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Beliefs Aggregation and Return Predictability

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ABSTRACT

We study return predictability using a model of speculative trading among competitive traders who agree to disagree about the precision of private information. Although traders apply Bayes' Law consistently, returns are predictable. In addition to trading on long-term fundamental value, traders also trade on perceived short-term opportunities arising from foreseen future disagreement, as in a Keynesian beauty contest. Contradicting conventional wisdom, this short-term speculation dampens price fluctuations and generates time-series momentum. Model calibration shows quantitatively realistic patterns of return dynamics. Consistent with empirical evidence, our model predicts more pronounced momentum for stocks with higher trading volume.

ALTHOUGH RESEARCHERS HAVE DOCUMENTED SHORT-RUN time-series momentum in equity returns, they have found it notoriously difficult to explain theoretically why this pattern can occur in a competitive market. We present a dynamic model of competitive trading based on flows of new private information. The model generates time-series momentum endogenously by relaxing the rationality assumption in only a minimal way.

As new information arrives, all traders apply Bayes' Law consistently and optimize correctly, with the single exception that each trader symmetrically assigns a higher precision to his own signal than to other traders' signals. As a result of this relative overconfidence, each trader anticipates that, upon observing new information in the future, others will correct their mistakes by adjusting their valuations toward unconditional levels and toward his own level. To profit from this perceived short-run return predictability, traders make speculative trades which bet against other traders' expectations. This

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dampens prices relative to traders' own long-term valuations and leads to time-series momentum even if traders' signals are empirically correct on average.

For example, suppose all traders have positive signals such that they all agree that the fundamental value is $30. Since each trader believes that others are putting too much weight on both their current and future information, each trader believes that other traders will revise their expectations downward in the future. As a result, each trader would want to trade on this short-term profit opportunity by selling today at price $30 and plan to buy at better (lower) prices later. This internalization of future disagreement reduces the price today below $30, even when all traders correctly agree that $30 is an accurate estimate of fundamental value. As Keynes puts it, "it is not sensible to pay 25 for an investment of which you believe the prospective yield to justify value of 30, if you also believe that the market will value it at 20 three months hence."

While this short-term trading faithfully reflects the logic of a Keynesian beauty contest, its empirical implication contradicts Keynes's unmodeled intuition that short-term speculation amplifies volatility and leads to mean reversion in prices. In contrast, our theory implies dampened price fluctuations, dampened volatility, and therefore momentum—exactly the opposite of what Keynes expected. We first confirm our insight using a two-period model with closed-form solution. We then obtain a quasi-closed-form numerical solution to an infinite-horizon discrete-time model. Price dampening is more pronounced in the infinite-horizon dynamic trading model than in the two-period model. The intuition that prices are dampened because of speculative trading suggests the hypothesis that increasing the frequency of trading will increase momentum. Indeed, we show that increased trading frequency magnifies speculative trading in a Keynesian beauty contest and leads to larger momentum.

With belief heterogeneity, empirical prices and quantities depend on both traders' beliefs and the empirically correct model specification which ultimately govern the dynamics of fundamentals and information. For example, when traders' beliefs about the decay rates of signals are different from the empirically correct parameter, the expected returns are described by a structural model which is linear in state variables including the current levels of prices and dividends as well as exponentially weighted averages of these variables in the past. We calibrate model parameters and obtain positive return autocorrelation over short periods of one to two years and negative autocorrelation over longer horizons, matching empirically observed return patterns. Time-series return momentum arises from price dampening, and return mean reversion arises when traders believe that the signals decay at a slower rate than the empirically correct parameter implies.

Our results contribute to the theoretical literature concerning whether differences in beliefs or expectations can generate price drift (defined as positive return autocorrelation). The rational expectations paradigm of Muth (1961) implies that price changes are unpredictable because prices aggregate fundamental information correctly when traders are correct on average, even when individual traders make mistakes. It may thus seem reasonable to attribute return predictability, including momentum, to irrational behavior or bounded rationality. However, when return anomalies are motivated by behavioral biases, Fama (1998) suggests that a Pandora's box is opened, undermining modeling parsimony by enabling one plethora of behavioral biases to explain another plethora of anomalies. We therefore explore the suggestion of Morris (1995) and drop "the common prior assumption from otherwise rational behavior," based on the idea that even rational agents may have heterogeneous beliefs. We do so by assuming overconfidence, as in Aumann (1976).

Allen, Morris, and Shin (2006) attribute price drift to iterating expectations in a market where traders are assumed to share a common prior but observe different information. Banerjee, Kaniel, and Kremer (2009) point out, however, that in such a noisy rational expectations model, when traders are allowed to learn from prices, returns should exhibit mean reversion, not momentum, due to the noise in prices.

Importantly, Banerjee, Kaniel, and Kremer (2009) also show that for heterogeneous beliefs to generate price drift, it is necessary for traders to disagree not only about the joint distribution of signals and fundamental value but also about the joint distribution of the signals themselves. Our two-period and infinite-horizon models satisfy both of these necessary conditions. Even when traders are correct on average, our model generates positive return autocorrelation by assuming that traders receive new signals over time, disagree about how to interpret the signals both in the present and in the future, and correctly understand how this disagreement plays out in prices. In our framework, these conditions are sufficient to generate price dampening and return momentum. The return predictability is related to perceived order flow predictability from traders' short-term speculative trading based on disagreement about the information content of future signals. Banerjee, Kaniel, and Kremer (2009), in their Example 3 (p. 3718), describe a model in which noisy asset supply has independent increments and traders receive signals only at date 0, correctly interpret their own signals and noisy supply, and believe that other traders' signals are uninformative; this example does not generate price drift (Kyle and Wang (2021)).[1]

Our model is similar to Kyle, Obizhaeva, and Wang (2018), who show that strategic traders rationally smooth their trading in continuous time to reduce permanent and temporary price impact. The models differ in that we assume a competitive model of trading in discrete time. Assuming perfect competition and discrete trading allows us to make direct comparisons with existing literature, which also assumes perfect competition and discrete time. Assuming discrete-time trading also allows us to show that momentum is more pronounced when trading opportunities are more frequent.

Hong and Stein (1999), Barberis and Shleifer (2003), and Greenwood and Shleifer (2014) generate return predictability by assuming that traders follow simple trading rules and do not extract information from prices.

[1] In private correspondence, Banerjee pointed out that positive return autocorrelation can be obtained if their Example 3 is modified to assume that supply has small variance and follows an AR(1) process rather than a random walk.

Gruber (1996), Lou (2012), and Vayanos and Woolley (2013) derive time-series momentum from the aggregate amount of money chasing returns. Detemple and Murthy (1994), Basak (2005), Jouini and Napp (2007), Dumas, Kurshev, and Uppal (2009), Xiong and Yan (2010), Cujean and Hasler (2017), Atmaz and Basak (2018), and Ottaviani and Sorensen (2015) derive return predictability from the interaction of beliefs aggregation and wealth effects. Andrei and Cujean (2017) study word-of-mouth communication instead of beliefs aggregation as a mechanism that generates return predictability. In our model, return predictability is unrelated to traders following simple rules, wealth effects, the flow of money into the market, and word-of-mouth communication. Instead, we assume that traders follow optimal strategies, there are no wealth effects, the asset is in zero net supply, and traders infer other traders' information from prices.

Consistent with the empirical evidence presented in Internet Appendix Section IV,[2] our model implies that time-series momentum is more significant when disagreement among traders is larger and assets are more actively traded. Lee and Swaminathan (2000) and Cremers and Pareek (2014) find momentum to be stronger for stocks with higher volume and more short-term trading. Zhang (2006) and Verardo (2009) show that momentum returns are larger for stocks with higher analyst disagreement. Moskowitz, Ooi, and Pedersen (2012) document time-series momentum in liquid futures contracts. Our findings may also be relevant for empirical research on return predictability in models with heterogenous beliefs, such as Greenwood and Shleifer (2014) and Buraschi, Piatti, and Whelan (2022).

This paper is structured as follows. Section I presents a two-period model and describes intuition behind price dampening. Section II considers a competitive dynamic model with discrete trading and further clarifies mechanisms generating price dampening. Section III analyzes predictions for holding-period return and calibrates the model parameters. Section IV concludes. Proofs are in the Appendix.

I. A Two-Period Model

To explain the economic intuition behind price dampening and time-series momentum, we first present a simple two-period model.

A. Model Setup

Consider a two-period model of competitive trading among N traders with different beliefs. A risky asset with random liquidation value $v \sim \mathcal{N}(0\ ,\ 1/\tau_v)$ is traded for a safe numeraire asset. The total supply of the risky asset is zero.

[2] The Internet Appendix may be found in the online version of this article.

Traders trade in periods 1 and 2, and the payoff v is realized at period 3. At periods $t = 1$ and $t = 2$, each trader n observes private signals i_{nt},

$$i_{nt} := \tau_n^{1/2} \left(\tau_v^{1/2} v\right) + e_{nt}, \tag{1}$$

with $e_{nt} \sim \mathcal{N}(0, 1)$. The asset payoff v and $2N$ private-signal errors e_{1t}, ..., e_{Nt} with $t = 1, 2$ are independently distributed.

In equilibrium, prices are functions of the average of traders' private signals.[3] Therefore, each trader n can infer from the equilibrium price the average of other traders' private signals, denoted i_{-nt}:

$$i_{-nt} := \tfrac{1}{N-1} \sum_{m=1, m\neq n}^{N} i_{mt}. \tag{2}$$

To generate trading, we assume that traders agree to disagree about the precisions of private signals τ_n. Each trader is "relatively overconfident," believing that his own signal has high precision $\tau_n = \tau_H$ and other traders' signals have low precision $\tau_m = \tau_L$ for $m \neq n$, with $\tau_H > \tau_L \geq 0$.[4]

Each trader has the same exponential utility function with constant absolute risk aversion (CARA) parameter A. At each period $t = 1, 2$, trader n's problem is to choose the optimal inventory S_{nt} to maximize his expected exponential utility function of terminal wealth at time $t = 3$,

$$\mathrm{E}_t^n\left[-\mathrm{e}^{-A\, W_{n3}}\right], \tag{3}$$

where $\mathrm{E}_t^n[\ldots]$ denotes trader n's expectation conditional on all signals at period $t = 1, 2$, and terminal wealth W_{n3} is

$$W_{n3} = W_{n1} + (P_2 - P_1)\, S_{n1} + (v - P_2)\, S_{n2}. \tag{4}$$

The prices P_1 and P_2 are set to clear markets each period.

Let $\mathrm{var}_t^n[\ldots]$ denote trader n's variance operator conditional on all signals at period $t = 1, 2$. The projection theorem for normally distributed random variables implies that the conditional error variance is related to signal precision

[3] For expositional simplicity, we assume there are no public signals in the two-period model. Including public signals does not change any of our main results.

[4] We deviate from the standard approach in the literature, which assumes that a signal has the form value-plus-noise and these informativeness is given by the variance of the noise term. Instead, in equation (1), we rescale signals by the standard deviation of noise terms $\tau_n^{1/2}$, so that the coefficient on the noise term is exactly one. If we assume $i_{nt} := (\tau_n + 1)^{-1/2}\big(\tau_n^{1/2}(\tau_v^{1/2} v) + e_{nt}\big)$, then it can be shown that $\mathrm{var}[i_{nt}] = 1$ and thus traders do not disagree about the variance of signals. We show that the dynamic dampening effect arises and returns still exhibit momentum in this setting because traders disagree about the coefficient on v in the signals. To make the two-period model comparable to our continuous-time setting where a trader can estimate the diffusion variance (quadratic variation) with high accuracy (see Footnote 9), we rescale signals so that the coefficient on the noise term is exactly one in the two-period setting.

τ_t at time $t = 1$ and 2 by

$$\tau_t := \left(\mathrm{var}_t^n[v]\right)^{-1} = \tau_v \left(1 + \left(\tau_H + (N-1)\,\tau_L\right) t\right). \tag{5}$$

Trader n's expected value of v at time $t = 1, 2$ is

$$\mathrm{E}_t^n[v] = \frac{\tau_v^{1/2}}{\tau_t} \left(\tau_H^{1/2} \sum_{s=1}^{t} i_{ns} + (N-1)\,\tau_L^{1/2} \sum_{s=1}^{t} i_{-ns} \right), \tag{6}$$

which puts weight proportional to $\tau_H^{1/2}$ on trader n's own private signals i_{nt} and to $\tau_L^{1/2}$ on others' private signals i_{mt}, $m \neq n$.

B. The Equilibrium with Myopic Conditional Mean-Variance Optimizers

Instead of maximizing terminal utility (3), we first solve for equilibrium prices when myopic traders hold conditional mean-variance optimal portfolios and have no hedging demand.[5]

THEOREM 1 (Equilibrium with Myopic Conditional Mean-Variance Optimizers): *At $t = 1, 2$, (i) trader n's myopic mean-variance optimal inventory is*

$$\check{S}_{n1} = \frac{E_1^n\left[\check{P}_2\right] - \check{P}_1}{A\ \mathrm{var}_1^n\left[\check{P}_2\right]}, \qquad \check{S}_{n2} = \frac{E_2^n[v] - \check{P}_2}{A\ \mathrm{var}_2^n[v]}, \tag{7}$$

and (ii) the prices $\check{P}_1$ and $\check{P}_2$ are

$$\check{P}_1 = \tfrac{1}{N} \sum_{n=1}^{N} E_1^n\left[\check{P}_2\right] = \check{C}_g \tfrac{1}{N} \sum_{n=1}^{N} E_1^n[v], \qquad \check{P}_2 = \tfrac{1}{N} \sum_{n=1}^{N} E_2^n[v], \tag{8}$$

where the coefficient $\check{C}_g$ satisfies $0 < \check{C}_g < 1$ and is given by

$$\check{C}_g := 1 - \frac{\tau_v}{\tau_2}\left(1 - \tfrac{1}{N}\right)\left(\tau_H^{1/2} - \tau_L^{1/2}\right)^2. \tag{9}$$

PROOF: The proof is in Appendix subsection A. □

Since traders do not have any hedging demand, equation (7) implies that each trader's optimal inventory at $t = 1$ is his short-term speculative position. Equation (8) implies that the equilibrium price at $t = 1$ is a weighted average of traders' expectations of the fundamental value v with weights summing to a constant $\check{C}_g$ that is less than one.

[5] The results are equivalent to the results if we assume that traders must consume their period t trading gains at period t ($t = 2, 3$), so they choose optimal inventory S_{nt} ($t = 1, 2$) to maximize $\mathrm{E}_1^n[-\mathrm{e}^{-A\,W_{n2}}]$ and $\mathrm{E}_2^n[-\mathrm{e}^{-A\,W_{n3}}]$, where W_{n2} and W_{n3} are $W_{n2} = W_{n1} + (P_2 - P_1)\,S_{n1}$ and $W_{n3} = (v - P_2)\,S_{n2}$.

In our setting, traders agree to disagree about the interpretation of signals in the present, and they also agree to disagree about how to interpret the signals arriving in the future. More specifically, each trader n observes both i_{nt} and the average of others' signal i_{-nt} $(t = 1, 2)$, and he believes that both his current and future signals have high precision and other traders' current and future signals have low precision. As a result, even if all traders happen to agree about the fundamental value at time $t = 1$ (so $\mathrm{E}_1^n[v] = \mathrm{E}_1^m[v] > 0$ for all $m \neq n$), trader n's current expectation of the average of others' future expectations $\mathrm{E}_1^n[\mathrm{E}_2^{-n}[v]]$ is lower than his own current and future expectations $\mathrm{E}_t^n[v]$, $t = 1, 2$, where the average of others' expectations at t, $\mathrm{E}_t^{-n}[v]$, is defined as

$$\mathrm{E}_t^{-n}[v] := \tfrac{1}{N-1} \sum_{m=1, m\neq n}^{N} \mathrm{E}_t^m[v]. \tag{10}$$

To see this, in Appendix subsection A we show that trader n's expectations of others' expectations next period can be written as

$$\mathrm{E}_1^n\big[\mathrm{E}_2^{-n}[v]\big] = \left(1 - \left(\tau_H^{1/2} - \tau_L^{1/2}\right)^2 \frac{\tau_v}{\tau_2}\right)\mathrm{E}_1^n[v] - \frac{\tau_1}{\tau_2}\left(\mathrm{E}_1^n[v] - \mathrm{E}_1^{-n}[v]\right). \tag{11}$$

At time $t = 1$, trader n's estimate of $\check{P}_2$ is

$$\mathrm{E}_1^n\left[\check{P}_2\right] = \mathrm{E}_1^n\left[\tfrac{1}{N} \sum_{n=1}^{N} \mathrm{E}_2^n[v]\right] = \frac{\tau_1}{\tau_2} \tfrac{1}{N} \sum_{n=1}^{N} \mathrm{E}_1^n[v] + \frac{\tau_v}{N\tau_2}\left(\tau_H^{1/2} + (N-1)\tau_L^{1/2}\right)^2 \mathrm{E}_1^n[v]. \tag{12}$$

If all traders happen to agree about the fundamental value at time $t = 1$ (so $\mathrm{E}_1^n[v] = \mathrm{E}_1^m[v] > 0$, for all $m \neq n$), we obtain

$$\mathrm{E}_1^n\left[\check{P}_2\right] = \left(1 - \frac{\tau_v}{\tau_2}\left(1 - \tfrac{1}{N}\right)\left(\tau_H^{1/2} - \tau_L^{1/2}\right)^2\right) \tfrac{1}{N} \sum_{n=1}^{N} \mathrm{E}_1^n[v] < \tfrac{1}{N} \sum_{n=1}^{N} \mathrm{E}_1^n[v]. \tag{13}$$

Equation (13) implies that each trader n expects the price to fall below fundamental value in the next period. In addition, at $t = 1$, each trader n expects his own optimal inventory next period to be

$$\mathrm{E}_1^n\left[\check{S}_{n2}\right] = \frac{(N-1)\tau_v}{A\,N}\left(\tau_H^{1/2} - \tau_L^{1/2}\right)^2 \mathrm{E}_1^n[v] + \frac{(N-1)\tau_1}{A\,N}\left(\mathrm{E}_1^n[v] - \mathrm{E}_1^{-n}[v]\right). \tag{14}$$

Equation (14) implies that, if all traders happen to agree about the fundamental value at time $t = 1$ (so $\mathrm{E}_1^n[v] = \mathrm{E}_1^m[v] > 0$ for all $m \neq n$), then each trader expects to buy next period because each trader expects prices to fall next period. Each trader believes that others make the mistake of attributing too much precision to their current signals. Each trader expects that, upon observing new information in the future, others will adjust their valuations toward unconditional levels, and thus each trader expects others to sell and prices to fall at $t = 2$.

Therefore, even if the price at $t=1$ were equal to the consensus fundamental valuation ($\check{P}_1 = \frac{1}{N}\sum_{n=1}^{N} \mathrm{E}_1^n[v]$), all traders would want to hold short positions ($\check{S}_{n1} < 0$) at $t=1$ because they would all expect prices to fall below fundamental value in the next period. With myopic mean-variance optimizers, the equilibrium price $\check{P}_1$ is traders' average expectations of next-period price $\check{P}_2$. Since all traders expect prices to fall below fundamental value in the next period (equation (13)), it follows that

$$\begin{aligned}\check{P}_1 &= \frac{1}{N}\sum_{n=1}^{N} \mathrm{E}_1^n\Big[\check{P}_2\Big] = \left(1 - \tfrac{\tau_v}{\tau_2}\left(1-\tfrac{1}{N}\right)\left(\tau_H^{1/2} - \tau_L^{1/2}\right)^2\right)\tfrac{1}{N}\sum_{n=1}^{N} \mathrm{E}_1^n[v] \\ &= \check{C}_g \tfrac{1}{N}\sum_{n=1}^{N} \mathrm{E}_1^n[v],\end{aligned} \tag{15}$$

where the dampening factor $\check{C}_g$ defined in equation (9) satisfies $0 < \check{C}_g < 1$.

To summarize, all traders internalize at $t=1$ their perceived disagreement about future valuations. If prices were equal to the consensus fundamental valuation, all traders would want to hold short positions at $t=1$. As a result, the equilibrium price is dampened in the first period relative to traders' average contemporaneous expectations about fundamental value, $\frac{1}{N}\sum_{n=1}^{N} \mathrm{E}_1^n[v]$. The price-dampening effect increases ($\check{C}_g$ falls) when the magnitude of disagreement τ_H/τ_L increases. Since traders have no hedging demand in this setting with myopic conditional mean-variance optimizers, the key driving force of price dampening is traders' short-term speculation due to their disagreement about future valuations, not hedging demand.

C. The Equilibrium of the Two-Period Model

We now solve for the general two-period trading model where trader n chooses the inventory S_{nt} ($t = 1, 2$) to maximize his expected utility (3). The following theorem characterizes the equilibrium in the two-period model.

THEOREM 2 (Equilibrium of the Two-Period Model): *(i) At $t=2$, trader n's optimal inventory S_{n2} and the price P_2 are*

$$S_{n2} = \frac{E_2^n[v] - P_2}{A\ \mathrm{var}_2^n[v]}, \qquad P_2 = \frac{1}{N}\sum_{n=1}^{N} E_2^n[v]. \tag{16}$$

(ii) At $t=1$, trader n's optimal inventory is

$$\begin{aligned} S_{n1} = &\frac{\mathrm{var}_1^n\left[E_2^n[v]-P_2\right]+\mathrm{var}_2^n[v]}{A\left(\mathrm{var}_1^n[P_2](\mathrm{var}_1^n\left[E_2^n[v]\right]+\mathrm{var}_2^n[v]) - \left(cov_1^n\left[P_2\,,\,E_2^n[v]\right]\right)^2\right)} \\ &\times\left(\left(E_1^n[P_2]-P_1\right) - \frac{cov_1^n\left[P_2\,,\,E_2^n[v]-P_2\right]}{\mathrm{var}_1^n\left[E_2^n[v]-P_2\right]+\mathrm{var}_2^n[v]}\left(E_1^n\left[E_2^n[v]-P_2\right]\right)\right). \end{aligned} \tag{17}$$

The price P_1 is a weighted average of traders' expectations of the fundamental value v with weights summing to a constant C_g, which is less than one:

$$
\begin{aligned}
P_1 &= \frac{1}{N}\sum_{n=1}^{N} E_1^n[P_2] - \frac{cov_1^n[P_2\ ,\ E_2^n[v]-P_2]}{\mathrm{var}_1^n[E_2^n[v]-P_2]+\mathrm{var}_2^n[v]}\left(\frac{1}{N}\sum_{n=1}^{N} E_1^n[v] - \frac{1}{N}\sum_{n=1}^{N} E_1^n[P_2]\right) \\
&= C_g \frac{1}{N}\sum_{n=1}^{N} E_1^n[v].
\end{aligned}
\tag{18}
$$

PROOF: The proof is in Appendix subsection B. Equation (A21) is a closed-form expression for the dampening factor C_g. □

In equation (17), trader n's short-term speculative position is increasing in his short-term expected price change, $\mathrm{E}_1^n[P_2] - P_1$. Hedging demand is decreasing in his expectation at $t = 1$ of the next period's return, $\mathrm{E}_1^n[\mathrm{E}_2^n[v] - P_2]$. Mathematically, this monotonicity results from the two factors expressed as ratios of variances and covariances both being positive:

$$
\begin{aligned}
&\mathrm{var}_1^n[P_2]\ \mathrm{var}_1^n[\mathrm{E}_2^n[v]] > \left(\mathrm{cov}_1^n[P_2\ ,\ \mathrm{E}_2^n[v]]\right)^2, \\
&\mathrm{cov}_1^n[P_2\ ,\ \mathrm{E}_2^n[v] - P_2] = \frac{\tau_v^2(N-1)}{\tau_1\tau_2^2N^2}\left(\tau_H^{1/2} + (N-1)\tau_L^{1/2}\right)^2\left(\tau_H^{1/2} - \tau_L^{1/2}\right)^2 > 0,
\end{aligned}
\tag{19}
$$

where the parameters τ_1 and τ_2 are defined as in equation (5). Economically, at $t = 1$, disagreement about the precision of future signals ($\tau_H \neq \tau_L$) generates positive perceived autocovariance of returns for trader n, $\mathrm{cov}_1^n[P_2\ ,\ \mathrm{E}_2^n[v] - P_2] > 0$. Therefore, in trader n's optimal demand of equation (17), the coefficient on next period's return $\mathrm{E}_1^n[\mathrm{E}_2^n[v] - P_2]$ is negative. Assume that trader n observes positive signals about the asset and thus would like to hold a long position. Due to disagreement about future valuation, each trader believes that others make the mistake of attributing too much precision to their signals and thus each trader expects the next period's return $\mathrm{E}_1^n[\mathrm{E}_2^n[v] - P_2]$ to be positive. Therefore, the hedging component in equation (17) is negative. The negative hedging demand reduces traders' demand, which, in equation (18), further pushes down the price at $t = 1$ relative to the case with myopic traders.

Without disagreement, hedging demand does not generate price dampening because a common prior ($\tau_H = \tau_L$) implies $\mathrm{cov}_1^n[P_2\ ,\ \mathrm{E}_2^n[v] - P_2] = 0$.

Equation (18) shows that the price P_1 is a weighted average of traders' estimates of the fundamental value of the asset ($\frac{1}{N}\sum_{n=1}^{N}\mathrm{E}_1^n[v]$) and traders' estimates of next period's price ($\frac{1}{N}\sum_{n=1}^{N}\mathrm{E}_1^n[P_2]$). Because traders expect prices to fall below fundamental value in the next period, the weight on $\frac{1}{N}\sum_{n=1}^{N}\mathrm{E}_1^n[v]$ is

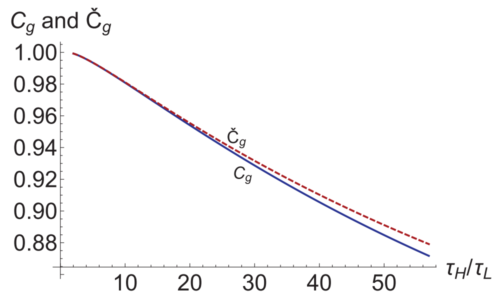

Figure 1. C_g and $\check{C}_g$ against τ_H/τ_L. (Color figure can be viewed at wileyonlinelibrary.com)

negative while the weight on $\frac{1}{N}\sum_{n=1}^{N}\mathrm{E}_1^n[P_2]$ is greater than one. We have

$$\begin{aligned} P_1 =& \frac{1}{N}\sum_{n=1}^{N}\mathrm{E}_1^n[P_2] - \frac{\mathrm{cov}_1^n\left[P_2\,,\,\mathrm{E}_2^n[v]-P_2\right]}{\mathrm{var}_1^n\left[\mathrm{E}_2^n[v]-P_2\right]+\mathrm{var}_2^n[v]}\left(\frac{1}{N}\sum_{n=1}^{N}\mathrm{E}_1^n[v] - \frac{1}{N}\sum_{n=1}^{N}\mathrm{E}_1^n[P_2]\right) \\ =& \check{C}_g\frac{1}{N}\sum_{n=1}^{N}\mathrm{E}_1^n[v] - \frac{\tau_v}{\tau_2}\left(1-\frac{1}{N}\right)\left(\tau_H^{1/2}-\tau_L^{1/2}\right)^2\frac{\mathrm{cov}_1^n\left[P_2\,,\,\mathrm{E}_2^n[v]-P_2\right]}{\mathrm{var}_1^n\left[\mathrm{E}_2^n[v]-P_2\right]+\mathrm{var}_2^n[v]}\,\frac{1}{N}\sum_{n=1}^{N}\mathrm{E}_1^n[v] \\ =& C_g\,\frac{1}{N}\sum_{n=1}^{N}\mathrm{E}_1^n[v]. \end{aligned} \tag{20}$$

Different from the case with myopic mean-variance optimizers (equation (15)), the first line of equation (20) states that price P_1 is the average expectation of the price P_2, adjusted for the extra trading due to hedging demand related to traders' expectations at $t = 1$ of the next period's return ($\mathrm{E}_1^n[\mathrm{E}_2^n[v] - P_2]$). The first term in the second line is the same as the second line of equation (15) in the myopic case. It shows that the average expectation of P_2 at $t = 1$ is dampened relative to the average expectation of fundamentals, $\frac{1}{N}\sum_{n=1}^{N}\mathrm{E}_1^n[v]$. The last line combines terms and shows that price P_1 is further dampened relative to the average expectations of fundamentals, $\frac{1}{N}\sum_{n=1}^{N}\mathrm{E}_1^n[v]$, compared to the myopic case; we prove analytically that the coefficient C_g satisfies $0 < C_g < 1$.

PROPOSITION 1: *The price-dampening factor C_g satisfies $0 < C_g < \check{C}_g < 1$. In addition, both C_g and $\check{C}_g$ decrease in the disagreement level τ_H/τ_L, holding fixed the total precision $\tau_H + (N-1)\tau_L$.*

PROOF: The proof is in Appendix subsection C. □

Figure 1 shows that price-dampening factors C_g and $\check{C}_g$ decrease in the disagreement level τ_H/τ_L.[6] This is because traders trade more aggressively

[6] Figures 1, 2, and 3 depend only on parameter values of τ_H, τ_L, and N. In all three figures, total precision is fixed, with $\tau_H + (N-1)\tau_L = 4$ and $N = 100$.

against one another with greater disagreement. As illustrated in Figure 1, the difference between C_g and $\check{C}_g$ is quite small for a large range of parameter values. This suggests that the magnitude of the additional dampening effect from hedging demand at $t = 1$ is small, and the main driving force of the dampening effect comes from traders' short-term speculation on their disagreement about future valuation.

As in the simple myopic case, traders agree to disagree about the interpretation of signals in the present, and also about how to interpret the signals arriving in the future. All traders internalize in the present their perceived disagreement about future valuations. In aggregate, the equilibrium price becomes dampened in the first period relative to traders' average contemporaneous expectations about the fundamental value, $\frac{1}{N}\sum_{n=1}^{N}\mathrm{E}_1^n[v]$. The price-dampening effect increases (C_g decreases) in the magnitude of disagreement τ_H/τ_L.

For example, even if all traders expect the fundamental value to be \$30 at $t = 3$, each trader also expects others to interpret new information incorrectly and thus to revise expectations upon receiving a new signal. If prices were equal to the consensus fundamental valuation of \$30, all traders would expect prices to fall in the short run and thus would trade on short-term profit opportunities by "selling" today to buy at better (lower) prices in the next round. The internalization of future disagreement dampens the price today by driving it below \$30, even when all traders correctly agree that \$30 is an accurate estimate of fundamental value.

There is also a static dampening effect. At $t = 1$, 2, traders' average valuation in equations (8) and (18), $\frac{1}{N}\sum_{n=1}^{N}\mathrm{E}_t^n[v]$, differs from what it would be in a model with a common prior where the same amount of information $\tau_H + (N-1)\,\tau_L$ is known to be split equally across traders. Each trader believes that signals of others have lower precision, and this disagreement pushes down the weight on each private signal in the average valuation $\frac{1}{N}\sum_{n=1}^{N}\mathrm{E}_t^n[v]$. Specifically, the average valuation can be expressed as

$$\frac{1}{N}\sum_{n=1}^{N}\mathrm{E}_t^n[v] = \frac{\tau_v^{1/2}}{\tau_t}\Big(\frac{1}{N}\,\tau_H^{1/2} + \frac{N-1}{N}\,\tau_L^{1/2}\Big)\sum_{n=1}^{N}\sum_{s=1}^{t} i_{ns} = \frac{\tau_v^{1/2}}{\tau_t}C_J\big(\frac{1}{N}\tau_H + \frac{N-1}{N}\tau_L\big)^{1/2}\sum_{n=1}^{N}\sum_{s=1}^{t} i_{ns}, \quad (21)$$

where

$$C_J := \Big(\frac{1}{N}\,\tau_H^{1/2} + \frac{N-1}{N}\,\tau_L^{1/2}\Big)\big(\frac{1}{N}\,\tau_H + \frac{N-1}{N}\,\tau_L\big)^{-1/2}. \quad (22)$$

When traders are relatively overconfident ($\tau_H > \tau_L$), Jensen's inequality implies $C_J < 1$. In Internet Appendix Section I, we show that static dampening of private signals in the averages of expectations shows up in an analogous one-period model; it is unrelated to dynamic trading. In Internet Appendix Section VII, we develop a model in which static dampening does not occur but prices are dampened and returns exhibit momentum patterns due to $C_g < 1$.

D. Return Autocorrelations

D.1. The Myopic Case with $\tau_L = 0$

To see how price dampening implies time-series return momentum, it is helpful to first study the myopic case with $\tau_L = 0$. In the special case of $\tau_L = 0$, the coefficient on v in the signal i_{mt} collapses to zero and the coefficient on the noise term stays at one. This case implies that each trader believes that all other traders trade on noise as if it were information. Using equation (11), it can be shown that each trader n's expectation at $t = 1$ of the average of others' expectations at $t = 2$ is

$$\mathrm{E}_1^n\big[\mathrm{E}_2^{-n}[v]\big] = \frac{\tau_1}{\tau_2}\mathrm{E}_1^{-n}[v]. \tag{23}$$

Since $\tau_1 = \tau_v(1 + \tau_H)$ and $\tau_2 = \tau_v(1 + 2\tau_H)$, equation (23) implies that trader n believes that others' average valuations tend to mean-revert toward the unconditional level of zero. We have

$$\begin{aligned} \mathrm{E}_1^n[\check{P}_2] &= \mathrm{E}_1^n\big[\tfrac{1}{N}\mathrm{E}_2^n[v] + \tfrac{N-1}{N}\mathrm{E}_2^{-n}[v]\big] = \tfrac{1}{N}\mathrm{E}_1^n[v] + \tfrac{N-1}{N}\tfrac{\tau_1}{\tau_2}\mathrm{E}_1^{-n}[v], \\ \check{P}_1 &= \tfrac{1}{N}\sum_{n=1}^{N}\mathrm{E}_1^n[\check{P}_2] = \big(\tfrac{1}{N} + \tfrac{N-1}{N}\tfrac{\tau_1}{\tau_2}\big)\big(\tfrac{1}{N}\mathrm{E}_1^n[v] + \tfrac{N-1}{N}\mathrm{E}_1^{-n}[v]\big). \end{aligned} \tag{24}$$

The coefficient on other traders' valuation $\mathrm{E}_1^{-n}[v]$ in $\mathrm{E}_1^n[\check{P}_2]$ is dampened by a factor of $\frac{\tau_1}{\tau_2}$, compared to the coefficient on $\mathrm{E}_1^{-n}[v]$ in the fundamental value of the asset, $\frac{1}{N}\mathrm{E}_1^n[v] + \frac{N-1}{N}\mathrm{E}_1^{-n}[v]$. The coefficient on trader n's own valuation $\mathrm{E}_1^n[v]$ in $\mathrm{E}_1^n[\check{P}_2]$ is not dampened and is the same as the coefficient on $\mathrm{E}_1^n[v]$ in the fundamental value $\frac{1}{N}\mathrm{E}_1^n[v] + \frac{N-1}{N}\mathrm{E}_1^{-n}[v]$. In the myopic model, the equilibrium price at $t = 1$ is the average of all traders' expectations of next period's price, $\check{P}_1 = \frac{1}{N}\sum_{n=1}^{N}\mathrm{E}_1^n[\check{P}_2]$, and hence it follows that the price at $t = 1$ is dampened relative to traders' average expectations about the fundamental value, $\frac{1}{N}\mathrm{E}_1^n[v] + \frac{N-1}{N}\mathrm{E}_1^{-n}[v]$. The dampening coefficient on $\frac{1}{N}\mathrm{E}_1^n[v] + \frac{N-1}{N}\mathrm{E}_1^{-n}[v]$ in $\check{P}_1$ is $\frac{1}{N} + \frac{N-1}{N}\frac{\tau_1}{\tau_2}$, which is greater than $\frac{\tau_1}{\tau_2}$ and less than one, implying that the coefficient on other traders' valuation $\mathrm{E}_1^{-n}[v]$ in $\check{P}_1$ is dampened by a smaller magnitude compared to that in $\mathrm{E}_1^n[\check{P}_2]$. It follows that trader n's expectation of the price change is

$$\mathrm{E}_1^n[\check{P}_2] - \check{P}_1 = \tfrac{(N-1)(\tau_2-\tau_1)}{N^2\tau_2}\big(\mathrm{E}_1^n[v] - \mathrm{E}_1^{-n}[v]\big). \tag{25}$$

Since traders are ex ante identical and the prior mean of v is zero, it follows that $\check{P}_0 = 0$. Under trader n's belief, it is straightforward to show that

$$\begin{aligned}\mathrm{cov}^n\left[\check{P}_2 - \check{P}_1\,,\ \check{P}_1 - \check{P}_0\right] &= \mathrm{E}_0^n\big[(\mathrm{E}_1^n[\check{P}_2] - \check{P}_1)(\check{P}_1 - \check{P}_0)\big] - \mathrm{E}_0^n\big[\check{P}_2 - \check{P}_1\big]\mathrm{E}_0^n\big[\check{P}_1 - \check{P}_0\big] \\ &= \mathrm{E}_0^n\big[(\mathrm{E}_1^n[\check{P}_2] - \check{P}_1)\check{P}_1\big] \\ &= \tfrac{(N-1)(\tau_2-\tau_1)((N-1)\tau_1+\tau_2)}{N^4\tau_2^2}\mathrm{E}_0^n\big[\big(\mathrm{E}_1^n[v] - \mathrm{E}_1^{-n}[v]\big)\big(\mathrm{E}_1^n[v] + (N-1)\mathrm{E}_1^{-n}[v]\big)\big].\end{aligned} \tag{26}$$

The second line of equation (26) follows from the law of iterated expectations and $\mathrm{E}_0^n\big[\check{P}_2 - \check{P}_1\big] = 0$. The third line of equation (26) follows from equation (25). Since each trader n believes that his signals have high precision and other traders' signals have low precision, the coefficient on v in the difference between trader n's valuation and the average of other traders' valuations is positive,

$$\mathrm{E}_1^n[v] - \mathrm{E}_1^{-n}[v] = \tfrac{\tau_v^{1/2}\tau_H^{1/2}}{\tau_1}(i_{n1} - i_{-n1}) = \tfrac{\tau_v^{1/2}\tau_H^{1/2}}{\tau_1}\Big(\tau_H^{1/2}\tau_v^{1/2}v + e_{n1} - \tfrac{1}{N-1}\sum_{m=1,m\neq n}^{N} e_{m1}\Big). \tag{27}$$

Equation (27) implies that the difference between traders' valuations $\mathrm{E}_1^n[v] - \mathrm{E}_1^{-n}[v]$ increases in v. In addition, $\mathrm{E}_1^n[v] + (N-1)\mathrm{E}_1^{-n}[v]$ also increases in the fundamental value v. As a result, the covariance between $\mathrm{E}_1^n[v] - \mathrm{E}_1^{-n}[v]$ and $\mathrm{E}_1^n[v] + (N-1)\mathrm{E}_1^{-n}[v]$ is positive,

$$\mathrm{E}_0^n\big[\big(\mathrm{E}_1^n[v] - \mathrm{E}_1^{-n}[v]\big)\big(\mathrm{E}_1^n[v] + (N-1)\mathrm{E}_1^{-n}[v]\big)\big] = \tfrac{\tau_v\tau_H^2}{\tau_1^2} > 0. \tag{28}$$

Substituting equation (28) into (26) yields

$$\mathrm{cov}^n\left[\check{P}_2 - \check{P}_1\,,\ \check{P}_1 - \check{P}_0\right] = \tfrac{(N-1)(\tau_2-\tau_1)((N-1)\tau_1+\tau_2)\tau_H^2\tau_v}{N^4\tau_1^2\tau_2^2} > 0. \tag{29}$$

To summarize, agreement to disagree about future valuations dampens the price in the first period relative to traders' average expectations about the fundamental value. In the simple myopic case, the price dampening implies that trader n's expectation at date 1 of the price change from date 1 to date 2 is proportional to the difference between traders' valuations $\mathrm{E}_1^n[v] - \mathrm{E}_1^{-n}[v]$, which tends to be positively correlated with the fundamental value v as well as with the price at date 1, implying time-series return momentum.

D.2. The General Case

Traders use models with correct structure but with possibly incorrect parameters. While traders' possibly incorrect parameters affect prices, the properties of return dynamics, such as autocorrelations at different horizons, also depend on empirically correct model specification of private information. Therefore, the properties of return dynamics are functions of both traders' (subjective) parameters *and* empirically correct (objective) parameters.

In this subsection, we first show that each trader believes that returns exhibit momentum in the general case in which traders may not be myopic and τ_L may not be zero. We then show that there is also positive return autocorrelation under empirically correct beliefs that traders are symmetrically informed and correct on average.

PROPOSITION 2: *Each trader believes that prices exhibit momentum, defined as the covariance of price changes being positive in both the myopic model and the general two-period model:*

$$\begin{aligned} cov^n\left[\check{P}_2 - \check{P}_1\ ,\ \check{P}_1 - \check{P}_0\right] > 0, \qquad & cov^n\left[v - \check{P}_2\ ,\ \check{P}_2 - \check{P}_1\right] > 0, \\ cov^n\left[P_2 - P_1\ ,\ P_1 - P_0\right] > 0, \qquad & cov^n\left[v - P_2\ ,\ P_2 - P_1\right] > 0. \end{aligned} \tag{30}$$

If traders are "correct on average" in the sense that the empirically correct precision of each trader's signal is $\hat{\tau}_n := \frac{1}{N}\tau_H + \frac{N-1}{N}\tau_L$, then returns exhibit momentum:

$$\begin{aligned} cov\left[\check{P}_2 - \check{P}_1\ ,\ \check{P}_1 - \check{P}_0\right] > 0, \qquad & cov\left[v - \check{P}_2\ ,\ \check{P}_2 - \check{P}_1\right] > 0, \\ cov\left[P_2 - P_1\ ,\ P_1 - P_0\right] > 0, \qquad & cov\left[v - P_2\ ,\ P_2 - P_1\right] > 0. \end{aligned} \tag{31}$$

PROOF: We include the proof of $\mathrm{cov}^n[\check{P}_2 - \check{P}_1\ ,\ \check{P}_1 - \check{P}_0] > 0$ and $\mathrm{cov}^n[P_2 - P_1\ ,\ P_1 - P_0] > 0$ below. The proof of $\mathrm{cov}^n[v - \check{P}_2\ ,\ \check{P}_2 - \check{P}_1] > 0$ and $\mathrm{cov}^n[v - P_2\ ,\ P_2 - P_1] > 0$ is similar and is in Appendix subsection D. Equations (A29) and (A30) present closed-form expressions for return covariances under empirically correct beliefs. □

Positive return autocovariance comes from the dampening factor C_g in the equation for price P_1 satisfying $0 < C_g < 1$, not $C_g = 1$. Assume traders observe positive signals. Each trader n anticipates that the price at $t = 2$ is lower than his own valuation ($\mathrm{E}_2^n[v]$) and expects to buy at $t = 2$ since trader n believes that others make the mistake of attributing too much precision to their signals. In aggregate, the equilibrium price becomes dampened in the first period relative to traders' average expectations about the fundamental value.

In the myopic case, using equation (12), it can be shown that

$$\mathrm{E}_1^n[\check{P}_2] - \check{P}_1 = \frac{(N-1)\left(\tau_H^{1/2} + (N-1)\tau_L^{1/2}\right)^2 \tau_v}{N^2 \tau_2}\left(\mathrm{E}_1^n[v] - \mathrm{E}_1^{-n}[v]\right). \tag{32}$$

Substituting equation (32) into the second line of equation (26), we have

$$\mathrm{cov}^n\left[\check{P}_2 - \check{P}_1\ ,\ \check{P}_1 - \check{P}_0\right] = \frac{\check{C}_g(N-1)\left(\tau_H^{1/2} + (N-1)\tau_L^{1/2}\right)^2 \tau_v}{N^3 \tau_2} \mathrm{E}_0^n\left[\left(\mathrm{E}_1^n[v] - \mathrm{E}_1^{-n}[v]\right)\left(\mathrm{E}_1^n[v] + (N-1)\mathrm{E}_1^{-n}[v]\right)\right]. \tag{33}$$

Similar to the special case $\tau_L = 0$, the coefficient on v in the difference between trader n's valuation and the average of other traders' valuations is positive. The covariance between $\mathrm{E}_1^n[v] - \mathrm{E}_1^{-n}[v]$ and $\mathrm{E}_1^n[v] + (N-1)\mathrm{E}_1^{-n}[v]$ is positive:

$$\mathrm{E}_0^n\big[\big(\mathrm{E}_1^n[v] - \mathrm{E}_1^{-n}[v]\big)\big(\mathrm{E}_1^n[v] + (N-1)\mathrm{E}_1^{-n}[v]\big)\big] = \tfrac{\tau_v}{\tau_1^2}\big(\tau_H^{1/2} - \tau_L^{1/2}\big)^2\big(\tau_H^{1/2} + (N-1)\tau_L^{1/2}\big)^2 > 0. \tag{34}$$

Substituting equation (34) into (33) yields

$$\mathrm{cov}^n\Big[\check{P}_2 - \check{P}_1\ ,\ \check{P}_1 - \check{P}_0\Big] = \frac{\check{C}_g(N-1)\Big(\tau_H^{1/2}+(N-1)\tau_L^{1/2}\Big)^4\Big(\tau_H^{1/2}-\tau_L^{1/2}\Big)^2\tau_v^2}{N^3\tau_1^2\tau_2} > 0. \tag{35}$$

In the general two-period model, due to the negative hedging demand, the equilibrium price P_1 is further dampened relative to the average expectations of fundamentals, compared to the myopic case. It follows that

$$\mathrm{E}_1^n[P_2] - P_1 = \mathrm{E}_1^n[\check{P}_2] - \check{P}_1 + \tfrac{\tau_v}{\tau_2}\Big(1 - \tfrac{1}{N}\Big)\Big(\tau_H^{1/2} - \tau_L^{1/2}\Big)^2 \frac{\mathrm{cov}_1^n[P_2\ ,\ \mathrm{E}_2^n[v]-P_2]}{\mathrm{var}_1^n[\mathrm{E}_2^n[v]-P_2]+\mathrm{var}_2^n[v]}\,\frac{1}{N}\sum_{n=1}^{N}\mathrm{E}_1^n[v]. \tag{36}$$

At $t = 1$, disagreement about the precision of future signals ($\tau_H \neq \tau_L$) generates positive perceived autocovariance of returns for trader n, $\mathrm{cov}_1^n[P_2\ ,\ \mathrm{E}_2^n[v] - P_2] > 0$. Therefore, the coefficient on v in $\mathrm{E}_1^n[P_2] - P_1$ is a larger positive number than that of the myopic model. Similar to the myopic case, we show that $\mathrm{cov}^n[P_2 - P_1\ ,\ P_1 - P_0] = \mathrm{E}_0^n\big[(\mathrm{E}_1^n[P_2] - P_1)P_1\big]$. Using equation (36), we show that the autocovariance of returns in the general two-period model is

$$\begin{aligned}&\mathrm{cov}^n[P_2 - P_1\ ,\ P_1 - P_0]\\ &= \tfrac{C_g}{\check{C}_g}\mathrm{cov}^n\Big[\check{P}_2 - \check{P}_1\ ,\ \check{P}_1 - \check{P}_0\Big] + \frac{C_g(N-1)\Big(\tau_H^{1/2}-\tau_L^{1/2}\Big)^2\tau_v}{N^3\tau_2}\,\frac{\mathrm{cov}_1^n[P_2\ ,\ \mathrm{E}_2^n[v]-P_2]}{\mathrm{var}_1^n[\mathrm{E}_2^n[v]-P_2]+\mathrm{var}_2^n[v]}\mathrm{var}_0^n\big[\mathrm{E}_1^n[v]\\ &\quad + (N-1)\mathrm{E}_1^{-n}[v]\big].\end{aligned} \tag{37}$$

From the myopic case, the first term of the second line of equation (37) is positive. The second term of the second line of equation (37) is also positive because the perceived autocovariance of returns for trader n, $\mathrm{cov}_1^n[P_2\ ,\ \mathrm{E}_2^n[v] - P_2]$, is positive. Therefore, the return covariance in the general two-period model, $\mathrm{cov}^n[P_2 - P_1\ ,\ P_1 - P_0]$, is positive.

To summarize, in the general two-period model, the price at date 1 is further dampened due to hedging demand. This implies that trader n's expectation at date 1 of the price change from date 1 to date 2 increases in v with a larger coefficient on v than in the myopic model. Therefore, trader n's expectation at date 1 of the price change from date 1 to date 2 is positively correlated with the price at date 1, implying time-series return momentum in the general two-period model.

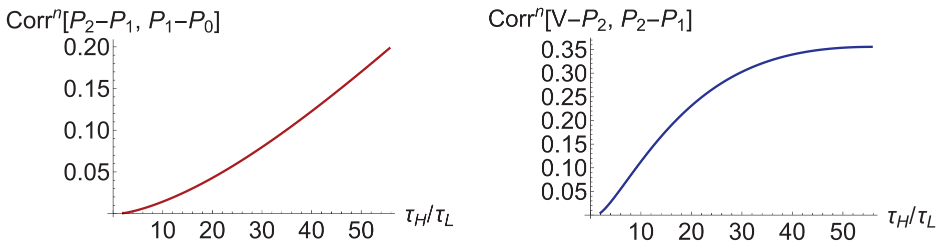

Figure 2. Autocorrelation of returns for different disagreement levels τ_H/τ_L. (Color figure can be viewed at wileyonlinelibrary.com)

The equilibrium prices and trading strategies are determined by traders' expectations based on their beliefs. Prices are the same functions of signals i_{nt} under both traders' (subjective) parameters *and* empirically correct (objective) parameters:

$$P_1 = C_g \tfrac{1}{N} \sum_{n=1}^{N} \mathrm{E}_1^n[v] = C_g \frac{\tau_v^{1/2}}{\tau_1}\left(\tau_H^{1/2} + (N-1)\tau_L^{1/2}\right) \tfrac{1}{N} \sum_{i=1}^{N} i_{n1},$$
$$P_2 = \tfrac{1}{N} \sum_{n=1}^{N} \mathrm{E}_2^n[v] = \frac{\tau_v^{1/2}}{\tau_2}\left(\tau_H^{1/2} + (N-1)\tau_L^{1/2}\right)\left(\tfrac{1}{N} \sum_{i=1}^{N} i_{n1} + \tfrac{1}{N} \sum_{i=1}^{N} i_{n2}\right). \tag{38}$$

Traders' distributions for signals are different from the objective distribution of signals. Under the empirically correct parameters that all private signals have the same precision, the average of private signals under the empirically correct belief is

$$\tfrac{1}{N} \sum_{n=1}^{N} \hat{i}_{nt} = \left(\tfrac{\tau_H + (N-1)\tau_L}{N}\right)^{1/2} \tau_v^{1/2} v + \tfrac{1}{N} \sum_{n=1}^{N} e_{nt}. \tag{39}$$

Substituting equation (39) into (38), we show analytically that, under empirically correct parameters, return covariances $\mathrm{cov}[v - P_2\ ,\ P_2 - P_1]$ and $\mathrm{cov}[P_2 - P_1\ ,\ P_1 - P_0]$ are also positive.

Figures 2 and 3 depict how return autocorrelation changes with the disagreement level τ_H/τ_L while holding fixed the total precision of the signals. Since the price-dampening effect increases (C_g decreases) in the magnitude of disagreement τ_H/τ_L, return autocorrelation tends to increase in the disagreement level under both traders' beliefs and the empirically correct beliefs, as illustrated in Figures 2 and 3.

E. Iterated Average Expectations

In this subsection, we discuss how price dampening is related to iterating average expectations. Let $\bar{\mathrm{E}}_t$ denote the average expectations operator $\bar{\mathrm{E}}_t[\cdot] := \frac{1}{N}\ \sum_{n=1}^{N} \mathrm{E}_t^n[\cdot]$. We say that "average expectations iterate" if $\bar{\mathrm{E}}_1\left[\bar{\mathrm{E}}_2[v]\right] = \bar{\mathrm{E}}_1[v]$.

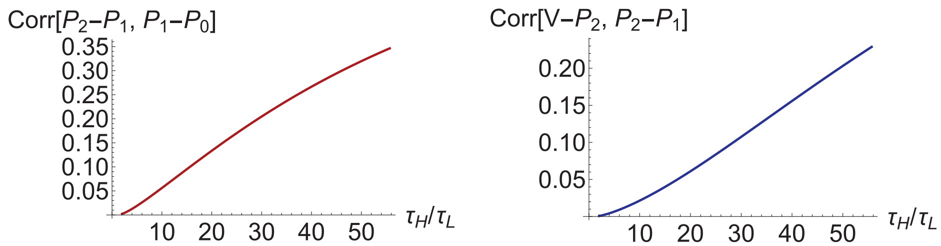

Figure 3. Autocorrelation of returns under empirically correct parameters for different τ_H/τ_L. (Color figure can be viewed at wileyonlinelibrary.com)

With myopic mean-variance optimizers, equation (8) implies that average expectations do not iterate when there is price dampening ($\check{C}_g < 1$):

$$\check{P}_1 = \frac{1}{N}\sum_{n=1}^{N} \mathrm{E}_1^n\left[\check{P}_2\right] = \bar{\mathrm{E}}_1\left[\bar{\mathrm{E}}_2[v]\right] = \check{C}_g\bar{\mathrm{E}}_1[v] < \bar{\mathrm{E}}_1[v]. \tag{40}$$

In the general two-period model, equation (20) similarly implies that average expectations do not iterate when there is price dampening ($C_g < 1$):

$$P_1 = \bar{\mathrm{E}}_1\left[\bar{\mathrm{E}}_2[v]\right] - \frac{\mathrm{cov}_1^n\left[P_2\,,\,\mathrm{E}_2^n[v]-P_2\right]}{\mathrm{var}_1^n\left[\mathrm{E}_2^n[v]-P_2\right]+\mathrm{var}_2^n[v]}\left(\bar{\mathrm{E}}_1[v]-\bar{\mathrm{E}}_1\left[\bar{\mathrm{E}}_2[v]\right]\right) = C_g\bar{\mathrm{E}}_1[v] < \bar{\mathrm{E}}_1[v]. \tag{41}$$

For expectations to iterate in equations (40) and (41), it must be the case that $\check{C}_g = C_g = 1$ and thus $\check{P}_1 = P_1 = \bar{\mathrm{E}}_1[v]$. This requires $\tau_H = \tau_L$, which implies that traders share a common prior and also share common information because each trader infers sufficient statistics for other traders' information from prices. Allen, Morris, and Shin (2006) find that average expectations do not iterate when traders condition on different noisy signals. In their noisy rational expectations model, returns exhibit mean reversion, not momentum.

In models without a common prior, the following proposition provides conditions under which average expectations do iterate.

PROPOSITION 3:

(i) If traders agree about the conditional distribution of v given signals, then $E_1^n\left[E_2^m[v]\right] = E_1^n[v]$ for any m and n and thus average expectations iterate, $\bar{E}_1\left[\bar{E}_2[v]\right] = \bar{E}_1[v]$.

(ii) If traders agree about the joint distribution of signals, then $E_1^n\left[E_2^m[v]\right] = E_1^m[v]$ for any m and n and thus average expectations iterate, $\bar{E}_1\left[\bar{E}_2[v]\right] = \bar{E}_1[v]$.

(iii) If traders agree about the conditional distribution of v given signals or they agree about the joint distribution of signals, then we have $P_1 = E_1^n[P_2]$ for any n and thus traders believe that there is no return predictability.

PROOF: The proof is in Appendix subsection E. □

Proposition 3 is similar to proposition 2 of Banerjee, Kaniel, and Kremer (2009, p. 3716), which says that if traders have correct beliefs about the joint distribution of signals but disagree about the joint distribution of signals and prices, then there is no price drift. Our Proposition 3 provides conditions under which traders believe there is no drift. If traders' beliefs are correct, this trivially also implies no autocorrelation under correct beliefs.

In general, when expectations do not iterate, there may or may not be price dampening and drift. In Internet Appendix Section VI, we construct a model in which the average of expectations do not iterate, but under some conditions there is no price dampening or price drift.

II. A Dynamic Model

We next describe a competitive model in which information arrives continuously but trading takes place at discrete intervals. The dynamic steady-state model can be calibrated to generate quantitatively realistic empirical patterns of return dynamics. In addition, extending to a dynamic trading model allows us to show that return momentum is more pronounced when traders trade more frequently.

The price aggregates traders' heterogeneous beliefs and private information. Given their beliefs, traders behave in a rational manner. They collect public and private information, construct signals from available information, apply Bayes' Law to predict returns, and calculate optimal holdings. However, traders are collectively irrational in that each of them is relatively overconfident, believing that the precision of his own private information flow is greater than other traders believe it to be.

A. Model Setup

Both fundamentals and information evolve continuously over the time interval $t \in (-\infty, \infty)$. Trading takes place at discrete dates $t = kh$, where $h > 0$ is the time interval between each round of trading, and k indexes time periods $k = \cdots, -2, -1, 0, 1, 2, \ldots$. Varying h allows us to examine how trading frequency affects equilibrium prices and quantities, with continuous information flows remaining the same. There are N risk-averse perfect competitors who trade a risky asset against a risk-free asset at price P_k at times $t = kh$. The risky security is in zero net supply, and the risk-free asset earns constant risk-free rate $r > 0$.

The risky asset pays dividends at continuous rate $D(t)$. Dividends follow a stochastic process with mean-reverting stochastic growth rate $G^*(t)$, constant instantaneous volatility $\sigma_D > 0$, and constant rate of mean reversion $\alpha_D > 0$,

$$\mathrm{d}D(t) := -\alpha_D\, D(t)\ \mathrm{d}t + G^*(t)\ \mathrm{d}t + \sigma_D\ \mathrm{d}B_D(t). \tag{42}$$

The growth rate $G^*(t)$ follows an AR-1 process with mean reversion α_G and volatility σ_G:[7]

$$\mathrm{d}G^*(t) := -\alpha_G\, G^*(t)\ \mathrm{d}t + \sigma_G\ \mathrm{d}B_G(t). \tag{43}$$

The dividend stream $D(t)$ is publicly observable. The growth rate $G^*(t)$, marked with a star superscript, is not observed by traders. Each trader uses available information to estimate its value. The model parameters α_D, σ_D, α_G, and σ_G are all common knowledge among traders.

If the dividend $D(t)$ and the growth rate $G^*(t)$ were observable, then the price would equal its fundamental value from a generalization of Gordon's growth formula,

$$F(t) = \frac{D(t)}{r+\alpha_D} + \frac{G^*(t)}{(r+\alpha_D)(r+\alpha_G)}. \tag{44}$$

Since traders do not observe $G^*(t)$, their valuations replace $G^*(t)$ with an estimate based on available information.

Each trader n observes a continuous stream of private information $\mathrm{d}I_n(t)$, with precision τ_n, about the unobservable growth rate $G^*(t)$,

$$\mathrm{d}I_n(t) := \tau_n^{1/2}\, \frac{G^*(t)}{\sigma_G\ \Omega^{1/2}}\ \mathrm{d}t + \mathrm{d}B_n(t). \tag{45}$$

Each increment $\mathrm{d}I_n(t)$ in equation (45) is a noisy observation of the unobserved scaled growth rate $G^*(t)$. The scaling factor $\Omega^{1/2}$ is defined in equation (51) so that τ_n can be interpreted as the rate at which traders learn from signal $\mathrm{d}I_n(t)$. Since our model is stationary, Ω is constant. Since traders learn from prices that reveal other traders' signals, the definition of Ω depends on how traders learn from both signals and prices.[8]

The symmetry of the model implies that traders infer from the equilibrium price the average of other traders' information flow about the growth rate,

$$\mathrm{d}I_{-n}(t) := \tfrac{1}{N-1} \sum_{m=1, m\neq n}^{N} \mathrm{d}I_m(t). \tag{46}$$

Each trader n is certain that his own private information $I_n(t)$ has high precision $\tau_n = \tau_H$, and the other traders' private information has low preci-

[7] We use a CARA-normal setting and thus $G^*(t)$ can be described as a growth rate in dollar terms (not a percentage growth rate). The unit of the dividend $D(t)$ is *dollars per share per time* ($\$/s/T$) and the unit of the growth rate $G^*(t)$ is *dollar per share per time squared* ($\$/s/T^2$).

[8] In addition to scaling the drift term by σ_G, we scale it by $\Omega^{1/2}$. The scaling is chosen so that linear regression of $G^*(t)$ on $\mathrm{d}I_n(t)$ has constant incremental instantaneous R^2 equal to $\tau_n\ \mathrm{d}t$. Traders learn from their own private signal at a constant rate τ_n, which measures the percentage rate at which new information reduces variance, reflected in equation (60). Without this scaling, if our model were nonstationary, traders would learn more about $G^*(t)$ in markets in which the prior variance of $G^*(t)$ is large and less in markets in which it is small.

sion $\tau_m = \tau_L$, for $m \neq n$, with $\tau_H > \tau_L \geq 0$.[9] Since this disagreement is common knowledge, relatively overconfident traders agree to disagree about the precisions of their signals. This disagreement generates trading. Traders believe that they can make profits at the expense of others, even though it is common knowledge that aggregate profits are equal to zero.

Each trader also makes inferences about the growth rate $G^*(t)$ from the publicly observable dividend stream $D(t)$ modeled in equation (42). To streamline notation for the information content of dividends, define $\mathrm{d}I_0(t) := (\alpha_D\, D(t)\ \mathrm{d}t + \mathrm{d}D(t))/\sigma_D$, where $\mathrm{d}B_0 := \mathrm{d}B_D$ and

$$\tau_0 := \frac{\Omega\, \sigma_G^2}{\sigma_D^2}. \tag{47}$$

Then, the stochastic process

$$\mathrm{d}I_0(t) := \tau_0^{1/2}\, \frac{G^*(t)}{\sigma_G\, \Omega^{1/2}}\ \mathrm{d}t + \mathrm{d}B_0(t) \tag{48}$$

is informationally equivalent to the dividend process $D(t)$ in equation (42). Assume that it is common knowledge that the Brownian motions $\mathrm{d}B_0$, $\mathrm{d}B_G$, $\mathrm{d}B_1, \ldots, \mathrm{d}B_N$ are independently distributed. Traders agree on the precision τ_0 of public information in equation (48).

Since each trader believes his own signal has high precision τ_H and others' signals have low precision τ_L, the symmetry implies that traders agree on the total precision τ of information flows in the model,

$$\tau := \tau_0 + \tau_H + (N-1)\,\tau_L. \tag{49}$$

Let $\mathrm{E}^n_{kh}[\cdots]$ denote the expectation of trader n calculated with respect to his beliefs about parameter values using information at time $t = kh$. This information consists of the history of continuous flows of public signals $\mathrm{d}I_0(j)$, continuous flows of private signals $\mathrm{d}I_n(j)$ with $j \in [-\infty, t]$, and discrete sequence of prices P_j, where $j \leq k$. Let

$$G_n(t) := \mathrm{E}^n_t\big[G^*(t)\big] \qquad \text{and} \qquad G_{n,k} := \mathrm{E}^n_{kh}\big[G^*(kh)\big] \tag{50}$$

[9] A typical way of modeling private information, $\mathrm{d}I_n(t) = G^*(t)\ \mathrm{d}t + \tau_n^{-1/2}\ \mathrm{d}B_n(t)$, would multiply the diffusion term $\mathrm{d}B_n(t)$ by $\tau_n^{-1/2}$, which measures both the standard deviation of noise and the precision of the signal. Our approach in equation (45) instead multiplies the drift term by its reciprocal $\tau_n^{1/2}$. Rescaling signals by the factor $\tau_n^{1/2}$ does not affect the equilibrium when traders agree about $\tau_n^{1/2}$ but does affect the equilibrium when traders disagree about its value. Using the typical approach in continuous time, a trader can estimate the diffusion variance, and therefore τ_n, with high accuracy by measuring the quadratic variation of $\mathrm{d}I_n(t)$ over short time intervals. This would conflict with our assumption that traders always disagree about the precision of their private signals. The information structure defined in equation (45) has the appealing feature that traders infer the correct diffusion variance var$[\mathrm{d}B_n(t)] = \mathrm{d}t$. By assuming that $\tau_n^{1/2}$ multiplies the drift term in equation (45), traders disagree about how to interpret parameters of the drift terms, which are much more difficult to estimate empirically.

denote trader n's estimate of the growth rate at time t and at discrete times $t = kh$, respectively. We use similar notation for other continuous and discrete variables.

Define Ω as the steady-state error variance of trader n's estimate of $G^*(t)$, scaled in units of the standard deviation of its innovation σ_G,

$$\Omega := \mathrm{var}\left[\frac{G^*(t) - G_n(t)}{\sigma_G}\right]. \tag{51}$$

The variable Ω has no time subscript k because it is constant in our steady-state model, where the total precision of traders' private information flow is fixed. Except for symmetrically disagreeing about the precisions τ_H and τ_L of signals, traders agree about all parameter values. Since symmetry implies that all traders agree about the variance Ω, it has no subscript n.

Each trader chooses optimal consumption and portfolio holdings to maximize an additively separable exponential utility function with risk aversion A and time preference ρ:

$$\mathrm{E}^n_{kh}\left[\int_{t=kh}^{\infty} \mathrm{e}^{-\rho(t-kh)}\, U(c_n(t))\ \mathrm{d}t\right]. \tag{52}$$

The optimization problem is complicated by the fact that consumption $c_n(t)$ is chosen continuously while portfolio holdings $S_{n,k}$ change only at discrete trading times $t = kh$. For analytical tractability, we simplify the problem slightly by assuming that when trading occurs at round j, each trader chooses both portfolio holdings $S_{n,j}$ and a consumption budget $hc_{n,j}$, which do not change until the next trading round. Thus, traders do not use new public or private information unfolding between trading rounds to adjust consumption between rounds. They cannot use other traders' private information between trading rounds either because there are no updated prices from which to infer the average of other traders' signals.

With these assumptions, the optimization problem becomes the discrete-time problem of choosing consumption budget $hc_{n,j}$ and holdings $S_{n,j}$ at trading times $t = jh$ to solve

$$\max_{[c_{n,j},S_{n,j}],\,j=k,k+1,\ldots,\infty} \quad \mathrm{E}^n_{kh}\left[\sum_{j=k}^{\infty} \mathrm{e}^{-\rho(j-k)h} U_{n,j}(c_{n,j})\right], \tag{53}$$

subject to the budget constraint at each trading round $j+1$,

$$W_{n,j+1} = \mathrm{e}^{rh}\big(W_{n,j} - hc_{n,j} - S_{n,j}P_j\big) + S_{n,j}\check{D}_{j+1} + S_{n,j}P_{j+1}, \tag{54}$$

where $\check{D}_{j+1}$ defines the future value of dividends between trading rounds j and $j+1$,

$$\check{D}_{j+1} = \mathrm{e}^{rh}\int_{jh}^{(j+1)h} \mathrm{e}^{-r(t-jh)} D(t)\,\mathrm{d}t, \tag{55}$$

and $U_{n,j}(c_{nj})$ solves the nested consumption subproblem between trading rounds j and $j+1$,

$$U_{n,j}(c_{n,j}) := \max_{c_n(\cdot)} \mathrm{E}^n_{jh}\left[- \int_0^h \mathrm{e}^{-\rho u}\, \mathrm{e}^{-Ac_n(jh+u)}\ \mathrm{d}u \right] \tag{56}$$

subject to consumption budget $hc_{n,j}$ allocated for the period between rounds j and $j+1$,

$$hc_{n,j} = \int_0^h \mathrm{e}^{-ru} c_n(jh+u)\,\mathrm{d}u. \tag{57}$$

Equations (53) to (57) summarize the optimization problem. As we prove in Lemma A1, Stratonovich-Kalman-Bucy filtering implies that trader n's estimate $G_{n,k}$ of the growth rate at period k can be conveniently written as the weighted sum of the three sufficient statistics $H_{0,k}$, $H_{n,k}$, and $H_{-n,k}$, which summarize the information content of dividends, trader n's private information, and other traders' private information, respectively, with

$$H_{n,k} := \int_{t=-\infty}^{kh} \mathrm{e}^{-(\alpha_G+\tau)\,(kh-t)}\ \mathrm{d}I_n(t), \qquad n = 0, 1, \ldots, N, \tag{58}$$

$$H_{-n,k} := \tfrac{1}{N-1} \sum_{m=1, m\neq n}^{N} H_{m,k}, \qquad n = 1, \ldots, N. \tag{59}$$

These formulas for sufficient statistics have an intuitive interpretation. The signal $H_{n,k}$ is a sufficient statistic for trader n's own information flow. The average signal $H_{-n,k}$ is a sufficient statistic for other traders' information flow. The importance of each bit of information $\mathrm{d}I_n$ decays exponentially at rate $\alpha_G+\tau$, the sum of the natural decay rate α_G of the growth rate and the speed τ at which traders learn about it. Although trader n does not observe other traders' private signals directly, the sufficient statistic $H_{-n,k}$ can be inferred from prices.

This filtering also implies that the steady-state error variance is

$$\Omega = \mathrm{var}\left[\frac{G^*(t) - G_n(t)}{\sigma_G}\right] = \frac{1}{2\,\alpha_G + \tau}, \tag{60}$$

and trader n's expected growth rate at $t = kh$ is

$$G_{n,k} := \mathrm{E}^n_{kh}\big[G^*(kh)\big] = \sigma_G\, \Omega^{1/2} \left(\tau_0^{1/2}\, H_{0,k} + \tau_H^{1/2}\, H_{n,k} + (N-1)\,\tau_L^{1/2}\, H_{-n,k}\right). \tag{61}$$

When forming his estimate, trader n assigns weight $\tau_0^{1/2}$ to the public signal $H_{0,k}$, weight $\tau_H^{1/2}$ to his own signal $H_{n,k}$, and a smaller weight $\tau_L^{1/2}$ to each of the other traders' signals $H_{m,k}$, for $m \neq n$. Trade occurs as a result of the symmetrically different weights used by traders in construction of their estimates.

Define the aggregate sufficient statistic H_k at period k, with common weights $\tau_I^{1/2}$, as

$$H_k := \tau_0^{1/2} H_{0,k} + \sum_{n=1}^{N} \tau_I^{1/2} H_{n,k}, \qquad \text{where} \qquad \tau_I^{1/2} := \tfrac{1}{N}\tau_H^{1/2} + \frac{N-1}{N}\tau_L^{1/2}. \quad (62)$$

Then the average estimate of the growth rate $\bar{G}_k$ in period k is proportional to H_k:

$$\bar{G}_k := \tfrac{1}{N}\sum_{n=1}^{N} G_{n,k} = \sigma_G\, \Omega^{1/2}\, H_k. \quad (63)$$

B. The Equilibrium of the Dynamic Trading Model

As in the two-period model, in Appendix subsection G, we solve for the equilibrium prices where myopic traders hold conditional mean-variance optimal portfolios. In this subsection, we solve for the dynamic trading model in which trader n chooses consumption and holdings to solve his optimal optimization problem (52). Each trader's optimal inventory is proportional to his own risk tolerance and the difference between his valuation and the average valuation of other traders, which he infers from prices. The following theorem characterizes equilibrium for the discrete-time dynamic model.

THEOREM 3 (Equilibrium of the Dynamic Trading Model): *There exists a steady-state competitive equilibrium with symmetric linear strategies and with positive trading volume if and only if the three polynomial equations (A85) to (A87) have a solution, and traders' demand curves are downward sloping. Such an equilibrium has the following properties:*

(i) There is an endogenously determined constant $C_L > 0$, defined in equation (A82), such that trader n's optimal inventories $S_{n,k}$ at period k are

$$S_{n,k} = C_L\,(H_{n,k} - H_{-n,k}). \quad (64)$$

(ii) There is an endogenously determined constant $C_G > 0$, defined in equation (A80), such that the equilibrium price at period k is

$$P_k = \frac{D_k}{r+\alpha_D} + C_G\,\frac{\bar{G}_k}{(r+\alpha_D)(r+\alpha_G)}. \quad (65)$$

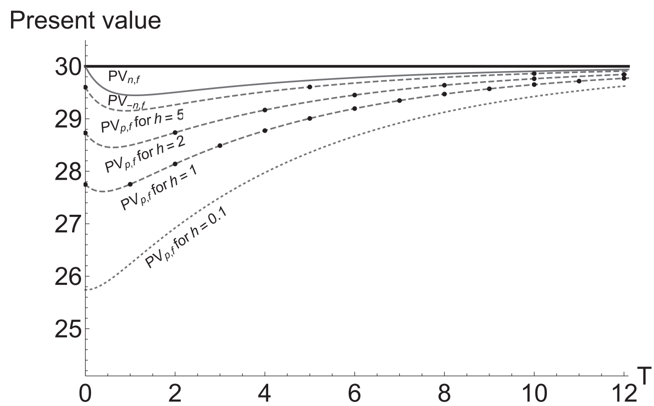

Figure 4. Present value of dividends and liquidation value from the perspective of a trader. The thick (thin) solid curve $PV_{n,f}$ ($PV_{-n,f}$) is the present value when trader n sells the stock at price equal to his own valuation of fundamentals (the average of other traders' valuations). The dashed curves $PV_{p,f}$ are the present values when trader n sells the stock at the market price at time $T = fh$ for trading frequencies $h = 0.1$, 1, 2, and 5.

PROOF: The dynamics of main state variables $H_{n,k}$ and $H_{-n,k}$ and the proof of Theorem 3 are presented in Appendix subsection H. □

Competitive traders immediately adjust inventories to optimal levels. With relative overconfidence ($\tau_H > \tau_L$), extensive numerical analysis suggests $C_G < 1$ in the dynamic model. In the limit $h \to 0$, we prove $C_G < 1$ when $\tau_H > \tau_L$. Pricing formula (65) implies that the price is dampened relative to traders' average valuation. Under trader n's information set, the stochastic processes for estimates $G_n(t)$ and $G_{-n}(t) := \frac{1}{N-1} \sum_{m=1,m\neq n}^{N} G_m(t)$ are

$$\mathrm{d}G_n(t) = -\alpha_G\, G_n(t)\, \mathrm{d}t + \text{“d}B\text{-terms”}, \tag{66}$$

$$\begin{aligned}\mathrm{d}G_{-n}(t) = &-\left(\alpha_G + \left(\tau_H^{1/2} - \tau_L^{1/2}\right)^2\right) G_{-n}(t)\, \mathrm{d}t \\ &+ \left(\tau_0 + \tau_L^{1/2}\left(2\tau_H^{1/2} + (N-2)\tau_L^{1/2}\right)\right)\left(G_n(t) - G_{-n}(t)\right) \mathrm{d}t + \text{“d}B\text{-terms”}.\end{aligned} \tag{67}$$

Trader n believes that his own growth rate estimate $G_n(t)$ will mean-revert to zero at rate α_G; his expectations satisfy the law of iterated expectations. Trader n believes that other traders' growth rate estimates $G_{-n}(t)$ follow a more complicated path, namely, that they will mean-revert to zero at rate $\alpha_G + (\tau_H^{1/2} - \tau_L^{1/2})^2 > \alpha_G$ and they will tend to drift toward trader n's estimate $G_n(t)$. If $G_n(t) = G_{-n}(t)$, then trader n believes that other traders' estimates $G_{-n}(t)$ will mean-revert to zero at rate $\alpha_G + (\tau_H^{1/2} - \tau_L^{1/2})^2 > \alpha_G$.

Figure 4 illustrates the intuition behind the dynamic price-dampening effect. For simplicity, we assume that at $t = 0$ all traders have the same buy-and-hold valuations of \$30.[10] However, the price is lower than the average of their valuations due to the dampening effect. The figure shows trader n's hypothetical expected buy-and-hold valuation at $t = 0$ under the assumption that he buys the asset at $t = 0$, holds it until time $T = fh$, and then sells it at some hypothetical price. While calculations are made under trader n's beliefs, they are symmetrically identical for all traders. Different buy-and-hold valuations $PV_{n,f}$, $PV_{-n,f}$, and $PV_{p,f}$ on the vertical axis are plotted against various holding horizons $T = fh$ on the horizontal axis (for trading frequencies $h = 0.1$, 1, 2, and 5). A formal analysis of expectations dynamics is provided in Internet Appendix Section II.

The thick solid horizontal line $PV_{n,f}$ is based on the assumption that trader n holds the risky asset until date $T = fh$ and then sells it for what he expects at $t = 0$ that his valuation of fundamentals will be at time T. Since, given his beliefs, trader n correctly applies Bayes' law, the martingale property of his valuation (the law of iterated expectations) implies that the present value $PV_{n,f}$ is a constant equal to \$30 for any horizon T.

The thin solid curve $PV_{-n,f}$, just below the line for $PV_{n,f}$, is the present value of the risky asset based on the assumption that trader n holds the asset until time $T = fh$ and then liquidates his position at what he expects at $t = 0$ the average of the other $N - 1$ traders' valuations of the fundamental value will be at time T. Trader n believes that the other traders' estimates of the growth rate will mean-revert to zero at rate $\alpha_G + (\tau_H^{1/2} - \tau_L^{1/2})^2$, which is faster than the mean-reversion rate α_G that he attributes to his own estimate. Thus, $PV_{-n,f}$ will first fall toward its unconditional level (of zero) and then in the long run rise back to catch up with trader n's own (correct, in his opinion) estimate of the fundamental value equal to \$30. These buy-and-hold valuations deviate from the horizontal line at \$30.

Figure 4 shows that prices are dampened relative to traders' valuations. The four dashed curves $PV_{p,f}$ correspond to the present values of the risky asset when trader n holds the risky asset until time $T = fh$ and then sells it at the market price at time T assuming trading frequencies $h = 0.1$, 1, 2, and 5. Consistent with the equilibrium result $0 < C_G < 1$, the initial price $P_0 := PV_{p,0}$ is lower than the consensus valuation of \$30, even if all traders agree about this valuation at $t = 0$. Indeed, if prices were equal to the consensus fundamental valuation of \$30, all traders would expect prices to fall in the short run and thus would want to hold short positions. As a result, the price P_0 at $t = 0$ is dampened relative to the average of traders' estimates of fundamental valuations. Traders agree to disagree about the dynamics of their future valuations, and internalization of this future disagreement dampens the market price in the present. Figure 4 also shows that price dampening becomes more pronounced with more frequent trading. When h becomes smaller, traders exploit profitable trading opportunities more frequently. In contrast, if $h \to \infty$

[10] The buy-and-hold value of \$30 corresponds to $G_n(0) = G_{-n}(0) = 0.105$, $D(0) = 1$. Parameter values $r, A, N, \alpha_D, \alpha_G, \sigma_D, \sigma_G, \tau_H$, and τ_L are given in Table II. Details of present-value calculations are given in equations (IA14), (IA15), and (IA18) in Internet Appendix Section II.

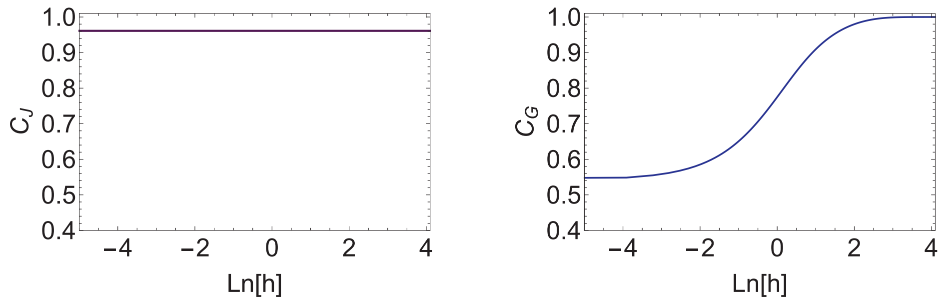

Figure 5. C_J and C_G against time interval ln[h]. (Color figure can be viewed at wileyonlinelibrary.com)

and traders can only buy and hold the risky asset, then $C_G = 1$ and the price is exactly equal to traders' average valuation of fundamentals.

As in the two-period model of Section I (see equation (22)), there is also a static dampening effect. The average valuation without a common prior differs from what it would be if the same precision $\tau_H + (N-1)\,\tau_L$ were known to be split equally across traders. The average of traders' estimates of the growth rate $\bar{G}_k$, defined in (63) and related to equilibrium prices (65), can then be expressed as

$$\bar{G}_k = \sigma_G\,\Omega^{1/2}\left(\tau_0^{1/2}\,H_{0,k} + C_J\left(\tfrac{1}{N}\tau_H + \tfrac{N-1}{N}\tau_L\right)^{1/2}\sum_{n=1}^{N} H_{n,k}\right), \tag{68}$$

where C_J is as defined in equation (22). The constant C_J measures the bias in how the weight of each private (but not public) signal is dampened in the average valuation $\bar{G}_k$ due to relative overconfidence among traders; it is defined as the ratio of the average of the square roots to the square root of the average precision. When traders are relatively overconfident ($\tau_H > \tau_L$), Jensen's inequality implies $C_J < 1$. The dynamic dampening due to $C_G < 1$ is quantitatively more important than the static dampening due to $C_J < 1$.

Figure 5 shows how the constants C_J and C_G depend on the time interval h between trading rounds.[11] The coefficient C_G increases with h and ultimately converges to one, as trading opportunities occur at less frequent intervals (h gets large) and it becomes more difficult for traders to take advantage of short-term opportunities. When h approaches zero, traders trade more aggressively against each other's perceived mistakes. The coefficient C_G becomes flat when $\ln(h) < -2$. Thus, the results of the discrete-time model converge to those of the continuous-time model approximately when $h < 0.135$ years, corresponding to about seven weeks in this example. As expected from its definition (22), the static coefficient C_J does not depend on h.

[11] In Figures 5 and 6, parameter values r, A, N, α_D, α_G, σ_D, and σ_G are given in Table II. In Figure 5, τ_H and τ_L are given in Table II. In Figure 6, $\tau = 2.2$, $h = 0.01$, 0.5, and 2.

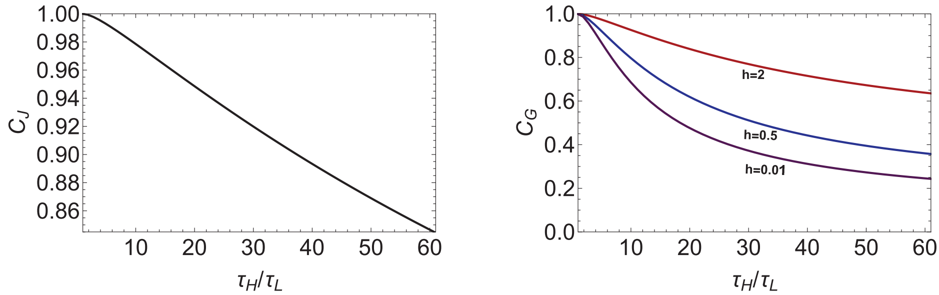

Figure 6. C_J **and** C_G **against** τ_H/τ_L **holding fixed** τ. (Color figure can be viewed at wileyonlinelibrary.com)

Figure 6 illustrates how constants C_J and C_G depend on the level of disagreement τ_H/τ_L. Holding total precision fixed, C_J and C_G gradually decrease as disagreement τ_H/τ_L increases. Both more disagreement and more trading opportunities lead to more pronounced price dampening $C_G < 1$ since traders have greater incentives to engage in short-term speculation. Figures 5 and 6 show that more disagreement also amplifies the effect of $C_J < 1$, since it magnifies the effect of Jensen's inequality. The price-dampening effect of $C_G < 1$ is usually much greater than the effect of $C_J < 1$.

PROPOSITION 4: *A common prior* ($\tau_H = \tau_L$) *implies no price dampening, with* $C_G = 1$ *and* $C_J = 1$. *Relative overconfidence* ($\tau_H > \tau_L$) *implies*

$$0 < \lim_{h \to 0} C_G \leq \left(1 + \frac{N-1}{N} \frac{\left(\tau_H^{1/2} - \tau_L^{1/2}\right)^2}{r + \alpha_G}\right)^{-1} < 1, \quad \textit{and} \quad 0 < C_J < 1. \tag{69}$$

PROOF: The proof is in Appendix subsection J. □

PROPOSITION 5: *The risk-tolerance parameter* $1/A$ *scales trading volume but has no effect on prices or on the constants* C_G *and* C_J.

PROOF: The proof is in Appendix subsection K. This proposition implies that price dampening arises due to specific features of information processing rather than risk-sharing. □

III. Return Predictability

We next derive a structural model for return dynamics and examine its time-series properties. To study return dynamics empirically, we introduce a set of empirically "correct" beliefs describing the true data-generating process.

A. Inference under Correct Beliefs

We first introduce empirically correct parameters. Let "hats" distinguish empirically correct parameter values from the possibly incorrect beliefs of traders.

In a symmetric model, it is natural to assume that all signals have the same precision, and thus no trader interprets information flow correctly. Therefore, assume traders correctly believe that total private precision is τ and public precision is τ_0, but—in contrast to traders' overconfident beliefs that their own signals have precision τ_H and other traders' signals have precision τ_L —all private signals $n = 1, \ldots, N$ have symmetrically the same empirically correct precision

$$\hat{\tau}_n = \hat{\tau}_I := \frac{\tau_H + (N-1)\,\tau_L}{N}. \tag{70}$$

In the general case, the empirically correct total precision is $\hat{\tau} = \hat{\tau}_0 + N\,\hat{\tau}_I$; this may differ from what traders believe the total precision to be, $\tau = \tau_0 + \tau_H + (N-1)\,\tau_L$. In Section III.C, we calibrate the model to generate both short-term momentum and long-run mean reversion by considering the more general case in which traders may have incorrect beliefs about the decay rate of the signal. Below we show that $\alpha_G + \tau < \hat{\alpha}_G + \hat{\tau}$ implies long-run mean reversion in returns.

In this subsection, except for τ_n, $n = 1, \ldots, N$, we assume that traders have empirically correct beliefs about all other model parameters ($\hat{\tau} = \tau$ and $\hat{\alpha}_G = \alpha_G$). In Appendix subsection L, we derive filtering formulas using an approach similar to that in Section II.A. The mathematical formulas are different because the analysis distinguishes between the true precision of signals $\hat{\tau}_I$ and traders' incorrect beliefs τ_H and τ_L. The history of each information flow $I_n(t)$ can be summarized by the same sufficient statistic $H_{n,k}$ as in equation (58) because traders use the correct decay rate $\alpha_G + \tau$ to deflate past information ($\hat{H}_{n,k} = H_{n,k}$). We can also define the aggregate sufficient statistic $\hat{H}_k$ as a linear combination of $H_{n,k}$, $n = 0, \ldots, N$,

$$\hat{H}_k = \tau_0^{1/2}\,H_{0,k} + \sum_{n=1}^{N} \hat{\tau}_I^{1/2}\,H_{n,k}. \tag{71}$$

The aggregate sufficient statistic $\hat{H}_k$ is defined similarly to H_k in equation (62), but the coefficient on $H_{n,k}$ is $\hat{\tau}_I^{1/2}$ in (70) rather than $\tau_I^{1/2}$ in (62).

Let $\hat{\mathrm{E}}_{kh}[\ldots]$ denote the empirically correct expectation operator given all information at time $t = kh$. Under correct beliefs, the estimate of the steady-state error variance is the same Ω as in equation (60), but the estimate of the growth rate at time $t = kh$ is

$$\hat{G}_k := \hat{\mathrm{E}}_{kh}\big[G^*(kh)\big] = \sigma_G\,\Omega^{1/2}\,\hat{H}_k. \tag{72}$$

The estimate of the growth rate $\hat{G}_k$ is defined like $\bar{G}_k$ in equation (63), but it depends on the aggregate sufficient statistic $\hat{H}_k$ rather than the aggregate sufficient statistic H_k.

Even though equilibrium prices and strategies depend only on traders' expectations based on their inconsistent beliefs, the objective empirical properties of prices also depend on empirically correct parameters. From equation (65), the price P_k at time kh depends on the dividend D_k and traders' sufficient statistics H_k about the growth rate; it does not depend on $\hat{H}_k$. But, to construct returns, one must project the dynamics of D_k and H_k into the future by modeling how information flows $\mathrm{d}I_n(t)$, $n = 0, 1, \ldots, N$, will unfold after time kh. This projection must be done under correct parameter values for precisions. The true evolution of these processes also depends on the correct forecasts of growth rate $\hat{G}_k$ from equation (72), which themselves depend on the sufficient statistics $\hat{H}_k$, defined in (71), not on H_k. Since the model is symmetric, the sufficient statistic $\hat{H}_k$ can be obtained from traders' statistics $H_{0,k}$ and H_k, which in turn can be inferred from current dividends and prices.

B. Return Autocorrelations

Let $R_{k,k+f}$ denote the cumulative net holding-period excess return, measured in dollars per share, from buying one share at trade date $t = kh$, financing the purchase at the risk-free rate, and selling the asset f trades dates later at $t + T = (k + f)h$,

$$R_{k,k+f} = P_{k+f} - \mathrm{e}^{rT} P_k + \mathrm{e}^{rT} \int_{t=kh}^{kh+T} \mathrm{e}^{-r(t-kh)} D(t)\, \mathrm{d}t. \tag{73}$$

The holding-period return sums capital gains and dividends. It can be expressed as a linear combination of current and past dividends and prices.

As explained in Section I.D of the two-period model, the properties of return dynamics are functions of both traders' (subjective) parameters *and* empirically correct (objective) parameters.[12]

THEOREM 4 (Expected Holding-Period Returns for the Case $\hat{\alpha}_G = \alpha_G$ and $\hat{\tau} = \tau$): *Assume $\hat{\alpha}_G = \alpha_G$ and $\hat{\tau} = \tau$. The expected holding-period return over period $T = fh$ from time $t = kh$ to $(k + f)h$ is*

$$\hat{E}_{kh}\left[R_{k,k+f}\right] = \beta_1(T)\left(P_k - \frac{D_k}{r + \alpha_D}\right) - \beta_0(T)\, H_{0,k}. \tag{74}$$

(i) *If traders have a common prior ($\tau_H = \tau_L$), then coefficients satisfy $\beta_0(T) = \beta_1(T) = 0$. The price is a martingale, and the expected return is zero.*

[12] In Internet Appendix Section VIII, we examine return autocorrelation as a function of traders' beliefs. We show that dynamic price dampening leads to time-series return momentum, and momentum is more substantial when traders can trade more frequently.

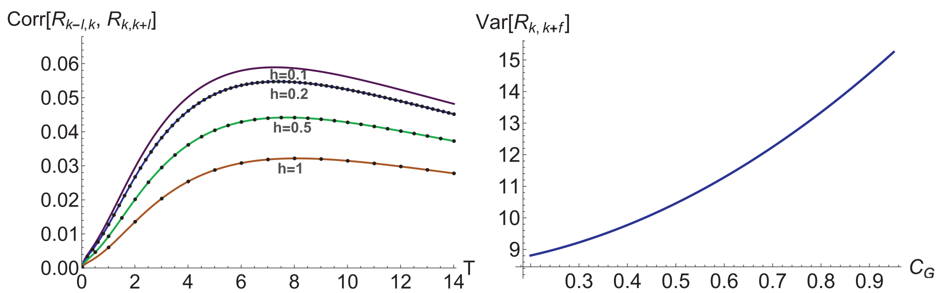

Figure 7. Autocorrelations and variance of holding-period returns. Autocorrelations of holding-period returns, corr[$R_{k-l,k}$, $R_{k,k+l}$], over holding periods $T = lh$ and variance of holding-period returns, var[$R_{k,k+f}$], against C_G for $T = fh$. (Color figure can be viewed at wileyonlinelibrary.com)

(ii) If traders are relatively overconfident ($\tau_H > \tau_L$), then coefficients satisfy $\beta_0(T) > 0$, and $\beta_1(T) > 0$ is monotonically increasing in the horizon T.

PROOF: The proof is in Appendix subsection L. In the benchmark case with common prior $\tau_H = \tau_L$, expected returns are zero for all horizons. □

Otherwise, holding-period excess returns $\hat{\mathrm{E}}_{kh}[R_{k,k+f}]$ depend on two terms capturing (i) time-series momentum related to the deviation of the current price P_k from its unconditional value $D_k/(r + \alpha_D)$, and (ii) overreaction to the history of public information.

The first term in equation (74) is related to price dampening and time-series momentum. In Appendix subsection L, we prove that the coefficient $\beta_1(T)$ is positive and can be decomposed into two terms, where the first term is positive due to static dampening ($C_J < 1$) and the second term is positive due to dynamic dampening ($C_G < 1$). Since $\beta_1(T)$ is positive, future returns are expected to be positive when the current price P_k in (65) is above its unconditional expectation, which suggests positive past returns, and future returns are expected to be negative when the price is below its unconditional expectation, which suggests negative past returns.

The second term is proportional to the public signal. Traders put too much weight on public signals relative to the dampened weights on private signals. This overreaction to public information $H_{0,k}$, aggregated over a long period of positive or negative news, is expected to be corrected going forward.

We next examine return autocorrelations corr[$R_{k-l,k}$, $R_{k,k+f}$], assuming for simplicity equal leads and lags $f = l$ or $T = T_l = T_f$. Derivations are in Internet Appendix Section V. The left panel of Figure 7 depicts return autocorrelations corr[$R_{k-l,k}$, $R_{k,k+l}$] for horizons $T = lh$; plots are presented for different trading intervals.[13] The return autocorrelations are positive and decreasing with trading frequency h, implying that the short-run momentum is more

[13] In Figure 7, parameter values r, A, N, α_D, α_G, σ_D, and σ_G are given in Table II. Parameters τ_H and τ_L are given in Table II and trading intervals are $h = 0.1$, 0.2, 0.5, and 1 in the left panel. Parameter $\tau = 2.2$ and the holding period is $T = 1/12$ in the right panel.

significant when traders trade more frequently. This result is consistent with Figures 4 and 6, where price dampening is also more substantial for smaller h. In Internet Appendix Section IV, we present empirical results showing that time-series momentum tends to be more pronounced in securities with higher trading volume. Lee and Swaminathan (2000) point out that momentum tends to be stronger for stocks with higher turnover. Cremers and Pareek (2014) find more substantial time-series momentum in stocks with more short-term trading. Moskowitz, Ooi, and Pedersen (2012) show that more liquid contracts in equity index, currency, commodity, and bond futures markets exhibit greater momentum. Zhang (2006) and Verardo (2009) find that momentum returns are larger for stocks with higher analyst disagreement. The predictions of our model are consistent with these stylized facts.

The right panel of Figure 7 plots the variance of holding-period returns $\text{var}[R_{k,k+f}]$ against C_G by varying the disagreement level τ_H/τ_L while holding τ fixed. The panel shows that return volatility is smaller when investors become more overconfident (τ_H/τ_L increases and thus C_G decreases).[14] This implies that traders' short-term speculative trading tends to dampen, rather than magnify, price fluctuations.

C. Model Calibration

In this subsection, we calibrate model parameter values to fit empirical return dynamics. To generate both short-term momentum and long-run mean reversion, we assume that traders may have incorrect beliefs about the decay rate of the signal ($\hat{\alpha}_G + \hat{\tau} \neq \alpha_G + \tau$), and they agree about the instantaneous volatility of the growth rate ($\sigma_G = \hat{\sigma}_G$). In this general case, the expected holding-period excess return from time t to $t + T$ is presented in the following theorem.[15]

THEOREM 5 (Expected Holding-Period Return for the Case $\hat{\alpha}_G + \hat{\tau} \neq \alpha_G + \tau$): *Assume $\hat{\alpha}_G + \hat{\tau} \neq \alpha_G + \tau$. The expected holding-period return over period $T = fh$ from time $t = kh$ to $(k + f)h$ is*

$$\hat{E}_t\big[R(t, t+T)\big] = \beta_1(T)\left(P(t) - \frac{D(t)}{r+\alpha_D}\right) - \beta_0(T)\int_{u=-\infty}^{t} e^{-(\hat{\alpha}_G+\hat{\tau})(t-u)}\, \mathrm{d}I_0(u) + \beta_2(T)\int_{u=-\infty}^{t}\left(P(u) - \frac{D(u)}{r+\alpha_D}\right) e^{-(\hat{\alpha}_G+\hat{\tau})(t-u)}\, \mathrm{d}u. \quad (75)$$

[14] It can be shown analytically that the instantaneous return variance increases in C_G from equation (A102).

[15] If $\hat{\alpha}_G + \hat{\tau} \neq \alpha_G + \tau$, then $\hat{H}_n(t) \neq H_n(t)$ and the relationship between the two sufficient statistics depends on the entire history of information flow (the history of publicly observable dividends and prices), as shown in equation (A95). Therefore, we assume a trading interval $h \to 0$ for the calibration.

Coefficients $\beta_1(T)$, $\beta_2(T)$, *and* $\beta_0(T)$ *in equation (75) are defined as*

$$\beta_1(T) := \left(\zeta_2(T)\, \frac{\hat{\tau}_I^{1/2}}{\tau_I^{1/2}} - \zeta_1(T) \right) \frac{(r+\alpha_D)\,(r+\alpha_G)}{C_G\, \sigma_G\, \Omega^{1/2}},$$

$$\beta_0(T) := \zeta_2(T)\, \frac{\hat{\tau}_I^{1/2}\, \tau_0^{1/2} - \tau_I^{1/2}\, \hat{\tau}_0^{1/2}}{\tau_I^{1/2}}, \tag{76}$$

$$\beta_2(T) := \zeta_2(T)\, \frac{\hat{\tau}_I^{1/2}}{\tau_I^{1/2}}\, \frac{(\alpha_G + \tau - \hat{\alpha}_G - \hat{\tau})\,(r+\alpha_D)\,(r+\alpha_G)}{C_G\, \sigma_G\, \Omega^{1/2}}, \tag{77}$$

where $\zeta_1(T) > 0$ *and* $\zeta_2(T) > 0$ *are defined in equation (A107).*

The coefficient $\beta_1(T)$ might be positive or negative depending on parameter values. The coefficient $\beta_0(T)$ is positive if and only if $\hat{\tau}_I^{1/2}\, \tau_0^{1/2} > \tau_I^{1/2}\, \hat{\tau}_0^{1/2}$. In addition, the coefficient $\beta_2(T)$ is negative if and only if $\alpha_G + \tau < \hat{\alpha}_G + \hat{\tau}$. A positive $\beta_1(T)$ or a positive $\beta_2(T)$ is related to time-series return momentum. A negative $\beta_1(T)$, a negative $\beta_2(T)$, or a positive $\beta_0(T)$ implies mean reversion in returns. Therefore, different from the case in which $\hat{\alpha}_G + \hat{\tau} = \alpha_G + \tau$ and thus $\beta_2(T) = 0$ (equation (74)), in the general case, the expected holding-period excess return also depends on the past deviations of prices from the unconditional valuation ($\beta_2(T) \neq 0$); the importance of each past component decays exponentially at rate $\hat{\alpha}_G + \hat{\tau}$. As a result, the general case can generate rich patterns of return dynamics.[16]

We next calibrate model parameter values to fit well-known empirical return autocorrelations. We consider a sample of all common stocks listed on the NYSE and Amex during the period January 1965 through December 2006 with at least two years of prior data.[17]

Let i subscript stocks and k subscript months. Let $R_{k-12,k,i}$ denote the annual return for the year prior to the end of month k, and $R_{k+f-12,k+f,i}$ denote the annual return over the year ending at the end of month $k+f$.

Column (1) of Table I presents time-series average slope coefficients from monthly cross-sectional regressions of one-year returns for one to five years ahead ($f = 12, 24, 36, 48, 60$ months) on lagged one-year returns,

$$\text{Model A} \qquad R_{k+f-12,k+f,i} = a_{f,12} + b_{f,12} R_{k-12,k,i} + \varepsilon_{k+f-12,k+f,i}. \tag{78}$$

[16] Our structural model essentially imposes testable nonlinear microfounded economic restrictions on VAR models of expected returns such as Goyal and Welch (2003), Ang and Bekaert (2007), Cochrane (2008), Van Binsbergen and Koijen (2010), and Rytchkov (2012). These restrictions are sufficiently flexible to be consistent with the rich patterns of short-term momentum and long-term mean reversion.

[17] We conduct our analysis using the sample period of 1965 to 2006 to exclude the financial crisis period. We also exclude companies incorporated outside the United States, Americus Trust Components (Primes and Scores), closed-end funds, and real estate investment trusts.

Table I

Regression Tests of Return Momentum and Reversal

This table reports time-series averages of slope coefficients estimated from monthly Fama-MacBeth cross-sectional regressions of Models A and B over 444 months from January 1965 to December 2001 ($k = 1, \ldots, 444$). Model A is a regression of one-year returns from month $k + f - 12$ to $k + f$ on one-year lagged returns from month $k - 12$ to k. Model B is a regression of cumulative returns over f months from month k to $k + f$ on lagged one-year returns from month $k - 12$ to k. The coefficients are time-series means. The t-statistics in parentheses use the Hansen-Hodrick correction.

	Time-Series Average Slope Coefficients $b_{f,12}$			
	Model A		Model B	
f (months)	Coeff (1)	t-Stat (2)	Coeff (3)	t-Stat (4)
12	0.0463	(2.9338)	0.0463	(2.9338)
24	−0.0302	(−1.6528)	0.0109	(0.3326)
36	−0.0190	(−1.0088)	−0.0110	(−0.2509)
48	−0.0265	(−1.5759)	−0.0549	(−1.1776)
60	−0.0366	(−2.1726)	−0.1076	(−1.5957)

The methodology follows Fama and MacBeth (1973). Standard errors are computed using the correction of Hansen and Hodrick (1980). Estimated slope coefficients are positive and significant for 12 months, negative and insignificant for 24, 36, 48 months, and negative and significant for 60 months. These results are similar to the estimates of Lee and Swaminathan (2000) (column (1) of table VIII). They suggest time-series momentum in year 1 and reversal by year 5.

Column (3) of Table I presents time-series average slope coefficients from monthly Fama-MacBeth cross-sectional regressions of cumulative returns $R_{k,k+f,i}$ for one to five years ($f = 12, 24, 36, 48, 60$ months) on lagged one-year returns $R_{k-12,k,i}$,

$$\text{Model B} \qquad R_{k,k+f,i} = a_{f,12} + b_{f,12} R_{k-12,k,i} + \varepsilon_{k,k+f,i}. \tag{79}$$

The coefficients reveal similar short-run time-series momentum and long-run mean reversion.

To calibrate the structural model, we find model parameters that match theoretical regression coefficients to corresponding empirically estimated values in column (3) of Table I.

Proposition 5 implies that the risk-aversion parameter A scales trading volume but has no effect on prices; we therefore assume $A = 1$. We assume $r = 0.01$ and $\alpha_D = 0.04$, implying $1/(r + \alpha_D) = 20$.[18] We also assume dividend volatility $\sigma_D = 0.5$ and instantaneous volatility of the growth rate $\sigma_G = \hat{\sigma}_G =$

[18] If the growth rate is zero, then $P(t) = D(t)/(r + \alpha_D)$. We assume that a firm pays out all earnings as dividends, then the P/E ratio is approximately $1/(r + \alpha_D)$. If $r = 0.01$ and $\alpha_D = 0.04$, and hence the implied P/E ratio is 20. The median of the Shiller P/E ratio for the S&P 500 is

Table II

List of Given and Estimated Parameters

The table reports the given parameter values for r , A , N, α_D, σ_D, α_G, σ_G, τ_H, and τ_L. The bottom two lines report the estimated parameters $\hat{\alpha}_G$ and $\hat{\tau}$.

Parameter	Description	Value
r	Risk-free rate	0.01
A	Risk aversion	1
N	Numbers of traders	100
α_D	Mean-reversion rate of dividend	0.04
σ_D	Instantaneous volatility of dividend	0.5
α_G	Traders' mean-reversion rate of $G(t)$	0.2
σ_G	Instantaneous volatility of $G(t)$	0.1
τ_H	Precision of trader n's signal	0.3
τ_L	Precision of others' signal	0.02
$\hat{\alpha}_G$	Empirically correct mean-reversion rate of $G(t)$	0.292
$\hat{\tau}$	Empirically correct total precision	6.034

0.1.[19] There remain six parameters: N, τ_H, τ_L, α_G, $\hat{\alpha}_G$, and $\hat{\tau}$. As intuition suggests, the magnitude and horizon of return momentum and reversal are determined by the level of disagreement τ_H/τ_L and the difference between empirically correct beliefs and traders' beliefs about the information decay rate ($\hat{\alpha}_G - \alpha_G$ and $\hat{\tau} - \tau$).

To further reduce the number of free parameters, we assume $N = 100$, $\tau_H = 0.3$, $\tau_L = 0.02$, and $\alpha_G = 0.2$; traders are overconfident and they actively engage in short-term trading against others in a Keynesian beauty contest. The mean-reversion rate $\alpha_G = 0.2$ implies that the growth rate $G(t)$ mean-reverts about $\ln(2)/0.2 \approx 3.5$ years.[20] We calibrate the two remaining parameters by matching empirically estimated values in column (3) of Table I to theoretically predicted regression coefficients. The calibrated parameter values are $\hat{\alpha}_G = 0.292$ and $\hat{\tau} = 6.034$. The calibrated parameter values imply that $\beta_1(T) > 0$, $\beta_2(T) < 0$, and $\beta_0(T) > 0$ in equation (75). Consistent with our previous discussion, a positive $\beta_1(T)$ is related to time-series return momentum, and a negative $\beta_2(T)$ and a positive $\beta_0(T)$ imply mean reversion in returns. The implied values of the two price-dampening factors are $C_J = 0.96$ and $C_G = 0.54$. Time-series momentum arises from both effects, but the dynamic dampening effect of C_G is much larger than the static dampening effect of C_J. Time-series mean reversion arises because traders think that the signals decay at a slower rate than the empirically correct parameter ($\alpha_G + \tau < \hat{\alpha}_G + \hat{\tau}$). Table II summarizes the exogenously given and calibrated parameter values.

15.76 and the current Shiller P/E ratio is 30.2. As illustrated in Internet Appendix Section III, the calibration results are not sensitive to the particular choice of α_D.

[19] Rountree, Weston, and Allayannis (2008) document that the average (median) standard deviation of quarterly earnings per share is 0.72 (0.19). Koren and Tenreyro (2007) document that the standard deviation of growth rates varies between 0.02 and 0.15.

[20] In Internet Appendix Section III, we also calibrate the model assuming $\alpha_G = 0.1$, implying that the growth rate mean-reverts in about $\ln(2)/0.1 \approx 7$ years.

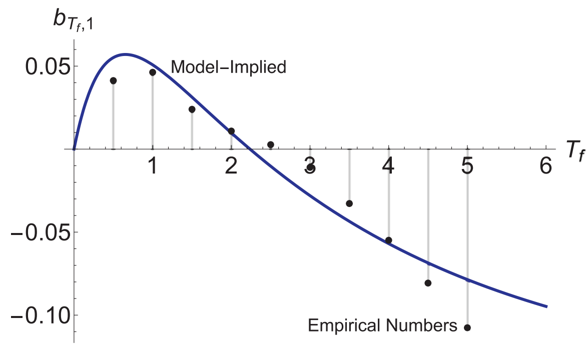

Figure 8. Theoretical regression coefficients and empirically estimated coefficients. Theoretical regression coefficients (solid curve) and empirically estimated coefficients (dots) for different holding periods of $T_f = f/12$ years (or f months). (Color figure can be viewed at wileyonlinelibrary.com)

Figure 8 shows that the regression coefficients in our calibrated model closely match the empirical numbers. The 10 dots correspond to 10 empirical slope estimates from regression Model B for $f = 6$, 12, 18, 24, 30, 36, 42, 48, 54, and 60 months. The solid curve depicts the theoretical regression coefficients for the calibrated model for $f = 6$, 12, 18, 24, 30, 36, 42, 48, 54, and 60 months. These theoretical predictions closely track all 10 empirical estimates.

In Internet Appendix Section III, we discuss how our calibration results are affected by parameter values. The calibrated parameter values for $\hat{\alpha}_G$ and $\hat{\tau}$ are affected by different values of disagreement τ_H/τ_L, α_G, and σ_G as expected. When exogenous values of τ_H, τ_L, N, α_G, and σ_G change, we show that the model can still generate empirically realistic patterns with different calibrated parameter values for $\hat{\alpha}_G$ and $\hat{\tau}$.

IV. Conclusion

When traders disagree about future valuations, a competitive market aggregates this disagreement in a manner that induces time-series momentum in returns. We clarify the underlying mechanism by first using a two-period model and then developing a dynamic trading model in which traders have heterogeneous beliefs about how to interpret continuous flows of privately observed information and trade at discrete trading rounds.

Even though all traders apply Bayes' law consistently, they believe that they regularly spot opportunities at the expense of others. Even though prices fully reflect the average of all signals at each point in time, the prices are dampened relative to the average of traders' buy-and-hold valuations. Price dampening occurs as a result of traders taking advantage of perceived short-term speculative trading opportunities rather than trading solely on long-term valuations of fundamentals. Contrary to the Keynesian intuition, this short-term speculative trading, when internalized by the market, leads to price dampening, not excessive volatility. When trading rounds occur less frequently, opportunities to engage in short-term speculation are reduced, and there is less dampening. We calibrate model parameter values to demonstrate that our model can generate quantitatively realistic empirical patterns of return dynamics.

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Appendix A: Proofs

A. *Proof of Theorem 1*

If each trader holds conditional mean-variance optimal portfolio at each period, then

$$\check{S}_{n1} = \frac{\mathrm{E}_1^n\left[\check{P}_2\right] - \check{P}_1}{A\,\mathrm{var}_1^n\left[\check{P}_2\right]}, \qquad \check{S}_{n2} = \frac{\mathrm{E}_2^n[v] - \check{P}_2}{A\,\mathrm{var}_2^n[v]}. \tag{A1}$$

The market-clearing condition $\sum_{n=1}^{N} \check{S}_{nt} = 0$ $(t = 1, 2)$ implies equilibrium prices

$$\check{P}_1 = \frac{1}{N} \sum_{n=1}^{N} \mathrm{E}_1^n\left[\check{P}_2\right], \qquad \check{P}_2 = \frac{1}{N} \sum_{n=1}^{N} \mathrm{E}_2^n[v]. \tag{A2}$$

Simple calculation shows that $\mathrm{E}_1^n[\check{P}_2]$ is given by

$$\mathrm{E}_1^n\left[\check{P}_2\right] = \mathrm{E}_1^n\left[\frac{1}{N} \sum_{n=1}^{N} \mathrm{E}_2^n[v]\right] = \frac{\tau_1}{\tau_2} \frac{1}{N} \sum_{n=1}^{N} \mathrm{E}_1^n[v] + \frac{\tau_v}{N\tau_2}\left(\tau_H^{1/2} + (N-1)\tau_L^{1/2}\right)^2 \mathrm{E}_1^n[v]. \tag{A3}$$

It follows that

$$\check{P}_1 = \frac{1}{N} \sum_{n=1}^{N} \mathrm{E}_1^n\left[\check{P}_2\right] = \left(1 - \frac{\tau_v}{\tau_2}\left(1 - \frac{1}{N}\right)\left(\tau_H^{1/2} - \tau_L^{1/2}\right)^2\right) \frac{1}{N} \sum_{n=1}^{N} \mathrm{E}_1^n[v] = \check{C}_g \frac{1}{N} \sum_{n=1}^{N} \mathrm{E}_1^n[v], \tag{A4}$$

where the coefficient $\check{C}_g$ satisfies $0 < \check{C}_g < 1$ and is given as in equation (9).

At $t = 1$, trader n's estimate of his own next-period estimate and the sum of all traders' next-period estimates are

$$\mathrm{E}_1^n\left[\mathrm{E}_2^n[v]\right] = \frac{\tau_1}{\tau_2} \mathrm{E}_1^n[v] + \frac{(\tau_H + (N-1)\tau_L)\tau_v}{\tau_2} \mathrm{E}_1^n[v] = \mathrm{E}_1^n[v], \tag{A5}$$

$$\mathrm{E}_1^n\big[\sum_{n=1}^{N}\mathrm{E}_2^n[v]\big] = \mathrm{E}_1^n\big[\mathrm{E}_2^n[v] + \sum_{m=1,m\neq n}^{N}\mathrm{E}_2^m[v]\big] = \mathrm{E}_1^n[v] + (N-1)\mathrm{E}_1^n\big[\tfrac{1}{N-1}\sum_{m=1,m\neq n}^{N}\mathrm{E}_2^m[v]\big]. \tag{A6}$$

In addition, since $\sum_{n=1}^{N}\mathrm{E}_2^n[v] = \frac{\tau_v^{1/2}}{\tau_2}(\tau_H^{1/2} + (N-1)\,\tau_L^{1/2})\sum_{n=1}^{N}(i_{n1}+i_{n2})$, we have

$$\begin{aligned}\mathrm{E}_1^n\big[\sum_{n=1}^{N}\mathrm{E}_2^n[v]\big] &= \frac{\tau_v^{1/2}}{\tau_2}\Big(\tau_H^{1/2} + (N-1)\,\tau_L^{1/2}\Big)\big(\sum_{n=1}^{N} i_{n1} + \mathrm{E}_1^n\big[\sum_{n=1}^{N} i_{n2}\big]\big)\\ &= \frac{\tau_1}{\tau_2}\sum_{n=1}^{N}\mathrm{E}_1^n[v] + \frac{\tau_v}{\tau_2}\Big(\tau_H^{1/2} + (N-1)\,\tau_L^{1/2}\Big)^2\mathrm{E}_1^n[v].\end{aligned} \tag{A7}$$

Equation (A6) and the last line of equation (A7) imply

$$\begin{aligned}\mathrm{E}_1^n[\mathrm{E}_2^{-n}[v]] &= \tfrac{1}{N-1}\frac{\tau_1}{\tau_2}\big(\mathrm{E}_1^n[v] + \sum_{m=1,m\neq n}^{N}\mathrm{E}_1^m[v]\big) + \tfrac{1}{N-1}\Big(\frac{\tau_v}{\tau_2}\Big(\tau_H^{1/2} + (N-1)\,\tau_L^{1/2}\Big)^2 - 1\Big)\mathrm{E}_1^n[v]\\ &= \Big(1 - \Big(\tau_H^{1/2} - \tau_L^{1/2}\Big)^2\frac{\tau_v}{\tau_2}\Big)\mathrm{E}_1^n[v] - \frac{\tau_1}{\tau_2}\Big(\mathrm{E}_1^n[v] - \mathrm{E}_1^{-n}[v]\Big).\end{aligned} \tag{A8}$$

This completes the proof of equation (11).

B. Proof of Theorem 2

At $t = 1, 2$, trader n chooses quantity S_{nt} to maximize the expected utility of terminal wealth $\mathrm{E}_t^n[-\mathrm{e}^{-A\,W_{n3}}]$. At $t = 2$, the optimal inventory is $S_{n2} = (\mathrm{E}_2^n[v] - P_2)/(A\,\mathrm{var}_2^n[v])$. The market-clearing condition $\sum_{n=1}^{N} S_{n2} = 0$ yields the market-clearing price at $t = 2$:

$$P_2 = \tfrac{1}{N}\sum_{n=1}^{N}\mathrm{E}_2^n[v] = \tfrac{\tau_v^{1/2}}{\tau_2}\Big(\tau_H^{1/2} + (N-1)\,\tau_L^{1/2}\Big)\tfrac{1}{N}\sum_{n=1}^{N}(i_{n1}+i_{n2}). \tag{A9}$$

At $t = 1$, trader n chooses optimal demand S_{n1} to maximize

$$\mathrm{E}_1^n\left[-\exp\left(-A\left(W_{n1} + (P_2 - P_1)\,S_{n1} + \frac{\big(\mathrm{E}_2^n[v] - P_2\big)^2}{2A\,\mathrm{var}_2^n[v]}\right)\right)\right], \tag{A10}$$

where $P_2 - P_1$ and $\mathrm{E}_2^n[v] - P_2$ can be expressed as

$$\begin{aligned}P_2 - P_1 &= (P_2 - \mathrm{E}_1^n[P_2]) + (\mathrm{E}_1^n[P_2] - P_1),\\ \mathrm{E}_2^n[v] - P_2 &= (\mathrm{E}_2^n[v] - \mathrm{E}_1^n[v]) - (P_2 - \mathrm{E}_1^n[P_2]) + (\mathrm{E}_1^n[v] - \mathrm{E}_1^n[P_2]).\end{aligned} \tag{A11}$$

Defining

$$y := \left[P_2 - \mathrm{E}_1^n[P_2]\ ,\ \mathrm{E}_2^n[v] - \mathrm{E}_1^n[v]\right]', \tag{A12}$$

the solution for problem (A10) can be obtained using

$$\mathrm{E}_1^n\left[\mathrm{e}^{-\alpha\left(\bar{A}+B'y+\frac{1}{2}y'Cy\right)}\right] = \frac{1}{\sqrt{|I+\alpha C\Sigma|}}\mathrm{e}^{-\alpha\left(\bar{A}-\frac{1}{2}\alpha B'\Sigma(I+\alpha C\Sigma)^{-1}B\right)}, \tag{A13}$$

where y is a 2×1 vector of zero-mean normally distributed random variables with covariance matrix Σ,

$$\Sigma := \begin{bmatrix} \mathrm{var}_1^n[P_2] & \mathrm{cov}_1^n\left[P_2\ ,\ \mathrm{E}_2^n[v]\right] \\ \mathrm{cov}_1^n\left[P_2\ ,\ \mathrm{E}_2^n[v]\right] & \mathrm{var}_1^n\left[\mathrm{E}_2^n[v]\right] \end{bmatrix}, \tag{A14}$$

$\bar{A}$ is a scalar, B is a 2×1 vector, C is a 2×2 symmetric matrix, and I is the 2×2 identity matrix. Define α, $\bar{A}$, B, and C as follows:

$$\alpha := A, \qquad \bar{A} := W_{n1} + \left(\mathrm{E}_1^n[P_2] - P_1\right)S_{n1} + \frac{\left(\mathrm{E}_1^n[v] - \mathrm{E}_1^n[P_2]\right)^2}{2A\ \mathrm{var}_2^n[v]}, \tag{A15}$$

$$B := \psi_B S_{n1} + \frac{\mathrm{E}_1^n[v] - \mathrm{E}_1^n[P_2]}{A\ \mathrm{var}_2^n[v]}\phi_B, \qquad C := \frac{1}{A\ \mathrm{var}_2^n[v]} \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \tag{A16}$$

where $\psi_B := [1\ ,\ 0]'$ and $\phi_B := [-1\ ,\ 1]'$. The first-order condition for S_{n1} implies

$$S_{n1} = \frac{\mathrm{E}_1^n[P_2] - P_1 - \frac{\mathrm{E}_1^n[v]-\mathrm{E}_1^n[P_2]}{\mathrm{var}_2^n[v]}\psi_B'\Sigma(I+AC\Sigma)^{-1}\phi_B}{A\psi_B'\Sigma(I+AC\Sigma)^{-1}\psi_B}. \tag{A17}$$

It can be shown that

$$\begin{aligned} \frac{\psi_B'\Sigma(I+AC\Sigma)^{-1}\phi_B}{\mathrm{var}_2^n[v]} &= \frac{\mathrm{cov}_1^n\left[P_2\ ,\ \mathrm{E}_2^n[v]-P_2\right]}{\mathrm{var}_1^n\left[\mathrm{E}_2^n[v]-P_2\right]+\mathrm{var}_2^n[v]}, \\ \psi_B'\Sigma(I+AC\Sigma)^{-1}\psi_B &= \frac{\mathrm{var}_1^n[P_2]\left(\mathrm{var}_1^n\left[\mathrm{E}_2^n[v]\right]+\mathrm{var}_2^n[v]\right)-\left(\mathrm{cov}_1^n\left[P_2\ ,\ \mathrm{E}_2^n[v]\right]\right)^2}{\mathrm{var}_1^n\left[\mathrm{E}_2^n[v]-P_2\right]+\mathrm{var}_2^n[v]}. \end{aligned} \tag{A18}$$

Substituting equation (A18) into (A17) yields trader n's optimal inventory as given in equation (17) in Theorem 2. The market-clearing condition $\sum_{n=1}^N S_{n1} = 0$ yields

$$P_1 = \frac{1}{N}\sum_{n=1}^N \mathrm{E}_1^n[P_2] + \frac{\mathrm{cov}_1^n\left[P_2\ ,\ \mathrm{E}_2^n[v]-P_2\right]}{\mathrm{var}_1^n\left[\mathrm{E}_2^n[v]-P_2\right]+\mathrm{var}_2^n[v]}\left(\frac{1}{N}\sum_{n=1}^N \mathrm{E}_1^n[P_2] - \frac{1}{N}\sum_{n=1}^N \mathrm{E}_1^n[v]\right). \tag{A19}$$

It can be shown that P_1 can be written as $C_g \times \frac{1}{N}\sum_{n=1}^N \mathrm{E}_1^n[v]$, where the coefficient C_g is

$$C_g = \frac{\tau_1}{\tau_2} + \frac{\tau_v}{\tau_2}\tfrac{1}{N}\left(\tau_H^{1/2} + (N-1)\tau_L^{1/2}\right)^2 \frac{N\tau_1\left(N\tau_2+(N-1)\tau_v\left(\tau_H^{1/2}-\tau_L^{1/2}\right)^2\right)}{N\tau_1\left(N\tau_2+(N-1)\tau_v\left(\tau_H^{1/2}-\tau_L^{1/2}\right)^2\right)+(N-1)^2\tau_v^2\left(\tau_H^{1/2}-\tau_L^{1/2}\right)^4} \quad \text{(A20)}$$

$$\leq \frac{\tau_1}{\tau_2} + \frac{\tau_v}{\tau_2}\tfrac{1}{N}\left(\tau_H^{1/2} + (N-1)\tau_L^{1/2}\right)^2 \leq \frac{\tau_1}{\tau_2} + \frac{\tau_v}{\tau_2}\left(\tau_H + (N-1)\tau_L\right) = 1. \quad \text{(A21)}$$

Equation (A21) implies that $C_g < 1$ if $\tau_H > \tau_L$ and $C_g = 1$ if $\tau_H = \tau_L$. Thus, P_1 is a weighted average of traders' valuations, with weights summing to less than one.

C. Proof of Proposition 1

From equation (20), we have

$$C_g = \check{C}_g - \tfrac{\tau_v}{\tau_2}\left(1 - \tfrac{1}{N}\right)\left(\tau_H^{1/2} - \tau_L^{1/2}\right)^2 \frac{\text{cov}_1^n\left[P_2\ ,\ \text{E}_2^n[v]-P_2\right]}{\text{var}_1^n\left[\text{E}_2^n[v]-P_2\right]+\text{var}_2^n[v]}, \quad \text{(A22)}$$

where $\text{cov}_1^n[P_2\ ,\ \text{E}_2^n[v] - P_2] > 0$ from equation (19). This implies $C_g < \check{C}_g$. From equation (A21), it is straightforward to see that C_g satisfies $0 < C_g < 1$ for $\tau_H \neq \tau_L$. Using direct calculation, we show analytically that $\text{d}C_g/\,\text{d}\tau_H < 0$ and $\text{d}\check{C}_g/\,\text{d}\tau_H < 0$, holding the total precision $\tau_H + (N-1)\tau_L$ fixed. This implies that both C_g and $\check{C}_g$ decrease in τ_H/τ_L holding the total precision fixed.

D. Proof of Proposition 2

Equations (35) and (37) of the paper present closed-form expressions for return covariance of the myopic model $\text{cov}^n[\check{P}_2 - \check{P}_1\ ,\ \check{P}_1 - \check{P}_0]$ and return covariance of the general two-period model $\text{cov}^n[P_2 - P_1\ ,\ P_1 - P_0]$.

Under traders' beliefs, the return autocovariance is given by

$$\begin{aligned}\text{cov}^n[v - P_2, P_2 - P_1] &= \text{E}_0^n\left[(\text{E}_2^n[v] - P_2)(P_2 - P_1)\right] - \text{E}_0^n[v - P_2]\text{E}_0^n[P_2 - P_1] \\ &= \tfrac{N-1}{N^2}\text{E}_0^n\left[(\text{E}_2^n[v] - \text{E}_2^{-n}[v])(\text{E}_2^n[v] + (N-1)\text{E}_2^{-n}[v]\right. \\ &\quad \left. - C_g(\text{E}_1^n[v] + (N-1)\text{E}_1^{-n}[v])\right]. \end{aligned} \quad \text{(A23)}$$

The coefficient on v is positive in $\text{E}_2^n[v] - \text{E}_2^{-n}[v]$,

$$\begin{aligned}\text{E}_2^n[v] - \text{E}_2^{-n}[v] &= \tfrac{\tau_v^{1/2}}{\tau_2}\left(\tau_H^{1/2} - \tau_L^{1/2}\right)(i_{n1} - i_{-n1} + i_{n2} - i_{-n2}) \\ &= \tfrac{\tau_v^{1/2}}{\tau_2}\left(\tau_H^{1/2} - \tau_L^{1/2}\right)\left(2\left(\tau_H^{1/2} - \tau_L^{1/2}\right)\tau_v^{1/2}v + (e_{n1} + e_{n2}) - \tfrac{1}{N-1}\sum_{m=1, m\neq n}^{N}(e_{m1} + e_{m2})\right).\end{aligned} \quad \text{(A24)}$$

In addition, $E_2^n[v]+(N-1)E_2^{-n}[v]$ increases in the fundamental value v. Therefore, the covariance between $E_2^n[v]-E_2^{-n}[v]$ and $E_2^n[v]+(N-1)E_2^{-n}[v]$ is positive,

$$\begin{aligned} &E_0^n\big[(E_2^n[v]-E_2^{-n}[v])(E_2^n[v]+(N-1)E_2^{-n}[v])\big] = \tfrac{4\tau_v}{\tau_2^2}\big(\tau_H^{1/2}-\tau_L^{1/2}\big)^2\big(\tau_H^{1/2}+(N-1)\tau_L^{1/2}\big)^2>0,\\ &E_0^n\big[(E_2^n[v]-E_2^{-n}[v])(E_1^n[v]+(N-1)E_1^{-n}[v])\big] = \tfrac{2\tau_v}{\tau_1\tau_2}\big(\tau_H^{1/2}-\tau_L^{1/2}\big)^2\big(\tau_H^{1/2}+(N-1)\tau_L^{1/2}\big)^2. \end{aligned} \tag{A25}$$

Substituting equation (A25) into (A23) yields

$$\text{cov}^n[v-P_2\ ,\ P_2-P_1] = \frac{2\tau_v}{N^2\tau_2}\left(\frac{2}{\tau_2}-\frac{C_g}{\tau_1}\right)\left(\tau_H^{1/2}+(N-1)\tau_L^{1/2}\right)^2(N-1)\left(\tau_H^{1/2}-\tau_L^{1/2}\right)^2. \tag{A26}$$

In the myopic model, we replace C_g by $\check{C}_g$ in equation (A26) and show that

$$\begin{aligned}\text{cov}^n\left[v-\check{P}_2,\check{P}_2-\check{P}_1\right] = \tfrac{2\tau_v^2(N-1)}{N^3\tau_1\tau_2^2}\left(N+(N-1)\left(\tau_H^{1/2}-\tau_L^{1/2}\right)^2\right)\Big(\tau_H^{1/2}\\ +\ (N-1)\tau_L^{1/2}\Big)^2\left(\tau_H^{1/2}-\tau_L^{1/2}\right)^2>0.\end{aligned} \tag{A27}$$

Under the empirically correct parameters that all private signals have the same precision, prices have the same expressions as those under traders' beliefs (equation (38)), but traders' private signals are

$$\begin{aligned} \hat{i}_{nt} &= \left(\tfrac{\tau_H+(N-1)\tau_L}{N}\right)^{1/2}(\tau_v^{1/2}v)+e_{nt},\\ \hat{i}_{mt} &= \left(\tfrac{\tau_H+(N-1)\tau_L}{N}\right)^{1/2}(\tau_v^{1/2}v)+e_{mt}, \qquad \text{for all } m\neq n. \end{aligned} \tag{A28}$$

Using the empirically correct parameters, the return autocovariances are

$$\begin{aligned} \text{cov}[P_2-P_1,P_1-P_0] &= \frac{\tau_v C_g}{\tau_1}\left(\left(\frac{2}{\tau_2}-\frac{C_g}{\tau_1}\right)\frac{\left(\tau_H^{1/2}+(N-1)\tau_L^{1/2}\right)^2(\tau_H+(N-1)\tau_L)}{N}\right.\\ &\quad\left.+\left(\frac{1}{\tau_2}-\frac{C_g}{\tau_1}\right)\frac{\left(\tau_H^{1/2}+(N-1)\tau_L^{1/2}\right)^2}{N}\right),\\ \text{cov}[v-P_2,P_2-P_1] &= \left(\frac{2}{\tau_2}-\frac{C_g}{\tau_1}\right)\frac{\tau_v}{N\tau_2}\Big(N^{1/2}(\tau_H+(N-1)\tau_L)^{1/2}\\ &\quad-\left(\tau_H^{1/2}+(N-1)\tau_L^{1/2}\right)\Big)\\ &\quad\times\left(\tau_H^{1/2}+(N-1)\tau_L^{1/2}\right)(1+2(\tau_H+(N-1)\tau_L)). \end{aligned} \tag{A29}$$

Simple calculation shows that $\text{cov}[v-P_2\ ,\ P_2-P_1]>0$ and $\text{cov}[P_2-P_1\ ,\ P_1-P_0]>0$. In the myopic model, we replace C_g by $\check{C}_g$ in equation (A29) and show that the return autocovariances are given by

$$\mathrm{cov}\Big[\check{P}_2-\check{P}_1\ ,\ \check{P}_1-\check{P}_0\Big]$$
$$=\frac{(N-1)\left(\tau_H^{1/2}-\tau_L^{1/2}\right)^2\left(\tau_H^{1/2}+(N-1)\tau_L^{1/2}\right)^2\left(N+2\tau_H+2(N-1)^2\tau_L+(N-1)\left(\tau_H^{1/2}+\tau_L^{1/2}\right)^2\right)\tau_v^2}{N^3\tau_1\tau_2^2}>0,$$
$$\mathrm{cov}\Big[v-\check{P}_2\ ,\ \check{P}_2-\check{P}_1\Big]$$
$$=\frac{\tau_v\left(N+(N-1)\left(\tau_H^{1/2}-\tau_L^{1/2}\right)^2\right)\left(\tau_H^{1/2}+(N-1)\tau_L^{1/2}\right)\left(N^{1/2}(\tau_H+(N-1)\tau_L)^{1/2}-\left(\tau_H^{1/2}+(N-1)\tau_L^{1/2}\right)\right)}{N^2\tau_1\tau_2}>0.$$
(A30)

E. Proof of Proposition 3

If traders agree about the conditional distribution of v given signals, then for $t=1,\ 2$, we have $\mathrm{E}_t^n[v]=\mathrm{E}_t^m[v]$. Thus, $\mathrm{E}_1^n\big[\mathrm{E}_2^m[v]\big]=\mathrm{E}_1^n\big[\mathrm{E}_2^n[v]\big]=\mathrm{E}_1^n[v]$, where the last equation follows from iteration of trader n's expectations. It follows that $\mathrm{E}_1^n\big[\frac{1}{N}\sum_{m=1}^N\mathrm{E}_2^m[v]\big]=\mathrm{E}_1^n[v]=\frac{1}{N}\sum_{m=1}^N\mathrm{E}_1^m[v]$, and thus $P_1=\mathrm{E}_1^n[P_2]$.

If traders agree about the joint distribution of signals, then $\mathrm{E}_1^n\big[\mathrm{E}_2^m[v]\big]=\mathrm{E}_1^m\big[\mathrm{E}_2^m[v]\big]$ because $\mathrm{E}_2^m[v]$ is some known function of observed signals that trader m uses in his filtering. It follows that $\mathrm{E}_1^n\big[\mathrm{E}_2^m[v]\big]=\mathrm{E}_1^m\big[\mathrm{E}_2^m[v]\big]=\mathrm{E}_1^m[v]$, where the last equation follows from iteration of trader m's expectations. We thus have that $\mathrm{E}_1^n\big[\frac{1}{N}\sum_{m=1}^N\mathrm{E}_2^m[v]\big]=\frac{1}{N}\sum_{m=1}^N\mathrm{E}_1^m[v]$, and hence $P_1=\mathrm{E}_1^n[P_2]$.

F. Filtering Formulas for the Dynamic Model

In this subsection, we derive Stratonovich-Kalman-Bucy filtering formulas for Ω and $G_{n,k}$.

LEMMA A1: *Define the total precision of information flows as $\tau:=\tau_0+\tau_H+(N-1)\tau_L$. Then the scaled steady-state error variance Ω and trader n's estimate $G_{n,k}$ of the growth rate at time t are*

$$\Omega^{-1}:=var^{-1}\left[\frac{G^*(t)-G_n(t)}{\sigma_G}\right]=2\,\alpha_G+\tau, \tag{A31}$$

$$G_n(t):=E_t^n\big[G^*(t)\big]=\sigma_G\,\Omega^{1/2}\left(\tau_0^{1/2}\,H_0(t)+\tau_H^{1/2}\,H_n(t)+(N-1)\tau_L^{1/2}\,H_{-n}(t)\right). \tag{A32}$$

PROOF: The stochastic process $dG^*(t)$, $dI_n(t)$, $dI_{-n}(t)$, and $dI_0(t)$ can be rewritten as

$$dG^*(t) := -\alpha_G G_n(t)\,dt - \alpha_G\left(G^*(t) - G_n(t)\right)\,dt + \sigma_G\,dB_G(t),$$
$$dI_n(t) := \tau_H^{1/2}\,\tfrac{G_n(t)}{\sigma_G\,\Omega^{1/2}}\,dt + \tau_H^{1/2}\,\tfrac{G^*(t)-G_n(t)}{\sigma_G\,\Omega^{1/2}}\,dt + dB_n(t), \tag{A33}$$

$$dI_0(t) := \tau_0^{1/2}\,\tfrac{G_n(t)}{\sigma_G\,\Omega^{1/2}}\,dt + \tau_0^{1/2}\,\tfrac{G^*(t)-G_n(t)}{\sigma_G\,\Omega^{1/2}}\,dt + dB_0(t),$$
$$dI_{-n}(t) := \tau_L^{1/2}\,\tfrac{G_n(t)}{\sigma_G\,\Omega^{1/2}}\,dt + \tau_L^{1/2}\,\tfrac{G^*(t)-G_n(t)}{\sigma_G\,\Omega^{1/2}}\,dt + \tfrac{1}{N-1}\sum_{m=1,m\neq n}^{N} dB_m(t). \tag{A34}$$

Since trader n's forecast of the error $G^*(t) - G_n(t)$ is zero given his information set, the last two terms in the information processes are independently distributed Brownian motions from the perspective of trader n. Let $I(u)$ denote trader n's information set. Trader n's estimate at time $t + dt$, $G_n(t + dt) := \mathrm{E}^n[G^*(t + dt) \mid I(u)|_{u=-\infty}^{t+dt}]$, can be decomposed into two terms:

$$G_n(t + dt) = \mathrm{E}^n\left[G^*(t) \mid I(u)\Big|_{u=-\infty}^{t+dt}\right] + \mathrm{E}^n\left[dG^*(t) \mid I(u)\Big|_{u=-\infty}^{t+dt}\right]. \tag{A35}$$

The first term reflects the expectation about the past growth rate $G^*(t)$ at time t given new information $dI_0(t)$, $dI_n(t)$, and $dI_{-n}(t)$. The second term reflects the expectation about the change in growth rate $dG^*(t) = G^*(t + dt) - G^*(t)$ at time t given new information $dI_0(t)$, $dI_n(t)$, and $dI_{-n}(t)$. The first term $\mathrm{E}^n[G^*(t) \mid I(u)|_{u=-\infty}^{t+dt}]$ can be calculated as follows:

$$\begin{aligned}
\mathrm{E}^n\left[G^*(t) \mid I(u)\Big|_{u=-\infty}^{t+dt}\right] &= \mathrm{E}^n\big[G_n(t) + (G^*(t) - G_n(t)) \mid G_n(t),\ dI_0(t),\\
&\quad dI_n(t),\ dI_{-n}(t)\big]\\
&= G_n(t) + \mathrm{E}^n\left[G^*(t) - G_n(t) \mid dI_0(t) - \frac{\tau_0^{1/2}\,G_n(t)}{\sigma_G\,\Omega^{1/2}}\,dt,\ dI_n(t) - \frac{\tau_H^{1/2}\,G_n(t)}{\sigma_G\,\Omega^{1/2}}\,dt,\right.\\
&\quad \left. dI_{-n}(t) - \frac{\tau_L^{1/2}\,G_n(t)}{\sigma_G\,\Omega^{1/2}}\,dt\right]
\end{aligned} \tag{A36}$$

$$= G_n(t) + \sigma_G \Omega^{1/2} \tau_0^{1/2} \left(\mathrm{d}I_0(t) - \frac{\tau_0^{1/2}\, G_n(t)}{\sigma_G\, \Omega^{1/2}}\; \mathrm{d}t \right) + \sigma_G \Omega^{1/2} \tau_H^{1/2} \left(\mathrm{d}I_n(t) - \frac{\tau_H^{1/2}\, G_n(t)}{\sigma_G\, \Omega^{1/2}}\; \mathrm{d}t \right) + \sigma_G \Omega^{1/2} (N-1) \tau_L^{1/2} \left(\mathrm{d}I_{-n}(t) - \frac{\tau_L^{1/2}\, G_n(t)}{\sigma_G\, \Omega^{1/2}}\; \mathrm{d}t \right). \tag{A37}$$

The last equation is obtained using the projection theorem for normal variables and equation (A34). The second term $\mathrm{E}^n[\, \mathrm{d}G^*(t) \mid I(u)|_{u=-\infty}^{t+\,\mathrm{d}t}]$ can be calculated as follows:

$$\mathrm{E}^n\Big[\, \mathrm{d}G^*(t) \mid I(u)\Big|_{u=-\infty}^{t+\,\mathrm{d}t}\Big] = \mathrm{E}^n\big[-\alpha_G G_n(t)\,\mathrm{d}t + \,\mathrm{d}G^*(t) + \alpha_G G_n(t)\,\mathrm{d}t \mid G_n(t), \mathrm{d}I_0(t),\ \mathrm{d}I_n(t)\,,\ \mathrm{d}I_{-n}(t)\big] = -\alpha_G G_n(t)\,\mathrm{d}t - \alpha_G \mathrm{E}^n\big[(G^*(t) - G_n(t))\,\mathrm{d}t \mid G_n(t), \mathrm{d}I_0(t)\,,\ \mathrm{d}I_n(t),\ \mathrm{d}I_{-n}(t)\big]. \tag{A38}$$

The second term in equation (A38) is as calculated in equation (A37). It can be ignored since it is of order smaller than $\mathrm{d}t$. Plugging equations (A37) and (A38) into equation (A35), we find that the estimate $G_n(t)$ is defined by the Itô differential equation

$$\mathrm{d}G_n(t) = -\alpha_G\, G_n(t)\;\; \mathrm{d}t + \sum_{n=0}^{N} \sigma_G\; \Omega^{1/2}\; \tau_n^{1/2} \left(\mathrm{d}I_n(t) - G_n(t)\, \frac{\tau_n^{1/2}}{\sigma_G\; \Omega^{1/2}}\;\; \mathrm{d}t \right). \tag{A39}$$

Rearranging terms yields

$$\mathrm{d}G_n(t) = -(\alpha_G + \tau)\, G_n(t)\;\; \mathrm{d}t + \sigma_G\; \Omega^{1/2} \Big(\tau_0^{1/2}\; \mathrm{d}I_0 + \tau_H^{1/2}\; \mathrm{d}I_n + (N-1)\tau_L^{1/2}\; \mathrm{d}I_{-n} \Big). \tag{A40}$$

From equation (A40), we obtain the solution for the estimate $G_n(t)$:

$$G_n(t) = \sigma_G \Omega^{1/2} \int_{u=-\infty}^{t} \mathrm{e}^{-(\alpha_G+\tau)\,(t-u)}\; (\tau_0^{1/2}\; \mathrm{d}I_0(u) + \tau_H^{1/2}\; \mathrm{d}I_n(u) + (N-1)\tau_L^{1/2}\; \mathrm{d}I_{-n}(u))\, \mathrm{d}u. \tag{A41}$$

Define the sufficient statistics or "signals" $H_n(t)$ by

$$H_n(t) := \int_{u=-\infty}^{t} \mathrm{e}^{-(\alpha_G+\tau)(t-u)} \, \mathrm{d}I_n(u), \qquad n = 0, 1, \ldots, N. \tag{A42}$$

The estimate $G_n(t)$ can be conveniently written as the weighted sum of these sufficient statistics as in equation (A32). The mean-square filtering error of the estimate $G_n(t)$, denoted by $\sigma_G^2\,\Omega(t)$, is defined by the Riccati differential equation

$$\sigma_G^2 \,\frac{\mathrm{d}\Omega(t)}{\mathrm{d}t} = -2\alpha_G\,\sigma_G^2\,\Omega(t) + \sigma_G^2 - \sigma_G^4\,\Omega(t)^2 \sum_{n=0}^{N} \left(\frac{\tau_n^{1/2}}{\sigma_G\,\Omega(t)^{1/2}} \right)^2. \tag{A43}$$

Using the steady-state assumption $\mathrm{d}\Omega(t)/\,\mathrm{d}t = 0$, solve this equation for the steady-state value Ω defined by $\Omega = \Omega(t)$ to obtain equation (A31). Equation (A32) implies that, in discrete notation, trader n's estimate of the growth rate at time $t = kh$ is

$$G_{n,k} := \mathrm{E}_{kh}^{n}\big[G^*(kh)\big] = \sigma_G \Omega^{1/2}\Big(\tau_0^{1/2}H_{0,k} + \tau_H^{1/2}H_{n,k} + (N-1)\tau_L^{1/2}H_{-n,k}\Big). \tag{A44}$$

□

G. The Equilibrium with Myopic Conditional Mean-Variance Optimizers

Similar to the two-period model, we first solve for the equilibrium prices when traders are myopic and hold the conditional mean-variance optimal portfolio:

$$\check{S}_{n,k} = \frac{\mathrm{E}_{kh}^{n}\Big[\check{P}_{k+1} + \check{D}_{k+1}\Big] - \mathrm{e}^{rh}\check{P}_k}{A\ \mathrm{var}_{kh}^{n}\Big[\check{P}_{k+1} + \check{D}_{k+1}\Big]}. \tag{A45}$$

The equilibrium price $\check{P}_k$ is given in the following theorem.

THEOREM A1 (Prices with Myopic Conditional Mean-Variance Optimizers): *Suppose each trader holds the conditional mean-variance optimal portfolio at each period. Then the price $\check{P}_k$ is*

$$\check{P}_k = \frac{D_k}{r+\alpha_D} + \check{C}_G\,\frac{\bar{G}_k}{(r+\alpha_D)(r+\alpha_G)}, \tag{A46}$$

where $\bar{G}_k$, defined in equation (63), is the average of traders' expected growth rates at time kh. The coefficient $\check{C}_G$ satisfies $0 < \check{C}_G < 1$ and is given by

$$\check{C}_G := \left(1 + \frac{1-e^{-\tau h}}{e^{(r+\alpha_G)h}-1}\,\frac{N-1}{N}\Big(\tau_H^{1/2} - \tau_L^{1/2}\Big)^2/\tau\right)^{-1}. \tag{A47}$$

PROOF: The proof of Theorem A1 is presented in Appendix subsection H. □

The pricing formula (A46) resembles an average of traders' valuations of fundamentals (44), with one important difference. The coefficient C_G satisfies $0 < \check{C}_G < 1$. This dampens prices relative to traders' average valuation. The endogenous parameter C_G measures price dampening just like the coefficients $\check{C}_g < 1$ and $C_g < 1$ in the two-period model in Section I.

PROPOSITION A1: *A common prior ($\tau_H = \tau_L$) implies no price dampening, with* $\check{C}_G = 1$. *Relative overconfidence ($\tau_H > \tau_L$) implies* $0 < \check{C}_G < 1$. *The coefficient* $\mathrm{C_G}$ *increases in the trading interval h and decreases in the disagreement τ_H/τ_L (holding total precision τ fixed).*

PROOF: The proof of Proposition A1 is in Appendix subsection I. □

Proposition A1 implies that price dampening becomes more pronounced with more frequent trading. When h becomes smaller, traders exploit profitable trading opportunities more frequently. If $h \to \infty$, then $\check{C}_G = 1$. This implies that if traders can only buy and hold the risky asset, then the price is exactly equal to traders' average valuations of fundamentals. The dampening effect $\check{C}_G < 1$ occurs because traders internalize their future disagreement about the growth rate. Each trader agrees to disagree with others about how to interpret private signals in the present and how to interpret private signals in the future. When internalized, this future disagreement dampens the current market price relative to average valuation. Each trader expects that, as time passes, other traders will revise their mistaken current valuations of fundamentals toward unconditional levels, before they eventually converge toward his own "correct" valuation. He tries to profit by trading ahead of anticipated short-term revisions of others' expectations, even when this means trading against his own long-term valuation. This short-term speculative trading dampens current prices relative to the average of traders' valuations. In equilibrium, this leads to a Keynesian beauty contest. Yet, contrary to common intuition, this short-term speculative trading tends to dampen, rather than magnify, price fluctuations.

We next discuss trader n's expectations about his and others' stock positions in the next period. Suppose trader n thinks that he can sell the stock at time $(k+1)h$ at price equal to his own valuation of fundamentals, $F_{n,k+1} = \frac{D_{k+1}}{r+\alpha_D} + \frac{G_{n,k+1}}{(r+\alpha_D)(r+\alpha_G)}$. Then his conditional mean-variance optimal portfolio at time kh is given as in equation (A45), replacing the price $\check{P}_{k+1}$ with $F_{n,k+1}$. It can be shown that the price at time kh is $\check{P}_k = \frac{D_k}{r+\alpha_D} + \frac{\bar{G}_k}{(r+\alpha_D)(r+\alpha_G)}$. At time $(k-1)h$, trader n expects his conditional mean-variance position at time kh, $\mathrm{E}^n_{(k-1)h}[\check{S}_{n,k}]$, is proportional to

$$\mathrm{E}^n_{(k-1)h}[G_{n,k} - G_{-n,k}] = \mathrm{e}^{-\alpha_G h}\left(\frac{1}{\tau}\left(\tau_H^{1/2} - \tau_L^{1/2}\right)^2\left(1 - \mathrm{e}^{-\tau h}\right)G_{n,k-1}\right.$$

$$+\,\mathrm{e}^{-\tau h}(G_{n,k-1}-G_{-n,k-1})\Big). \tag{A48}$$

If all traders happen to agree about the fundamental valuation at time $(k-1)h$ (i.e., $G_{n,k-1}=G_{-n,k-1}>0$), then equation (A48) is reduced to

$$\mathrm{E}^{n}_{(k-1)h}[G_{n,k}-G_{-n,k}]=\tfrac{1}{\tau}\left(\tau_H^{1/2}-\tau_L^{1/2}\right)^2\mathrm{e}^{-\alpha_G h}(1-\mathrm{e}^{-\tau h})G_{n,k-1}>0. \tag{A49}$$

Each trader n expects to buy ($\mathrm{E}^{n}_{(k-1)h}[\check{S}_{n,k}]>0$), and others to sell, next period. The reason is that each trader expects prices to fall in the next period, each trader believes that others make the mistake of attributing too much precision to their current and future signals, and each trader expects that, upon observing new information in the future, others will adjust their valuations toward unconditional levels. Thus, each trader expects others to sell and prices to fall.

Trader n's conditional mean-variance optimal portfolio, $\check{S}_{n,k-1}$ is proportional to

$$\mathrm{E}^{n}_{(k-1)h}\left[\mathrm{e}^{-rh}\left(\frac{D_k}{r+\alpha_D}+\frac{\bar{G}_k}{(r+\alpha_D)(r+\alpha_G)}+\check{D}_{k+1}\right)\right]-\check{P}_{k-1}$$
$$=\frac{D_{k-1}}{r+\alpha_D}+\frac{G_{n,k-1}}{(r+\alpha_D)(r+\alpha_G)}-\frac{N-1}{N}\mathrm{e}^{-rh}\frac{\mathrm{E}^{n}_{k-1}[G_{n,k}-G_{-n,k}]}{(r+\alpha_D)(r+\alpha_G)}-\check{P}_{k-1}. \tag{A50}$$

Equations (A49) and (A50) imply that, if traders agree about the fundamental valuation at time $(k-1)h$ (so $G_{n,k-1}=G_{-n,k-1}>0$) and the price $\check{P}_{k-1}$ is equal to their consensus fundamental valuation, $\check{P}_{k-1}=\frac{D_{k-1}}{r+\alpha_D}+\frac{\bar{G}_{k-1}}{(r+\alpha_D)(r+\alpha_G)}$, then all traders would want to hold short positions at time $(k-1)h$ because all traders would expect prices to fall below fundamental value at time kh. In equilibrium, this short-term speculative trading dampens current prices relative to the average of traders' valuations. The price dampening becomes more pronounced when traders exploit profitable trading opportunities more frequently.

H. Proof of Theorems 3 and A1

We first derive the dynamics of key state variables. We then solve for the equilibrium.

H.1. The Dynamics of Key State Variables

We next derive conditional expectations of key state variables and their variance-covariance matrix. Define $N+1$ processes $\mathrm{d}B_0^n(t)$, $\mathrm{d}B_n^n(t)$, and

$dB^n_m(t)$, $m = 1, \ldots, N$, $m \neq n$, by

$$\begin{aligned} dB^n_0(t) &= \tau_0^{1/2} \frac{G^*(t) - G_n(t)}{\sigma_G\ \Omega^{1/2}}\ dt +\ dB_D(t), \\ dB^n_n(t) &= \tau_H^{1/2} \frac{G^*(t) - G_n(t)}{\sigma_G\ \Omega^{1/2}}\ dt +\ dB_n(t), \\ dB^n_m(t) &= \tau_L^{1/2} \frac{G^*(t) - G_n(t)}{\sigma_G\ \Omega^{1/2}}\ dt +\ dB_m(t). \end{aligned} \tag{A51}$$

The superscript n indicates conditioning on beliefs of trader n. Since trader n's forecast of the error $G^*(t) - G_n(t)$ is zero given his information set, these $N+1$ processes are independently distributed Brownian motions from the perspective of trader n. In terms of these Brownian motions, trader n believes that sufficient statistics $H_0(t), H_n(t)$, and $H_{-n}(t)$ change as follows:

$$\begin{aligned} dH_0(t) &= -(\alpha_G + \tau)\, H_0(t)\ dt + \tau_0^{1/2} \frac{G_n(t)}{\sigma_G\ \Omega^{1/2}}\ dt +\ dB^n_0(t), \\ dH_n(t) &= -(\alpha_G + \tau)\, H_n(t)\ dt + \tau_H^{1/2} \frac{G_n(t)}{\sigma_G\ \Omega^{1/2}}\ dt +\ dB^n_n(t), \end{aligned} \tag{A52}$$

$$dH_{-n}(t) = -(\alpha_G + \tau)\, H_{-n}(t)\ dt + \tau_L^{1/2} \frac{G_n(t)}{\sigma_G\ \Omega^{1/2}}\ dt + \tfrac{1}{N-1} \sum_{m=1, m\neq n}^{N} dB^n_m(t). \tag{A53}$$

Combine the three sufficient statistics into the two sufficient statistics

$$H^c_n(t) := H_n(t) + \hat{a}\, H_0(t), \qquad H^c_{-n}(t) := H_{-n}(t) + \hat{a}\, H_0(t), \qquad \text{with}$$
$$\hat{a} := \tfrac{\tau_0^{1/2}}{\tau_H^{1/2} + (N-1)\tau_L^{1/2}}. \tag{A54}$$

Define $y(t) = [D(t)\ ,\ H^c_n(t)\ ,\ H^c_{-n}(t)]'$ as a continuous three-vector stochastic process of all state variables. Using equation (A52), write this stochastic vector in matrix form as

$$dy(t) = K\, y(t)\ dt + C_z\ dZ(t), \tag{A55}$$

where K is a 3×3 matrix and C_z is a 3×3 matrix given by

$$K = \begin{bmatrix} -\alpha_D & \sigma_G \Omega^{1/2} \tau_H^{1/2} & \sigma_G \Omega^{1/2} (N-1) \tau_L^{1/2} \\ 0 & -\alpha_G - \tau + \tau_H^{1/2}(\tau_H^{1/2} + \hat{a}\tau_0^{1/2}) & (N-1)\tau_L^{1/2}(\tau_H^{1/2} + \hat{a}\tau_0^{1/2}) \\ 0 & \tau_H^{1/2}(\tau_L^{1/2} + \hat{a}\tau_0^{1/2}) & -\alpha_G - \tau + (N-1)\tau_L^{1/2}(\tau_L^{1/2} + \hat{a}\tau_0^{1/2}) \end{bmatrix},$$

$$C_z = \begin{bmatrix} \sigma_D & 0 & 0 \\ \hat{a} & 1 & 0 \\ \hat{a} & 0 & \frac{1}{\sqrt{N-1}} \end{bmatrix}, \tag{A56}$$

and $\mathrm{d}Z(t) = [\,\mathrm{d}B_0^n(t),\ \mathrm{d}B_n^n(t),\ \frac{1}{\sqrt{N-1}}\sum_{m=1,m\neq n}^{N}\ \mathrm{d}B_m^n(t)]'$ is a three-dimensional Brownian motion.

We can also represent the process $y_{k+1} = [D_{k+1}, H^c_{n,k+1}, H^c_{-n,k+1}]'$ as an integral,

$$y_{k+1} = \mathrm{e}^{Kh}\, y_k + \int_{kh}^{(k+1)h} \mathrm{e}^{K((k+1)h-t)}\, C_z\ \mathrm{d}Z(t). \tag{A57}$$

This yields the following equations for conditional expectations and the variance of y_{k+1}:

$$\mathrm{E}^n_{kh}[y_{k+1}] = \mathrm{e}^{Kh}[D_k, H^c_{n,k}, H^c_{-n,k}]', \qquad \mathrm{var}_{kh}[y_{k+1}] = \int_0^h \mathrm{e}^{K(h-t)} C_z C_z' \mathrm{e}^{K'(h-t)}\ \mathrm{d}t. \tag{A58}$$

Since trading is discrete, we also need to derive dynamics for variable $\breve{D}_{k+1}$, defined in equation (55). In particular, we solve for $\mathrm{E}^n_{kh}[\breve{D}_{k+1}]$, $\mathrm{var}_{kh}[\breve{D}_{k+1}]$, and $\mathrm{cov}[\breve{D}_{k+1}\ ,\ y_{k+1}]$. Define

$$\breve{y}_{k+1} := \mathrm{e}^{rh} \int_{kh}^{(k+1)h} \mathrm{e}^{-r(t-kh)} y(t)\ \mathrm{d}t. \tag{A59}$$

It can be shown that

$$\breve{y}_{k+1} = (K - rI)^{-1}\Bigg(\Big(\mathrm{e}^{(K-rI)h} - I\Big)\mathrm{e}^{rh} y_k + \int_{kh}^{(k+1)h} \mathrm{e}^{K((k+1)h-t)} C_z\ \mathrm{d}Z(t) - \int_{kh}^{(k+1)h} \mathrm{e}^{-r(t-(k+1)h)} C_z\ \mathrm{d}Z(t)\Bigg). \tag{A60}$$

The expectation $\mathrm{E}^n_{kh}[\breve{D}_{k+1}]$ is given by the first element in the 4×1 vector $\mathrm{e}^{rh}(K - rI)^{-1}(\mathrm{e}^{(K-rI)h} - I)y_k$ in the equation above. We can then derive

$$\begin{aligned}
\mathrm{cov}[y_{k+1}\ ,\ \breve{y}_{k+1}] &= \mathrm{var}_{kh}[y_{k+1}](K' - rI)^{-1} - (K + rI)^{-1}\Big(\mathrm{e}^{(K+rI)h} - I\Big)C_z C_z'(K' - rI)^{-1}, \\
\mathrm{cov}[\breve{y}_{k+1}\ ,\ \breve{y}_{k+1}] &= \Big((K - rI)^{-1}\mathrm{var}_{kh}[y_{k+1}] - (K - rI)^{-1}C_z C_z'(\mathrm{e}^{(K+rI)h} - I)'(K' + rI)^{-1}\Big)(K' - rI)^{-1} \\
&\quad - \left((K^2 - r^2 I)^{-1}(\mathrm{e}^{(K+rI)h} - I) + \frac{1 - \mathrm{e}^{2rh}}{2r}(K - rI)^{-1}\right)C_z C_z'(K' - rI)^{-1}.
\end{aligned} \tag{A61}$$

It follows that $\text{var}_k[\breve{D}_{k+1}]$ is the (1,1) entry of the matrix $\text{cov}[\breve{y}_{k+1}\ ,\ \breve{y}_{k+1}]$, and $\text{cov}[\breve{D}_{k+1}\ ,\ y_{k+1}]$ is given by the first column of the matrix $\text{cov}[y_{k+1}\ ,\ \breve{y}_{k+1}]$.

H.2. Solution of the General Dynamic Model

We next solve for the equilibrium. We conjecture that the price in period k is a linear function of D_k and $\bar{G}_k$, of the form

$$P_k = \frac{D_k}{r+\alpha_D} + C_G\,\frac{\bar{G}_k}{(r+\alpha_D)(r+\alpha_G)}. \tag{A62}$$

Trader n's problem (52) can be rewritten in discrete-time form as (53), where $U_{n,j}$ is obtained by solving the maximization problem (56) subject to constraint (57). The first-order condition yields

$$c(jh+t) = -\frac{1}{A}\left((\rho-r)t + \ln\frac{\lambda}{A}\right), \tag{A63}$$

where λ is the Lagrange multiplier. If $t=0$ in equation (A63), then $c(jh) = -\frac{1}{A}\ln\frac{\lambda}{A}$. Thus,

$$c(jh+t) = c(jh) + \frac{1}{A}(r-\rho)t. \tag{A64}$$

Substituting (A64) into constraint (57), we have

$$c(jh) = \frac{rh}{1-\mathrm{e}^{-rh}}c_{n,j} - \frac{r-\rho}{Ar(1-\mathrm{e}^{-rh})}\left(1-(rh+1)\mathrm{e}^{-rh}\right). \tag{A65}$$

From equations (A64), (A65), and (56), we get

$$U_{n,j} = -h\ \exp\left(-A\frac{rh}{1-\mathrm{e}^{-rh}}c_{n,j}\right)\phi(r,\rho,h), \tag{A66}$$

where $\phi(r,\rho,h)$ is defined as

$$\phi(r,\rho,h) = \frac{1-\mathrm{e}^{-rh}}{rh}\exp\left(\frac{r-\rho}{r(1-\mathrm{e}^{-rh})}\left(1-(1+rh)\mathrm{e}^{-rh}\right)\right) = 1-\tfrac{1}{2}\rho h + O(h^2). \tag{A67}$$

Trader n's problem (53) is then equivalent to

$$\max_{\substack{[c_{n,j}],j=k,k+1,\ldots,\infty\\ [S_{n,j}],j=k,k+1,\ldots,\infty}} \quad \mathrm{E}^n_{kh}\sum_{j=k} -h\mathrm{e}^{-\rho(j-k)h}\exp\left(-A\frac{rh}{1-\mathrm{e}^{-rh}}c_{n,j}\right) \tag{A68}$$

subject to the budget constraint (54).

We conjecture and verify that the value function has the quadratic exponential form

$$V_k(W_{n,k}\ ,\ H^c_{n,k}\ ,\ H^c_{-n,k}) = -\exp\big(\psi_0 + \psi_W W_{n,k} + \tfrac{1}{2}\psi_{nn}(H^c_{n,k})^2 + \tfrac{1}{2}\psi_{xx}(H^c_{-n,k})^2 + \psi_{nx}H^c_{n,k}H^c_{-n,k}\big). \tag{A69}$$

The five constants ψ_0, ψ_W, ψ_{nn}, ψ_{xx}, and ψ_{nx} have values consistent with a steady-state equilibrium. The terms ψ_{nn}, ψ_{xx}, and ψ_{nx} capture the value of future trading opportunities based on current public and private information. The value of trading on innovations to future information is built into the constant term ψ_0.

The Hamilton-Jacobi-Bellman (HJB) equation for the discrete problem is

$$V_k(W_{n,k}\ ,\ H^c_{n,k}\ ,\ H^c_{-n,k}) = \max_{c_{n,k}S_{n,k}}\left[-h\exp\left(-\frac{Arh}{1-\mathrm{e}^{-rh}}c_{n,k}\right) + \mathrm{e}^{-\rho h}\mathrm{E}^n_{kh}V_{k+1}(W_{n,k+1}\ ,\ H^c_{n,k+1}\ ,\ H^c_{-n,k+1})\right], \tag{A70}$$

where the dynamics of wealth satisfies (54) and the dynamics of $H^c_{n,k}$ and $H^c_{-n,k}$ can be obtained from equations (A52) and (A54). Define

$$x_{k+1} := \Big[D_{k+1}\ ,\ \breve{D}_{k+1}\ ,\ H^c_{n,k+1}, H^c_{-n,k+1}\Big]' - \Big[\mathrm{E}^n_{kh}[D_{k+1}]\ ,\ \mathrm{E}^n_{kh}\Big[\breve{D}_{k+1}\Big], \mathrm{E}^n_{kh}\big[H^c_{n,k+1}\big]\ ,\ \mathrm{E}^n_{kh}\big[H^c_{-n,k+1}\big]\Big]', \tag{A71}$$

where $\breve{D}_{k+1}$ is as defined in equation (55). Then, $\mathrm{E}^n_{kh}[V_{k+1}]$ can be obtained using

$$\mathrm{E}^n_{kh}\left[\mathrm{e}^{-\alpha\left(\bar{A}+B'x+\frac{1}{2}x'Cx\right)}\right] = \frac{1}{\sqrt{|I+\alpha C\Sigma|}}\mathrm{e}^{-\alpha\left(\bar{A}-\frac{1}{2}\alpha B'\Sigma(I+\alpha C\Sigma)^{-1}B\right)}, \tag{A72}$$

where x is an $n\times 1$ normal vector with mean zero and covariance matrix Σ, $\bar{A}$ is a scalar, B is an $n\times 1$ vector, C is an $n\times n$ symmetric matrix, and I is the $n\times n$ identity matrix. The value of Σ can be obtained from the previous section. More specifically, $\mathrm{var}_k[\breve{D}_{k+1}]$ is the (1,1) entry of the matrix $\mathrm{cov}[\breve{y}_{k+1}\ ,\ \breve{y}_{k+1}]$ in equation (A61), $\mathrm{cov}[\breve{D}_{k+1}\ ,\ y_{k+1}]$ is given by the first column of the matrix $\mathrm{cov}[y_{k+1}\ ,\ \breve{y}_{k+1}]$ in equation (A61), and $\mathrm{var}_k[y_{k+1}]$ is given in equation (A58). Define α, $\bar{A}$, B, and C by

$$\bar{A} = \psi_0 + \psi_W\mathrm{e}^{rh}(W_{n,k} - hc_{n,k} - S_{n,k}P_k) + \psi_W S_{n,k}\Big(\mathrm{E}^n_{kh}\Big[\breve{D}_{k+1}\Big] + \tfrac{\mathrm{E}^n_{kh}[D_{k+1}]}{r+\alpha_D} + \tfrac{C_G\sigma_G\Omega^{1/2}(\tau_H^{1/2}+(N-1)\tau_L^{1/2})}{N(r+\alpha_D)(r+\alpha_G)}\big(\mathrm{E}^n_{kh}\big[H^c_{n,k+1}\big] + (N-1)\mathrm{E}^n_{kh}\big[H^c_{-n,k+1}\big]\big)\Big) + \tfrac{1}{2}\psi_{nn}\big(\mathrm{E}^n_{kh}\big[H^c_{n,k+1}\big]\big)^2 + \tfrac{1}{2}\psi_{xx}\big(\mathrm{E}^n_{kh}\big[H^c_{-n,k+1}\big]\big)^2 + \psi_{nx}H^c_{n,k+1}H^c_{-n,k+1}, \tag{A73}$$

$$\alpha = -1, \qquad B = \psi_W \psi_B S_{n,k} + \varphi_B, \qquad C = \begin{bmatrix} 0_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & c_{2\times 2} \end{bmatrix}, \qquad \text{where}$$

$$c_{2\times 2} = \begin{bmatrix} \psi_{nn} & \psi_{nx} \\ \psi_{nx} & \psi_{xx} \end{bmatrix}, \qquad \psi_B = \psi_{B1} + \tfrac{C_G \sigma_G \Omega^{1/2}(\tau_H^{1/2} + (N-1)\tau_L^{1/2})}{N(r+\alpha_D)(r+\alpha_G)} \psi_{B2}, \tag{A74}$$

$$\psi_{B1} = \left[\tfrac{1}{r+\alpha_D}\ ,\ 1\ ,\ 0\ ,\ 0 \right]', \qquad \psi_{B2} = [0\ ,\ 0\ ,\ 1\ ,\ N-1]',$$

$$\varphi_B = \left[0\ ,\ 0\ ,\ \psi_{nn} \mathrm{E}^n_{kh}\left[H^c_{n,k+1}\right] + \psi_{nx} \mathrm{E}^n_{kh}\left[H^c_{-n,k+1}\right]\ ,\ \psi_{xx} \mathrm{E}^n_{kh}\left[H^c_{-n,k+1}\right] + \psi_{nx} \mathrm{E}^n_{kh}\left[H^c_{n,k+1}\right] \right]'. \tag{A75}$$

Taking the first-order condition with respect to $c_{n,k}$ in the HJB equation (A70) yields

$$c^*_{n,k} = -\tfrac{1-\mathrm{e}^{-rh}}{Arh} \left((r-\rho)h + \ln\left(\tfrac{\psi_W (1-\mathrm{e}^{-rh})}{Arh} \mathrm{E}^n_{kh} V_{k+1} \right) \right). \tag{A76}$$

Substituting (A76) into the HJB equation (A70) yields

$$V_k = \mathrm{e}^{-\rho h} \left(1 - \tfrac{\psi_W (\mathrm{e}^{rh}-1)}{Ar} \right) \mathrm{E}^n_{kh}\left[V_{k+1}\right]. \tag{A77}$$

Taking the first-order condition with respect to $S_{n,k}$ yields

$$S^*_{n,k} = \frac{\mathrm{E}^n_{kh}\left[P_{k+1} + \breve{D}_{k+1}\right] - \mathrm{e}^{rh} P_k + \varphi_B' \Sigma (I - C\Sigma)^{-1} \psi_B}{-\psi_W \psi_B' \Sigma (I - C\Sigma)^{-1} \psi_B}. \tag{A78}$$

It can be shown that optimal quantity $S^*_{n,k}$ is linear in the state variables $H^c_{n,k}$ and $H^c_{-n,k}$. Define constants c_{d1}, c_{d2}, c_{n1}, c_{x1}, c_{n2}, and c_{x2} by

$$c_{d1} := \frac{\sigma_G \Omega^{1/2}\left(\mathrm{e}^{-\alpha_G h} - \mathrm{e}^{-\alpha_D h} \right)}{\alpha_D - \alpha_G}, \qquad c_{d2} := \frac{\sigma_G \Omega^{1/2}\left((r+\alpha_G)\mathrm{e}^{-(r+\alpha_D)h} - (r+\alpha_D)\mathrm{e}^{-(r+\alpha_G)h} + \alpha_D - \alpha_G \right)}{(\alpha_D - \alpha_G)(r+\alpha_D)(r+\alpha_G)},$$

$$c_{n1} := \frac{\mathrm{e}^{-(\alpha_G+\tau)h}\left(\mathrm{e}^{\tau h}\left(\tau_H^{1/2}(\tau_0+\tau_H) + (N-1)\tau_H \tau_L^{1/2} \right) + (N-1)\tau_L^{1/2}\left(\tau_0 + \tau_H^{1/2}\tau_L^{1/2} + (N-1)\tau_L \right) \right)}{\left(\tau_H^{1/2} + (N-1)\tau_L^{1/2} \right)\tau}, \tag{A79}$$

$$c_{n2} := \frac{\mathrm{e}^{-(\alpha_G+\tau)h}\left(\mathrm{e}^{\tau h}-1\right)\tau_H^{1/2}\left(\tau_0+\tau_H^{1/2}\tau_L^{1/2}+(N-1)\tau_L\right)}{\left(\tau_H^{1/2}+(N-1)\tau_L^{1/2}\right)\tau},$$

$$c_{x1} := \frac{\mathrm{e}^{-(\alpha_G+\tau)h}\left(\mathrm{e}^{\tau h}-1\right)(N-1)\tau_L^{1/2}\left(\tau_0+(N-1)\tau_H^{1/2}\tau_L^{1/2}+\tau_H\right)}{\left(\tau_H^{1/2}+(N-1)\tau_L^{1/2}\right)\tau},$$

$$c_{x2} := \frac{\mathrm{e}^{-(\alpha_G+\tau)h}\left(\mathrm{e}^{\tau h}(N-1)\tau_L^{1/2}\left(\tau_0+\tau_H^{1/2}\tau_L^{1/2}+(N-1)\tau_L\right)+\tau_H^{1/2}(\tau_0+\tau_H)+(N-1)\tau_H\tau_L^{1/2}\right)}{\left(\tau_H^{1/2}+(N-1)\tau_L^{1/2}\right)\tau}.$$

Then φ_B, defined in equation (A75), can be written as $\varphi_B = \varphi_{B1}H^c_{n,k} + \varphi_{B2}H^c_{-n,k}$, where $\varphi_{B1} := [0\ ,\ 0\ ,\ \psi_{nn}c_{n1} + \psi_{nx}c_{n2}\ ,\ \psi_{xx}c_{n2} + \psi_{nx}c_{n1}]'$ and $\varphi_{B2} := [0\ ,\ 0\ ,\ \psi_{nn}c_{x1} + \psi_{nx}c_{x2}\ ,\ \psi_{xx}c_{x2} + \psi_{nx}c_{x1}]'$. The market-clearing condition $\sum_{n=1}^N S^*_{n,k} = 0$ and equation (A78) imply

$$C_G = \frac{\mathrm{e}^{-\alpha_G h}-\mathrm{e}^{rh}-\frac{(r+\alpha_D)(r+\alpha_G)}{\sigma_G\Omega^{1/2}\left(\tau_H^{1/2}+(N-1)\tau_L^{1/2}\right)}(\varphi'_{B1}+\varphi'_{B2})\Sigma(I-C\Sigma)^{-1}\psi_{B1}}{\frac{N\tau_0+\left(\tau_H^{1/2}+(N-1)\tau_L^{1/2}\right)^2}{N\tau}\mathrm{e}^{-\alpha_G h}-\mathrm{e}^{rh}+\frac{(N-1)\left(\tau_H^{1/2}-\tau_L^{1/2}\right)^2}{N\tau}\mathrm{e}^{-(\alpha_G+\tau)h}+\frac{1}{N}(\varphi'_{B1}+\varphi'_{B2})\Sigma(I-C\Sigma)^{-1}\psi_{B2}}. \tag{A80}$$

Then, from equations (A78) and (A80), the optimal inventory for trader n is given by

$$S^*_{n,k} = C_L\left(H^c_{n,k} - H^c_{-n,k}\right), \tag{A81}$$

where the constant C_L is defined as

$$\begin{aligned} C_L =& \frac{1}{rAN\tau(r+\alpha_D)(r+\alpha_G)\psi'_B\Sigma(I-C\Sigma)^{-1}\psi_B}\Big(N\tau\Big(\sigma_G\Omega^{1/2}\tau_H^{1/2}(\mathrm{e}^{rh}-\mathrm{e}^{-\alpha_G h}) \\ &+(r+\alpha_D)(r+\alpha_G)\varphi'_{B1}\Sigma(I-C\Sigma)^{-1}\psi_{B1}\Big) \\ &+C_G\sigma_G\Omega^{1/2}\Big(\tau\left(\tau_H^{1/2}+(N-1)\tau_L^{1/2}\right)\varphi'_{B1}\Sigma(I-C\Sigma)^{-1}\psi_{B2}+\mathrm{e}^{-\alpha_G h}\tau_H^{1/2} \\ &\left(N\tau_0+\left(\tau_H^{1/2}+(N-1)\tau_L^{1/2}\right)^2\right) \\ &\qquad -\mathrm{e}^{rh}\left(\tau_H^{1/2}+(N-1)\tau_L^{1/2}\right)\tau-\mathrm{e}^{-(\alpha_G+\tau)h}(N-1)\left(\tau_H^{1/2}-\tau_L^{1/2}\right) \\ &\left(\tau_0+\tau_H^{1/2}\tau_L^{1/2}+(N-1)\tau_L\right)\Big)\Big). \end{aligned} \tag{A82}$$

Equations (A77) and (A69) imply

$$\begin{aligned} \ln(-\mathrm{E}^n_{kh}V_{k+1}) =& \psi_0 + \psi_W W_{n,k} + \tfrac{1}{2}\psi_{nn}(H^c_{n,k})^2 + \tfrac{1}{2}\psi_{xx}(H^c_{-n,k})^2 + \psi_{nx}H^c_{n,k}H^c_{-n,k} \\ &+\rho h - \ln\left(1 - \frac{\psi_W(\mathrm{e}^{rh}-1)}{Ar}\right). \end{aligned} \tag{A83}$$

Substituting (54), (A58), (A62), value function (A69), (A76), (A81), and (A83) into the HJB equation (A70) and setting the constant term and the coefficients on $W_{n,k}$, $(H^c_{n,k})^2$, $(H^c_{-n,k})^2$, and $H^c_{n,k}\,H^c_{-n,k}$ to zero yields five equations, which can be solved for the five unknown parameters ψ_0, ψ_W, ψ_{nn}, ψ_{nx}, and ψ_{xx}. Setting the constant term and coefficient on $W_{n,k}$ to zero yields

$$\psi_W = -rA, \qquad \psi_0 = \frac{(r-\rho)h - (e^{rh}-1)\ln\frac{1-e^{-rh}}{h} - \ln\sqrt{|I-C\Sigma|}}{e^{rh}-1}. \tag{A84}$$

By setting the coefficients on $(H^c_{n,k})^2$, $(H^c_{-n,k})^2$, and $H^c_{n,k}\,H^c_{-n,k}$ to zero, we obtain three polynomial equations in the three unknowns ψ_{nn}, ψ_{xx}, and ψ_{nx}. These three equations in three unknowns can be written as the following system of equations:

$$\begin{aligned} 0 = &-\tfrac{1}{2}e^{rh}\psi_{nn} + rAC_LC_G\Big(e^{rh} - c_{n1} - (N-1)c_{n2}\Big)\tfrac{\sigma_G\Omega^{1/2}(\tau_H^{1/2}+(N-1)\tau_L^{1/2})}{N(r+\alpha_D)(r+\alpha_G)} \\ &- rAC_L\Big(e^{rh}c_{d2} + \tfrac{c_{d1}}{r+\alpha_D}\Big)\tau_H^{1/2} + \tfrac{1}{2}\psi_{nn}c_{n1}^2 + \tfrac{1}{2}\psi_{xx}c_{n2}^2 + \psi_{nx}c_{n1}c_{n2} \\ &+ \tfrac{1}{2}r^2A^2C_L^2\psi_B'\Sigma(I-C\Sigma)^{-1}\psi_B - rAC_L\psi_B'\Sigma(I-C\Sigma)^{-1}\varphi_{B1} + \tfrac{1}{2}\varphi_{B1}'\Sigma(I-C\Sigma)^{-1}\varphi_{B1}, \end{aligned} \tag{A85}$$

$$\begin{aligned} 0 = &-\tfrac{1}{2}e^{rh}\psi_{xx} + rAC_LC_G\Big(-e^{rh}(N-1) + c_{x1} + (N-1)c_{x2}\Big)\tfrac{\sigma_G\Omega^{1/2}(\tau_H^{1/2}+(N-1)\tau_L^{1/2})}{N(r+\alpha_D)(r+\alpha_G)} \\ &+ rAC_L\Big(e^{rh}c_{d2} + \tfrac{c_{d1}}{r+\alpha_D}\Big)(N-1)\tau_L^{1/2} + \tfrac{1}{2}\psi_{xx}c_{x2}^2 + \psi_{nx}c_{x1}c_{x2} \\ &+ \tfrac{1}{2}r^2A^2C_L^2\psi_B'\Sigma(I-C\Sigma)^{-1}\psi_B + rAC_L\psi_B'\Sigma(I-C\Sigma)^{-1}\varphi_{B2} + \tfrac{1}{2}\varphi_{B2}'\Sigma(I-C\Sigma)^{-1}\varphi_{B2}, \end{aligned} \tag{A86}$$

$$\begin{aligned} 0 = &\ e^{rh}\psi_{nx} + rAC_LC_G\Big(e^{rh}(N-2) - c_{x1} - (N-1)c_{x2} + c_{n1} + (N-1)c_{n2}\Big) \\ &\frac{\sigma_G\Omega^{1/2}(\tau_H^{1/2} + (N-1)\tau_L^{1/2})}{N(r+\alpha_D)(r+\alpha_G)} \\ &- rAC_L\Big(e^{rh}c_{d2} + \tfrac{c_{d1}}{r+\alpha_D}\Big)\Big((N-1)\tau_L^{1/2} - \tau_H^{1/2}\Big) + \psi_{nn}c_{n1}c_{x1} + \psi_{xx}c_{n2}c_{x2} \\ &+ \psi_{nx}(c_{n1}c_{x2} + c_{x1}c_{n2}) \\ &- r^2A^2C_L^2\psi_B'\Sigma(I-C\Sigma)^{-1}\psi_B - rAC_L\psi_B'\Sigma(I-C\Sigma)^{-1}(\varphi_{B2} - \varphi_{B1}) \\ &+ \varphi_{B1}'\Sigma(I-C\Sigma)^{-1}\varphi_{B2}. \end{aligned} \tag{A87}$$

To summarize, optimal consumption is defined in (A76), the optimal strategy is defined in (A81), and the endogenous coefficient C_L is defined in (A82). The equilibrium price is defined in (A62), and the endogenous coefficient C_G is defined in (A80). The quadratic value function is defined by five parameters. Parameters ψ_W and ψ_0 are presented in (A84). Parameters ψ_{nn}, ψ_{nx}, and ψ_{xx} are solved numerically from the system of equations (A85) to (A87). This concludes the proof of Theorem 3.

H.3. The Case with Myopic Conditional Mean-Variance Optimizers

Trader n's myopic conditional mean-variance optimal holding at time kh is given in equation (A45). Similar to the proof of the general dynamic model case (equation (A78)), it can be shown that optimal quantity $\check{S}_{n,k}$ is linear in the state variables $H^c_{n,k}$ and $H^c_{-n,k}$. The market-clearing condition $\sum_{n=1}^{N} \check{S}_{n,k} = 0$ and equation (A45) imply

$$\check{C}_G = \frac{e^{-\alpha_G h} - e^{rh}}{\left(\frac{\tau_0}{\tau} + \frac{1}{N\tau}\left(\tau_H^{1/2} + (N-1)\tau_L^{1/2}\right)^2\right) e^{-\alpha_G h} - e^{rh} + \frac{N-1}{N\tau}\left(\tau_H^{1/2} - \tau_L^{1/2}\right)^2 e^{-(\alpha_G+\tau)h}}, \tag{A88}$$

which can be simplified as equation (A47). This completes the proof of Theorem A1.

I. Proof of Proposition A1

The endogenous parameter $\check{C}_G$ is defined in equation (A47). From equation (A47), we can see that a common prior ($\tau_H = \tau_L$) implies $\check{C}_G = 1$. Relative overconfidence ($\tau_H > \tau_L$) implies $0 < \check{C}_G < 1$. Fixing the total precision τ, we have that $\check{C}_G$ decreases in the disagreement τ_H/τ_L. Using a Taylor series expansion, it can be shown that $\frac{1-e^{-\tau h}}{e^{(r+\alpha_G)h}-1}$ decreases in trading interval h. It follows that $\check{C}_G$ increases in the trading interval h.

J. Proof of Proposition 4

Assuming $\tau_H = \tau_L$, then $\psi_{nn} = \psi_{nx} = \psi_{xx} = 0$ solves equations (A85) to (A87). We also get $C_G = 1$ and $C_L = 0$ from equations (A80) and (A82). There is no trading. If $\tau_H = \tau_L$, then $C_J = 1$ in equation (22); if $\tau_H > \tau_L$, then Jensen's inequality implies $0 < C_J < 1$.

In the case with relative overconfidence of $\tau_H > \tau_L$, information cannot have negative value in the value function (A69) since traders can always ignore it. Therefore, the 2×2 matrix

$$\begin{bmatrix} \psi_{nn} & \psi_{nx} \\ \psi_{nx} & \psi_{xx} \end{bmatrix} \tag{A89}$$

must be negative semidefinite. This implies $\psi_{nn} \le 0$, $\psi_{xx} \le 0$, and $\psi_{nx}^2 \le \psi_{nn}\psi_{xx}$. It follows that $\psi_{nn} + \psi_{xx} + 2\psi_{nx} \le 0$. In the continuous-time model ($h \to 0$), we can show

$$C_G = \frac{N(r+\alpha_G)\left(\sigma_G \Omega^{1/2} + \frac{\sigma_D \hat{a}(\psi_{nn}+\psi_{xx}+2\psi_{nx})}{\tau_H^{1/2}+(N-1)\tau_L^{1/2}}\right)}{\sigma_G \Omega^{1/2}\left(N(r+\alpha_G) + (N-1)\left(\tau_H^{1/2} - \tau_L^{1/2}\right)^2 - (1+N\hat{a}^2)(\psi_{nn}+\psi_{xx}+2\psi_{nx})\right)}. \tag{A90}$$

It can then be shown that $\lim_{h\to 0} C_G \le \lim_{h\to 0} \check{C}_G < 1$. This concludes the proof.

K. *Proof of Proposition 5*

Let a vector $[\psi^*_{nn}\ ,\ \psi^*_{nx}\ ,\ \psi^*_{xx}]$ be a solution to the system (A85) to (A87) for exogenous parameters A, σ_D, σ_G, r, α_G, α_D, τ_0, τ_L, and τ_H. If risk aversion is rescaled by factor F from A to A/F and other exogenous parameters are kept unchanged, then it is straightforward to show that the vector $[\psi^*_{nn}\ ,\ \psi^*_{nx}\ ,\ \psi^*_{xx}]$ is still the solution to the system (A85) to (A87). From equations (A80) and (A82), it then follows that C_L becomes $C_L\,F$, but C_G remains the same.

L. *Proof of Theorems 4 and 5*

We first calculate the empirically correct estimate of the growth rate. We then derive the holding-period excess return. In terms of true parameters, the growth rate in equation (43) becomes

$$\mathrm{d}G^*(t) := -\hat{\alpha}_G\, G^*(t)\ \mathrm{d}t + \hat{\sigma}_G\ \mathrm{d}B_G(t). \tag{A91}$$

Similarly, using true parameters, the process for signals $\mathrm{d}I_n(t)$ and $\mathrm{d}I_0(t)$ in equations (45) and (48) becomes

$$\begin{aligned}\mathrm{d}I_n(t) &:= \hat{\tau}_I^{1/2}\,\frac{G^*(t)}{\hat{\sigma}_G\ \hat{\Omega}^{1/2}}\ \mathrm{d}t +\ \mathrm{d}\hat{B}_n(t), \quad n = 1,\ldots,N,\\ \mathrm{d}I_0(t) &:= \hat{\tau}_0^{1/2}\,\frac{G^*(t)}{\hat{\sigma}_G\ \hat{\Omega}^{1/2}}\ \mathrm{d}t +\ \mathrm{d}B_0(t),\end{aligned} \tag{A92}$$

$$\text{with}\quad \hat{\tau}_0 := \frac{\hat{\Omega}\ \hat{\sigma}_G^2}{\sigma_D^2}\quad \text{and}\quad \mathrm{d}\hat{B}_n(t) =\ \mathrm{d}B_n(t) + \left(\frac{\tau_n^{1/2}}{\sigma_G\Omega^{1/2}} - \frac{\hat{\tau}_n^{1/2}}{\hat{\sigma}_G\hat{\Omega}^{1/2}}\right)G^*(t)\ \mathrm{d}t, \tag{A93}$$

and $\mathrm{d}B_G(t)$, $\mathrm{d}B_0(t)$, $\mathrm{d}\hat{B}_1(t)$, …, $\mathrm{d}\hat{B}_N(t)$ are independent Brownian motions.

The sufficient statistic under the correct parameters is defined as

$$\hat{H}_n(t) := \int_{u=-\infty}^{t} \mathrm{e}^{-(\hat{\alpha}_G+\hat{\tau})\,(t-u)}\ \mathrm{d}I_n(u), \qquad n = 0, 1, \ldots, N. \tag{A94}$$

The sufficient statistic $\hat{H}_n(t)$ and $H_n(t)$ relate to each other as follows:

$$\hat{H}_n(t) = H_n(t) + (\alpha_G + \tau - \hat{\alpha}_G - \hat{\tau}\,) \int_{u=-\infty}^{t} \mathrm{e}^{-(\hat{\alpha}_G+\hat{\tau}\,)\,(t-u)}\, H_n(u)\ \mathrm{d}u. \tag{A95}$$

If traders use the empirically correct parameters ($\alpha_G = \hat{\alpha}_G$) and empirically correct total precisions of signals ($\tau = \hat{\tau}$), then we obtain $\hat{H}_n(t) = H_n(t)$. If $\alpha_G + \tau \neq \hat{\alpha}_G + \hat{\tau}$, then $\hat{H}_n(t) \neq H_n(t)$. Define the aggregate sufficient statistic as the linear combination of $\hat{H}_n(t)$, $n = 0, 1, \ldots, N$,

$$\hat{H}(t) := \hat{\tau}_0^{1/2}\hat{H}_0(t) + \sum_{n=1}^{N} \hat{\tau}_I^{1/2}\hat{H}_n(t). \tag{A96}$$

We then redo the calculation of Lemma A1 in Appendix subsection F to derive Stratonovich-Kalman-Bucy filtering formulas using empirically correct parameters. The empirically correct estimate of the growth rate at $t = kh$ is $\hat{G}_k := \hat{\mathrm{E}}_{kh}[G^*(kh)] = \sigma_G\, \hat{\Omega}^{1/2}\, \hat{H}_k$, and the steady-state error variance is

$$\hat{\Omega} := \hat{\mathrm{var}}\Big[\tfrac{G^*(t) - \hat{G}(t)}{\hat{\sigma}_G}\Big] = \tfrac{1}{2\,\hat{\alpha}_G + \hat{\tau}}. \tag{A97}$$

Plugging equation (63) into the continuous version of equation (65) yields price $P(t)$ at time t:

$$P(t) = \frac{D(t)}{r + \alpha_D} + C_G\, \frac{\sigma_G\, \Omega^{1/2}}{(r + \alpha_D)(r + \alpha_G)}\, H(t). \tag{A98}$$

The price depends on dividend $D(t)$ and traders' sufficient statistics $H(t)$ about the growth rate; it does not depend on empirically correct sufficient statistics $\hat{H}(t)$. The dynamics of state variables is determined by empirically correct parameters. The return dynamics thus also depend on both traders' sufficient statistics $H(t)$ and empirically correct sufficient statistics $\hat{H}(t)$. Specially, when we calculate $\mathrm{d}P(t)$ using equation (A98), we plug in $\mathrm{d}H_n(t)$ using equation (58), and plug in the correct empirical specification of the dynamics of $\mathrm{d}I_n(t)$ from equation (A92) and the correct estimate $\hat{G}(t)$ from equation (72). We can show that the equilibrium instantaneous return process is a linear combination of the two statistics $H(t)$ and $\hat{H}(t)$,

$$\mathrm{d}P(t) + D(t)\ \mathrm{d}t - r\, P(t)\ \mathrm{d}t = \Big(b\, \hat{H}(t) - a\, H(t)\Big)\ \mathrm{d}t + \ \mathrm{d}\hat{B}_r(t), \tag{A99}$$

where constants a, b, and $\mathrm{d}\hat{B}_r(t)$ are defined as follows:

$$a := \frac{\sigma_G\, C_G\, \Omega^{1/2}}{(r + \alpha_D)(r + \alpha_G)}\,(\alpha_G + r + \tau), \qquad b := \frac{\sigma_G\, \hat{\Omega}^{1/2}}{r + \alpha_D} + \frac{\sigma_G\, C_G\, \Omega^{1/2}}{(r + \alpha_D)(r + \alpha_G)}$$
$$\Big(\tau_0^{1/2}\hat{\tau}_0^{1/2} + N\tau_I^{1/2}\ \hat{\tau}_I^{1/2}\Big),$$

$$\mathrm{d}\hat{B}_r(t) := \frac{\sigma_D}{r + \alpha_D}\ \mathrm{d}B_0^*(t) + \frac{\sigma_G\, C_G\, \Omega^{1/2}}{(r + \alpha_D)(r + \alpha_G)}\Big(\tau_0^{1/2}\ \mathrm{d}B_0^*(t) + \tau_I^{1/2}\, N\ \mathrm{d}\bar{B}^*(t)\Big). \tag{A100}$$

The diffusion term $d\hat{B}_r(t)$ depends on the processes $d\bar{B}^*(t)$ and $dB_0^*(t)$:

$$\begin{aligned} d\bar{B}^*(t) &:= \hat{\tau}_I^{1/2}\left(\sigma_G\,\hat{\Omega}^{1/2}\right)^{-1}\left(G^*(t)-\hat{G}(t)\right)\,dt + \frac{1}{N}\sum_{n=1}^{N} d\hat{B}_n(t), \\ dB_0^*(t) &:= \hat{\tau}_0^{1/2}\left(\sigma_G\,\hat{\Omega}^{1/2}\right)^{-1}\left(G^*(t)-\hat{G}(t)\right)\,dt + dB_0(t). \end{aligned} \tag{A101}$$

Since $\hat{G}(t)$ is the best estimate of $G^*(t)$, both $d\bar{B}^*(t)$ and dB_0 are Brownian motions under empirically correct beliefs, with variances var[$dB_0^*(t)$] = dt and var[$d\bar{B}^*(t)$] = $1/N$. So the instantaneous variance of the excess return is given by

$$\hat{\text{var}}\left[\frac{d\hat{B}_r(t)}{dt^{1/2}}\right] = \left(\frac{\sigma_D}{r+\alpha_D} + \frac{\sigma_G\,\Omega^{1/2}\,C_G\,\tau_0^{1/2}}{(r+\alpha_D)(r+\alpha_G)}\right)^2 + \frac{(\sigma_G\,\Omega^{1/2}\,C_G)^2\,N\,\tau_I}{(r+\alpha_D)^2(r+\alpha_G)^2}. \tag{A102}$$

Define a continuous two-vector stochastic process $y_H(t) = [H(t)\,,\ \hat{H}(t)]'$. Using the definitions of $H(t)$ and $\hat{H}(t)$ in equations (62) and (71), $y_H(t)$ can be shown to satisfy the linear stochastic differential equation

$$dy_H(t) = K_H\,y_H(t)\ dt + C_H\ dZ_H(t), \tag{A103}$$

where K_H is a 2×2 matrix and C_H is a 2×2 matrix given by

$$K_H = \begin{bmatrix} -\alpha_G-\tau & \hat{\tau}_0^{1/2}\tau_0^{1/2} + N\,\hat{\tau}_I^{1/2}\,\tau_I^{1/2} \\ 0 & -\hat{\alpha}_G \end{bmatrix}, \qquad C_H = \begin{bmatrix} \tau_0^{1/2} & \sqrt{N}\,\tau_I^{1/2} \\ \hat{\tau}_0^{1/2} & \sqrt{N}\,\hat{\tau}_I^{1/2} \end{bmatrix}. \tag{A104}$$

Under empirically correct beliefs, the vector $dZ_H(t) = [\,dB_0^*(t)\,,\ \sqrt{N}\,d\bar{B}^*(t)]'$ is a 2×1-dimensional Brownian motion, with var[$dB_0^*(t)$] = dt and var[$d\bar{B}^*(t)$] = dt/N. Integrating equation (A99) over time yields the holding-period return over f periods from $t = kh$ to $t + T = (k+f)h$:

$$\begin{aligned} R_{k,k+f} &= \int_{u=kh}^{kh+T} e^{r(kh+T-u)}\big(\,dP(u) + D(u)\,du - rP(u)\,du\big) \\ &= P_{k+f} - e^{rT}P_k + e^{rT}\int_{u=kh}^{kh+T} e^{-r(u-kh)}D(u)\,du. \end{aligned} \tag{A105}$$

Using equation (A103), we can obtain recursive formulas for the stochastic vector $y_H(u) = [H(u)\,,\ \hat{H}(u)]'$ for time $u \geq kh$ as a function of $y_H(kh) = [H_k\,,\ \hat{H}_k]'$. Plugging these recursive formulas into equation (A105), we obtain the cumulative holding-period return $R_{k,k+f}$ over period $T = fh$ between time kh and $(k+f)h$ as a linear function of H_k and $\hat{H}_k$:

$$R_{k,k+f} = \zeta_2(T)\,\hat{H}_k - \zeta_1(T)\,H_k + \bar{B}_{k,k+f}. \tag{A106}$$

The coefficients $\zeta_1(T) > 0$ and $\zeta_2(T) > 0$ in the equation above are defined as

$$\zeta_1(T) := \frac{\sigma_G\, C_G\, \Omega^{1/2}}{(r+\alpha_D)(r+\alpha_G)} \mathrm{e}^{rT}\left(1-\mathrm{e}^{-(r+\alpha_G+\tau)T}\right) > 0,$$

$$\zeta_2(T) := \frac{\mathrm{e}^{rT}-\mathrm{e}^{-\hat{\alpha}_G T}}{r+\hat{\alpha}_G} \frac{\hat{\sigma}_G\, \hat{\Omega}^{1/2}}{r+\alpha_D} + \frac{\sigma_G C_G \Omega^{1/2}\left(\tau_0^{1/2}\hat{\tau}_0^{1/2} + N\hat{\tau}_I^{1/2}\tau_I^{1/2}\right)}{(\tau+\alpha_G-\hat{\alpha}_G)\,(r+\alpha_G)(r+\alpha_D)} \left(\mathrm{e}^{-\hat{\alpha}_G T} - \mathrm{e}^{-(\alpha_G+\tau)T}\right) > 0, \tag{A107}$$

and the Brownian motion term $\bar{B}_{k,k+f}$ is defined as

$$\bar{B}_{k,k+f} := \int_{s=kh}^{kh+T} \int_{u=s}^{kh+T} [-a, b] \mathrm{e}^{r(kh+T-u)+K_H(u-s)} C_H \;\; \mathrm{d}u \;\; \mathrm{d}Z_H(s) + \int_{s=kh}^{kh+T} \mathrm{e}^{r(kh+T-s)}\, \mathrm{d}\hat{B}_r(s). \tag{A108}$$

If traders have correct model parameters $\hat{\alpha}_G = \alpha_G$, $\hat{\sigma}_G = \sigma_G$, and $\hat{\tau} = \tau$, then $\zeta_1(T)$ and $\zeta_2(T)$ defined in equation (A107) can be written as

$$\zeta_1(T) = \frac{C_G \sigma_G \Omega^{1/2}}{(r+\alpha_G)(r+\alpha_D)}\left(\mathrm{e}^{rT} - \mathrm{e}^{-(\alpha_G+\tau)T}\right), \tag{A109}$$

$$\zeta_2(T) = \frac{\sigma_G \Omega^{1/2} \mathrm{e}^{-(\alpha_G+\tau)T}}{(r+\alpha_G)\,(r+\alpha_D)\tau} \left(\left(\mathrm{e}^{(r+\alpha_G+\tau)T} - \mathrm{e}^{\tau T}\right)\tau + C_G\left(\tau_0 + N\hat{\tau}_I^{1/2}\tau_I^{1/2}\right)\left(\mathrm{e}^{\tau T}-1\right)\right). \tag{A110}$$

Since both $H(t)$ and $\hat{H}(t)$ can be recovered from the history of prices $P(t)$ and dividends $D(t)$, the expected holding-period return $\hat{\mathrm{E}}_{kh}[R_{k,k+f}]$ can be expressed as a linear combination of P_k, D_k, and $H_{0,k}$, as in equation (74), with corresponding coefficients $\beta_0(T)$ and $\beta_1(T)$:

$$\beta_1(T) := \left(\zeta_2(T)\,\frac{\hat{\tau}_I^{1/2}}{\tau_I^{1/2}} - \zeta_1(T)\right) \frac{(r+\alpha_D)\,(r+\alpha_G)}{C_G\, \sigma_G\, \Omega^{1/2}}, \qquad \beta_0(T) := \zeta_2(T)\left(\frac{\hat{\tau}_I^{1/2}}{\tau_I^{1/2}} - 1\right)\tau_0^{1/2}. \tag{A111}$$

A common prior ($\tau_H = \tau_L$) implies $C_G = C_J = 1$ and $\hat{\tau}_I^{1/2} = \tau_I^{1/2} = 0$, and thus $\beta_0(T) = 0$ and $\beta_1(T) = 0$. Direct computations show that relative overconfidence ($\tau_H > \tau_L$) implies $\beta_1(T) > 0$, $\mathrm{d}\beta_1(T)/\,\mathrm{d}T > 0$, and $\beta_0(T) > 0$. It can also be shown that the term $\zeta_2(T)\hat{\tau}_I^{1/2} - \zeta_1(T)\tau_I^{1/2}$ in the definition of $\beta_1(T)$ in equation (A111) can be written as

$$\zeta_2(T)\hat{\tau}_I^{1/2} - \zeta_1(T)\tau_I^{1/2} = \frac{\sigma_G \Omega^{1/2}\mathrm{e}^{rT}}{(r+\alpha_D)(r+\alpha_G)}\Bigg(\hat{\tau}_I^{1/2}(1-C_G)\left(1-\mathrm{e}^{-(r+\alpha_G)T}\right)$$

$$+C_G \frac{\hat{\tau}_I^{1/2} - \tau_I^{1/2}}{\tau} \left(N \hat{\tau}_I \left(1 - \mathrm{e}^{-(r+\alpha_G)T} \right) + \tau_0 \left(1 - \mathrm{e}^{-(r+\alpha_G+\tau)T} \right) \right) \Bigg). \quad \text{(A112)}$$

Thus, the coefficient $\beta_1(T) > 0$ can be decomposed into two terms. The first term in large parenthesis with $1 - C_G > 0$ results from the price-dampening effect of the Keynesian beauty contest, and the second term with $\hat{\tau}_I^{1/2} - \tau_I^{1/2} > 0$ results from the price-dampening effect of $C_J < 1$. This concludes the proof of Theorem 4.

In the continuous-trading limit $h \to 0$, both $H(t)$ and $\hat{H}(t)$ can be recovered from the history of prices $P(t)$ and dividends $D(t)$. If traders have incorrect model parameters and total precisions of signals ($\alpha_G \neq \hat{\alpha}_G$ and $\hat{\tau} \neq \tau$), then the continuous version of equation (A106) implies that the expected holding-period excess return, $\hat{\mathrm{E}}_t[R(t, t+T)]$, can be expressed as in equation (75). The coefficients $\zeta_1(T) > 0$ and $\zeta_2(T) > 0$ are defined in equation (A107). In this general case, there are four possible patterns of the return dynamics: (i) only momentum, (ii) first momentum and then mean reversion, (iii) first mean reversion and then momentum, and (iv) only mean reversion for four specific combinations of different parameter values.

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Supporting Information

Additional Supporting Information may be found in the online version of this article at the publisher's website:

Appendix S1: Internet Appendix.
Replication Code.