Computational Macroeconomics

Module 3, 2018-2019

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Course description

Many, if not most, dynamic models used in modern macroeconomics do not have analytical (closed-form) solutions. For this reason, numerical methods and computer programming have become indispensable tools of the macroeconomic research. In this course we will discuss the main computational algorithms of the dynamic optimization problem. We will start with overview of the basic results from dynamic programming. Then, we will study the main numerical algorithms for its solution with application to a simple neoclassical growth model. In particular, we will consider linear-quadratic approximation, value and policy functions iterations algorithms and their modifications, perturbation and projection methods. Finally, we will discuss more complicated algorithms for solving heterogeneous agents' models, optimal policy and dynamic contracting problems. This course requires a basic knowledge of MATLAB, however programming skills in Fortran 90 or Python will be a plus.

Course requirements, grading, and attendance policies

There will be a few (maximum 4) home assignments (50% of the grade) asking for writing a code in MATLAB (or in GNU Octave, Fortran 90, Python, C++, etc.) to solve a simple dynamic programming problem. The exam (50% of the grade) will contain questions on a published macroeconomic article handed out in advance. All these components (including all home assignments), as well as at least 70% attendance, are mandatory for getting a passing grade.

Course contents

1. **Review of dynamic programming:** mathematical preliminaries, contraction mapping theorem, Blackwell's sufficient conditions, theorem of the maximum, dynamic programming under uncertainty

2. **Discrete-state dynamic programming:** value function iteration algorithm and its improvements, policy function iteration, interpolations and splines

3. **Linear approximation methods:** linear-quadratic (LQ) approximation algorithm, first-order perturbation methods

4. **Higher-order perturbation methods**

5. **Projection methods:** finite elements method, spectral methods (Chebyshev polynomials)

6. **Parameterized expectations algorithm**

7. **Heterogeneous agents models and incomplete market economies:** computation of stationary equilibrium, transitional dynamics, aggregate uncertainty in heterogeneous agents models, Krussel-Smith algorithm

8. **Extensions:** solving non-optimal economies, Ramsey policy under and without commitments, limited commitment and incentive problems

Sample tasks for course evaluation

Problem 1: Deterministic Growth with Human Capital

Consider the deterministic version of the neoclassical growth model with human capital accumulation. Assume that the worker can accumulate human capital h_{t+1} only by reducing the time he works n_t. Consumption good c_t is produced with human capital h_t and hours worked n_t as inputs. Representative agent solves the following sequence problem:

$$\max_{\{c_t, n_t, h_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{SP}$$

subject to

$$\begin{aligned} & c_t \leq f(n_t, h_t), \\ & h_{t+1} = (\lambda + \delta)(1 - n_t)h_t + (1 - \delta)h_t, \\ & c_t \geq 0, 0 \leq n_t \leq 1, h_{t+1} \geq 0, \forall t = 0, 2, \ldots \end{aligned}$$

with $h_0 > 0$ given.

Assume that utility and production functions are given by $u(c) = c^\gamma/\gamma$ and $f(n, h) = (nh)^\alpha$. Let $\beta = 0.96$, $\alpha = 0.3$, $\gamma = -1$, $\delta = 0.02$ and $\lambda = 0.03$.

1. Write down the Bellman equation for this problem. Be precise in specifying the state space X, flow payoff function $F(x, x')$ and feasibility correspondence $\Gamma(x)$.

2. Assume that $\beta(1 + \lambda) < 1$. Show that there exists a unique continuous function v satisfying the Bellman equation. *(1 point)*

3. Show that Bellman equation has an analytical solution with the following functional form: $v(h) = Ah^{\alpha\gamma}$. Show that the optimal policy is a constant growth rate for human capital: $g(h) = \theta h$. Find A and θ.

4. Let $M = 100$ and $\mathcal{H} = \{h_1, h_2, \ldots, h_M\}$ where $h_0 = 0$ and $h_M = 1$ and the distance between two consecutive points in $\mathcal{H}$ is constant. Write MATLAB code to compute the value function $v_{\mathcal{H}}(h)$ and the optimal decision rule $h' = g_{\mathcal{H}}(h)$ by iterating on the Bellman's operator associated to discretized version of the model.

5. Using plots compare the analytical and numerical solutions to value and policy functions of this problem. Plot the absolute values of the Euler errors.

6. Refine your algorithm by exploiting the properties of value and policy functions and using the linear interpolation of value function between the grid points.

Problem 2: Stochastic Growth Model

Consider the stochastic version of the neoclassical growth model:

$$v(k, z) = \max_{k' \in [0, f(k,z)]} \{u(f(k, z) - k') + \beta E\{v(k', z')|z\}\} \tag{FE}$$

Assume that utility and production functions are given by $u(c) = c^\gamma/\gamma$ and $f(k, z) = \exp(z)k^\alpha + (1-\delta)k$ and productivity shock z follows AR(1) stochastic process: $z' = \rho z + \epsilon'$ where ϵ' is i.i.d. $N(0, \sigma^2)$. Let $\beta = 0.9$, $\gamma = -1$, $\alpha = 0.3$, $\delta = 0.1$, $\rho = 0.85$ and $\sigma = 0.05$. Let $M = 100$ and $\mathcal{K} = \{k_1, k_2, \ldots, k_M\}$ where $k_0 = 0.01\bar{k}$ and $k_M = 1.5\bar{k}$ and $\bar{k}$ is the deterministic steady state level of capital and the distance between two consecutive points in $\mathcal{K}$ is constant.

1. Use Tauchen's method (see Tauchen, G (1986) 'Finite State Markov-Chain Approximations to Univariate and Vector Autoregressions', Economics Letters: 20) to construct a 3-state Markov chain over $\mathcal{Z} = \{z_1, z_2, z_3\}$ that approximates the AR(1) process for z. Write down the Bellman equation for the discretized version of the model.

2. Write MATLAB code that solves numerically the stochastic growth model by iterating over the value function. Provide two algorithms: with and without linear interpolation of the value function. Plot the computed value and policy functions.

3. Assume that the model period is a quarter. Simulate the economy for 50 years, that is for 200 periods. Plot one stochastic realization of this economy, $\{y_t, c_t, i_t, k_t\}_{t=0}^{199}$ starting from $z_0 = 0$ and $k_0 = \bar{k}$. Use Hodrick-Prescott (HP) filter with $\lambda = 1600$ to compute and plot the deviations of the logs of simulated data $\{y_t, c_t, i_t, k_t\}$ about their HP(1600) trends.

4. Obtain 100 independent stochastic realizations of this economy and write MATLAB code to compute the statistics that describe the business cycle fluctuations of $\{y_t, c_t, i_t, k_t\}$ (see Hansen, G.B. (1985) 'Indivisible Labor and the Business Cycle',

Journal of Monetary Economics: 16; and Kydland F. E. and Edward C. Prescott (1990) 'Business Cycles: Real Facts and a Monetary Myth', Federal Reserve Bank of Minneapolis Quarterly Review (Spring): 318).

Problem 3: LQ Approximation problem

Consider stochastic growth model with the *divisible* labor described in Hansen (1985):

$$\max_{\{c_t,n_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, 1-n_t) \tag{1}$$

subject to

$$\begin{aligned} &c_t + i_t \leq z_t k_t^{\theta} n_t^{1-\theta}, \\ &k_{t+1} = i_t + (1-\delta) k_t, \\ &z_{t+1} = \gamma z_t + \epsilon_{t+1}, \ln(\epsilon_t) \sim N(\mu, \sigma^2) \end{aligned}$$

where $c_t \geq 0$, $0 \leq n_t \leq 1$ and k_0 is given. Assume that utility function is given by $u(c_t, 1-n_t) = \log(c_t) + A\log(1-n_t)$. Let $\beta = 0.99$, $A = 2$, $\theta = 0.36$, $\delta = 0.025$ and $\gamma = 0.95$.

Note, that productivity shocks have log-normal distribution with mean $E(\epsilon_t) = m = 1-\gamma$ and variance $Var(\epsilon_t) = v = 0.00712^2$. So, $\ln(\epsilon_t) \sim N(\mu, \sigma^2)$ where $\sigma^2 = \ln(\frac{v}{m^2}+1)$ and $\mu = \ln(m) - \frac{1}{2}\sigma^2$.

1. Write MATLAB program that uses LQ approximation algorithm to replicate statistics reported for the *divisible* labor economy (third and fourth columns of Table 1 in the paper).

2. Modify your program to replicate the statistics reported for the *indivisible* labor economy (fifth and sixth columns of Table 1 in the paper). Note, that representative agent in this version of the model has utility function given by $u(c_t, n_t) = \log(c_t) + B(1-n_t)$, where $B = -A\frac{\log(1-h_0)}{h_0}$ and $h_0 = 0.53$.

Problem 4: Log-Linear Approximations with Dynare

Problem 6.3 in Heer & Maussner: Consider the following model with a variable utilization rate of capital u_t and a second shock that represent exogenous variations in the price of imported oil p_t (this is adapted from Finn, 1995). The representative agent solves:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left(\ln C_t + \theta \ln(1-N_t)\right) \tag{2}$$

subject to

$$\begin{aligned}
&K_{t+1} = (Z_t N_t)^\alpha (u_t K_t)^{1-\alpha} + (1-\delta(u_t))K_t - C_t - p_t Q_t,\\
&\delta(u_t) = \frac{u_t^\gamma}{\gamma},\\
&\frac{Q_t}{K_t} = \frac{u_t^\zeta}{\zeta},\\
&\ln Z_t = \ln Z + \ln Z_{t-1} + \epsilon_t^Z, \epsilon_t^Z \sim N(0, \sigma^Z),\\
&\ln p_t = \rho^p \ln p_{t-1} + \epsilon_t^p, \epsilon_t^p \sim N(0, \sigma^p)
\end{aligned}$$

with K_0 given. C_t denotes consumption in period t, N_t are working hours, K_t is the stock of capital, and Q_t is the quantity of oil imported at the price p_t. A more intense utilization of capital increases the amount of energy required per unit of capital. Thus, if the price of oil rises, capital utilization will decrease.

In this model, labor augmenting technical progress follows a random walk with the drift rate $\ln Z$. Define the following stationary variables $c_t = \frac{C_t}{Z_t}$, $k_t = \frac{K_t}{Z_{t-1}}$, and $z_t = \frac{Z_t}{Z_{t-1}}$. The state variables of the model are k_t, z_t and p_t. Use the following parameter values taken from Finn (1995): $\beta = 0.9542$, $\theta = 2.1874$, $\alpha = 0.7$, $\gamma = 1.4435$, $\zeta = 1.7260$, $\rho^p = 0.9039$, $\sigma^p = 0.0966$, $Z = 1.0162$, $\sigma^Z = 0.021$.

1. Write down equilibrium conditions characterizing this model. Stationarize the model.
2. Write down the log-linear approximation of the model around a steady state.
3. Solve the model using first-order perturbation algorithm realized in Dynare. Write down computed optimal decision rules for the state and control variables.
4. Plot impulse responses of the endogenous variables to oil price and productivity shocks, ϵ_t^p and ϵ_t^Z. Demonstrate that if the price of oil rises, capital utilization will decrease. Interpret the results.
5. Compute the business cycle statistics of the model.

Problem 5. Individual household's policy function

Let labor efficiency s evolves according to Markov chain with two states $\{e = 1, u = 0.5\}$ and transition matrix:

$$\Pi = \begin{bmatrix} 0.97 & 0.03 \\ 0.5 & 0.5 \end{bmatrix}$$

If the realization of the process at t is $\bar{s}_i$, then at time t the household receives labor income $w\bar{s}_i l_t$, where l_t is labour supply.

We constrain holdings of a single asset to a grid $\mathcal{K} = [0, k_1, k_2, \ldots, k_{\max}]$. For given values of (w, r) and given initial values (k_0, s_0) the household chooses $\{k_{t+1}, c_t, l_t\}_{t=0}^{\infty}$ to maximize:

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{[c_t^{\gamma}(1-l_t)^{1-\gamma}]^{1-\sigma}}{1-\sigma} \tag{3}$$

subject to

$$\begin{aligned} c_t + k_{t+1} = (1+r)k_t + ws_t l_t \\ k_{t+1} \in \mathcal{K} \end{aligned} \tag{4}$$

where $\beta = 0.99$, $\sigma = 2$, $\gamma = 0.6$ and $k_{\max} = 10$.

1. Write Bellman equation for this problem.
2. Write MATLAB function computing for any values of (w, r) policy functions $k' = g^k(s, k; r, w)$, $c = g^c(s, k; r, w)$ and $l = g^l(s, k; r, w)$ using value function iteration (VFI) algorithm.

Problem 6. Invariant distributions of Markov chains

Define the unconditional distribution of (k_t, s_t) pairs, $\lambda_t(k, s) = P(k_t = k, s_t = s)$. The exogenous Markov chain Π on s and the optimal policy function $k' = g^k(k, s; w, r)$ induce a law of motion for the distribution λ_t, namely,

$$\lambda_{t+1}(k', s') = \sum_k \sum_s \lambda_t(k, s)\Pi(s, s')\mathcal{I}(k', s, k) \tag{5}$$

where we define the indicator function $\mathcal{I}(k', s, k) = 1$ if $k' = g^k(k, s; w, r)$, and 0 otherwise.

1. Write MATLAB function computing for any values of (w, r) invariant distribution $\lambda(k, s)$ by iterating equation (5).
2. Write MATLAB function constructing for any values of (w, r) Markov chain for the joint evolution of (k, s) and compute its invariant distribution. Compare the results of both algorithms.

Problem 7. Stationary equilibrium of Aiyagari model

The aggregate production function determines the rental rates on capital $r+\delta$ and labor w from the marginal conditions

$$\begin{aligned} w\left(\frac{K}{L}\right) &= (1-\alpha)\left(\frac{K}{L}\right)^{\alpha} \\ r\left(\frac{K}{L}\right) &= \alpha\left(\frac{K}{L}\right)^{\alpha-1} - \delta \end{aligned} \tag{6}$$

where $\alpha = 0.3$ is capital income share, K is aggregate capital, L is aggregate labour and $\delta = 0.01$ is depreciation rate.

Write MATLAB program computing stationary equilibrium of Aiyagari model with endogenous labour, i.e. aggregate capital K, aggregate labour L, real interest rate r and real wage w, policy functions of households $k' = g^k(k,s)$, $l = g^l(k,s)$ and invariant distribution $\lambda(k,s)$, such that:

$$\begin{aligned} K &= \sum_{k,s} g^k(k,s)\lambda(k,s) \\ L &= \sum_{k,s} s g^l(k,s)\lambda(k,s) \end{aligned}$$

1. For fixed value of $\frac{K}{L} = k = k_j$ with $j = 0$, compute (w,r) from equation (6), then solve the household's optimum problem. Use the optimal policy $g_j^k(k,s)$ to deduce an associated stationary distribution $\lambda_j(k,s)$.

2. Compute the average values of capital and labour associated with $\lambda_j(k,s)$, namely,

$$\begin{aligned} K_j^* &= \sum_{k,s} g_j^k(k,s)\lambda_j(k,s) \\ L_j^* &= \sum_{k,s} s g_j^l(k,s)\lambda_j(k,s) \end{aligned}$$

3. For a fixed relaxation parameter $\xi \in (0,1)$, compute a new estimate of k from

$$k_{j+1} = \xi k_j + (1-\xi)\frac{K_j^*}{L_j^*} \tag{7}$$

4. Iterate on this scheme to convergence.

The Problems 8-11 are based on the following paper: Guerrieri & Iacoviello (2015) *'OccBin: A toolkit for solving dynamic models with occasionally binding constraints easily'*, Journal of Monetary Economics, 70, 22-38

Problem 8.

To find a full nonlinear solution of the RBC model with a constraint on investment this paper uses a value function iteration algorithm on a very fine grid of capital stocks.

1. Write down the Bellman equation for the RBC model with a constraint on investment, as described in Section 4.1 of this paper. Be precise in specifying state space, payoff function and feasibility correspondence.
2. Provide a sketch of the value function iteration algorithm to compute value and policy functions of this model.
3. Discuss the main advantages and disadvantages of this approach comparing to the piecewise linear solution proposed in the paper.

Problem 9.

This paper proposes an algorithm to obtain a piecewise linear solution of the models with occasionally binding constraints.

1. Why is the first-order perturbation approach not applicable for the models with occasionally binding constraints?
2. Write down the linearized systems (M1) and (M2) and associated functions f and g for a simple linear model described in Section 2.4. Be precise in specifying the matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$ and $\mathcal{A}^*, \mathcal{B}^*, \mathcal{C}^*, \mathcal{D}^*, \mathcal{E}^*$.
3. The algorithm in this paper has six main steps. Describe them very briefly. What exactly have we to guess and verify in this algorithm?
4. What are the main sources of approximation error in this algorithm? Which of them are the most problematic?

Problem 10.

This paper uses two approaches to assess numerical accuracy of the linear, piecewise linear and full nonlinear algorithms.

1. What are these two approaches? Describe them very briefly.
2. What is the best algorithm in terms of accuracy? The worst? Explain.
3. Explain why a superior approximation to the solution of the model should yield a higher level of utility.

Problem 11.

Consider a simple model of small open economy. The representative household maximizes an expected life-time utility:

$$E_0 \sum_{t=0}^{\infty} \beta^t \log(c_t)$$

subject to budget and borrowing constraints:

$$c_t + b_{t+1} + \psi(b_{t+1}) = y_t + (1+r)b_t$$
$$b_{t+1} \geq -\phi y^{ss}$$

where c_t denotes consumption, b_{t+1} are holdings of risk-free international asset (negative values mean foreign debt), $\psi(b) = \frac{1}{2}\psi b^2$ denotes debt adjustment costs, r is a risk-free international real interest rate (with $r = \frac{1}{\beta} - 1$) and output y_t follows AR(1) stochastic process:

$$\log(y_t) = (1-\rho)\log(y^{ss}) + \rho \log(y_{t-1}) + u_t, u_t \sim N(0, \sigma^2)$$

Assume that $\beta = 0.96$, $\psi = 0.001$, $\phi = 0.1$, $\rho = 0.9$, $\sigma = 0.02$ and $y^{ss} = 5$.

1. Write down the first-order conditions characterizing an equilibrium in this model.

Download here the OccBin toolkit realizing the piecewise linear algorithm. Note that you will need Dynare to use it. Read carefully `readme.pdf` file and look at the examples of models solved in OccBin.

2. Using this toolkit write the code and solve the model both with and without borrowing constraint. Illustrate the impulse responses (both with and without borrowing constraint) of output y_t, consumption c_t, asset holdings b_{t+1} and Lagrange multiplier λ_t to 3 standard deviations negative shock in u_t equal to -0.06. What is about the effects of positive shock of the same size?

3. Explain the differences in the results of these two models. Why does consumption fall more initially but recover faster afterwards after the negative shock in the model with borrowing constraints?

Additional materials

1. Marimon, Ramon & Scott, Andrew, *Computational Methods for the Study of Dynamic Economies*, Oxford University Press, 1999

2. Adda, Jerome & Cooper, Russell W., *Dynamic Economics: Quantitative Methods and Applications*, The MIT Press, 2003

3. Judd, Kenneth L., *Numerical Methods in Economics*, The MIT Press, 1998

4. Stokey, Nancy L., Lucas, Robert E. & Prescott, Edward C., *Recursive Methods in Economic Dynamics*, Harvard University Press, 1989

5. DeJong, David N. & Dave, Chetan, *Structural Macroeconometrics*, Princeton University Press, 2nd ed., 2011

I will also provide a reading list of papers applying the quantitative methods discussed in the class, with the rate of about 2-3 per week.

Academic integrity policy

Cheating, plagiarism, and any other violations of academic ethics at NES are not tolerated.