Optimal Investment in Development of New Technology (the exponential distribution case)

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Abstract

In this paper a one-output growth model with endogenous transition to a new technology is considered. This model in general form is presented in [5], where the authors carefully considered the case of a uniform distribution governing the random transition to a new technology. This master thesis follows the main article [5], and our problem is to investigate the case when the probability of transition to a new technology is governed by an exponential distribution.

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1 Introduction

Qualitative analysis of trajectories is one of the most important directions for investigation of dynamic economic models. This approach involves examining trajectory qualities only with the help of the structural qualities of the model, without direct knowledge of a given trajectory. It allows us to make predictions and assessments about the future growth rate and structural breaks in economies. Qualitative analysis also improves our understanding about the nature of technological choice and growth in an economy.

The theoretical basis for our analysis is represented by "turnpike" qualities of dynamic economic models, which state that for almost all planning periods optimal trajectories are close to some very simple trajectory (e.g. trajectories of maximal stationary growth in von Neumann's models or a golden ray in models involving consumption) and this path does not depend on the planning horizon, nor the initial conditions, nor the concrete requirements of the optimality criterion, and so on.

It is reasonable to investigate the question of whether analogous characteristics are valid for dynamic economic models which consider transition to a new technology. Actually we are interested only in the processes of transition, i.e. in optimal trajectories from the beginning until the moment of transition to the new technology (since after transition trajectory behaviour becomes trivial), and in the corresponding investment in the development of the new technology.

In our work a one-output growth model with endogenous transition to a new technology is considered. This model in general form is presented in [5], where the authors carefully considered the case of a uniform distribution governing the random transition to a new technology (see below section 2.1). My master thesis will follow the main article [5], and our problem is to investigate the case when the probability of transition to a new technology is governed by an exponential distribution.

2 Construction of model

2.1 Specification of model

Let us consider the following linear model in discrete time. The "state" is described by a nonnegative number x, which evolves with rate α , i.e., during a unit time period x is transformed into αx. The state x can be interpreted as the current monetary value of a multiproduct economic system, and linear development can be explained by the "turn pike" quality, which states that the optimal trajectories of economic models with large planning horizons are close to some ray (i.e., grow proportionally at the von Neumann rate). Thus, the technological opportunities of the system in our model are described by the rate α.

There is a traditional technology with rate $\alpha_0 \geq 1$. Further, it is known that there is an opportunity to transform to a new (more advanced) technology with growth rate $\alpha_1 > \alpha_0$. (We will denote these technologies as α_0 and α_1 correspondingly.) At the beginning, α_0 is the operational technology and for the transition of the system to α_1 some investment is needed. The investments made each period accumulate to create a fund of project development (FPD) for the new technology. The dynamics of this fund is described by the following procedure: the current investment c transforms the state of the fund $z \in R$ to the new level $z' := z + c$ for one period.

Let ξ characterize the minimal level of the fund which is needed to create the new technology. Let us also suggest that, from the point of view of the planner, ξ is random with known distribution function $\pi(z) = \Pr(\xi < z), z \geq 0$. The time of transition to the new technology is considered to be the first moment at which the FPD is greater than ξ.

Thus, the evolution of the whole system in discrete time can be described in the following way. The state of the system is given by a pair $(x, z) \in R^2_+$. If the decision to invest amount c $(0 \leq c \leq x)$ is taken in the state (x, z), then in the next moment the state of the system will be (x', z'), with

$$x' = \alpha_0(x - c), \quad z' = z + c. \tag{1}$$

Transition to the new technology during the current period happens with probability

$$\beta := \frac{\pi(z') - \pi(z)}{1 - \pi(z)}. \tag{2}$$

If transition does occur, then after that the system develops at the rate α_1, having x' as an initial value.(After transition investment ceases: we consider only a single transition.) If the system remains in α_0 (with probability $1 - \beta$) then the process repeats with the new state given by x' and z'.

2.2 Stationary model, fundamental equation

As shown in [5], a stationary dynamic approach can be used for describing our model. The Bellman function of this approach is the solution to the following functional equation

$$F(x, z) = \mu \max_{0 \le c \le x} [(1 - \beta)F(x', z') + \beta x'], \ \ (x, z) \in R^2_+ \tag{3}$$

where $\mu := \alpha_1^{-1}$, (x', z') and β can be derived from (1) and (2).

It was proved that this equation fully describes the asymptotic behaviour of the solutions to the problem of maximizing the average state of the system at the end of the planning interval: $\mathbf{E}x_T \to \max$, when planning horizons are large and equal T. In particular, the optimal choice of investment in new technology (i.e., optimal investment as a function of the current state of the system) is close to investment $c^* = C(x, z)$, which maximizes the right side of equation (3).

Now let's make some assumptions about the function π and the level ξ, which sets the time of transition to the new technology. The case when π is a uniform distribution is carefully considered in [5]. We will consider the case when π is an exponential distribution, i.e., $\pi(z) = 1 - e^{-\Theta z}$, $z \ge 0$. This represents a situation when the probability of transition to the new technology depends only on current investment and does not depend on accumulated funds, i.e., rewriting (2), $\beta = \beta(c) = 1 - e^{-\Theta c}$.

Then the functional F also does not depend on z, and equation (3) can be written as

$$F(x) = \mu \max_{0 \le c \le x} \left[(1-\beta)F(\alpha_0(x-c)) + \beta\alpha_0(x-c) \right], \;\; x \ge 0. \tag{4}$$

Let's denote the solution to (4) as Φ.

Lemma 1: The function Φ possesses the following qualities:

a) $\Phi(0) = 0$

b) $\Phi(x)$ is nondecreasing in x

c) Φ is convex, i.e. $\Phi'(x)$ is nondecreasing in x

d) $\Phi(x) \le \lambda x$ when $x \ge 0$, where $\lambda := \mu\alpha_0 < 1$.

The proof of Lemma 1 is given in [5].

The function Φ depends on three parameters: α_0, α_1 and λ. We can diminish the number of parameters by one. Let $\varphi(x) := \Theta\alpha_0^{-1}\Phi(x/\Theta)$; the function φ possesses the qualities a, b and c of Lemma 1, and from qualities c and d it follows that there is a limit γ, where

$$\gamma := \lim_{x\to\infty} \frac{\varphi(x)}{x} = \lim_{x\to\infty} \varphi'(x) \le \mu. \tag{5}$$

After changing the maximization variable to $c := c/\Theta$, equation (4) transforms to

$$\varphi(x) = \mu \max_{0 \le c \le x} \left[(1-\beta)\varphi(\alpha_0(x-c)) + \beta(x-c) \right], \;\; x \ge 0 \tag{6}$$

where now $\beta = 1 - e^{-c}$.

With the help of (d) from Lemma 1 we can assess $\varphi \le \mu x$. In equation (6) the variables $x, c, \alpha_0, \beta, \alpha_1$ and φ are all unit free. It is clear that φ is not identically zero (we can substitute $c := x/2$ in (6)). Substituting also $v := x - c$, $0 \le c \le x$, $\beta = 1 - e^{-(x-v)}$ in (6) we have

$$\varphi(x) = \mu \max_{0 \le v \le x} \left(v - e^{-x} f(v) \right), \;\; f(v) := e^v(v - \varphi(\alpha_0 v)) \tag{7}$$

and we will refer to this expression as the fundamental equation.

Since the optimal choice $v = V(x) \ne 0$ for any $x > 0$, then all semi-axes R_+ are divided into two regions:

1) Region RE, in which

$$V(x) = x \Longrightarrow \varphi(x) = \mu\varphi(\alpha_0 x); \tag{8}$$

2) Region RI, where $0 < V(x) < x$, and then $v = V(x)$ is an interior optimum, which can be derived from the condition

$$\frac{d}{dv}(v - e^{-x}f(v)) = 1 - e^{-x}f'(v) = 0$$

i.e.

$$f'(v) = e^{v}(v - \varphi(\alpha_0 v) + 1 - \alpha_0\varphi'(\alpha_0 v)) \mid_{v=V(x)} = e^{x}. \tag{9}$$

Lemma 2: There is x' such that $V(x) < x$ when $x > x'$. *Proof:* If the lemma is not true, then there exists a sequence $x_n \to \infty$, such that $V(x_n) = x_n$, i.e., $\varphi(x_n) = \mu\varphi(\alpha_0 x_n)$. Then, from (5) we get

$$\gamma = \lim_{n\to\infty}\frac{\varphi(x_n)}{x_n} = \lambda \lim_{n\to\infty}\frac{\varphi(\alpha_0 x_n)}{\alpha_0 x_n} = \lambda\gamma,$$

which is impossible. □

Lemma 3: Optimal choice $v = V(x)$ is an increasing function.

Proof: In region RE it is clear. In region RI by differentiating (2.7) by x, we can find (with the help of (2.9))

$$\varphi'(x) = \mu e^{-x}f(v) \mid_{v=V(x)} . \tag{10}$$

By using (2.7), we get

$$\varphi(x) + \varphi'(x) = \mu V(x) \tag{11}$$

Since $\varphi(x)$ and $\varphi'(x)$ are growing in x (φ is convex), the Lemma is proved. □

Function $V(x)$ determines the *transitional mapping* $x \to Y(x) := \alpha_0 V(x)$, which gives birth to the trajectory $x_{t+1} = Y(x_t), t = 0, 1, ...$

Because of Lemma 2, the transitional mapping $y = Y(x)$ is an increasing function. Hence, trajectories are monotone: if $x_1 > x_0$, then the trajectory is growing, otherwise it is decreasing.

The construction and investigation of the functions $V(x)$ and $\varphi(x)$, and also trajectory behaviour analysis, which depends on the parameters α_0 and α_1, are the main problems considered here.

Note 1.

As in [5], we will assume that the volume of investment $C(x) = x - V(x)$ is nondecreasing in x; this assumption is satisfied by all the examples for which calculations on a computer were made. From this fact it follows that the region RE, where $C(x) \equiv 0$ (*the state of noninvestment*), is represented by the segment $[0, x^0]$, and possibly $x^0 = 0$. Thus,

$$RE := \{x \mid V(x) = x\} = [0, x^0], RI := \{x \mid V(x) < x\} = (x^0, \infty), \tag{12}$$

and also from Lemma 2 we get $x^0 < \infty$. □

3 Detailed analysis

3.1 Stationary points

Point x^* is stationary if $Y(x^*) = x^*$. For finding such points we will use the "vekovoe" equation from V.Z.Belen'kii's article [6]. In our case, for a stationary point x the following equations are valid: $v = x/\alpha_0$, $\varphi(x) = \psi(x)$ and $\varphi'(x) = \psi'(x)$, where

$$\psi(x) := \frac{x(1 - e^{-\sigma x})}{\alpha_0(\alpha_1 - e^{-\sigma x})}, \quad \sigma := 1 - 1/\alpha_0, \tag{1}$$

and then equation (2.9) can be transformed to

$$1 - \frac{(\alpha_1 - 1)(\sigma x + \alpha_0 - 1)}{(\alpha_0 - 1)(\alpha_1 e^{\sigma x} - 1)} + \frac{(\alpha_1 - 1)\sigma x}{(\alpha_1 e^{\sigma x} - 1)^2} = 0. \tag{2}$$

Let's denote $s := \sigma x$, $\delta := \alpha_1 - 1$ and $\varepsilon := \alpha_0 - 1$; therefore $0 < \varepsilon < \delta$ and (2) can be rewritten as

$$1 - \frac{\delta(s + \varepsilon)}{\varepsilon((\delta + 1)e^s - 1)} + \frac{\delta s}{((\delta + 1)e^s - 1)^2} = 0, \tag{3}$$

Thus, a stationary point can be found using the formula $x^* = s/\sigma$, where s is a solution of the "vekovoe" equation (3).

3.1.1 Investigation of "vekovoe" equation

Introducing the variable $y = e^s - 1$, transform equation (3) to

$$f(y) := ky + \frac{by}{y + q} - \ln(1 + y) = 0, \quad y \geq 0 \tag{4}$$

$$q := 1 - \lambda, \;\; k := \frac{\varepsilon(1+\varepsilon)}{q+\varepsilon}, \;\; b := \frac{\lambda\varepsilon^2}{q+\varepsilon}.$$

We are interested in nontrivial (strictly positive) solutions to equation (4). If m is the number of such solutions then because $f(0) = 0$, m is less than or equal to the number of roots of the derivative $f'(y)$. This derivative can be transformed to $f'(y) = \frac{g(y)}{(1+y)(y+q)^2}$, where

$$g(y) := (k(y+1) - 1)(y+q)^2 + qb(y+1) \tag{5}$$

is a polynomial of the third degree in y; so $m \leq 3$.

The precise number of solutions depends on ratios of some of the model parameters. It is convenient to conduct our research in the parametric space

$$\Pi := \{\pi = (\varepsilon, q) | \varepsilon > 0, q \in (0,1)\}. \tag{6}$$

The essential role is played by the sign of f's derivative when $y = 0$

$$\chi := f'(0) = k - 1 + \frac{b}{q} = \frac{\varepsilon}{q} - 1$$

and, correspondingly, by the diagonal $\varepsilon = q$ in the space Π (see picture 1). Since $f(\infty) = \infty$, there are four cases, as shown on picture 2. Along the diagonal $q = \varepsilon$, polynomial (5) can be written as

$$\begin{aligned} g(y)\mid_{q=\varepsilon} &= (\tfrac{1+\varepsilon}{2}y - \tfrac{1-\varepsilon}{2})(y+\varepsilon)^2 + \tfrac{\varepsilon^2(1-\varepsilon)}{2}(y+1) = \\ &= \tfrac{y}{2}[(1+\varepsilon)y + 2\varepsilon](y + 2\varepsilon - 1); \end{aligned}$$

This expression is equal to zero when $y = 0$, and it is positive when $y > 0$ and $\varepsilon > 1/2$, or, it has one solution $y = 1 - 2\varepsilon$ when $y > 0$ and $\varepsilon < 1/2$. At the parametric point $\pi = (1/2, 1/2)$, $g(y)$ has the double root $y = 0$, and is positive when $y > 0$. Therefore, if the parametric point π is on the diagonal $q = \varepsilon$, then $\chi = 0$ and the function f_π does not have any positive solutions when $\varepsilon \in [1/2, 1]$, or, has exactly one positive solution when $\varepsilon \in [0, 1/2)$.

The region that lies to the right of the diagonal in the space π corresponds to the right part of picture 2. It has two subregions, $m(\pi) = 0$ and $m(\pi) = 2$. The boundary between them, Γ, corresponds to the case when the function f_π is tangent to the y axis at some positive and unique point, which serves as a double root. Thus, the boundary Γ is determined by the following condition: the system of two equations $f(y) = 0$, $f'(y) = 0$ has a positive solution, i.e.

$$\Gamma = \{\pi | \exists y > 0 : f_\pi(y) = 0, \;\; g_\pi(y) = 0\} \tag{7}$$

This argument likewise applies to the left part of picture 2, i.e., for the region to the left of the diagonal. Thus, formula (7) characterizes the boundary Γ in the whole space Π.

3.1.2 Construction of the boundary Γ

Note that if $k \geq 1$, then using (5), $g(y) > 0 \; \forall y > 0$, and therefore, the boundary Γ lies in the region where $k < 1$, i.e.

$$0 < 1 - k = 1 - \frac{\varepsilon(1+\varepsilon)}{q+\varepsilon} = \frac{q - \varepsilon^2}{q+\varepsilon} \sim \varepsilon^2 < q < 1 \tag{8}$$

Equation (7) can be seen as a parametric way to describe the curve Γ, where the variable $y > 0$ is considered in the capacity of a parameter and a parametric point $\pi = (\varepsilon, q) \in \Pi$ is a function of y. Condition (7) affords the determination of explicit analytic expressions for the functions $\varepsilon(y)$ and $q(y)$; therefore the curve Γ can be calculated with the help of a computer.

So, let's move forward. Since we are only interested in strictly positive solutions of the system $f(y) = f'(y) = 0$, the function f can be replaced by the function $v(y) := f(y)/y$, and instead of (7) we can write the equivalent conditions $v(y) = 0$, $v'(y) = 0$, i.e.,

$$\begin{cases} \frac{L}{y} = k + \frac{b}{y+q} \\ -\frac{d}{dy}\left(\frac{L(y)}{y}\right) = A(y) := \frac{(1+y)L - y}{y^2(1+y)} = \frac{b}{(y+q)^2}, \end{cases} \tag{9}$$

where $L = L(y) := \ln(1+y)$.

Using the equality

$$\frac{L}{y} = yA(y) + \frac{1}{1+y} \tag{10}$$

and the second equation of (9), write (9) as

$$\begin{cases} k = \frac{L}{y} - \frac{b}{y+q} = yA(y) + \frac{1}{1+y} - (y+q)A(y) = \frac{1}{1+y} - qA(y) \\ b = (y+q)^2 A(y). \end{cases} \tag{11}$$

Substituting for k and b their expressions in terms of the main parameters (ε, q) from (4), we can transform the system of equations (11) to

$$\begin{cases} \varepsilon^2 + B\varepsilon - q(1-B) = 0 \\ \varepsilon^2 - C\varepsilon - qC = 0, \end{cases} \tag{12}$$

where

$$B := qA(y) + \frac{y}{1+y} \ \ and \ \ C := \frac{1}{\lambda}(q+y)^2 A(y).$$

Note that B and do not depend on ε. Subtracting the first equation of system (12) from the second equation we have

$$\varepsilon = q\left(\frac{1}{H} - 1\right), \ \ H := B + C. \tag{13}$$

By substituting the expression for ε, for instance, into the second equation of (2), we can derive a formula which includes only q and y:

$$H(C + 2q - qH) = q. \tag{14}$$

In Appendix 1, section 1 it is shown how this formula can be transformed into the following quadratic equation relative to q

$$\begin{gathered} q^2[1 + (2+4y)E] - q[1 - y(1+4y)E] + Ey^3 = 0 \\ E = E(y) := (1+y)A(y), E > 0. \end{gathered} \tag{15}$$

The discriminant of the quadratic equation has the following form

$$D(y) = (1+y)^2 M(y), \ \ M(y) := \left(2 - \frac{L}{y}\right)^2 - 4L.$$

The function $M(y)$ is decreasing (see Appendix 1, section 2) and has a unique root $y* = 0.3985743...$(all digits are significant). Therefore, when $y \leq y^*$ equation (15) has two positive roots q_1 and q_2, which can be derived explicitly as functions of y.

Computer calculations show that the smaller root $q_1(y)$ grows in $y \in [0, y^*]$ from $q_1 = 0$ to $q_1 = q^* = 0.104...$, and that the larger root $q_2(y)$ decreases from $q_2 = 1/2$ to q^*. Therefore, the two-component function $q_{1,2}(y)$ has a simple and smooth (differentiable) inverse function $y = Y(q), q \in [0, 1/2]$. Substituting it in (13) gives ε as a function of q. This function, according to the examples for which computer calculations were made, is monotonic (moreover $\varepsilon(q = 0) = 0$, $\varepsilon(q = 1/2) = 1/2$) and therefore it has an inverse function $q = Q(\varepsilon)$, $\varepsilon \in [0, 1/2]$, which corresponds to the boundary curve Γ defined in (7).

Let's consider some of the features of the function $Q(\varepsilon)$.

Γ*1.* As was noted, curve Γ lies in region (8). It is important for us that the whole curve lies below the diagonal: $Q(\varepsilon) < \varepsilon$, when $\varepsilon \in (0, 1/2)$.

Γ*2.* $Q(\varepsilon) \approx \sqrt{2}\varepsilon^{3/2}$ when $\varepsilon \to +0$.

Γ*3.* $Q(\varepsilon) \approx \varepsilon - \frac{3}{4}(1/2 - \varepsilon)^2$ when $\varepsilon \to 1/2 - 0$.

Γ*4.* Function

$$q = \begin{cases} Q(\varepsilon), & \text{when} \quad 0 \leq \varepsilon \leq 1/2 \\ \varepsilon, & \text{when} \quad 1/2 < \varepsilon \leq 1 \end{cases}$$

is continuous and has a continuous derivative in the entire segment $\varepsilon \in [0, 1]$.

Γ*5.* When $\varepsilon = q = 1/2$ point $y^* = 0$ serves as a triple root of function (4):

$$f_\pi(y) = \frac{y(3y+2)}{4y+2} - \ln(1+y) \approx \frac{2}{3}y^3 - \frac{7}{4}y^4 ... y \to 0, \;\; \pi = (1/2, 1/2)$$

Furthermore, the function $X^*(\varepsilon) := s(\varepsilon)/\sigma = \frac{1+\varepsilon}{\varepsilon} \ln(1 + Y(Q(\varepsilon)))$ (twofold stationary point as a function of ε) is decreasing in $\varepsilon \in (0, 1/2)$ from $+\infty$ to zero, moreover

$$X^*(\varepsilon) \approx \sqrt{2}\varepsilon^{-1/2} \;\; \varepsilon \to 0, \;\; X^*(\varepsilon) \approx \frac{9}{2}(1/2 - \varepsilon) \;\; \varepsilon \to 1/2 - 0.$$

Functions $Q(\varepsilon)$ and $X^*(\varepsilon)$ are sketched in figure 3; and tables of these functions are given in Appendix 3.

3.2 Behaviour of functions for large x

By virtue of Lemma 2, formula (2.11) is valid for large x; from this it follows that $V(x) \to \infty$ when $x \to \infty$. From (2.5) and (2.9) we have (note that in this case $v = V(x)$)

$$\lim_{x\to\infty} \frac{e^{x-v}}{v} = \lim_{v\to\infty} (1 - \frac{\varphi(\alpha_0 v)}{v}) = 1 - \alpha_0\gamma =: a, \tag{16}$$

and

$$e^{x-v} = av + o(v) \Rightarrow x = v + \ln(av) + o(1). \tag{17}$$

Further, we get correspondingly

$$\lim_{x\to\infty} \frac{x}{v} = \lim_{v\to\infty} \frac{1}{v}(v + \ln(av)) = 1,$$

$$x = v + \ln(ax) + \ln\frac{v}{x} + o(1) = v + \ln(ax) + o(1); \tag{18}$$

from (2.11),(4) and (16) it follows that

$$\gamma = \mu, \;\; a = 1 - \lambda = q. \tag{19}$$

Now, from (18) we can derive

$$\begin{aligned} V(x) &= x - \ln(qx) + o(1) \\ C(x) &:= x - V(x) = \ln(qx) + o(1); \end{aligned} \tag{20}$$

and, from (2.10),(2.11) and (20),

$$\begin{aligned} \varphi(x) &= \mu(x - \ln(qx) - 1) + o(1) \\ \varphi'(x) &= \mu(1 - \tfrac{1}{x}) + o(1/x). \end{aligned} \tag{21}$$

3.3 Behaviour of functions for small x

When $x \to 0$ both of the cases $V(x) = x$ and $V(x) < x$ are relevant, require separate attention.

3.3.1 Case $V(x) < x$.

Here equation (2.11) is applied, from which it follows that $\varphi(x) = \varphi'(x) \mid_{x=0} = 0$. Assuming that $\varphi''(x) \mid_{x=0}$ exists, we have

$$\varphi(x) = k\frac{x^2}{2} + o(x^2), \;\; \varphi'(x) = kx + o(x), \;\; x \to 0; \tag{22}$$

where k is some constant; applying equations (2.11) and (2.9) we get

$$v(x) = \frac{k}{\mu}x + o(x) \Longrightarrow k \leq \mu \tag{23}$$

$$e^x = f'(v) = 1 + (2 - k\alpha_0^2)v + o(x) = 1 + (2 - k\alpha_0^2)\frac{k}{\mu}x + o(x),$$

from which follows $(2 - k\alpha_0^2)\frac{k}{\mu} = 1$, i.e. k will be a root of the equation

$$k^2\alpha_0^2 - 2k + \mu = 0.$$

It is possible only if

$$\lambda\alpha_0 = \mu\alpha_0^2 \leq 1 \;\; \sim \;\; \nu := \frac{\ln\alpha_1}{\ln\alpha_0} \geq 2 \tag{24}$$

and, with the help of (23),

$$k = \frac{1 - \sqrt{1 - \lambda\alpha_0}}{\alpha_0^2} = \frac{\mu}{1 + \sqrt{1 - \lambda\alpha_0}}. \tag{25}$$

3.3.2 Case $V(x) = x$.

This case is relevant when

$$1 < \nu < 2,$$

where ν is determined in (24). The corresponding region is represented by segment $RE = [0, x^0]$ (see (2.12)), in which formula (2.8) is valid and demonstrates a state of noninvestment, $C(x) \equiv 0$. This formula does not depend on the distribution function $\pi(z)$, and therefore the analysis, conducted in [5] for the uniform distribution, can be applied in our case.

In [5] it was shown that in the state of noninvestment the functional φ grows at rate ν (i.e. $\varphi(x) \approx x^\nu$), which, when $\nu < 2$ is faster than quadratic growth (22).

4 Synthesis of the overall solution

The detailed analysis accomplished in the previous sections gives us an opportunity to construct a synthesis of the overall solution to the optimal investment path problem.

4.1 The Parametric space

According to the results of section 3.1, the parametric space (3.6) can be divided into five subregions, $\Pi 1, ..., \Pi 5$, [1], as shown in picture 4; with every subregion is associated a specific type of behavior of trajectories, which are represented by the corresponding fragments in picture 5.

Since the boundary Γ lies entirely above the diagonal $q = \varepsilon$, the number m of roots of the function (3.4) in region $q > \varepsilon$ is constant. From the left part of picture 2 it can be seen that $m = 1$ or $m = 3$. In practice it appears that $m = 1$, and the case of $m = 3$ is not relevant. Thus, in subregion $\Pi 3$ there is a single stationary point x^*, such that $Y(x) > x$ when $x > x^*$ and $Y(x) < x$ when $x < x^*$. So, point x^* is unstable: trajectories move away from it in different directions. Furthermore, in the neighborhood of the point $x = 0$, the functional is quadratic, since $\nu > 2$ (see section 3.3.1). This is illustrated by fragment 3 of picture 5.

The subregion above the diagonal is divided by the boundary Γ and the line $\nu = 2$ into four parts. On the right of Γ (subregions $\Pi 1, \Pi 2$) there is no stationary point ($m = 0$), on the left (subregions $\Pi 4, \Pi 5$) there are stationary points x_1^* and x_2^*; when $\nu > 2$ (subregions $\Pi 2, \Pi 4$), the asymptote $x \to 0$ corresponds to the situation examined in section 3.1.1; when $\nu < 2$ ($\Pi 1, \Pi 5$) it corresponds to the situation examined in section 3.3.2. This is illustrated by the corresponding fragments in picture 5.

Let us briefly describe the types of system behavior, depending on which subregion $\Pi 1, ..., \Pi 5$ the parametric point $\pi = (\varepsilon, q)$ is situated.

In *subregion* $\Pi 1$ the rate α_0 of the traditional technology is sufficiently high, and the difference in the rates α_0 and α_1 is not large. In this case, $Y(x) > x \ \forall x$, and every

[1] Unlike [5], we do not use parameters (α_0, λ), but rather, the coordinates $(\varepsilon, q) = (\alpha_0 - 1, 1 - \lambda)$.

trajectory $x_t, \ t = 0, 1, ...$ grows with time, $x_t \to \infty$. Moreover, from any initial state the system transforms to the new technology with probability one (since $C(x)$ grows in x and $z_t = \sum_{k=0}^{t} C(x_t) \to \infty$ and $P(\tau > t) = 1 - \pi(z_t) = e^{-z_t} \to 0, \ as \ t \to \infty$).

Note that when $x_0 < x^0$ (see picture 5.1), at the beginning we have a state of noninvestment with duration L, which can be derived from the condition $x_0 \alpha_0^L \approx x^0$, i.e.

$$L \simeq \ln{(x^0/x_0)}/\ln{\alpha_0}; \tag{1}$$

when $t < L, \ (x_t) = 0$ and investing in the project begins at the moment $t = L$. Correspondingly,

$$\begin{aligned} &\varphi(x_0) = \mu^L \varphi(x_L) \simeq \mu^L \varphi(x^0) = e^{-L \ln{\alpha_1}} \varphi(x^0) = \\ &= e^{-\nu \ln{(x^0/x_0)}} \varphi(x^0) = (\tfrac{x_0}{x^0})^\nu \varphi(x^0) = A x_0^\nu, \ \ x_0 < x^0, \end{aligned} \tag{2}$$

where

$$A := \frac{\varphi(x^0)}{(x^0)^\nu}.$$

In [5] it is shown, that in a model with continuous time (when L is not restricted to be an integer), (1) and (2) are represented by strict equalities, i.e. (see section 3.3.2),

$$\varphi(x) = A x^\nu, \ \ x < x^0. \tag{3}$$

Assume that at point x^0 the function φ is twice differentiable. We could determine the constant A in (3) from the condition of conjunction

$$\varphi, \varphi', \varphi'' \mid_{x^0 - 0} = \varphi, \varphi', \varphi'' \mid_{x^0 + 0};$$

in Appendix 1, section 3 it is shown that point x^0 and constant A can be derived from these conditions, i.e.,

$$x^0 = \frac{\nu(2-\nu)}{\nu - 1}, \qquad A = \mu(2-\nu)(x_0)^{1-\nu}, \qquad 1 < \nu < 2. \tag{4}$$

In *subregion* $\Pi 2$ the system's behaviour differs from the behaviour in $\Pi 1$ only in the sense that there is no state of noninvestment and investing begins from the first moment, independently of the initial state $x_0 > 0$. The asymptote of the functional behaviour near point $x = 0$ is described in section 3.3.1.

Note that on the boundary of subregions Π1 and Π2, $\nu = 2$, and when $\nu \to 2-0$, in accordance with (4),

$$A \to \lim_{\nu \to 2-0} \left[\mu(2-\nu)(\frac{\nu(2-\nu)}{\nu-1})^{1-\nu} \right] = \frac{\mu}{2} \lim_{s \to +0} s^s = \frac{\mu}{2}, \tag{5}$$

i.e., $\varphi(x) \equiv \frac{\mu}{2} x^2$, which coincides with asymptote (3.22) when $\nu = 2$.

Subregion Π3 (the triangular region $0 < \varepsilon < q < 1$) has already been described above. Let's mention here only that on the boundary of the subregions Π3 and Π2 (the segment $q = \varepsilon$, $\varepsilon \in [1/2, 1]$), stationary point x^* which is strictly positive inside Π3 becomes zero, i.e., Π3 also disappears (the type of behaviour turns into that associated with region Π2). The peculiarity of region Π3 is *the zone of risk*: when $x_0 < x^*$, trajectory x_t converges to zero. With positive probability the process of investing could continue infinitely and the system never transforms to the new technology.

Subregions Π1, Π2, Π3 are similar to the corresponding subregions Λ1, Λ2, Λ3 from article (6), and the boundaries among them (the diagonal $q = \varepsilon$ and the line $\nu = 2$) in our case and in the case of uniform distribution (considered in 5) coincide.

New in our case is the presence of *subregions* Π4 and Π5, created by the boundary Γ as constructed in section 3.1.2; there are no such subregions in [5], where the three types mentioned above wholly describe the model: Λ1 = Π1 + Π5, Λ2 = Π2 + Π4, Λ3 = Π3. The peculiarity of subregions Π4, Π5 is that they have two stationary points x_1^* and x_2^*. The larger one (x_2^*) is similar to the stationary point x^* of subregion Π3 and in the process of transition of Π3 → Π4 (segment $q = \varepsilon$, $\varepsilon \in (0, 1/2]$), point x_2^* serves as a successor to x^* (x_2^* is also unstable). Point x_1^*, absent in Π3, at the time of transition from Π3 to Π4 appears starting from $x = 0$ and is stable (see picture 5, fragments 4,5). When the parameter π moves from the diagonal, $q = \varepsilon$ x_1^* grows, and x_2^* decreases. When π reaches the boundary Γ, x_1^* and x_2^* turn into a double stationary point, which disappears after the transition.

The evolution of the system in subregions Π4 and Π5 has features typical of the subregions П1 and П2 - with unit probability the system transforms to the new technology - and also typical for the subregion П3 - the presence of a zone of *stagnation*, when x_t decreases with time. Stagnation occurs when the initial state x_0 is situated between the stationary points, $x_0 \in (x_1^*, x_2^*)$: see picture 5. If $x_0 > x_2^*$, then $x_t \to \infty$, else $x_t \to x_1^*$. So, unlike П3, even if stagnation occurs, the transition to the new technology will still happen sooner or later.

The difference between subregions П4 and П5 is only that in П5, there is a zone of noninvestment RE, while in П4 there is no such zone.

Note 2. In picture 2 it is denoted, which fragment corresponds to which subregion i.e., the subregion where point π is situated. □

4.2 Algorithm of solution's construction for fixed parametric point

In this section the algorithm to construct the solution is described. The algorithm is written in PASCAL.

In accordance with section 3.2 (for any parametric point) for large x, $V(x)/x \approx 1$, and so $Y(x) > x$. Let us consider a *retrotrajectory* (trajectory in the past) i.e., the sequence $x_0,\ x_{-1},\ x_{-2}, ...$ such that $Y(x_{-(t+1)}) = x_{-t},\ t = 0, 1, ...$, beginning at some large point. Because $x_{-1} < x_0$ and as long as $Y(x)$ is an increasing function, the sequence $\{x_{-t}\}$ is decreasing. In future for convenience instead of x_{-t} we will write y_t, so

$$y_{t+1} = Y^{-1}(y_t),\ \ t = 0, 1, ... \tag{6}$$

Let's show that the retrotrajectory can be constructed explicitly. For this purpose we introduce the sequence $\zeta_t := \{y_t, v_t, \varphi_t, \varphi 1_t\}$ where

$$\{y, v, \varphi, \varphi 1\}_t := \{y, V(y), \varphi(y), \varphi'(y)\}_{y=y_t}.$$

With y_0 is large enough, the sequence ζ_0 can be calculated on the basis of the asymptotic formulas from section 3.2, and therefore, we will consider ζ_0 as known.

Further, the process works iteratively: we can describe the transition $t \to t+1$, assuming that ζ_t is known. Since $y_t = Y(y_{t+1}) = \alpha_0 V(y_{t+1}) = \alpha_0 v_{t+1}$, we can assume $v_{t+1} := y_t/\alpha_0$. Formula (2.9) turns into the new equation

$$e^v(v - \varphi(y) + 1 - \alpha_0\varphi'(y)) \mid_{v=v_{t+1}, y=y_t} = e^x \mid_{x=y_{t+1}} . \tag{7}$$

The left side of equation (7) is already known, and therefore we equate $y_{t+1} :=$ $\ln(\text{left side of (7)})$. Further, apply formula (2.10); that is, presume that $\varphi 1_{t+1} :=$ $\mu e^{-x} f(v) \mid_{x=y_{t+1}, v=v_{t+1}}$. From (2.11), we find $\varphi_{t+1} := \mu V(x) - \varphi'(x) \mid_{x=y_{t+1}} = \mu v_{t+1} -$ $\varphi 1_{t+1}$. Thus, the sequence ζ_{t+1} is determined in full.

In subregion Π2 the process described above leads to the origin (i.e., $y_t \to 0$ when $t \to \infty$) and the constructed set of points, graphed on a screen, gives additional understanding into the solution; thus, when $\pi \in \Pi 2$ the solution is derived.

If $\pi \in \Pi 1$ then the iterative process reaches the point x^0, where $V(x) = x$, and further transforms in regime (2.8), i.e.

$$\zeta_{t+1} := \{y_t/\alpha_0, y_t/\alpha_0, \mu\varphi_t, \lambda\varphi 1_t\}. \tag{8}$$

So, for this subregion the solution is also constructed; point x^0 itself can be derived from iterative process by checking the condition

$$y_t - v_t < \varepsilon, \tag{9}$$

where ε is of some specified precision ($\varepsilon \sim 10^{-6}$).

If $\pi \in \Pi 3$, then the iterative process reaches point x^*; this fact is revealed in the process of computing the following condition

$$y_{t-1} - y_t < \varepsilon. \tag{10}$$

In the interval $(0, x^*)$, the solution is similarly constructed: a retrotrajectory starts from the origin (i.e. from point $x_0 = \delta$, where δ is a small number, $\delta \approx 10^{-4}$); the initial conditions for the process (the sequence ζ_0) are constructed on the basis of asymptote from section 3.3.1. The retrotrajectory always runs in the direction opposite to a straight trajectory (”straight” directions are shown in picture 5), therefore in this case it comes to the same point x^*, but from the left side. The coincidence of the values x^*_{left} and x^*_{right} is evidence helping to confirm our theoretical approach.

More complicated is the procedure to construct solution in the subregions Π4 and Π5, which are specific to the exponential model (see section 4.1). In these cases, an iterative process initiated at $y_0 = \infty$, comes to a stationary point x_2^*, and the point x_1^* can not be derived with the help of the algorithm described above, because retro-trajectories move away from it (straight trajectories, on the contrary, come towards x_1^*, since this point is stable; see picture 5, cases 4 and 5). But in this case the point x_1^*, as a root of the function (3.4), lies in the segment $[0, x_2^*]$ (see picture 2) and easily can be calculated with the method of bisection. After determining point x_1^*, we can construct the solution in the interval $(0, x_2^*)$, by initiating from x_1^* left and right retro-trajectories (i.e., equating $y_0 = x_1^* - \delta$ and $y_0 = x_1^* + \delta$; initial data ζ_0 are constructed on the basis of section 3.1). As above, determining the relevant case (where the point π is situated) can be done by checking conditions (9) and (10).

5 Conclusions

The main results of our model in the case of an exponential distribution function are:

1. The Bellman equation is applied and analyzed. The properties of its solutions are outlined (lemmas 1-3, section 2.2).

2. A detailed analytical investigation of the solution's characteristics, depending on model parameters, is carried out.

3. A division of the parametric space into regions is made. Every region corresponds to a specific type of system behaviour. In comparison with [5], which has only three types of system behavior, in our case we obtain five different types, and correspondingly five different regions.

4. An algorithm for model calculations and its corresponding program in the PASCAL language is constructed. We also compare the empirical results with theoretical derivations.

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A Proof of formulas

A.1 Derivation of equation (3.15).

We have

$$\begin{aligned} H = B + C = \left[q + \tfrac{1}{\lambda}(q+y)^2\right] A(y) + \tfrac{y}{1+y} = \\ = \left[1 + \tfrac{q}{\lambda}(\tfrac{1+y}{y})^2\right] L - \tfrac{q}{\lambda}\tfrac{1+y}{y}, \end{aligned}$$

which is linear in q;

$$\begin{aligned} C + 2q - qH = C + 2q - q(B+C) = \lambda C - qB + 2q = \\ = \left[(q+y)^2 - q^2\right] A - q\tfrac{y}{1+y} + 2q = (2qy + y^2)A - q\tfrac{y}{1+y} + 2q, \end{aligned}$$

which is also linear in q.

Further transformation maps into quadratic equation (3.15).

A.2 Decreasing of the function $M(y)$.

We have

$$\begin{aligned} A(y) = (\ln(1+y) - \tfrac{y}{1+y})/y^2 = \tfrac{1}{y^2}\left(\textstyle\int_0^y (\tfrac{1}{1+t} - \tfrac{1}{1+y})dt\right) = \\ = \tfrac{1}{(1+y)y^2}\textstyle\int_0^y \tfrac{y-t}{1+t}dt > 0 \end{aligned}$$

Thus, the function $L(y)/y$ decreases. Moreover,

$$E(y) = (1+y)A(y) = \frac{1}{y^2}\int_0^y \left(\frac{y-t}{1+t}\right) dt < \frac{1}{y^2}\int_0^y (y-t)dt = 1/2;$$

therefore

$$\begin{aligned} M'(y) = -2\left(2 - \tfrac{L}{y}\right)\left(\tfrac{L}{y}\right)' - \tfrac{4}{1+y} = 2\left(2 - \tfrac{L}{y}\right)A - \tfrac{4}{1+y} = \\ = -\tfrac{2}{1+y}\left(2 - \left(2 - \tfrac{L}{y}\right)E\right) = -\tfrac{2}{1+y}\left[2(1-E) + \tfrac{L}{y}E\right] < 0. \end{aligned}$$

A.3 Proof of equations (4.4).

To the right of x^0 formula (2.11) is valid. Moreover, it could be differentiated (from the assumption that $\varphi''(x)$ is continuous at point x^0), furthermore $V'(x^0)$ exists and is equal to $V'(x^0 - 0) = 1$; thus, we have

$$\left\{ \begin{array}{l} \varphi(x) + \varphi'(x) = \mu x \\ \varphi'(x) + \varphi''(x) = \mu \end{array} \right. \Big|_{x = x^0 + 0} .$$

From conditions of conjunction these equalities must be valid on the left of x^0, i.e. $\varphi(x) = Ax^{\nu}$ when $x = x^0 - 0$. These two equations with two variables A and x^0 yield the values in (4.4).

B Flow-chart of the algorithm

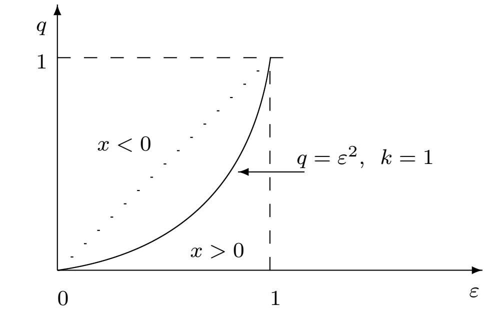
Available on request

Notes: $\varepsilon = 10^{-6}$, $\delta = 10^{-4}$, $\infty := max[100, 2/q]$,

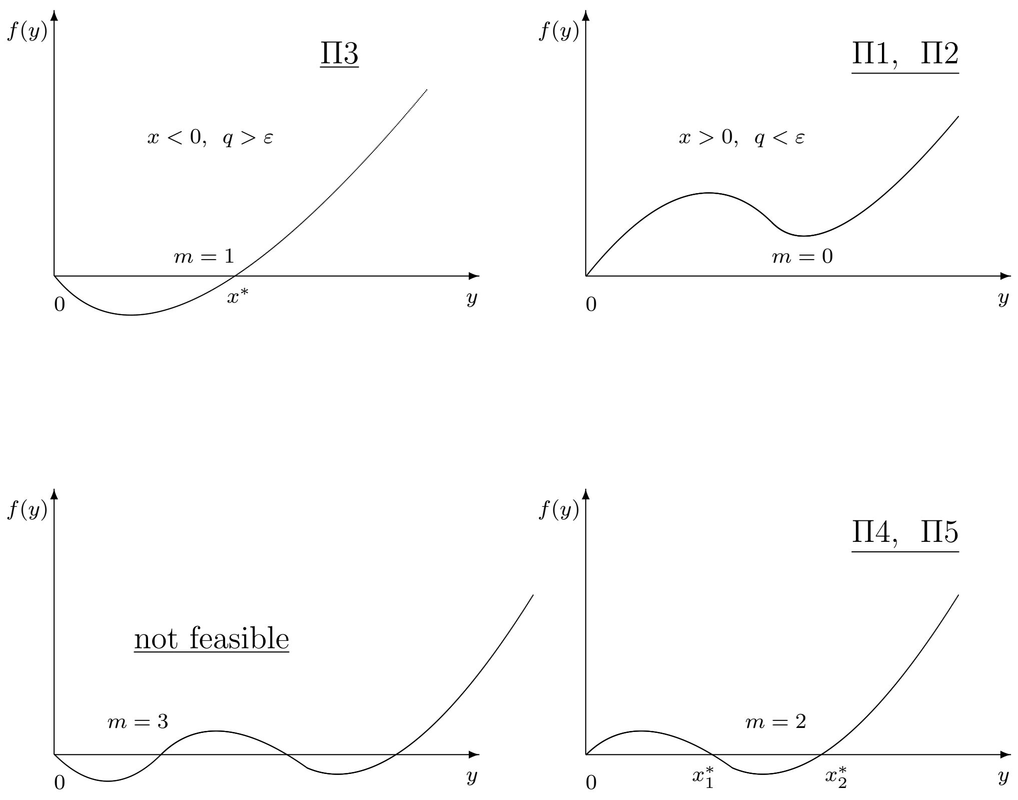
$$\zeta(y_0) := \begin{cases} \text{Asymptotic } formulas \ s.3.2, \ \ y_0 = \infty \\ \text{Asymptotic } formulas \ s.3.3.1, \ \ y_0 = \delta \\ \text{Formulas of stationary point } \ s.3.1, \ \ \text{else.} \end{cases}$$

$N = 1, ...5$ is the number of region, $N = 0$ is a stationary point, $\{\alpha_0 + \lambda = 2\} = \{\text{diagonal } q = \varepsilon\}$, root x_1^* can be calculated in segment $[\delta, x_{right}^* - \delta]$

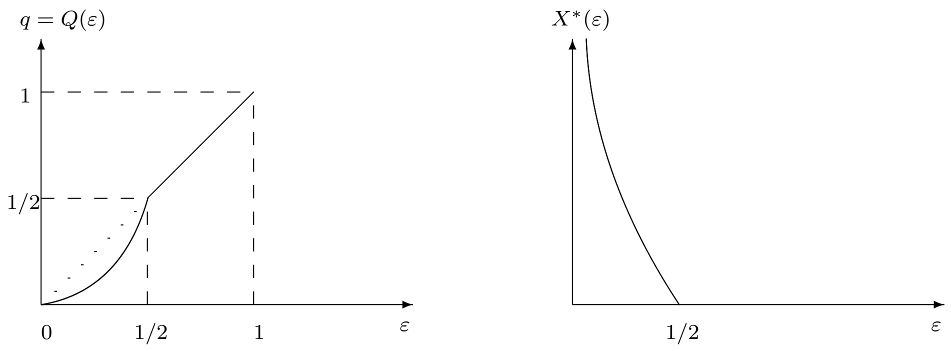
C Pictures



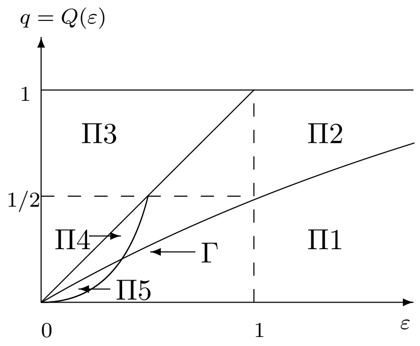
Picture 1. Parametric space Π



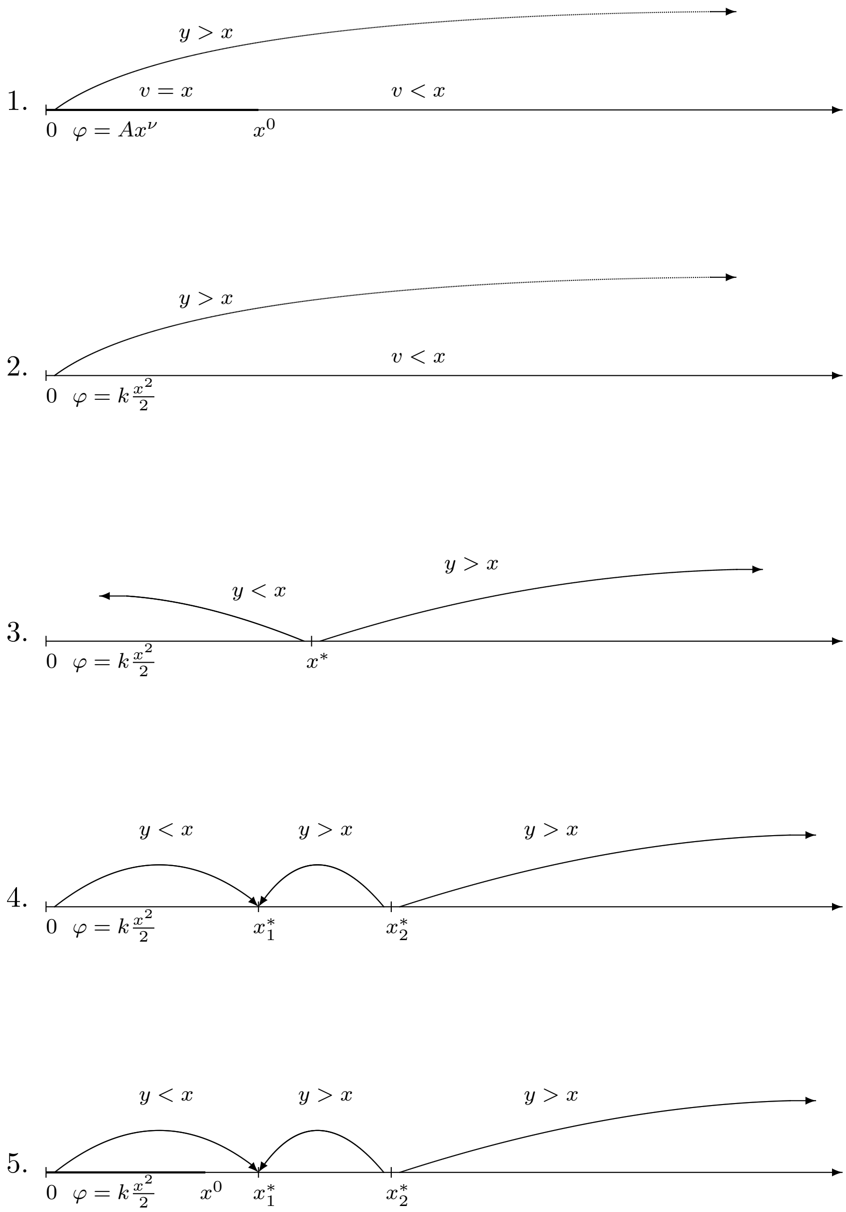
Picture 2. Four cases how roots of "vekovoe" equation can be situated
(see Notes 2 in the end of section 4.1)



Picture 3. Graphs of functions $Q(\varepsilon)$ and $X^*(\varepsilon)$



Picture 4. Laying out of parametric space



Picture 5. Five types of system's behaviour
Arrows show directions of trajectory's moving,
when x_0 is situated in some subregion of R_+.