theta}^{\bar{\theta}} T(\theta') f(\theta') d\theta' - \theta T(\theta) F(\theta) \qquad \text{(A12)}$$

Since all the terms of (A12) except the last one are differentiable and $F(\theta)$ is also differentiable, $T(\theta)$ must be differentiable too. □.

Proof of Lemma 4: Consider a multiplicative shift in θ, that is $\tilde{\theta} = \theta + \delta$: $\tilde{F}(\tilde{\theta}) = F(\theta); \delta > 0$. From (2.12) we have

$$\tilde{T}(\tilde{\theta}) = \Delta B \int_{\tilde{\theta}}^{\tilde{\bar{\theta}}} \frac{f(\tilde{x})}{F(\tilde{x})} \frac{d\tilde{x}}{\tilde{x}} = \Delta B \int_{\theta}^{\bar{\theta}} \frac{f(x)}{F(x)} \frac{dx}{x+\delta} < T(\theta) \qquad \text{(A13)}$$

Since $\tilde{F}(\tilde{\theta}) \tilde{f}(\tilde{\theta}) d\tilde{\theta} = F(\theta) f(\theta) d\theta$, $\tilde{T}^E < T^E$.

Consider an increase in $\Delta\theta$ for an unchanged $\underline{\theta}$, that is $\tilde{\theta} = \underline{\theta} + \lambda(\theta - \underline{\theta})$: $\tilde{F}(\tilde{\theta}) = F(\theta); \lambda > 1$ and, therefore, $\tilde{f}(\tilde{\theta}) d\tilde{\theta} = f(\theta) d\theta$. From (2.12) we have

$$\tilde{T}(\tilde{\theta}) = \Delta B \int_{\tilde{\theta}}^{\tilde{\bar{\theta}}} \frac{f(\tilde{x})}{F(\tilde{x})} \frac{d\tilde{x}}{\tilde{x}} = \Delta B \int_{\theta}^{\bar{\theta}} \frac{f(x)}{F(x)} \frac{dx}{\underline{\theta} + \lambda(x - \underline{\theta})} < T(\theta) \qquad \text{(A14)}$$

Again, since $\tilde{F}(\tilde{\theta}) \tilde{f}(\tilde{\theta}) d\tilde{\theta} = F(\theta) f(\theta) d\theta$, $\tilde{T}^E < T^E$.

Any increase in the costs of the "war" with unchanging distribution function can be represented as a successive combination of the increases considered above. Therefore, the result of Lemma 4 holds. □.

Proof of Lemma 5: As $\theta \to \underline{\theta}$, $F(\theta) \approx f(\underline{\theta})(\theta - \underline{\theta})$. Therefore, from (2.12), for $\theta \to \underline{\theta}$ $T(\theta) \approx \frac{\Delta B}{\underline{\theta}} \ln\left[\frac{1}{\theta - \underline{\theta}}\right]$ if $\underline{\theta} > 0$ and $T(\theta) \approx \frac{\Delta B}{\theta}$ if $\underline{\theta} = 0$. □.

Proof of Lemma 6: Since $T(\overline{\theta}) = 0$, if $T(\theta)$ is continuous there must exist $\tilde{\theta} \in [\underline{\theta}, \overline{\theta}) : T(\tilde{\theta}) = T_M$ and $T(\theta) < T_M$ for $\theta \in (\tilde{\theta}, \overline{\theta}]$. Consider $\theta = \tilde{\theta} + \varepsilon, \varepsilon > 0$. Then $T(\theta) = T_M - \delta$, where $\delta \to 0$ for $\varepsilon \to 0$. If no one has conceded before $T(\theta)$, the player θ concedes at $T(\theta)$ and obtains

$$U(T(\theta)|T(\theta)) = b - \theta(T_M - \delta) \tag{A15}$$

However, if the player waits until T_M, his expected payoff will be

$$U(T_M|T(\theta)) \geq \frac{b+B}{2} - \theta T_M \tag{A16}$$

Since $\delta \to 0$ for $\varepsilon \to 0$, $U(T_M|T(\theta)) > U(T(\theta)|T(\theta))$ and playing $T(\theta)$ is not optimal for the player θ. □.

Proof of Lemma 7: The proof of this lemma closely resembles the proof of Lemma1'. Let $T < T_M$ be a mass point of one of the agents' equilibrium strategy. In this case, the other agent will never play *T* since

$\lim_{\delta \to 0} U_i(T+\delta, \theta_i) - U_i(T, \theta_i) = \Pr[T_j(\theta_j) = T]\frac{\Delta B}{2} > 0$ for any θ_i and agent *i* will be better off setting his concession time just above *T*. Therefore, *T* cannot be a mss point of a symmetric equilibrium strategy. □.

Proof of Lemma 8: The proof is almost the same as the proof of Theorem 2.

Proof of Lemma 9: From (3.3) we have

$$T\left(\tilde{\theta}\right)+\frac{\Delta B}{2\tilde{\theta}}=T_M \qquad \text{(A17)}$$

Since the left-hand side of (A17) is a decreasing function of $\tilde{\theta}$, $\tilde{\theta}(T_M)$ is decreasing in T_M. From Lemma 5, $T\left(\tilde{\theta}\right)\to +\infty$ as $\tilde{\theta}\to \underline{\theta}$. Since $\frac{\Delta B}{2\tilde{\theta}}<\infty$ for $\tilde{\theta}>\underline{\theta}$, $\tilde{\theta}\to\underline{\theta}$ as $T_M \to +\infty$. □.

Proof of Lemma 10: Form (3.3) we have

$$\frac{T_M-\tilde{T}}{T_M}=\frac{\Delta B}{2\tilde{\theta}T_M}=\frac{\Delta B}{2\tilde{\theta}T(\tilde{\theta})+\Delta B} \qquad \text{(A18)}$$

From (2.12) we have

$$\frac{d}{d\theta}\left(\theta T(\theta)\right)=\Delta B\left[\int_{\theta}^{\bar{\theta}}\frac{f(x)}{F(x)}\frac{dx}{x}-\frac{f(\theta)}{F(\theta)}\right] \qquad \text{(A19)}$$

Since $f\left(\underline{\theta}\right)>0$, $\frac{f(\theta)}{F(\theta)}\to\frac{1}{\theta-\underline{\theta}}$ as $\theta\to\underline{\theta}$. Thus, $\frac{d}{d\theta}\left(\theta T(\theta)\right)<0$ for both $\underline{\theta}=0$ and $\underline{\theta}>0$. Therefore, (A18) is decreasing in T_M for $T_M \to +\infty$.

If $\underline{\theta}>0$, then, according to Lemma 5, $\tilde{\theta}T\left(\tilde{\theta}\right)\to +\infty$ for $\tilde{\theta}\to\underline{\theta}$ and $\frac{T_M-\tilde{T}}{T_M}\to 0$ when $T_M \to +\infty$. If $\underline{\theta}=0$, then $T(\theta)\approx\frac{\Delta B}{\theta}$ for $\theta\to\underline{\theta}$ (see the proof of Lemma 5). Therefore, $\tilde{\theta}T\left(\tilde{\theta}\right)\to\Delta B$ for $\tilde{\theta}\to\underline{\theta}$ and, from (A18), $\frac{T_M-\tilde{T}}{T_M}\to\frac{1}{3}$ when $T_M \to +\infty$. □.

Proof of Lemma 11: According to (3.1), $T^E=T_M$ for $T_M\le\frac{\Delta B}{2\bar{\theta}}$, which is obviously increasing in T_M.

For $T_M > \frac{\Delta B}{2\bar{\theta}}$, according to (3.5),

$$\frac{dT^E}{dT_M} = \left[F\left(\tilde{\theta}\right)\right]^2 + 2\left(T_M - T\left(\tilde{\theta}\right)\right)F\left(\tilde{\theta}\right)f\left(\tilde{\theta}\right)\frac{d\tilde{\theta}}{dT_M} \tag{A20}$$

From (3.3) we can obtain

$$T_M - T\left(\tilde{\theta}\right) = \frac{\Delta B}{2\tilde{\theta}} \tag{A21}$$

and from here, using (2.12),

$$-\Delta B\left[\frac{f\left(\tilde{\theta}\right)}{F\left(\tilde{\theta}\right)} + \frac{1}{2\tilde{\theta}}\right]\frac{d\tilde{\theta}}{dT_M} = \tilde{\theta} \tag{A22}$$

Substituting (A21) and (A22) into (A20), we finally obtain

$$\frac{dT^E}{dT_M} = \frac{\left[F\left(\tilde{\theta}\right)\right]^3}{F\left(\tilde{\theta}\right) + 2\tilde{\theta}f\left(\tilde{\theta}\right)} \tag{A23}$$

Thus, $\frac{dT^E}{dT_M} > 0$ for $T_M > \frac{\Delta B}{2\bar{\theta}}$. (Remember that we assume $f(\theta) > 0$) for $\theta \in (\underline{\theta}, \bar{\theta})$ throughout the paper. Thus, $\tilde{\theta} > \underline{\theta}$ for any $T_M < \infty$.) Therefore, T^E is strictly increasing in T_M for any $T_M \geq 0$.

As $T_M \to +\infty$, $\tilde{\theta} \to \underline{\theta}$ (see Lemma 9) and, therefore, we have

$T\left(\tilde{\theta}\right) \approx \frac{\Delta B}{\underline{\theta}} \ln\left[\frac{1}{\tilde{\theta} - \underline{\theta}}\right]$ for $\underline{\theta} > 0$, $T\left(\tilde{\theta}\right) \approx \frac{\Delta B}{\tilde{\theta}}$ for $\underline{\theta} = 0$ (see the proof of Lemma 5),

and $F\left(\tilde{\theta}\right) \approx f(\underline{\theta})\left(\tilde{\theta} - \underline{\theta}\right)$.

From (A17) we have

$$T_M\left[F\left(\tilde{\theta}\right)\right]^2 = \left(T\left(\tilde{\theta}\right) + \frac{\Delta B}{2\tilde{\theta}}\right)\left[F\left(\tilde{\theta}\right)\right]^2 \approx \begin{cases} \frac{3}{2}\Delta B\left(f(\underline{\theta})\right)^2\tilde{\theta} & \text{for } \underline{\theta} = 0 \\ \frac{\Delta B}{\underline{\theta}}\ln\left[\frac{1}{\tilde{\theta} - \underline{\theta}}\right]\left[f(\underline{\theta})\left(\tilde{\theta} - \underline{\theta}\right)\right]^2 & \text{for } \underline{\theta} > 0 \end{cases} \tag{A24}$$

Therefore, $T_M\left[F\left(\tilde{\theta}\right)\right]^2 \to 0$ as $\tilde{\theta} \to \underline{\theta}$ in both cases. □.

Proof of Lemma 12: Denote $\overline{T}_i(\theta_i) = E_{-i}\left[T(\theta_i,\theta_{-i})\right]$ and $\overline{P}_i(\theta_i) = E_{-i}\left[P(\theta_i,\theta_{-i})\right]$, which are the expected time of the end of the "war" and the expected probability to win the "war" for player *i* with type θ_i. Then the expected utility of player *i* is

$$U_i(\theta_i) = b + \Delta B\overline{P}_i(\theta_i) - \theta_i\overline{T}_i(\theta_i) \tag{A25}$$

As in proof of Lemma 3, it can be easily shown that

(i) $\overline{T}_i(\theta_i)$ is nonincreasing in θ_i and

(ii) $$U_i(\theta_i) = U_i(\overline{\theta}_i) + \int_{\theta_i}^{\overline{\theta}_i} \overline{T}_i(x)dx \tag{A26}$$

Then from (A25), (A26), and the fact that $P_1(\theta_1,\theta_2) + P_2(\theta_1,\theta_2) = 1$ for any (θ_1,θ_2), it can be easily obtained that

$$ESW = \sum_{i=1}^{2} U_i(\overline{\theta}_i) + \int_{\underline{\theta}_1}^{\overline{\theta}_1}\int_{\underline{\theta}_2}^{\overline{\theta}_2} T(\theta_1,\theta_2)\left[F_1(\theta_1)f_2(\theta_2) + F_2(\theta_2)f_1(\theta_1)\right]d\theta_1 d\theta_2 \tag{A27}$$

□.

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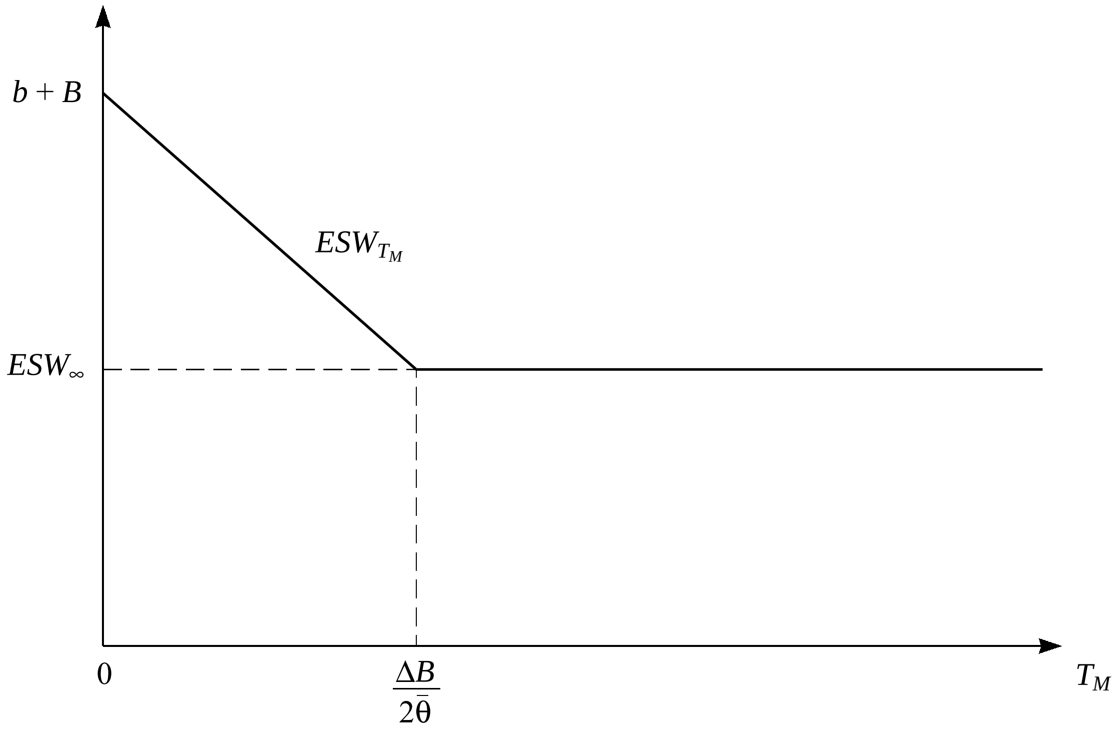

Figure 1a. The expected social welfare of the finite horizon war of attrition with uniform over $[0, \bar{\theta}]$ distribution of players' types.

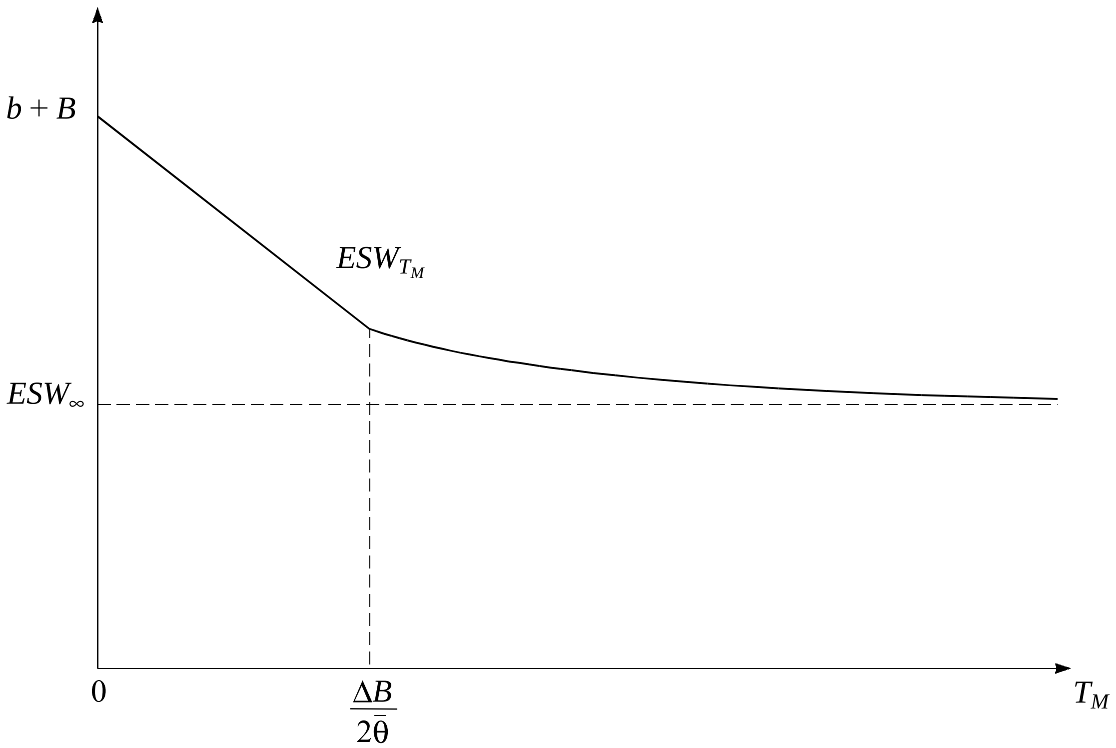

Figure 1b. The expected social welfare of the finite horizon war of attrition with uniform over $[\underline{\theta}, \bar{\theta}]$ ($\underline{\theta} > 0$) distribution of players' types.

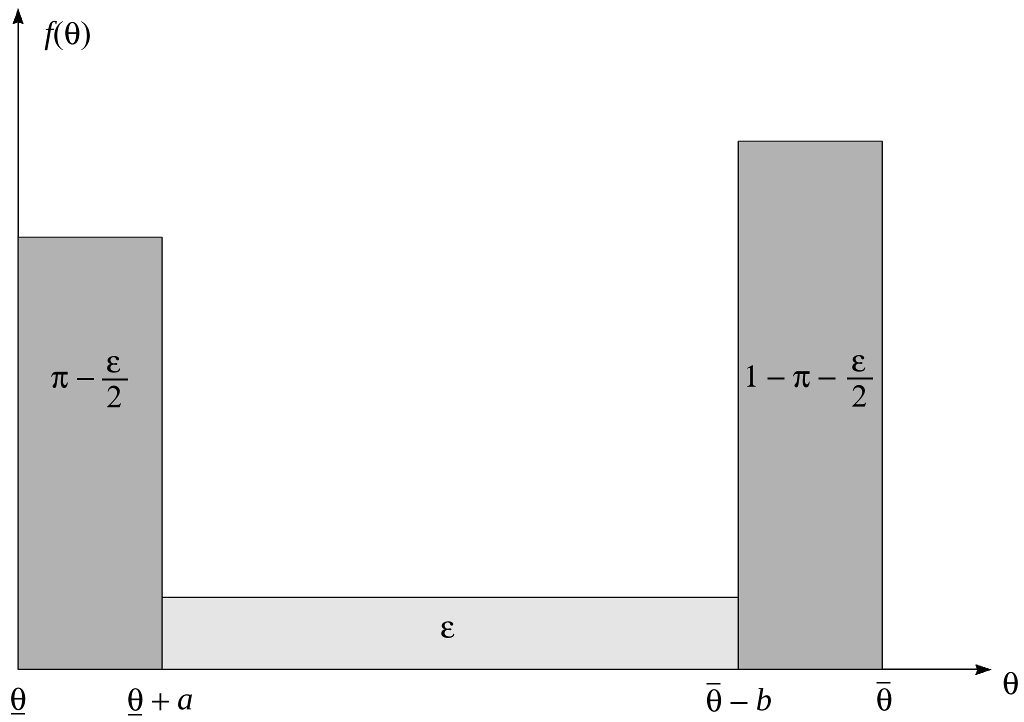

Figure 2. Players' types density function for Example 2.

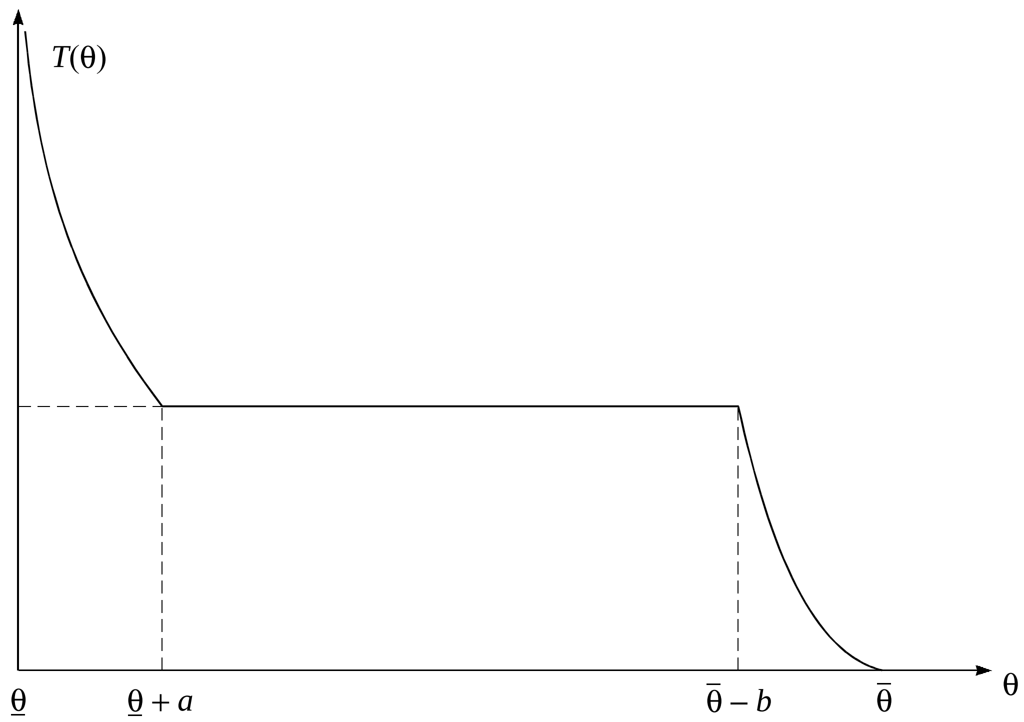

Figure 3. The symmetric perfect Bayesian equilibrium strategy for the density function in Figure 2 with ε=0.

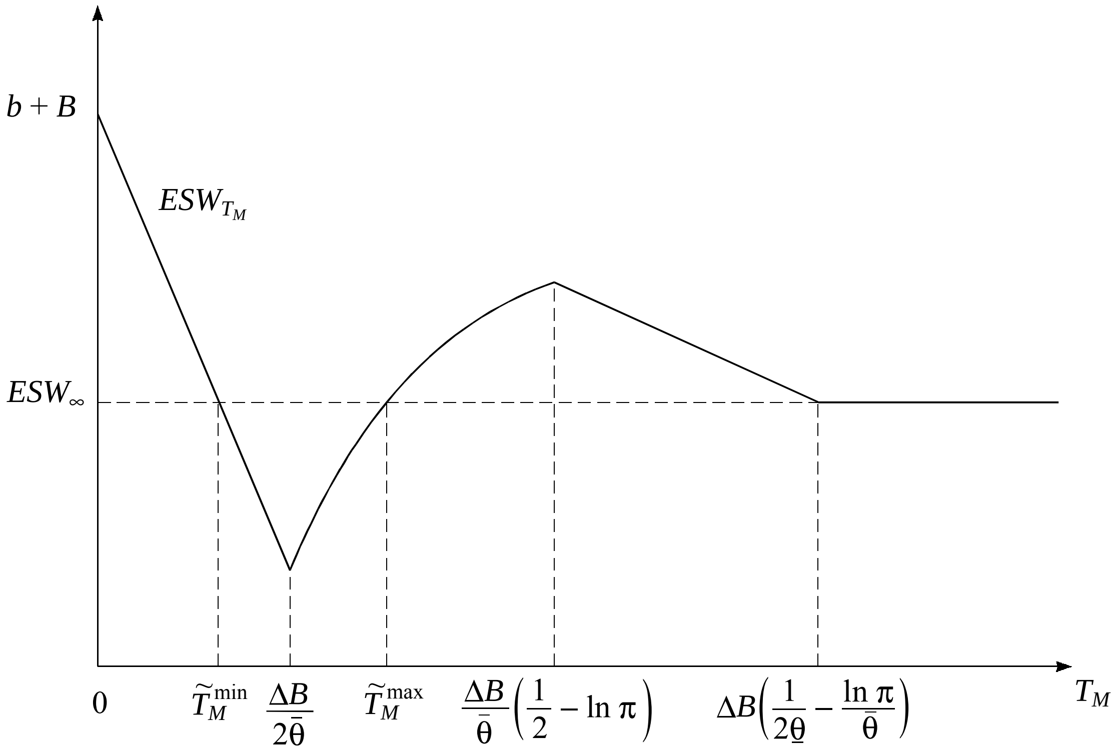

Figure 4. The expected social welfare of the finite horizon war of attrition with a degenerate distribution function of players' types: $\theta = \underline{\theta}$ with probability π and $\theta = \bar{\theta}$ with probability $1-\pi$, $\pi<1/2$.