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Abstract

In many environments, including credit and online markets, past records about participants are collected, published, and erased after some time. We study the effects of erasing past records in a dynamic market where sellers' quality follows a Markov process and buyers leave feedback about sellers to an information intermediary. When the average quality of sellers is low, unlimited records lead to a market breakdown in the long run. We consider the general information design problem and characterize information policies that can sustain trade and that maximize welfare. These policies hide some information from the market in order to foster socially desirable experimentation. We show that these outcomes can be implemented by appropriately deleting observable past records. Crucially, positive and negative records play opposite roles with different intensity and must have different length: negative records must be sufficiently long, and positive records sufficiently short.

JEL Codes: D82, D53, G20, G28, K35, L14, L15.

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1 Introduction

In many environments information intermediaries collect and make publicly available information on agents' history. Electronic platforms like eBay and Amazon, collect feedback about buyers and sellers, credit bureaus record credit history for borrowers, the legal system keeps track of criminal records, and Internet search engines effectively record information about an individual's past. Because of privacy laws, regulations, or intermediaries' own rules, after some time access to part or all of this information is denied. A number of natural questions arise: Can it be beneficial to limit the disclosure of past information to market participants? If so, for how long should past information be available? And do the answers depend on the type of recorded information?

This paper studies how a disclosure policy of past records affects the amount of trade and information prevailing in a market in the long run. We develop a continuous-time model of a dynamic market populated by sellers (borrowers, employees) of unknown quality, and competitive buyers (lenders, employers). At any point in time the seller can be of "high" or "low" quality and his quality changes according to a Markov process. After a purchase, the buyer learns the seller's quality and can leave "positive" or "negative" feedback with an information intermediary that makes the record accessible to future cohorts of buyers. We first consider general information disclosure policies and their effects on stationary equilibria and welfare in the long run. Then we show that simple limited records policies with different lengths of public memory for positive and negative records can implement the same outcomes as general policies. Distinguishing the memory of positive from that of negative records turns out to be critical to understanding the effects of limited records on market outcomes.[1]

We start out showing that when market conditions are poor (negative expected value of buying with no information), unlimited memory of past records has dramatic negative consequences in the long run: it necessarily leads to a market breakdown (Proposition 1). This happens even if the market starts with full information about sellers. The reason is that as soon as a seller gets a negative record, his expected quality drops and stays below the low population average, so that the buyers' willingness to pay for his product is below the seller's reservation value/cost. Since sellers' types change over time, in the long run each

[1]The distinction between positive and negative records is self-evident in financial markets, where credit records may include "black" (credit remarks, past arrears, defaults and bankruptcies) and "white" information (patterns of repayments, open and closed credit accounts, new loans, debt maturity, guarantees and assets); and in some electronic markets (e.g. eBay's positive and negative feedback); but as clarified in the last section, it applies to most economic environments.

good seller becomes bad and gets a negative record. Because of unlimited memory, from that moment on such sellers are out of the market forever. As time goes to infinity, each seller is almost surely excluded from trading.

We then show that, under the same poor market conditions, a stationary equilibrium with a constant, positive fraction of sellers trading at each moment in time can instead be sustained if the intermediary uses a suitable information policy. We consider the general information design problem where, in the spirit of the Bayesian persuasion literature, the intermediary can to commit to an information policy. We show that without loss of generality one can focus on threshold recommendation policies in which the intermediary does not disclose past feedback but instead recommends a seller for trade or not depending on the his posterior probability of being of high type: only sellers with posteriors above a certain threshold are recommended for trade. We characterize a range for the recommendation threshold that can sustain trade in the long run (Theorem 1). Intuitively, the threshold should be sufficiently high, so that the average quality of recommended sellers with posterior above the threshold is such that the resulting market price covers the seller's cost. The threshold must not be too high either, because if too few sellers are recommended there is not enough experimentation to compensate for the natural information loss due to the Markovian nature of sellers' types (past information becomes obsolete). If the average quality of sellers is high, instead, a wide range of policies can sustain trade, which can take place even if all sellers are recommended to trade indiscriminately.

We proceed to characterize the recommendation policy that maximizes long run social welfare (Theorem 2). The unique optimal threshold balances informational gains from lowering the threshold - recommending a larger mass of sellers generates more feedback information that can be used for future recommendations - with the losses from trades with sellers whose posterior probability of being high quality is low but just above the threshold.

The reason why the optimal policy is different from full disclosure is that feedback information has a positive social value that is not internalized by buyers. As a result, with full disclosure buyers would not buy from sellers that have old negative feedback even if this is socially optimal because they may have turned good. Hiding past feedback and issuing recommendations to trade with all sellers with posteriors above the threshold is therefore beneficial: since past negative feedback is not disclosed, all sellers that are recommended to trade are pooled together and can sell, including sellers with old negative feedback.

We then consider a class of commonly observed limited record policies where all records about past feedback are disclosed but deleted after a certain time span. We show that the same market outcomes we characterized with general policies can be achieved if past records are deleted in a specific way (Theorem 3). When average seller quality is low, market breakdown can be avoided by retaining past negative records for a sufficiently long time while deleting positive records quickly or not recording them at all. This is because a longer memory of negative records allows the bad sellers to be identified and kept out of the pool of sellers with no records (empty history) for a longer time, thereby improving the average quality of that pool. This encourages buyers to offer higher prices to sellers with no records and allows them to trade and get feedback. A longer memory of positive records, instead, allows more good sellers to separate from the pool with no records, thereby discouraging trade with non-rated sellers and the production of information on their quality. This makes stationary equilibria with trade harder to sustain: if the average quality of sellers is low then positive past records should necessarily be deleted faster than the negative ones (Proposition 3). The proposed short limits for positive records and long limits for negative ones encourage buyers to "experiment" by buying from sellers with no records, producing new information and improving long-run outcomes. We also find that positive and negative records differ in terms of the intensity of their effects: negative records simply prevent the seller from trading for a given amount of time, while positive records have a nonlinear, self-enforcing effect because they induce further trade and new records right away.

Finally, we develop a number of extensions of the main model. We consider profit maximizing intermediaries, patient/strategic sellers, and moral hazard. We find that our results are qualitatively robust: the same information policies sustain trade and maximize welfare in the long run, yet, due to the additional frictions the parameter space of feasible policies can be smaller.

We believe our results have important policy implications. Informational problems are among the main sources of market and government failures. Increasing the amount of information available is often regarded as a natural remedy. Privacy concerns, on the other hand, have been at the center of several recent debates on electronic markets and the Internet. To some, a privacy regulation that, for example, mandates the removal of data from the public domain is simply unjustified because it opens the door to more fraud.[2] Many disagree with this view, and the EU has introduced a highly debated, far reaching privacy regulation,

[2] From Posner's blog, 8th May 2005. http://www.becker-posner-blog.com/2005/05/index.html
See also Posner (1983) and Nock (1993) .

the General Data Protection Regulation (GDPR), that among other things mandates the cancellation by any entity - wherever located - of any personal data on European citizens after a limited time span.We believe our results bring novel, theoretically grounded arguments to this important debate. As discussed in the conclusions, our results are relevant for many markets. In the case of credit markets, there are empirical studies showing that removals of bankruptcy flags from borrowers' credit history and changes in record retention limits have significant effects on credit availability (Musto (2004), Bos and Nakamura (2014) and Liberman et al. (2017)), and can even have sizable spillover effects on labor markets (Bos et al. (2018)).

Related Literature

Our paper contributes to several strands of literature. The closest in terms of economic mechanisms is that on social learning. In particular we are close to Kremer et al. (2014) and Che and Hörner (2018), that focus on how an informed planner can optimally induce socially efficient but privately costly short-term product experimentation by over-recommending less attractive new alternatives to early users. We deal with an analogous informational externality, as the social value of experimenting by trading with sellers with no records is higher than its private value, but the focus of our paper is rather different. We study the effects of the retention policy about user generated past feedback in a stationary equilibrium, rather than strategic recommendations to induce short-term experimentation with a new product. The question of optimal information retention does not arise in these previous papers since types are fixed, and once the planner receives sufficient information about the unknown product, the experimentation stops and it is never optimal to hide this information. In our paper, instead, sellers' type change in time. As a result, it is optimal to hide any kind of information after an appropriate interval of time, and permanent experimentation is necessary to produce information and avoid market breakdown. Also, differently from these papers, we explicitly distinguish between positive and negative feedback and show that it is crucial, as they have fundamentally different effects on market dynamics, so that retention policies for positive and negative feedback should optimally be different.[3]

Optimal partial disclosure by the information intermediary relates our paper to the liter-

[3]Our paper is also related to the literature on strategic experimentation in markets with market power, including Bergemann and Välimäki (1996, 2000), Bolton and Harris (1999), Keller and Rady (1999), and Keller et al. (2005), among others. These papers do not have an information intermediary and do not analyze the effects of information disclosure, that our paper focuses on.

ature on information design and Bayesian persuasion, where a principal influences agent's behavior through strategic information disclosure. See Kamenica and Gentzkow (2011), Rayo and Segal (2010), and Ostrovsky and Schwarz (2010) for examples of static models, and Ely et al. (2015), Ely (2017), Orlov et al. (2018), and Romanyuk and Smolin (2018) for examples of dynamic ones. Differently from this literature, in our paper the information is not exogenous, but is produced by market participants and is endogenous to the information disclosure policy of the intermediary. The focus on memory and the information externality among buyers also makes our paper quite distinct from previous work in this literature.

Our paper shares the main research questions with the literature studying the effects of limiting access to past history in reputation models, including Vercammen (1995), Padilla and Pagano (2000), Ekmekci (2011), Elul and Gottardi (2015), Liu and Skrzypacz (2014) and Hörner and Lambert (2020). Differently from reputation models, that study the effects of the information asymmetry about agents' type on their effort choices, we focus on the positive information externality generated by market transactions through the production of information on sellers' types. Also, differently from these papers, we consider the different role played by positive and negative past signals and the different limits these records require. Our analysis therefore identifies novel, complementary mechanisms to those identified in the reputation literature, that may lead to quite different policy insights.[4] The literature on repeated games with restricted memory is also related in spirit, but does not have uncertainty about agents' types, so it is further away from what we study.[5] Our paper is also related to models on dynamic trade with adverse selection, such as Fuchs and Skrzypacz (2010), Guerrieri and Shimer (2014), and Chiu and Koeppl (2016). Also in these papers the quality on the market is endogenously determined, though from past prices and sale decisions by informed agents rather than by disclosure policies of an information intermediary.

Finally, in terms of policy implications our work appears highly relevant to the economic literature on privacy, surveyed in Acquisti et al. (2015).

The structure of the paper is as follows. Section 2 presents our model, and in Section 2.1 we study the case of unlimited past records. In Section 3 we consider general recommendation

[4]For instance, Elul and Gottardi (2015) argue in favor of forgetting defaults while retaining information about successful repayments forever. In our model such a policy leads to a market breakdown when average quality is low, and the opposite policy is optimal: short memory for repayments (positive records) and long memory for defaults (negative ones).

[5]Bhaskar and Thomas (2018), for example, shows in a pure moral hazard model that erasing defaults allows lenders to commit to an optimal punishment. Dellarocas (2006) offers an early analysis in this spirit. See also Barlo et al. (2009) and Doraszelski and Escobar (2012).

policies of the information intermediary, and characterize feasible and optimal policies. In Section 4 we show how feasible and optimal recommendation policies can be implemented with simple limited records, i.e. rules that restrict access to information after a particular time span. Section 5 presents extensions while Section 6 discusses policy implications and concludes.

2 Environment

Consider an economy populated by sellers, buyers and an information intermediary who interact in continuous time $t \in [0, \infty)$.

Sellers. There is a unit mass of infinitely lived sellers $i \in [0, 1]$. At each instant, a seller may be active on the market (i.e. may have a product to sell) or not. Seller i is active whenever there is a jump in a counting process $\{N_{it}\}_{t \geq 0}$ with Poisson intensity $m > 0$. Markets with many (few) transactions per unit of time can be described by a process with a high (low) intensity m. For instance, in the context of Ebay one can think that with a certain probability, a person may decide to sell an old gadget. Similarly, in the context of a credit market, with a certain probability a potential borrower (seller of debt) may need to borrow from (sell debt to) a bank (buyer of debt).

We normalize the value of the product to the seller to $c = 1$ (it can be the value the seller derives from alternative use, the cost of production or, in the case of the credit market, the amount of necessary investment). The product price p_{it} is determined by the buyers' willingness to pay, which in turn depends on their expectation of the seller's quality. The seller decides whether to sell the product ($s_i = 1$) or not ($s_i = 0$), his gain from trade being $s_i(p_{it} - c)$. In the main analysis we assume that the seller is impatient (myopic) as in Kremer et al. (2014) and Che and Hörner (2018), caring only about his instantaneous payoff. In Section 5.2 we develop an extension with patient sellers.

Product quality. The buyers' valuation of seller i's product (product quality) θ_{it} is stochastic: it can be high ($\theta^H > 1$) or low ($\theta^L = 0$). We refer to θ_i as seller i's type; a good seller's product has quality θ^H and a bad seller's product has quality θ^L. To make the population stationary we assume that the fraction of good sellers in the population is constant and equal to μ for any $t \in [0, \infty)$. Note that we do not specify whether the seller knows his type or not, as our analysis holds in both cases. The quality of each seller may change over time. For instance, there can be innovations in products offered, changes in the seller's management or ownership, or an evolution of buyers' preferences. Seller i's product quality follows an exogenous time-homogeneous Markov process θ_{it}, $t \in [0, \infty)$.[6] Denote by π_{it} the probability of seller $i \in [0, 1]$ being of high type at $t \geq 0$. We introduce the following assumption:

Assumption 1. *For any $t \geq 0$ for any seller $i \in [0, 1]$*

$$\frac{d\pi_{it}}{dt} = \varphi(\mu - \pi_{it}),$$

where $\varphi \in (0, \infty)$ parametrizes the intensity of type changes.

With time, seller's type θ_{it} changes as follows: with Poisson intensity φ his type is reset, in which case with probability μ the new type is θ^H and with probability $1 - \mu$ it is zero.

Buyers. At each moment $t \in [0, \infty)$, many competitive risk-neutral buyers are matched to active sellers. A buyer is never matched to the same seller twice. Alternatively, we could assume that buyers consume only once in their lifetime, or that they are short-lived, so in each instant buyers are different. We do not model competition between buyers explicitly, but follow Holmström (1999) and Mailath and Samuelson (2001) in simply assuming that buyers are ready to buy a product for a price equal to its expected quality. If the price is above the seller's valuation, the seller sells the product to one of the buyers that he chooses randomly. Before the buyer purchases the product from seller i, he does not know its quality and relies on a message r_{it} provided by an information intermediary (described below). Buyers have no other information about the seller except r_{it}, use Bayes' rule and believe that the seller is high quality with probability $\hat{\pi}_i(r_{it})$. After purchasing from seller i, the buyer learns the quality θ_{it} and derives utility θ_{it}.

Information intermediary. After the purchase the buyer can leave his feedback f_{it} on the seller with the information intermediary. We abstract from the buyer's motivation to leave honest feedback, and simply introduce the following:

Assumption 2. *After purchasing a high quality product the buyer leaves positive feedback $f_{it} = S$ (Satisfied, Solvent) to the information intermediary, and after purchasing a low quality product he leaves negative feedback $f_{it} = D$ (Dissatisfied, Defaulted).*[7]

[6] In Section 5.3 we introduce moral hazard and allow the seller to affect his type through costly effort as in of Board and Meyer-ter Vehn (2013).

[7] In Online Appendix we show that our results do not change qualitatively if buyers were to leave feedback only with some probability $\lambda \in (0, 1]$.

If the seller does not trade there is no feedback $f_{it} = N$. The information intermediary records the history of past feedback $h_{it} : [0, t) \rightarrow \{S, D, N\}$ for each seller $i \in [0, 1]$. The information available to the intermediary at any $t \geq 0$ is the collection of feedback histories of all sellers, which we denote by h_t. We denote by H_t the set of all possible histories that could have happened by time t.

At $t = 0$ the intermediary commits to an information policy σ_t, which at any t for any realization of $h_t \in H_t$ produces message (rating) $r_{it} \in M_t$ for each seller $i \in [0, 1]$. The set of feasible messages M_t depends on the particular application or policy studied.[8]

Market equilibrium. Formally, for any information policy σ_t and prior information about sellers π_{i0}, $i \in [0, 1]$, an equilibrium at each $t \in [0, \infty)$ is characterized by the feedback history h_t observed by the intermediary; messages r_{it} published by the information intermediary for sellers $i \in [0, 1]$; buyers' beliefs about sellers' types $\hat{\pi}_i(r_{it})$; realizations of Poisson shocks dN_{it} that determine the sellers active at t; prices $p_{it} \in R^+$ offered by buyers to each active seller; the optimal selling decision $s_{it} \in \{0, 1\}$ for each active seller; and feedback $f_{it} \in \{S, D, N\}$ recorded for each seller $i \in [0, 1]$ by the information intermediary.

Let us describe the equilibrium. If a Poisson shock hits seller i at time t, he becomes active and gets matched with many competitive buyers. Buyers use the rating r_{it}, update their belief about the seller $\hat{\pi}_i(r_{it})$, and offer a price $p_{it} = \hat{\pi}_i(r_{it})\theta^H$. Having observed the price, each active seller chooses $s_i \in \{0, 1\}$ in order to maximize his instantaneous payoff $s_i(p_{it} - 1)$, selling ($s_i = 1$) whenever $p_{it} \geq 1$.[9] If the seller sells, the buyer perfectly learns the product's quality and leaves the corresponding feedback $f_{it} = S$ if $\theta_{it} = \theta^H$ and $f_{it} = D$ if $\theta_{it} = 0$. If the seller is not active or if he does not sell, there is no feedback, $f_{it} = N$.

As one can see, the equilibrium behavior of all agents can be easily described once one knows the buyers' beliefs $\hat{\pi}_i(r_{it})$ that in turn depend on the information policy σ_t of the intermediary mapping feedback history h_t into messages M_t. This is the key object of our analysis. As a benchmark, we consider a perfect record policy which discloses all past feedback about each seller, and then turn to the analysis of general information policies.

[8]For instance, in Section 2.1 we illustrate a perfect record policy which discloses all past feedback for each seller, i.e. $r_{it} = h_{it}$, $i \in [0, 1]$, and $M_t = H_t$ for any $t \geq 0$. In Section 4 we consider limited record policies: recent history of each seller is disclosed, but all feedback older than a specified time span is hidden. Our main analysis in Section 3 focuses on a general information policy.

[9]Note that the seller's payoff does not depend on his type, i.e. both types of sellers sell whenever they can get a price of at least one. In the end of Section 2 we assume different costs for different types and study adverse selection.

2.1 Perfect records

Let us assume that the information intermediary discloses all past information about sellers, so that buyers observe $r_{it} = h_{it}$ for any, $i \in [0, 1]$, $t \geq 0$. Suppose the average quality of sellers in the population is low, $\mu\theta^H < 1$, so that there can't be trade unless the buyers are able to tell apart at least some good sellers from the bad ones. In such a situation, one may expect that providing information to the buyers and retaining it indefinitely would be beneficial. For instance, one may think that the availability of perfect information about sellers at $t = 0$ and of the full history of feedback at any $t > 0$ would facilitate trade. It turns out this is not the case in the long run.

Proposition 1 (**Market breakdown**). *If $\mu\theta^H < 1$ and the intermediary provides full history of past feedback for sellers in the long run the market breaks down: the fraction of sellers trading in equilibrium converges to zero with time.*

All omitted proofs can be found in the appendix. The finding that the full provision of past information is detrimental for trade in the long run even starting with complete information appears striking, but the logic behind Proposition 1 is very simple. With time, good sellers happen to become bad and get negative feedback. From that moment they are excluded from the market forever because buyers' posterior about their quality never exceeds the unconditional probability of a high-quality seller in population μ and, the unconditional expected quality of a seller is low ($\mu\theta^H < 1$). As time goes to infinity, therefore, each seller is almost surely excluded from trading.

3 General information policies

In this sectio we take an information design approach and allow the intermediary to use an arbitrary set of messages M_t to communicate his information to buyers.

3.1 Preliminary analysis

The intermediary observes the full history of feedback for all sellers h_t, and his posterior belief about seller i is $\pi_i^t(h_t)$. The intermediary's information policy σ is as follows: at time t conditional on feedback history h_t, the intermediary issues message $r \in M_t$ about seller i with probability $\sigma_i^t(r|h_t)$.

Buyers matched to an active seller i use the message r to update their belief $\hat{\pi}_{it}(r)$, and offer the price $p_{it} = \hat{\pi}_{it}(r)\theta^H$. The seller trades ($s_{it} = 1$) if the offered price p_i^t is at least as high as their reservation value $c = 1$; their equilibrium selling decision is effectively determined by the message r, and we can write $s_{it}(r)$ for brevity.

Intermediary's objective. We start our analysis by solving for a socially optimal recommendation policy which maximizes long-run welfare. We define the intermediary's objective as the sum of all parties' payoffs:[10]

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\int_0^1\int_{M_t} dN_{it}(\pi_{it}(h_t)\theta^H - 1)s_{it}(r)\sigma_i^t(r|h_t)drdidt. \tag{1}$$

For instance, consider seller i who became active at t. If the intermediary issues a message r which leads to a sale $s_i^t(r) = 1$, then expected surplus $\pi_{it}(h_t)\theta^H - 1$ is generated.[11] If the intermediary issues a message r', which leads to no trade, then no surplus is generated. One has to integrate this surplus over possible messages for seller i at time t taking into account the probabilities of messages $\sigma_i(r|h_t)$ prescribed by the information policy. Since each seller i can become active at any moment ($dN_{it} = 1$) with Poisson intensity m one has to integrate over all agents $i \in [0, 1]$. Finally, one has to integrate over time to compute long-run welfare.

Binary recommendations. The intermediary's objective does not depend on prices because they are transfers between agents and do not affect total surplus, and only selling decisions are important. Hence, all messages $r \in M_t$ that lead to trade ($s_{it} = 1$) are equivalent from the intermediary's point of view, and analogous reasoning holds for messages that lead to no trade ($s_{it} = 0$). This implies that a version of a Revelation principle holds in our case, i.e. it is without loss of generality to consider information policies with binary messages (recommendations) $M_t = \{0, 1\}$, where $r_{it} = 1$ is a "buy" recommendation which results in $s_{it} = 1$, and $r_{it} = 0$ is a "not buy" recommendation which leads to no trade:

Lemma 1. *For any information policy with messages $r \in M_t$, there exists an information policy with binary recommendations $r \in \{0, 1\}$ which delivers the same welfare given by (1).*

From now on we focus on binary recommendation policies without loss of generality.

The intermediary's payoff given by equation (1) depends on posteriors about sellers' types $\pi_{it}(h_t)$, but not on the full history of their feedback h_t per se. Moreover, all sellers with the

[10] We focus on the long-run welfare because we are interested in stationary equilibria and long-run effects of information policies. Later we show that a monopolistic profit maximizing intermediary extracts all surplus and chooses a socially optimal policy.

[11] Note that the intermediary's posterior $\pi_{it}(h_t)$ in general differs from the buyers' posterior $\hat{\pi}_{it}(r)$.

same $\pi_i^t = \pi$ are identical from the intermediary's point of view, and he only needs to know the mass of sellers with different posteriors $\pi \in [0, 1]$. One can summarize the intermediary's payoff-relevant information about sellers at any t with a distribution of posteriors about sellers in the market F_t. Here $F_t(\pi) \in [0, 1]$ is the measure of sellers with $\pi_t \leq \pi$.

We consider Markov recommendation policies that depend only on the payoff relevant information F_t: the intermediary's recommendation policy for any seller with a posterior π is characterized by the probability of the intermediary issuing recommendation $r = 1$: $\sigma(\pi, F_t) = \Pr[r = 1|\pi, F_t]$. The intermediary's objective can be concisely rewritten integrating over $dF(\pi)$, $\pi \in [0, 1]$:

$$W(\sigma) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \int_0^1 m(\pi\theta^H - 1)\sigma(\pi, F_t)dF_t(\pi)dt. \tag{2}$$

The intermediary's recommendation policy σ must maximize $W(\sigma)$, taking into account how recommendations affect trades, and the information produced by these trades.

Buyers' beliefs. At $t = 0$ buyers know the distribution of the intermediary's posteriors F_0 but at any $t > 0$ they do not know F_t as it depends on previously generated feedback. Buyers form rational beliefs $\hat{F}_t$ about the intermediary's information that must be correct in equilibrium.[12]

Buyers also do not know the intermediary's posterior π_i about an individual seller $i \in [0, 1]$. Their beliefs about an active seller depend on the recommendation and can depend on the seller's decision $s \in \{0, 1\}$ if the seller is informed, so that $\hat{\pi}_t(r, s)$. For simplicity we assume that their beliefs do not react to the seller's decision s, as would be natural if the seller were uninformed, i.e. $\hat{\pi}_t(0, s) = \hat{\pi}_t(0)$ and $\hat{\pi}_t(1, s) = \hat{\pi}_t(1)$ for any $s \in \{0, 1\}$. Their beliefs are given by the Bayes' rule:

$$\hat{\pi}_t(1) = \frac{\int_0^1 \pi\sigma(\pi, \hat{F}_t)d\hat{F}_t(\pi)}{\int_0^1 \sigma(\pi, \hat{F}_t)d\hat{F}_t(\pi)}, \ \hat{\pi}_t(0) = \frac{\int_0^1 \pi(1 - \sigma(\pi, \hat{F}_t))d\hat{F}_t(\pi)}{\int_0^1 (1 - \sigma(\pi, \hat{F}_t))d\hat{F}_t(\pi)}. \tag{3}$$

Sellers' incentives. In equilibrium, the intermediary's recommendations should be incentive compatible: a "buy" recommendation $r = 1$ should lead to a trade, and a "not buy" recommendation $r = 0$ should lead to no trade. Myopic sellers sell whenever $p_t(r) \geq c = 1$,

[12] When all agents follow their equilibrium strategies the evolution of F_t can be computed as we show in Lemma 2.

hence recommendations are incentive compatible if

$$\hat{\pi}_t(1)\theta^H \geq 1, \tag{4}$$

and

$$\hat{\pi}_t(0)\theta^H \leq 1. \tag{5}$$

In what follows we assume that sellers follow recommendations, and later check under which conditions this is indeed the case.

Information Evolution. Unlike in the standard Bayesian persuasion problem with exogenous information, in our model the intermediary's information F_t is endogenous, and depends on the recommendation policy σ. Hence, on top of the standard incentive compatibility constraints, the intermediary has to take into account a new information evolution constraint. To simplify notation, we introduce a stochastic process $x_t \in \{0,1\}$ which mimics a seller's type, i.e. $x_t = 1$ whenever $\theta_t = \theta^H$, and $x_t = 0$ whenever $\theta_t = \theta^L$. For a given recommendation policy σ the intermediary's posterior about any seller evolves then as follows

$$d\pi_t = \varphi(\mu - \pi_t)dt + (x_t - \pi_t)dN_t s_t. \tag{6}$$

If the seller does not trade, no information is produced and the posterior about him drifts towards the mean (μ). If the seller is hit by a Poisson shock dN_t (with intensity m) he becomes active and can sell. If the seller sells ($s = 1$), his type is revealed, and the posterior about him jumps to one if his type is high ($x_t = 1$), or to zero if his type is low ($x_t = 0$). Note that in equilibrium the seller follows recommendations, i.e. $s = 1$ with probability $\sigma(\pi_t, F_t)$.

The intermediary starts with some exogenous information F_0, and his information in subsequent periods F_t, $t > 0$ is endogenously determined by his recommendation policy σ. In equilibrium, Buyers' beliefs about seller's information must be correct: $\hat{F}_t = F_t$ for any $t \geq 0$. Provided that recommendations are incentive compatible, the recommendation policy σ determines the evolution of F_t according to the following Lemma:

Lemma 2. *Given F_0 and an incentive compatible recommendation policy σ, for $t \geq 0$ the* *intermediary's information F_t evolves as follows: for $\pi \in [0,\mu) \cup (\mu,1]$ one has*

$$\frac{F_{t+dt}(\pi) - F_t(\pi)}{dt} = -f_t(\pi)\varphi(\mu-\pi) - m\int_0^{\pi} \sigma(\pi, F_t)dF_t(\pi) + m\int_0^{1}(1-\pi)\sigma(\pi, F_t)dF_t(\pi), \quad (7)$$

and for $\pi = \mu$ one has

$$\frac{dF_{t+dt}(\mu) - dF_t(\mu)}{dt} = -m\sigma(\mu, F_t)dF_t(\mu). \quad (8)$$

The interpretation of this result is natural. If $\sigma(\pi_t, F_t) = 0$, the seller does not trade, and his posterior converges to μ according to (6). This effect is missing for sellers with $\pi = \mu$ because their posterior is stable. Moreover, posteriors $\pi \neq \mu$ drift towards μ but never reach it, i.e. no seller with $\pi \neq \mu$ can end up with $\pi_t = \mu$ at some point, hence the mass $dF_t(\mu)$ can only decrease with time. Indeed, if $\sigma(\mu, F_t) > 0$ sellers with posterior μ trade, their posterior jumps to 0 or 1, and the mass $dF_t(\mu)$ decreases according to (8). The effect of trade on sellers with $\pi \in [0,\mu) \cup (\mu,1]$ is more complex but is also intuitive.[13]

3.2 Socially optimal recommendation policy

In this section we characterize the socially optimal recommendation policy. To do so, we first show that the optimal recommendation policy must be a simple threshold policy that recommends sellers with posteriors above some threshold. Then, we characterize threshold policies that can sustain trade in the long run and solve for the optimal threshold policy.

The socially optimal recommendation policy maximizes the long run welfare, taking into account the initial information of the intermediary, incentive compatibility constraints, and information evolution constraints:

$$\max_{\sigma} W(\sigma), \; s.t. \; F_0, \; (4), (5), (7), (8). \quad (9)$$

Lemma 3. *An optimal recommendation policy σ is a threshold policy: for any $t \geq 0$, F_t there is a threshold $z(F_t)$, such that $\sigma = 0$ for $\pi < z(F_t)$, $\sigma = 1$ for $\pi > z(F_t)$, and $\sigma \in [0,1]$ for $\pi = z(F_t)$.*

[13] Consider sellers with posteriors below π with total mass $F_t(\pi)$. Each dt mass $mdt\int_0^{\pi}\sigma(\pi, F_t)dF_t(\pi)$ of them trades and flows out of $F_t(\pi)$. At the same time all bad sellers with any posterior $\pi \in [0,1]$ that trade, end up with the posterior $\pi = 0$ and join $F_t(\pi)$ from the bottom, their mass is $mdt\int_0^{1}(1-\pi)\sigma(\pi, F_t)dF_t(\pi)$. The net flow of sellers due to trade is described by last two terms in (7).

This result implies that we can consider threshold policies without loss of generality.

Stationary Equilibrium. The information intermediary has a long-term objective and we are interested in stationary equilibria that characterize long-run market outcomes. In a stationary equilibrium the intermediary's information ($F_t = F$) and his information policy does not change with time ($z_t = z$) [14]

Let us first establish an analog of Proposition 1 about market performance in the long run, when the average quality of seller is low ($\mu\theta^H \leq 1$) and the intermediary's policy is strict, so that threshold z is higher than the fraction of good sellers in population μ.

Proposition 1' (Market breakdown). *If $\mu\theta^H \leq 1$, a policy with a threshold $z \geq \mu$ leads to no trade in the long run.*

We omit the formal proof, because it is similar to that of Proposition 1, as is the intuition behind it. Sellers' types are Markov: each seller will turn bad at some point, trade and get a negative feedback. From that moment on the seller will never be recommended to trade again, as the intermediary's posterior about his type will converge to the mean (μ) from below, which is lower than the recommendation threshold if $z \geq \mu$. With time the mass of sellers with a posterior above μ converges to zero, and the market collapses.

Clearly, when $\mu\theta^H > 1$ there is always trade independently of the policy z. Yet, even in this case strict policies are not optimal as our next result shows:

Lemma 4. *If $\mu\theta^H > 1$, a recommendation policy with a threshold $z > 1/\theta^H$ is not optimal.*

The proof is immediate. The policy $z > 1/\theta^H$ is not optimal because the intermediary can choose a policy $z' < z$, such that $z'\theta^H = 1$, which generates higher welfare, and trivially satisfies incentives constraints. Moreover, the new policy z' produces more information than the original policy z: with the new policy the intermediary can "forget" what he learns about sellers with $\pi \in [z', z)$ recommended at any t, so that he ends up with the same information as under the old policy.

Proposition 1' and Lemma 4 imply, that we can restrict attention to policies with

$$z \leq \overline{z} = \min\{\mu, \frac{1}{\theta^H}\}. \tag{10}$$

[14]Stable information policies are common in practice: privacy regulations, credit and criminal records rules typically require information to be deleted after a certain fixed time span, which does not depend on calendar time. Implicitly, we assume that in a stationary equilibrium $F_0 = F$, that is the intermediary has the exactly right information at $t = 0$. One can think that the intermediary conducts a test at $t = 0$ at some cost which produces F.

Before solving for an optimal policy, let us characterize policies that can be sustained in a stationary equilibrium with trade. We first assume that the incentive constraints hold and characterize the stationary distribution F for a given policy $z \leq \overline{z}$, for details see Lemma 5 in the Appendix. Then we use resulting stationary distribution F and check that incentive constraints (4) and (5) are satisfied.

Theorem 1 (Stationary equilibrium with trade). *If $\mu\theta^H < 1$ a stationary equilibrium with trade can be implemented if and only if*

$$\underline{z} \equiv \frac{\varphi}{m}\frac{1-\mu\theta^H}{\theta^H - 1} < \mu, \tag{11}$$

with a threshold $z \in [\underline{z}, \mu)$.

If $\mu\theta^H \geq 1$ a stationary equilibrium with trade can be implemented with any $z \leq 1/\theta^H$.

When the average quality of sellers is high, $\mu\theta^H > 1$, any $z \leq 1/\theta^H$ is incentive compatible, and with the corresponding F given by Lemma 5 it implements the stationary equilibrium with trade. Indeed, (4) holds because $E_F[\pi|\pi > z]\theta^H \geq \mu\theta^H > 1$, and (5) holds because $E_F[\pi|\pi \leq z]\theta^H \leq E_F[\pi|\pi \leq \overline{z}]\theta^H < \overline{z}\theta^H = 1$.

When $\mu\theta^H < 1$, condition (11) has an intuitive interpretation: for intermediary's recommendations based on past feedback to have any value, the sellers' types must be persistent enough, i.e. φ must not be too large. Indeed, when the average quality of sellers is low, the market would not not work unless the information intermediary excludes some bad sellers from trading. If the sellers' types change very quickly, past feedback becomes obsolete very quickly and the information intermediary does not have enough information to exclude sufficiently many sellers with low posteriors in order to restore trade. For a similar reason high intensity of trade m corresponds to high flow of feedback information to the intermediary, which allows him to separate bad and good sellers and sustain trade. Condition (11) guarantees that whenever an intermediary recommends a seller, i.e. $\pi \geq z \geq \underline{z}$, the seller trades because the average quality of recommended sellers is high.

The fact that coarse binary recommendations can lead to better long-run outcomes (stationary equilibrium with trade, Theorem 1) than full disclosure of past feedback (collapse of trade, Proposition 1) has a clear and robust rationale behind it. When trading, buyers produce a positive informational externality - the feedback on the seller. With full disclosure of past feedback, buyers do not experiment enough - they only buy from sufficiently good sellers. If the average quality of sellers is low, $\mu\theta^H \leq 1$, because of low experimentation and the Markov nature of sellers' types, past feedback becomes obsolete with time, and in the long run the market ends up with no information but the unconditional fraction of good sellers in the market μ. Binary recommendations create two pools of sellers whose exact feedback history is unknown to buyers: recommended sellers and not recommended sellers. If the average quality of recommended sellers is not too low, buyers will be ready to experiment by buying from these sellers and produce information. By pooling relatively good sellers (with a posterior above a threshold) and separating them from relatively bad sellers, the intermediary can induce the market to produce enough new information to countervail the natural loss of information due the Markov nature of sellers' types, so that sufficient information is available to the market to sustain trade in the long run.

Having shown how a threshold recommendation policy can sustain trade, we can characterize a policy maximizing the flow of welfare in a stationary equilibrium.

Optimal Threshold. Suppose the stationary equilibrium with trade is feasible and consider z which implements such an equilibrium according to Theorem (1). Lemma 5 in the Appendix characterizes the stationary distribution F for a given threshold policy z. Substituting for F one can express the long-run social welfare in a stationary equilibrium:[15]

$$W(z) = \int_z^1 m(\pi\theta^H - 1)dF(\pi) = \frac{(\theta^H - 1)(\varphi + mz) - \varphi(1-\mu)\theta^H}{\varphi + mz + m(1-\mu)\ln\frac{\mu}{\mu - z}}. \qquad (12)$$

The optimal threshold policy maximizes $W(z)$, and is characterized in the next Theorem.

Theorem 2 (**Optimal policy**). *If condition (11) is satisfied, the stationary equilibrium with the highest welfare can be implemented with a unique threshold $z^* < 1/\theta^H$, which solves*

$$\ln\frac{\mu}{\mu - z^*} = \frac{1}{\mu - z^*}\left(z^* + \frac{\varphi}{m}\frac{z^*\theta^H - 1}{\theta^H - 1}\right). \qquad (13)$$

The intuition behind the result is the following. In a stationary equilibrium with trade it is beneficial to trade with sellers with high posteriors ($\pi\theta^H > 1$) because such trades generates a positive flow of surplus. It is also beneficial to exclude sellers with sufficiently low posteriors ($\pi < z^*$) from trade, because trading with them would significantly reduce the flow surplus $\pi\theta^H - 1 < 0$. At the same time, the optimal recommendation policy induces trade with sellers that have posteriors $\pi \in [z^*, 1/\theta^H)$, even though trade with these sellers is associated with a direct surplus loss $\pi\theta^H - 1 < 0$, because of indirect benefit from learning

[15]For details see the proof of Theorem 2.

their actual types. This information has a positive social value as it enables the intermediary to distinguish the sellers of different types and recommend future trade only with those that have sufficiently high expected quality. In essence, there is a positive option value of trading with any seller, even if $\pi\theta^H < 1$. For a seller with a posterior z^* the option value of learning his type exactly compensates for the expected direct loss generated by the trade.

3.3 Adverse selection

For simplicity we have assumed that the seller's cost was the same ($c = 1$) independently of the seller's quality. In this section we introduce adverse selection by assuming that the cost of a high quality seller is $c > 1$, while the cost of a low quality seller is 1 (little would change if we assumed that the cost of a low quality seller is smaller than 1). As in the main analysis we consider threshold recommendation policies and characterize thresholds that implement stationary equilibria with trade.

Proposition 2. *If $\mu\theta^H < c$ a stationary equilibrium with trade can be implemented if and only if*

$$\underline{z}(c) \equiv \frac{\varphi}{m}\frac{c - \mu\theta^H}{\theta^H - c} < \mu, \tag{14}$$

with a threshold $z \in [\underline{z}(c), \mu)$.

If $\mu\theta^H \geq c$ a stationary equilibrium with trade can be implemented with any $z \leq c/\theta^H$.

The proposition shows that qualitatively, the results in Theorem 1 continue to apply in the presence of adverse selection, as long as it is not too strong.

Intuitively, with adverse selection the high quality seller has a higher cost than a low quality seller and is less willing to trade. In order to encourage him to trade the market price and, hence, the average quality of sellers recommended by the intermediary should be higher than the cost c. That is why the minimal posterior for the seller to be recommended to trade is increasing in c as can be seen from (14). Note that $\underline{z}(c) < 0$ if the average quality of sellers is high ($\mu\theta^H < c$) and any policy $z \geq c/\theta^H$ implements a stationary equilibrium with trade.

Naturally, adverse selection makes it harder to sustain trade: an increase in c shrinks the set of thresholds that implement the stationary equilibrium with trade. If c is such that $z(c) \geq \mu$ no information policy can sustain trade in the long run and the market breaks down. Similarly, adverse selection would quantitatively alter the socially optimal threshold characterized in Theorem 2, so that the optimal threshold with $c > 1$ would be higher than with $c = 1$. Yet, adverse selection does not affect our results qualitatively. Therefore, in the subsequent analysis we maintain the simplifying assumption $c = 1$.

4 Implementing the optimal policy

The optimal recommendation policy studied in Section 3 is characterized in it's simplest and most concise form: sellers with posteriors above a threshold z^* are recommended to trade, while sellers with posteriors below the threshold are not. One may think that this threshold policy is rather artificial. However, as we argued before, considering such policies is without loss of generality, and many more realistic or more sophisticated information policies are equivalent to this simple recommendation policy. In this section we illustrate how the optimal threshold policy can be implemented through the limited records policies common to many markets: disclosing the most recent feedback about sellers, but hiding sufficiently old feedback.

To this aim we assume that the information intermediary discloses limited past feedback about sellers to the buyers, instead of issuing binary recommendations. Any record of negative or positive feedback about a seller is disclosed to buyers for time spans T^- and T^+ respectively. After the time limit the feedback is made unavailable to buyers. For technical reasons we consider $T^+ \leq T^-$.[16]

4.1 Relevant records and stationary equilibria

First, we note that the Markov nature of the seller's stochastic quality θ_{it} guarantees that just the latest record available at t contains the sufficient information to determine buyers' belief about the seller $\hat{\pi}_{it} = \Pr[\theta_{it} = \theta^H | h_{it}]$. Indeed, consider a seller at t whose latest feedback was positive (S) and was left at $t - \tau$, then buyers' belief about the seller is

$$\hat{\pi}_{it}(S, \tau) = \mu + (1 - \mu)e^{-\varphi\tau},\ \tau \in [0, T^+].$$

Analogously, buyers' belief about a seller whose latest feedback was negative (D) and was left at $t - \tau$ is

[16] Later we show that if $\mu\theta^H < 1$ considering $T^+ \leq T^-$ is not restrictive, because $T^+ > T^-$ leads to a market breaks down. If $\mu\theta^H > 1$ and $T^+ > T^-$, the analysis is rather complicated. In this case for some sellers the deletion of a negative record after the timespan T^- may result in the latest visible record being positive, and it becomes difficult to characterize the joint distribution of sellers' types and relevant records.

$$\hat{\pi}_{it}(D,\tau) = \mu - \mu e^{-\varphi\tau},\ \tau \in [0, T^-]. \tag{15}$$

Finally, the seller may not have any S or D record, because his past records were deleted and he received no new feedback, in which case we say that the seller has record N, and buyers' belief about him is

$$\hat{\pi}_{it}(N).$$

At any t the equilibrium is characterized by a joint distribution of sellers' types and relevant records $\boldsymbol{\Delta}_t$, which pins down buyers' beliefs about sellers. As before, we study the stationary equilibrium.

Stationary equilibrium. In a stationary equilibrium, buyers' beliefs about sellers with different records should only depend on records, but not on time t. Now we show that for any stationary equilibrium which can be implemented with a binary recommendation policy z one can find limited record policies that implement an equivalent stationary equilibrium, i.e. an equilibrium which generates the same trades and the same welfare.

Theorem 3 (**Limited records**). *For any threshold recommendation policy z which implements a stationary equilibrium with trade according to Theorem 1, one can find a limited records policy $T^-(z)$, $T^+(z)$ which implements an equivalent stationary equilibrium. The corresponding policy for negative records is pinned down by the threshold z as follows:*

$$\mu - \mu e^{-\varphi T^-(z)} = z. \tag{16}$$

Any policy for positive records such that $T^+ \leq T(z)$ together with $T^-(z)$ implements a stationary equilibrium equivalent to the equilibrium induced by the threshold z, where $T(z)$ solves

$$\frac{(1-\mu)\varphi\theta^H}{m+\varphi} + \frac{(1-\mu)m\theta^H}{m+\varphi}e^{-(m+\varphi)T(z)} + (\mu\theta^H - 1)e^{-mT(z)} = \frac{(1-\mu)\varphi}{\mu\varphi + mz}. \tag{17}$$

The intuition behind this result is as follows. First, in equilibrium sellers with negative records do not trade, while in an equilibrium with a threshold recommendation policy sellers with a posterior below z do not trade. For the two equilibria to be equivalent the same sellers should be prevented from trading, i.e. the posterior of the seller for whom the negative record is deleted $\hat{\pi}_{it}(D, T^-)$ should be equal to z, and substituting for $\hat{\pi}_{it}(D, T^-)$ from (15) we get (16). Second, sellers with positive records and with an N record should trade for an equilibrium to be equivalent to the one with the threshold recommendation policy, where sellers with posteriors above z trade. It is clear that a seller with a positive record can trade, yet this is not so clear for a seller with an N record. Indeed, if positive records are not deleted for a long time (T^+ is high), many good sellers would have visible positive records, and the proportion of good sellers with an N record would be below a threshold which allows them to sell. Essentially, when T^+ is too long, the pool of sellers with an N record can become a "black hole" for the market: if these sellers do not sell, the stationary equilibrium with trade is not sustainable. The condition $T^+ \leq T(z)$ guarantees that the expected quality of sellers with an N record is high enough so they can trade. Clearly, a policy which immediately deletes positive feedback, i.e. $T^+(z) = 0$, would support a stationary equilibrium with trade. In this case, all sellers with a posterior above z would have an N record, and the resulting stationary distribution would be equivalent to the one under the threshold policy z.

4.2 Differences between positive and negative records

Theorem 3 characterizes limited records policies compatible with stationary equilibria in the general case. If the average quality of sellers is low we obtain an additional result.

Proposition 3 (Positive vs negative records). *If $\mu\theta^H < 1$ then a limited records policy can implement a stationary equilibrium with trade if $\underline{z} < \mu$ and the limit to negative records is sufficiently long $T^- \in [\underline{T}, \infty)$, while the the limit for positive records is shorter $T^+ < T^-$.*

In equilibrium sellers with posteriors below z do not trade, i.e. negative records should be deleted only after $T^-(z) = \frac{1}{\varphi} \ln \frac{\mu}{\mu - z}$ according to (16). Possible values for $z \in [\underline{z}, \mu)$ according to Theorem 1 determine the corresponding values for $T^- \in [\underline{T}, \infty)$, where $\underline{T} = \frac{1}{\varphi} \ln \frac{\mu}{\mu - \underline{z}}$. Intuitively, in an equilibrium with trade sellers with an N records should be able to trade, i.e. their average quality should be high enough. Long limits for negative feedback $T^- \geq \underline{T}$ keeps low quality sellers outside of the pool of sellers with N record, thereby increasing the average quality of this pool. Long limit for positive feedback T^+ does the opposite, it keeps away good sellers from the pool with N record, and decreases the average quality of this pool. In order to sustain trade in a market where the average quality of sellers is low, positive records should be deleted sooner than the negative ones ($T^+ < T^-$).

The intuition behind Proposition 3 highlights the opposite effect positive and negative records have on the possibility of a seller with an N record to trade. Yet, there is another key difference between positive and negative records. We illustrate it for the special case where the sellers' types are almost permanent, i.e. their types change with a very low intensity $\varphi \to 0$. Substitute for z in (17) from (16), and take the limit $\varphi \to 0$ in order to obtain a restriction on $T^+ \leq T(z)$, which boils down to:

$$(1 + mT^-(z))e^{-mT^+} \geq \frac{1-\mu}{\mu(\theta^H - 1)} \, .$$

If $\mu\theta^H < 1$, this condition is necessary to sustain trade. It is easy to see that longer T^- relaxes this condition, while longer T^+ has the opposite effect. However, the strength of the two effects is also different.

Remark: *While the positive effect of T^- is linear, the negative effect of T^+ is exponential.*

To see why this is the case, consider a good and a bad seller that have an N record and happen to trade at a given moment. The bad seller gets a negative feedback after selling the product and is effectively excluded from trade for the timespan the negative feedback is retained T^-, after which the feedback is deleted and he enters back into the pool of sellers with an N record. Now consider the good seller who gets positive feedback after selling the product. Given that he has a good record, he can potentially trade again and get new positive feedback. The positive feedback is retained for the timespan T^+ and if the seller trades before the positive feedback is deleted, he gets a new positive feedback which will be retained for another timespan T^+, and he will be able to trade again. By repeating this argument, one can see that once a good seller leaves the pool of sellers with an N record, he can spend much longer than T^+ periods outside of this pool before entering again. This is why retaining positive records has a stronger negative effect on the average quality of the sellers with an N compared to the positive effect of retaining negative records. This is an additional reason why it is important to treat positive and negative records separately when analyzing models with feedback.

4.3 Optimal limited records

In Section 3 we have shown that any stationary optimal recommendation policy is equivalent to a binary recommendation policy with a threshold z^*, which we characterized in Theorem 2. Now we can use Theorem 3 and show that the optimal threshold policy can be implemented with a limited record policy.

Corollary 1. *If condition (11) is satisfied, the stationary equilibrium with the highest welfare* *can be implemented with a limited records policy $T^- = T^-(z^*)$, and $T^+ \leq T(z^*)$. The optimal threshold z^* is given by Theorem 2, and the corresponding limited records are described in Theorem 3.*

The proof is immediate and is omitted for brevity. Because limited records and threshold policies can implement equivalent stationary equilibria with the same level of welfare, the equilibrium with the highest welfare can be implements in both cases.

5 Robustness and extensions

In this section we investigate the robustness of our findings by studying three extensions of the model. First, we consider a profit maximizing intermediary. Second, we study patient sellers, who might strategically deviate from the intermediary's recommended trade. Finally, we introduce moral hazard, using a set-up with investment in quality similar to Board and Meyer-ter Vehn (2013).

5.1 Profit maximizing intermediary

In the main analysis we have assumed the intermediary to care about the social welfare. In this section, instead, we characterize information policies of a profit maximizing intermediary. We consider a monopolistic intermediary who charges a fee $\gamma \geq 0$ to the seller whenever he sells his product.[17]

The intermediary uses a threshold recommendation policy: he recommends sellers with a posterior above z. We assume that condition (11) in Theorem 1 holds, i.e. stationary equilibrium with trade can be implemented, and we can consider threshold policies $z \in [\underline{z}, \overline{z}]$ without loss of generality. Any such policy induces a stationary equilibrium with trade, and a stationary distribution of sellers' posteriors F_z, characterized by Lemma 5.

We assume that the intermediary cares about long term profits and maximizes the flow of profits in a stationary equilibrium. Only sellers with a posterior $\pi \geq z$ trade and the intermediary's maximization problem is

$$\max_{z \in [\underline{z}, \overline{z}],\, f \geq 0} \int_z^1 mf dF_z(\pi), \tag{18}$$

[17]Such fees are common for online platforms, for instance Amazon, Ebay, Uber, Airbnb, etc. charge a fee which is a fraction of the transaction value.

provided that sellers agree to pay the fee f, i.e. $p(1) = E[\pi|\pi \geq z]\theta^H \geq 1+\gamma$, which can be written as

$$\frac{\int_z^1 \pi\theta^H dF_z(\pi)}{1-F_z(z)} \geq 1+\gamma.$$

It is easy to see, that the intermediary's payoff is increasing in γ, and for a given z the profit maximizing fee $\gamma(z)$ is the highest fee satisfying the above constraint. Substituting for $\gamma(z)$, the intermediary's problem can be rewritten as

$$\max_{z\in[\underline{z},\overline{z}]} m \int_z^1 (\pi\theta^H - 1)dF_z(\pi). \tag{19}$$

Expression (19) coincides with the social welfare $W(z)$ defined by (12), because the monopolistic intermediary extracts the full surplus from market participants through fees. We can, therefore, use theorem Theorem 2 and conclude that the monopolistic intermediary will choose the welfare maximizing threshold recommendation policy z^*.

Corollary 2. *If condition (11) is satisfied the monopolistic information intermediary chooses the welfare maximizing threshold recommendation policy z^* given by (13).*

It follows that a monopolistic intermediary would induce socially efficient amount of trade and experimentation in the market.

5.2 Patient sellers

In our main analysis we followed Kremer et al. (2014) and Che and Hörner (2018) in considering myopic sellers who only cared about their instantaneous payoffs. In this section, we show that our results hold when sellers are patient.

Assumption 3. *Sellers discount future payoffs at a rate $\rho \to 0$.*

Note that this case is the exact opposite of myopic sellers and puts the odds against us.[18] Our results for myopic sellers hold whether they are informed about their types or not. Yet in many applications of our model it is natural to think that sellers are privately informed about their types, that is why here we consider informed sellers.

Assumption 4. *At any t a seller $i \in [0,1]$ is privately informed about his type θ_{it}.*

[18]Unfortunately, the general case with $\rho \in (0,\infty)$ turned out to be non tractable.

Whenever the seller is active ($dN_{it} = 1$), he decides whether to sell ($s_{it} = 1$) or not ($s_{it} = 0$) depending on his type θ_{it}, the intermediary's recommendation r_{it}, and the history of his feedback h_{it} which is summarized by the intermediary's posterior π_{it}.

The intermediary's information and policy. The intermediary observes the full history of past feedback and selling decisions for all sellers and forms a posterior belief π_{it} about each seller $i \in [0, 1]$. As in the main analysis the intermediary uses a binary threshold recommendation policy $r \in \{0, 1\}$: he recommends a seller i to trade if $\pi_{it} \geq z$.[19] We also focus on a symmetric equilibrium and drop index i for brevity. The evolution of the intermediary's posterior about the seller is the same as in the main analysis and is given by the expression (6), and the evolution of the intermediary's information (F_t) about all sellers is described in Lemma 2.

Buyers' beliefs. Buyers' belief about the type of a given seller depends on the intermediary's recommendation $r \in \{0, 1\}$, the seller's selling decision $s \in \{0, 1\}$, and on their belief about the intermediary's information $\hat{F}_t$. We focus on stationary Markov Perfect Equilibrium, in which the intermediary's information does not change with time, i.e. $F_t = \hat{F}_t = F$ for $t \geq 0$. If the seller follows the recommendation the beliefs $\hat{\pi}_t(r, s)$ are determined by the Bayes' rule similar to (3):

$$\hat{\pi}_t(1,1) = \frac{\int_z^1 \pi d\hat{F}_t(\pi)}{1 - \hat{F}_t(z)}, \; \hat{\pi}_t(0,0) = \frac{\int_0^z \pi d\hat{F}_t(\pi)}{\hat{F}_t(z)}. \tag{20}$$

The price is determined by the buyer's willingness to pay $p_t(r, s) = \hat{\pi}_t(r, s)\theta^H$. Buyers' beliefs when the seller deviates from the recommendation are off equilibrium path, and can be arbitrary. For simplicity, we consider pessimistic beliefs, so that a deviating seller is believed to be of low quality $\hat{\pi}_t(0, 1) = 0$.

Sellers' incentives. If a seller is active at date t ($N_t = 1$) his decision to sell depends on his information θ_t, π_t, and his value function can be written as follows:

$$V_{\theta_t}(\pi_t) = \max_{s_t \in \{0,1\}} s_t[p_t(r_t, s_t) - 1]dN_t + (1 - \rho dt)E_t[V_{\theta_{t+dt}}(\pi_{t+dt})|\theta_t, \pi_t, s_t]. \tag{21}$$

From the above expression one can obtain Bellman equations for both types of the seller,

[19] Note that in principle, the recommendation policy can depend on the seller's past actions. For instance, the intermediary can punish a seller who deviates from his recommendation by not recommending this seller in the future. Clearly, such a policy would prevent deviations by the seller and would make it easy to implement the desired outcome. Yet, we do not allow such punishments here because we want our results to be consistent wit the limited memory rules for past feedback records in real world applications.

and in turn check the seller's incentives depending on his type and posterior. It turns out that the low type seller always follows the recommendations, but the high type seller may want to deviate and sell when he is not recommended to (for details see the proof of Proposition 4). If he would deviate buyers would offer zero price and he would sell at a loss, but then he would get a positive feedback which would reveal his high type to the intermediary.

Clearly, in some applications sellers may not be able to "sell at a loss" and such deviations are not possible. This may be due to credit constraints or other reasons. Consider one of our main applications, the credit market, where the seller is a borrower who is issuing debt in order to finance a project. In this application the borrower is credit constrained by definition, that is why he is borrowing in the first place, and he can't start a project if he can't raise enough money to cover the investment cost.

Remark 1. If the seller can't afford to have a negative short-term profit then Theorem 1 holds with patient sellers. In the rest of our analysis we assume that sellers can have negative short-term profits.

Yet, if sellers are able to sell at a loss, we get an additional restriction on recommendation policies as our next result illustrates.

Proposition 4. *If $\mu\theta^H < 1$ stationary equilibrium with trade can be implemented if $\underline{z} < \mu$ and*

$$\underline{z} \leq z \leq \min\left[\frac{\varphi}{m}\frac{1}{\theta^H - 1}, \mu\right]. \tag{22}$$

If $\mu\theta^H \geq 1$ stationary equilibrium with trade can be implemented with $z \leq \min\left[\frac{\varphi}{m}\frac{1}{\theta^H-1}, \frac{1}{\theta^H}\right]$.

It is easy to see that $\underline{z} = \frac{\varphi}{m}\frac{1-\mu\theta^H}{\theta^H-1} < \frac{\varphi}{m}\frac{1}{\theta^H-1}$, and the set of compatible policy thresholds is not empty if $\underline{z} < \mu$. The higher is the minimal threshold z for a seller to be recommended, the higher is the average quality and profits of trading sellers, and the higher is the value for the high type seller from deviating and revealing his type. Therefore, only thresholds below the limit specified in (22) are incentive compatible. This result is consistent with Theorem 1 for myopic sellers, yet the the set of policies compatible with a stationary equilibrium with trade is smaller when sellers are patient and can sell at a loss.

5.3 Moral hazard

Thus far we have considered the quality of each seller to be an exogenous process which the seller could not control. In this section we study moral hazard in a reputation model similar to Board and Meyer-ter Vehn (2013), and show that our results continue to hold as long as the effort cost are not too large. We consider patient sellers as in Section 5.2 with a discount rate $\rho \to 0$.

The seller's type follows a Markov process: it is constant unless a Poisson shock with intensity φ hits. When the shock hits, the seller's new type ($\theta_t \in \{0, \theta^H\}$) depends on his effort $a_t \in [0, \mu]$: with probability a_t his new type is high, and with probability $1 - a_t$ his new type is low. The seller has to constantly exert effort a_t at a cost $ca_t > 0$ in order to increase his chance of being high quality when the Poisson shock hits.

The rest of the model remains the same. The information intermediary follows a threshold recommendation policy, he does not observe the seller's effort choice and in equilibrium believes that the seller exerts an effort level $\tilde{a}_t$. An analog of (6) determines the intermediary's posterior about the seller's type

$$d\pi_t = \varphi(\tilde{a}_t - \pi_t)dt + (x_t - \pi_t)dN_t s(\pi_t). \tag{23}$$

Here as before $x_t = 1$ if $\theta_t = \theta^H$, and $x_t = 0$ otherwise. Following Board and Meyer-ter Vehn (2013), we consider a stationary Markov Perfect Equilibrium assuming that the intermediary's beliefs about the seller's effort choice $\tilde{a}_t$ are Markovian and depend on calendar time and history of feedback only via the left-sided limit $\pi_{t-} = \lim_{\varepsilon \to 0} \pi_{t-\varepsilon}$.[20]

The buyers are short-lived and their information and behavior are the same as in Section 5.2.

Seller's effort choice. Given the beliefs $\tilde{a}$ we can write down the seller's value function

$$V_{\theta_t}(\pi_t) = \sup_{a,s=\{a_\tau, s_\tau\}_{\tau \geq t}} E^{a,s,\theta_t} \int_{\tau=t}^{\infty} e^{-\rho(\tau-t)} \left[(p_\tau(r_\tau, s_\tau) - 1)dN_\tau s_\tau - ca_\tau\right] d\tau.$$

In a stationary equilibrium prices depend only on the rating and seller's decision but not on time, i.e. $p_t(r_t, s_t) = p(r_t, s_t)$. Denoting $V_H(\pi) = V_{\theta^H}(\pi)$ and $V_L(\pi) = V_{\theta^L}(\pi)$ we can rewrite the value function by truncating the integral by the first time the seller's quality switches

$$V_\theta(\pi_0) = \sup_{a,s} E^\theta \int_0^\infty e^{-(\rho+\varphi)t} \left[(p(r_t, s_t) - 1)dN_t s_t + \varphi(a_t V_H(\pi_t) + (1 - a_t)V_L(\pi_t)) - ca_t\right] dt.$$

[20] We also impose the same technical restrictions on $\tilde{\mu}(\pi_t)$ as in Board and Meyer-ter Vehn (2013) to ensure that $\dot{\pi}_t = \varphi(\tilde{a}_t - \pi_t)$ admits a solution.

Note that the effort choice a_t does not affect the evolution of the posterior π given by (23), buyers' beliefs $\hat{F}_t$ and the price that they offer $p(r_t, s_t)$. This implies that it is optimal for the seller to choose a_t which maximizes his payoff pointwise, and the optimal strategy is the same for both types of sellers

$$a^*(\pi) = \mu, \text{iff } c \leq \varphi(V_H(\pi) - V_L(\pi)). \tag{24}$$

Intuitively, a seller with a posterior π chooses high effort if the value function of a high type exceeds that of a low type by a significant amount. It turns out that the difference between value functions increases with π and to guarantee high effort it is enough to check (24) for a seller with the lowest posterior $\pi = 0$. We first assume $\tilde{a} = a^* = \mu$, and then compute the resulting value functions and find a parameter range for which the high-effort condition (24) holds for any $\pi \in [0, 1]$.

Seller's selling decision. Finally, we need to check that the seller follows the recommendations of the intermediary and sells only when he is recommended to do so ($s = r$). Since the choice of effort is independent from the selling decision s, the seller's optimal selling decisions when $a^* = \mu$ are the same as in Section 5.2, and Proposition 4 provides the necessary and sufficient conditions for the seller to follow recommendations.

Proposition 5. *A recommendation policy with a threshold z is compatible with a high effort stationary equilibrium with trade if conditions in Proposition 4 are satisfied and the effort cost is below a threshold*

$$c \leq \overline{c} = \frac{z(\mu - z)m^2}{\mu} \left(\frac{\theta^H}{\varphi + mz} - \frac{1}{\mu\varphi + mz} \right). \tag{25}$$

Since $\overline{c}$ is strictly positive for $z \in (\underline{z}, \mu)$, as long as the cost of effort is below that level our results continue to apply in the presence of moral hazard. The parameter space where they do, however, is restricted by the additional need to provide incentives to exert effort.

The set of incentive compatible policies expands (weakly) when μ goes up, because $\overline{c}$ given by (25) increases in μ, and constraint (22) relaxes. The result is very intuitive: μ determines the average quality of sellers in the market, equilibrium prices, and sellers' profits. When μ goes up, sellers' expected profits go up, their incentives to invest in quality increase and the range of incentive compatible policies expands.

Optimal Policy. In Theorem 2 we characterized the optimal threshold policy z^*, when sellers are myopic and their types are independent of their effort. Clearly, if this policy z^* satisfies conditions in Proposition 5 it is also optimal in an environment with moral hazard and patient informed sellers.

Implementation with limited records. In Section 4 we have shown that any threshold recommendation policy that sustains trade in the long run can be implemented with a limited records policy when sellers are myopic and there is no moral hazard. A similar result holds with moral hazard.

Proposition 6. *For any threshold recommendation policy z which implements a stationary equilibrium with trade and high effort according to Proposition 5 one can find a limited records policy with $T^-(z)$ given by (16) and $T^+ = 0$ which implements an equivalent stationary equilibrium.*

The above result is intuitive, if only negative records are recorded $T^- > 0$ but positive ones are not $T^+ = 0$, then N record becomes the best record a seller can have, i.e. it becomes equivalent to $r = 1$ recommendation under the threshold policy. Clearly, default records (D, τ), $\tau \in [0, T^-]$ are different from the no trade recommendation $r = 0$, yet they generate the same incentives for the seller, so that resulting trades, effort, and welfare are the same.

6 Conclusions and Policy Implications

We studied a dynamic market where each seller's quality is uncertain and changes stochastically with time. In the model, as in many real markets, an information intermediary collects past feedback on sellers and publicly reports it in order to improve the information available to buyers. We derived information policies that sustain trade in the market, characterized the optimal information policy and showed that it can be implemented through limited records. The model suggests that even in the absence of moral hazard or adverse selection, when sellers' quality is unknown one has to think carefully about the length of the "public memory" of past feedback.

A central finding is that it is crucial to distinguish between positive and negative past records, as they affect the market equilibria in opposite ways, and with different intensity. To sustain trade and maximize welfare the information intermediary should limit access to both positive and negative past records, but to a very different extent. As argued in the analysis, positive records should be erased early or not disclosed at all as they may prevent buyers from experimenting with non-rated sellers and lead to suboptimal outcomes or a market collapse. Long negative records, instead, help sustaining trade in the long run by increasing incentives to trade with non-rated sellers, although if they are too long, the level of trade and learning also becomes suboptimal.

We believe these results provide a novel perspective, relevant to a number of important contexts and useful to better understand the possible aggregate effects of information retention rules and rating systems.

Until recently, little economic theory was available to guide policy on how long "public memory" should be in different environments and for different types of records. Regulation on data retention, on the other hand, has been in place for quite some time in many countries, so it is not surprising that it has remained rather heterogeneous across countries and markets.

Credit markets exemplify the wide variation in adopted retention policies. Figure 1 plots the number of years after which positive and negative information about borrowers must be erased by credit bureaus for a handful of countries. Positive information generally contains the pattern of repayments, open and closed credit accounts and new loans, while negative information is about defaults, bankruptcies, delinquencies, arrears. As one can see from Figure 1, retention limits differ substantially even among similar countries.

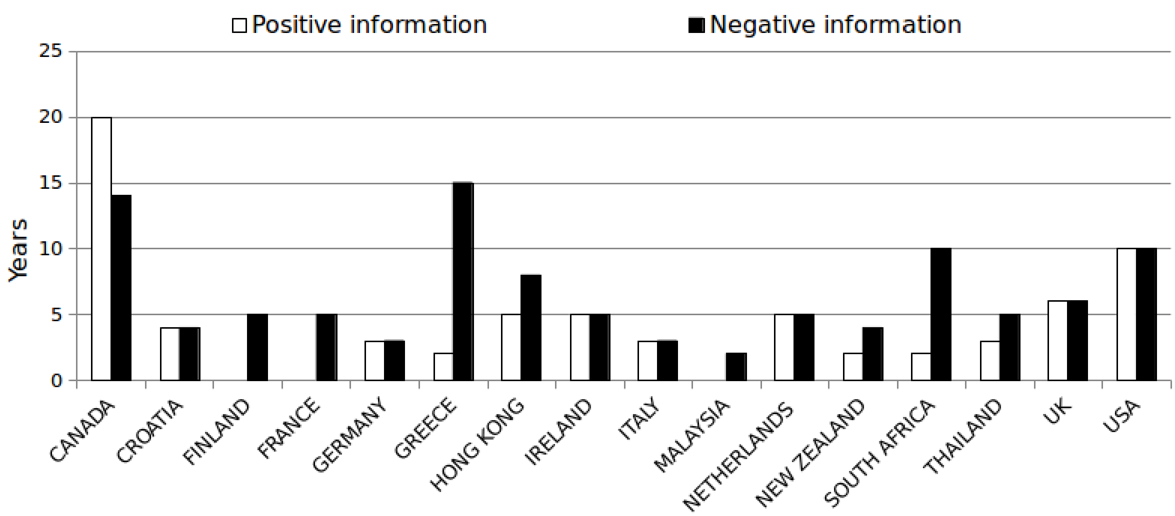

Figure 1: Retention periods for credit records

Our results suggest that for countries or credit market segments where lending is risky because the average quality of borrowers or the success chances of their project is poor, limiting credit bureau's ability to collect and distribute positive information, may be a good policy option for reducing the risk of market breakdowns. In terms credit bureaus' policies, our results suggest that the policy of France and New Zealand may be safer than that of Australia or the US.

Internet platforms collecting feedback on participants also have to decide for how long to leave past records accessible, and how much memory to assign to summary indicators. For example, in 2008 eBay changed its reputational indicator and made it a function of feedback left in the last year only, instead of all past records, while still leaving all past feedback accessible but at the cost of some search effort.

Our results suggest that eBay should at least have considered the possibility of assigning a different memory to positive and negative feedback in its new reputation indicator (and possibly limiting the overall history of the feedbacks it makes accessible to buyers).

Criminal records typically do not contain "positive" past records. Police and court databases tend to automatically store only negative past information, and in many countries there are (different) rules that limit for how long these can be accessed. These policies appear closer to what our model suggests could be optimal.[21]

For other environments, it may appear more difficult to distinguish, ex ante, positive from negative information. This, however, does not mean that our results do not apply. For instance, our results seem relevant for the debate on "the right to be forgotten" on the Internet, even though information is not already classified as "good" and "bad". For example, on May 13, 2014, the European Court of Justice ruled against Google in favor of Costeja (Judgement in case C-131/12), a Spanish man who requested the removal of a link to a digitized 1998 article in the La Vanguardia newspaper about an auction for his foreclosed home due to debt that he had subsequently repaid. The court ruled that search engines are responsible for the content they point to and that individuals have the right to ask them to remove links with personal information about them if the information is considered "excessive in relation to the purposes for which they were processed and in the light of the time that has elapsed." Its fundamental principle has then been fully incorporated in Article 17 of the EU GDPR.

[21] Still, there is a wild variation in the memory of criminal records, even within the US. For example, the minimum time after which misdemeanors can be expunged ranges from the one year of Hawaii to the five years of California, Kentucky and Mississipi, the ten years of Florida and the never of several other states (see http://ccresourcecenter.org/state-restoration-profiles/50-state-comparisonjudicial-expungement-sealing-and-set-aside/).

Suppose now that it is possible to erase past records from search results after some time, upon request (in practice this might be very costly). People have incentives to request cancellation of their negative records, which may be good according to our results, but none will have incentives to require the cancellation of positive records, which appears problematic in the light of our findings.

More generally, although excessively long memory of records may have the negative aggregate/social effects we described, in many environments these records are still valuable to individual buyers, and private incentives exist for intermediaries to collect and distribute them. Therefore, our model suggests that privacy regulation limiting data retention of negative and positive records separately may be desirable in markets plagued by informational problems.

Our analysis also suggests that the current regulation, where present, may not be optimal, at least in markets where average quality is poor. For example, the current trend of credit bureaus increasingly collecting positive records (in addition to the usual negative ones) may end up having harmful long-term consequences in high risk credit market segments. These positive past records may make it very difficult for borrowers without any record to enter (or re-enter) the market and obtain credit in the first place.

Appendix

Proof of Proposition 1. Buyers observe full history of past feedback and at any t they offer a price $p_{it} = \hat{\pi}(h_{it})\theta^H = \pi(h_{it})\theta^H$ to each active seller i. The seller sells whenever $p_{it} \geq 1$. Under Assumption 1, for any $t > 0$ without any feedback, the probability of seller $i \in [0, 1]$ being of high type is given by

$$\pi_{it} = \pi_{i0}e^{-\varphi t} + \mu(1 - e^{-\varphi t}).$$

At $t = 0$ sellers with $\pi_{i0} < \frac{1}{\theta^H}$ do not sell because $p_{i0} = \pi_{i0}\theta^H < 1$. These sellers will never trade and will never get any feedback because $\mu\theta^H < 1$ guarantees $\pi_{it} theta^H < 1$. If at $t = 0$ buyers' prior about all sellers is low $\pi_{i0} < \frac{1}{\theta^H}$ for all $i \in [0, 1]$, then clearly there is no trade ever. Suppose that some sellers at $t = 0$ have high prior probability of being high quality $\pi_{i0} \geq \frac{1}{\theta^H}$. Denote the total mass of these sellers by $x \leq 1$. Note that as soon as a seller sells a low-quality product, he gets negative feedback and is revealed to be of low type. From that moment the seller is excluded from trading forever. Each high quality seller from $t = 0$ onward may randomly become low quality, and then sell and get negative feedback. This happens with intensity $(1 - \mu)\varphi m$. At $t = 0$ the total mass of sellers that could trade is x, therefore for any $t \geq 0$, the mass of sellers that can trade does not exceed $\overline{\mu}_t x e^{-(1-\mu)\varphi m t}$. As $t \to \infty$, the mass $\overline{\mu}_t \to 0$, therefore the fraction of sellers trading in equilibrium converges to zero with time.

QED.

Proof of Lemma 1. The intermediary can replace all messages in M_t that induce $s_{it} = 1$ with a recommendation $r_{it} = 1$, and replace all messages that induce $s_{it} = 0$ with a recommendation $r_{it} = 0$. These recommendations will induce the same trades and the same surplus, as the original messages. Feedback produced by trades will be exactly identical, as well as the information available to the intermediary after the trades. Hence, the intermediary can replace any recommendation policy with a binary recommendation, and obtain the same expected payoff.

QED.

Proof of Lemma 2. First, consider sellers with posterior $\pi = \mu$, that have mass $dF_t(\mu)$. Note that (6) has a stable point $\pi = \mu$ towards which posteriors $\pi \neq \mu$ converge if no trade happens, but never reach it. This implies that the mass of sellers with the posterior μ can only decrease, indeed, these sellers become active with intensity m and are recommended with probability $\sigma(\mu)$, hence $dF_{t+dt}(\mu) - dF_t(\mu) = -mdt\sigma(\mu, F_t))dF_t(\mu)$ which is equivalent to (8).

Consider sellers with $\pi \in [0, \mu) \cup (\mu, 1]$. All sellers who face a trade opportunity and get recommended by the intermediary, trade and get their types reported by buyers: They jump to the top or to the bottom of the posterior distribution F_t. If a seller has a posterior $\pi = 0$, this posterior will evolve according to (6). Consider a positive $d\pi = \varphi(\mu - \pi)dt$.

$$F_{t+dt}(\pi + d\pi) = F_t(\pi) - dtm \int_0^{\pi} \sigma(\pi)dF_t(\pi) + dtm \int_0^1 (1 - \pi)\sigma(\pi)dF_t(\pi).$$

From t to $t + dt$ the posterior of sellers below π changes by $d\pi = \varphi(\mu - \pi)dt$, unless they trade and jump to posterior 0 or 1. At the same time each moment mass $m \int_0^1 (1 - \pi)\sigma(\pi)dF_t(\pi)$ of sellers is revealed to be of low type and joins the distribution $F(\pi)$ at the bottom. Substituting $F_{t+dt}(\pi + d\pi) = F_{t+dt}(\pi) + f_t(\pi)\varphi(\mu - \pi)dt$, where f_t is the corresponding density function, one obtains (7):

$$F_{t+dt}(\pi) - F_t(\pi) + f_t(\pi)\varphi(\mu - \pi)dt = -dtm \int_0^{\pi} \sigma(\pi)dF_t(\pi) + dtm \int_0^1 (1 - \pi)\sigma(\pi)dF_t(\pi).$$

QED.

Proof of Lemma 3. Let us first prove the following property of the optimal policy: if $\sigma(\pi', F_t)dF_t(\pi') > 0$ for some $\pi' \in [0, 1]$, then for any $\pi > \pi'$ such that $dF_t(\pi) > 0$ one must have $\sigma(\pi, F_t) = 1$. The proof is by contradiction. Suppose $\sigma(\pi, F_t) < 1$, then the information intermediary can lower $\sigma(\pi', F_t)$ by small ε' and increase $\sigma(\pi, F_t)$ by ε, such that $\varepsilon(1 - \pi)dF_t(\pi) = \varepsilon'(1 - \pi')dF_t(\pi')$, so that the modified policy recommends exactly the same mass of bad sellers as the original one. The modified policy recommends a larger mass of good sellers than the original one because

$$\varepsilon\pi dF_t(\pi) = \pi\varepsilon' \frac{1 - \pi'}{1 - \pi} dF_t(\pi') > \varepsilon'\pi' dF_t(\pi'), \tag{26}$$

and $\pi(1 - \pi')/(1 - \pi) > \pi'$ for $\pi > \pi'$. This implies that the modified policy generates higher flow of welfare at t than the original one, and at the same time satisfies incentive compatibility constraints (4), (5).

To prove that the modified policy dominates the original one, it remains to show that it does not make the intermediary's information "worse" according to the information evolution constraints. The difference between two policies, is that in a short dt the original one reveals types of sellers that have total mass $mdt\varepsilon' dF_t(\pi')$, while the modified one reveals types of sellers that have total mass $mdt\varepsilon dF_t(\pi)$. With the original policy, the intermediary learns that mass $\pi' dt\varepsilon' dF_t(\pi')$ of sellers are good, and mass $(1-\pi')dt\varepsilon' dF_t(\pi')$ sellers are bad. With the modified policy he learns that mass $\pi dt\varepsilon dF_t(\pi)$ of sellers are good, and mass $(1-\pi)dt\varepsilon dF_t(\pi)$ sellers are bad. Since ε is such that $\varepsilon(1-\pi)dF_t(\pi) = \varepsilon'(1-\pi')dF_t(\pi')$, both policies reveal the same mass of bad agents, but according to (26), the modified policy reveals a larger mass of good agents than the original one.

Suppose the intermediary implements the modified policy, and "forgets" information about a mass $mdt[\varepsilon\pi dF_t(\pi) - \varepsilon'\pi' dF_t(\pi')] > 0$ of good sellers, and mixes them with a mass $mdt\varepsilon' dF_t(\pi')$ of sellers with a posterior π', so his posterior $\hat{\pi}$ about sellers in this combined group of total mass $mdt\varepsilon\pi dF_t(\pi) + mdt\varepsilon'(1-\pi')dF_t(\pi')] = mdt\varepsilon dF_t(\pi)$ is given by a Bayes' rule

$$\hat{\pi} = \frac{mdt[\varepsilon\pi dF_t(\pi) - \varepsilon'\pi' dF_t(\pi')] + mdt\varepsilon'\pi' dF_t(\pi')}{mdt\varepsilon dF_t(\pi)} = \pi.$$

That is, the modified policy combined with the forgetting decision reveals a mass $mdt\varepsilon'\pi' dF_t(\pi')$ of good agents, and a mass $mdt\varepsilon'(1-\pi')dF_t(\pi')$ of bad agents, i.e. this policy is equivalent to the original policy in terms of information production.

The modified policy generates higher flow of welfare, satisfies incentive compatibility constraints, and allows to exactly replicate the information of the original policy by "forgetting", i.e. destroying some information. Therefore, the original policy is not optimal, a contradiction, hence we conclude that if $\sigma(\pi', F_t)dF_t(\pi') > 0$ for some $\pi \in [0,1]$, then for any $\pi > \pi'$ such that $dF_t(\pi) > 0$ one must have $\sigma(\pi, F_t) = 1$.

Take the lowest $z \in [0,1]$ such that $\sigma(z, F_t)dF_t(z) > 0$, then for any $\pi > z$ such that $dF_t(\pi) = 0$ the recommendation policy is irrelevant, and one can set $\sigma(\pi) = 1$. Combined with the above reasoning, an optimal recommendation policy is equivalent to a policy with $\sigma(\pi) = 1$ for $\pi > z$. Finally, for $\pi < z$, either $\sigma(\pi, F_t) = 0$ or $dF_t(\pi) = 0$, and this policy is equivalent to a policy with $\sigma(\pi, F_t) = 0$ for $\pi < z$.

This proves, that an optimal recommendation policy is equivalent to a threshold policy $z(F_t)$, such that $\sigma(\pi, F_t) = 1$ for $\pi > z$, and $\sigma(\pi, F_t) = 0$ for $\pi < z$.

QED.

Lemma 5. *If an information policy $z \leq \overline{z}$ satisfies incentive compatibility constraints (4) and (5), it induces the following stationary distribution of sellers' posteriors:*

$$F(\pi) = \frac{m(1-\mu)\ln\frac{\mu}{\mu-\pi}}{\varphi + mz + m(1-\mu)\ln\frac{\mu}{\mu-z}},\ \pi \in [0, z), \tag{27}$$

$$F(\pi) = \frac{1-\mu}{\varphi + mz + m(1-\mu)\ln\frac{\mu}{\mu-z}}\left(m\ln\frac{\mu}{\mu-z} + \varphi - \varphi\left(\frac{\mu-\pi}{\mu-z}\right)^{\frac{m}{\varphi}}\right),\ \pi \in [z, \mu), \tag{28}$$

$$F(\pi) = \left(\frac{\pi-\mu}{1-\mu}\right)^{\frac{m}{\varphi}} + \frac{\varphi(1-\mu) + m(1-\mu)\ln\frac{\mu}{\mu-z}}{\varphi + mz + m(1-\mu)\ln\frac{\mu}{\mu-z}}\left(1 - \left(\frac{\pi-\mu}{1-\mu}\right)^{\frac{m}{\varphi}}\right),\ \pi \in [\mu, 1]. \tag{29}$$

Proof. Suppose the recommendation policy with a threshold $z \in [0, \mu)$ satisfies incentive compatibility constraints (4) and (5), then the evolution of F_t is characterized by (8) and (7). Assuming that the distribution is stationary ($F_t = F$) we solve differential equations (8) and (7) for different intervals of $\pi \in [0, 1]$ and characterize the density function f of the stationary distribution.

Case 1. Take $\pi = \mu$, a threshold $z < \mu$ according to (8) implies that there are no sellers with posterior μ in a stationary equilibrium, i.e. $f(\mu) = 0$.

Case 2. Consider now $\pi \neq z$ starting with $\pi \leq z$, equation (7) implies

$$f(\pi) = \frac{m}{\varphi}\left(1 - F(z) - \int_z^1 xdF(x)\right)\frac{1}{\mu-\pi},$$

and integrating one obtains the corresponding CDF

$$F(\pi) = \frac{m}{\varphi}\left(1 - F(z) - \int_z^1 xdF(x)\right)ln\frac{\mu}{\mu-\pi}. \tag{30}$$

One can express

$$F(z) = \frac{m\ln\frac{\mu}{\mu-z}}{\varphi + m\ln\frac{\mu}{\mu-z}}\left(1 - \int_z^1 xdF(x)\right) \tag{31}$$

Case 3. For $\pi \in (z, \mu) \cup (\mu, 1]$, equation (7) implies

$$f(\pi) = \frac{m}{\varphi}\left(1 - F(\pi) - \int_z^1 xdF(x)\right)\frac{1}{\mu-\pi}. \tag{32}$$

For $\pi \in (z, \mu)$ solution to this differential equation determines the CDF up to a constant $B > 0$

$$F(\pi) = 1 - \int_z^1 x dF(x) - B(\mu - \pi)^{\frac{m}{\varphi}}.$$

Continuity of F at μ implies

$$F(\mu) = 1 - \int_z^1 x dF(x). \tag{33}$$

Continuity of F at z and (31) imply

$$B = \frac{\varphi F(\mu)}{\varphi + m \ln \frac{\mu}{\mu - z}} (\mu - z)^{-\frac{m}{\varphi}},$$

so that the CDF for $\pi \in (z, \mu)$ can be expressed as

$$F(\pi) = F(\mu) \left(1 - \frac{\varphi}{\varphi + m \ln \frac{\mu}{\mu - z}} \left(\frac{\mu - \pi}{\mu - z} \right)^{\frac{m}{\varphi}} \right). \tag{34}$$

Consider now $\pi \in (\mu, 1]$, solution to (32) determines the CDF up to a constant $A > 0$

$$F(\pi) = F(\mu) + A(\pi - \mu)^{\frac{m}{\varphi}}.$$

Border condition $F(1) = 1$ implies $A = (1 - F(\mu))(1 - \mu)^{-\frac{m}{\varphi}}$, and we obtain

$$F(\pi) = F(\mu) + (1 - F(\mu)) \left(\frac{\pi - \mu}{1 - \mu} \right)^{\frac{m}{\varphi}}. \tag{35}$$

In remains to find $F(\mu)$ to fully characterize the stationary distribution for a given policy z. Recall that

$$\mu = \int_0^1 x dF(x) = \int_0^z x dF(x) + \int_z^1 x dF(x).$$

Using $1 - \int_z^1 x dF(x) = F(\mu)$, one can express

$$\int_0^z x dF(x) = \mu - 1 + F(\mu). \tag{36}$$

Using (30) one can obtain the density function for $\pi \leq z$

$$f(\pi) = \frac{1}{\mu - \pi} \frac{mF(\mu)}{\varphi + m \ln \frac{\mu}{\mu - z}}.$$

Compute

$$\int_0^z x dF(x) = F(\mu) \frac{m}{\varphi + m \ln \frac{\mu}{\mu - z}} \int_0^z \frac{x}{\mu - x} dx = \left(\mu \ln \frac{\mu}{\mu - z} - z \right) \frac{mF(\mu)}{\varphi + m \ln \frac{\mu}{\mu - z}},$$

which together with (36) allows to express $F(\mu)$ as a function of z

$$F(\mu) = (1 - \mu) \frac{\varphi + m \ln \frac{\mu}{\mu - z}}{\varphi + mz + m(1 - \mu) \ln \frac{\mu}{\mu - z}}. \quad (37)$$

Substituting for $F(\mu)$ in (30), (34), and (35) one obtains (27), (28), and (29), correspondingly, that fully characterize F for a given z.

QED.

Proof of Theorem 1. If $\mu\theta^H < 1$, the incentive constraint (5) is trivially satisfied for any z because $E_F[\pi|\pi < z]\theta^H \leq \mu\theta^H < 1$.

The incentive constraint (4) requires $E_F[\pi|\pi \geq z]\theta^H \geq 1$. Using (33) one can express

$$E_F[\pi|\pi \geq z] = \frac{1 - F(\mu)}{1 - F(z)}.$$

Substituting for $F(z)$ from (31), and for $F(\mu)$ from (37) one obtains

$$E_F[\pi|\pi \geq z] = \frac{1 - F(\mu)}{1 - \frac{mF(\mu)\frac{\mu}{\mu - z}}{\varphi + m \ln \frac{\mu}{\mu - z}}} = 1 - \frac{\varphi(1 - \mu)}{\varphi + mz}. \quad (38)$$

Condition $E_F[\pi|\pi \geq z]\theta^H \geq 1$ can be rewritten as

$$z \geq \underline{z} = \frac{\varphi}{m} \frac{1 - \mu\theta^H}{\theta^H - 1}.$$

According to Proposition 1, any threshold $z \geq \mu$ leads to no trade, hence, a stationary equilibrium with trade is feasible only if $\underline{z} < \mu$. If this condition holds, for any threshold $z \in [\underline{z}, \mu)$ one can find the corresponding initial distribution F_0, which coincides with the stationary distribution F induced by the threshold z (characterized in Lemma 5). Any such pair z, F_0 implements a stationary equilibrium with trade.

If $\mu\theta^H \geq 1$, any threshold $z \in [0, 1/\theta^H]$ satisfies the incentive constraints. Indeed $E[\pi|\pi <$

$z]\theta^H < E[\pi|\pi < 1/\theta^H]\theta^H < 1$ and constraint (5) is satisfied. The incentive constraint (4) is also satisfied because $E_F[\pi|\pi \geq z]\theta^H \geq 1$ is equivalent to $z \geq \frac{\varphi}{m}\frac{1-\mu\theta^H}{\theta^H-1}$, which holds for any $z \geq 0$ when $\mu\theta^H \geq 1$.

QED.

Proof of Theorem 2. In a stationary equilibrium $F_t = F$, $t \in [0, \infty)$, and the payoff of the information intermediary can be expressed as

$$W(z) = \int_z^1 (\pi\theta^H - 1)dF(\pi) = (E_F[\pi|\pi \geq z]\theta^H - 1)(1 - F(z)).$$

Using (31) and (38) one obtains

$$W(z) = \frac{(\theta^H - 1)(\varphi + mz) - \varphi(1-\mu)\theta^H}{\varphi + mz + m(1-\mu)\ln\frac{\mu}{\mu - z}}.$$

We look for a maximum of $W(z)$. If $\mu\theta^H > 1$ we consider $z \in [0, 1/\theta^H]$, and if $\mu\theta^H \leq 1$ we consider $z \in [\underline{z}, \mu)$. Note that in either case $z < \mu$. The first order condition is

$$\frac{\partial W}{\partial z} = m(1-\mu)\frac{m(\theta^H - 1)(\ln\frac{\mu}{\mu-z} - \frac{z}{\mu-z}) + \varphi\left(\theta^H\frac{1-z}{\mu-z} - (\theta^H - 1)\frac{1}{\mu-z}\right)}{\left(\varphi + mz + m(1-\mu)\ln\frac{\mu}{\mu-z}\right)^2} = 0.$$

Denote $h(z) = m(\theta^H - 1)(\ln\frac{\mu}{\mu-z} - \frac{z}{\mu-z}) + \varphi\left(\theta^H\frac{1-z}{\mu-z} - (\theta^H - 1)\frac{1}{\mu-z}\right)$.

The necessary condition for the maximum is $h(z^*) = 0$, which requires

$$m(\theta^H - 1)(\mu - z^*)\ln\frac{\mu}{\mu - z^*} - z^*(\varphi\theta^H + m(\theta^H - 1)) + \varphi = 0,$$

which in turn is equivalent to (13).

If $h(z)$ is decreasing for z, then (13) delivers the maximum. Compute

$$\frac{\partial h(z)}{\partial z} = m(\theta^H - 1)\frac{-z}{(\mu - z)^2} + \varphi\left(\theta^H\frac{(1-z) - (\mu - z)}{(\mu - z)^2} - (\theta^H - 1)\frac{1}{(\mu - z)^2}\right).$$

The derivative is negative if

$$\varphi\theta^H(1-\mu) - (\theta^H - 1)(\varphi + mz) = \varphi(1 - \mu\theta^H) - (\theta^H - 1) < 0.$$

If $\mu\theta^H \geq 1$ the above condition holds. If $\mu\theta^H < 1$ the above condition is equivalent to $z > \underline{z}$, i.e. $h(z)$ decreases for $z > \underline{z}$ and z^* corresponds to a maximum.

Finally, we need to check that z^* is within the relevant range. If $\mu\theta^H > 1$ we must have $z^* \in [0, 1/\theta^H]$, and if $\mu\theta^H \leq 1$ we must have $z^* \in [\underline{z}, \mu)$.

Consider $\mu\theta^H > 1$. Compute $h(0) = \varphi/\mu > 0$. Express $h(1/\theta^H) = m(\theta^H - 1)(\ln \frac{\mu\theta^H}{\mu\theta^H - 1} - \frac{1}{\mu\theta^H - 1})$. Denote $y = \frac{1}{\mu\theta^H - 1}$ and recall that $\ln(1 + y) - y < 0$ for any $y > 0$, which implies $h(1/\theta^H) < 0$. Because $h(z)$ is decreasing, and $h(0) > h(z^*) = 0 > h(1/\theta^H)$, we conclude that $z^* \in [0, 1/\theta^H]$, and because $\mu\theta^H > 1$ we have $z^* < 1/\theta^H < \mu$, i.e. z^* is feasible.

Consider $\mu\theta^H \leq 1$ and $z \in [\underline{z}, \mu)$. Substitute for $\underline{z}$ and compute

$$h(\underline{z}) = \frac{1}{\mu - \underline{z}} \left(m(\theta^H - 1)(\mu - \underline{z}) \ln \frac{\mu}{\mu - \underline{z}} + \varphi\theta^H(\mu - \underline{z}) \right) > 0.$$

This implies $z^* > \underline{z}$ because $h(z)$ is decreasing for $z > \underline{z}$. Now consider $z \to \mu - 0$ and express

$$\begin{aligned}
\lim_{z \to \mu - 0} h(z) &= \lim_{z \to \mu - 0} \frac{1}{\mu - z} \left(m(\theta^H - 1)(\ln \frac{\mu}{\mu - z}(\mu - z) - \mu) + \varphi(1 - \mu\theta^H) \right) \\
&= \lim_{z \to \mu - 0} \frac{1}{\mu - z} \left(-m(\theta^H - 1)\mu + \varphi(1 - \mu\theta^H) \right) < 0,
\end{aligned}$$

because (11) implies $m(\theta^H - 1)\mu > \varphi(1 - \mu\theta^H)$. Given that $h(z)$ is decreasing, and $h(\underline{z}) > h(z^*) = 0 > h(\mu)$, we conclude that there is unique $z^* \in [\underline{z}, \mu)$, which implements a stationary equilibrium with the highest welfare.

QED.

Proof of Proposition 2. The proof is very similar to the proof if Theorem 1. In an equilibrium with trade the high quality seller must be willing to sell, i.e. (4) must hold $p(1) = E_F[\pi|\pi \geq z]\theta^H \geq c$ .Substituting for the stationary distribution F from Lemma 5 we compute $E_F[\pi|\pi \geq z] = 1 - \frac{\varphi(1-\mu)}{\varphi + mz}$ and obtain a necessary condition for the high quality seller to be willing to trade $z \geq \underline{z}(c)$, where $z(c)$ is given by (14). If the average quality of sellers is low $\mu\theta^H < c$, then any threshold $z \geq \mu$ leads to no trade. In this case trade can be sustained if and only if $\underline{z}(c) < \mu$ and the threshold should be $z \in [\underline{z}, \mu)$. If the average quality is high ($\mu\theta^H \geq c$), then any threshold $z \leq c/\theta^H$ can implement the stationary equilibrium with trade. Note that an analog of Lemma 4 holds, and we can focus on thresholds $z \leq c/\theta^H$ without loss of generality. One can easily check that the low quality seller is willing to trade when he is recommended to do so, and no seller wants to trade when he is not recommended to do so.

QED.

Proof of Theorem 3. We need to prove that a policy with a limited memory $T^{-}(z)$ for negative records, and a limited memory $T^{+} \leq T(z)$ for positive records implements a stationary equilibrium equivalent to the equilibrium induced by the threshold policy z, where $T(z)$ is given by (17).

Take a policy z which implements a stationary equilibrium with trade according to Theorem 1, this policy prevents sellers with posteriors below z from trading, and allows sellers with posteriors above z to trade. A limited records policy (T^{+}, T^{-}) implements an equivalent equilibrium if it allows trading only for sellers with posteriors above z. Let us find such policies.

With limited records, there are three groups of sellers: 1) sellers whose latest feedback is negative D, τ, with $\tau \in [0, T^{-}]$, 2) sellers who have an N record, i.e. an empty history, and 3) sellers whose latest feedback is positive S, τ, with $\tau \in [0, T^{+}]$.

Buyers' posterior about a seller with D, τ record is $\hat{\pi}(D, \tau) = \mu - \mu e^{-\varphi \tau}$, it grows with τ and one can find $T^{-}(z)$ such that $\pi(D, T^{-}(z)) = z$, because $z < \mu$. Sellers with records $\hat{\pi}(D, \tau)$, $\tau < T^{-}(z)$ do not trade since buyers are ready to pay $p(D, \tau) = \hat{\pi}(D, \tau)\theta^{H} \leq z\theta^{H} < 1$, because either $\mu\theta^{H} < 1$ and $z < \mu$, or because $\mu\theta^{H} \geq 1$ and $z < 1/\theta^{H}$. Sellers with posteriors above z must be able to sell, hence negative records must be deleted after a time interval $T^{-} = T^{-}(z)$, where $T^{-}(z)$ given by (16). Therefore,

$$z = \hat{\pi}(D, T^{-}) = \mu - \mu e^{-\varphi T^{-}},$$

is the highest posterior a seller with a D record can have before it is erased and the seller joins the pool of sellers with an N record. Let us analogously denote the lowest posterior a seller with an S record can have, before it is erased and the seller joins the pool of sellers with an N record

$$z^{+} = \hat{\pi}(S, T^{+}) = \mu + (1 - \mu)e^{-\varphi T^{+}}.$$

Sellers with an N record have a posterior $\hat{\pi}(N) \in [z, z^{+}]$. In a stationary equilibrium they must be able to sell, i.e. $\hat{\pi}(N)\theta^{H} \geq 1$. Otherwise with time S and D records would be erased and the pool of sellers with an N record would grow, which can't happen in a stationary equilibrium. We assume $\hat{\pi}(N)\theta^{H} \geq 1$ and check when it is indeed the case.

Sellers with S, τ record can trade because $\hat{\pi}(S, \tau) = \mu + (1 - \mu)e^{-\varphi \tau} \geq z^{+} \geq \hat{\pi}(N)$ for any $\tau \in [0, T^+]$. In the equilibrium with a threshold recommendation policy z described in the main analysis in Section 2 all sellers with posteriors above z where pooled together in a group with the same recommendation to sell ($r = 1$). With limited records, the situations is a little different, posteriors of seller above z^+ are revealed to buyers, yet these sellers sell with the same intensity m as in the main analysis, and the stationary distribution of sellers' posteriors $F(\pi)$ is the same as described in Lemma 5.

Using the distribution $F(\pi)$ we can compute the buyers' posterior belief about sellers with an N record in the following way

$$\hat{\pi}(N) = \frac{\int_z^{z^+} \pi dF(\pi)}{F(z^+) - F(z)} = \frac{F(z^+)z^+ - F(z)z - \int_z^{z^+} F(\pi)d\pi}{F(z^+) - F(z)}. \tag{39}$$

Equations (34) and (35) define $F(\pi)$ on intervals $[z, \mu)$ and $[\mu, 1]$ correspondingly, using them we express

$$\begin{aligned}\int_z^{z^+} F(\pi) = &\int_z^{\mu} \left(F(\mu) - F(\mu) \frac{\varphi}{\varphi + m \ln \frac{\mu}{\mu - z}} \left(\frac{\mu - \pi}{\mu - z} \right)^{\frac{m}{\varphi}} \right) d\pi \\ &+ \int_{\mu}^{z^+} \left(F(\mu) + (1 - F(\mu)) \left(\frac{\pi - \mu}{1 - \mu} \right)^{\frac{m}{\varphi}} \right) d\pi,\end{aligned} \tag{40}$$

where $F(\mu)$ is given by equation (37):

$$F(\mu) = (1 - \mu) \frac{\varphi + m \ln \frac{\mu}{\mu - z}}{\varphi + mz + m(1 - \mu) \ln \frac{\mu}{\mu - z}}.$$

Integrating (40) one obtains

$$\int_z^{z^+} F(\pi) = F(\mu)(z^+ - z) + F(\mu) \frac{\varphi}{\varphi + m \ln \frac{\mu}{\mu - z}} \frac{\varphi(\mu - z)}{m + \varphi} + (1 - F(\mu)) \frac{(z^+ - \mu)^{\frac{m+\varphi}{\varphi}}}{(1 - \mu)^{\frac{m}{\varphi}}} \frac{\varphi}{m + \varphi}.$$

Using the above expression, and substituting for $F(z^+)$ from (34) and for $F(z)$ from (35) in (39) we get

$$\hat{\pi}(N) = \frac{(1 - F(\mu)) \left(\frac{z^+ - \mu}{1 - \mu} \right)^{\frac{m}{\varphi}} \frac{mz^+ + \varphi\mu}{m + \varphi} + F(\mu) \frac{\varphi}{\varphi + m \ln \frac{\mu}{\mu - z}} \frac{mz + \varphi\mu}{m + \varphi}}{(1 - F(\mu)) \left(\frac{z^+ - \mu}{1 - \mu} \right)^{\frac{m}{\varphi}} + F(\mu) \frac{\varphi}{\varphi + m \ln \frac{\mu}{\mu - z}}},$$

which can be rewritten as

$$\hat{\pi}(N) = \mu + \frac{m}{m+\varphi} \frac{\left(\frac{z^+-\mu}{1-\mu}\right)^{\frac{m}{\varphi}} (z^+ - \mu) + \frac{F(\mu)}{1-F(\mu)} \frac{\varphi}{\varphi + m \ln \frac{\mu}{\mu - z}} (z - \mu)}{\left(\frac{z^+-\mu}{1-\mu}\right)^{\frac{m}{\varphi}} + \frac{F(\mu)}{1-F(\mu)} \frac{\varphi}{\varphi + m \ln \frac{\mu}{\mu - z}}}.$$

Substituting for $F(\mu)$ from (37) one gets

$$\hat{\pi}(N) = \mu + \frac{m}{m+\varphi} \frac{\left(\frac{z^+-\mu}{1-\mu}\right)^{\frac{m}{\varphi}} (z^+ - \mu) + \frac{\varphi(1-\mu)}{\mu\varphi + mz} (z - \mu)}{\left(\frac{z^+-\mu}{1-\mu}\right)^{\frac{m}{\varphi}} + \frac{\varphi(1-\mu)}{\mu\varphi + mz}},$$

Further substituting for $z = \mu - \mu e^{-\varphi T^-}$ and $z^+ = \mu + (1-\mu)e^{-\varphi T^+}$ one obtains

$$\hat{\pi}(N) = \mu + \frac{m}{m+\varphi} \frac{e^{-mT^+}(1-\mu)e^{-\varphi T^+} - \frac{\varphi(1-\mu)\mu e^{-\varphi T^-}}{\mu\varphi + m\mu(1-e^{-\varphi T^-})}}{e^{-mT^+} + \frac{\varphi(1-\mu)}{\mu\varphi + m\mu(1-e^{-\varphi T^-})}}.$$

Sellers with an N record must be able to trade, i.e. we must have $\hat{\pi}(N)\theta^H \geq 1$, which can be written as

$$\hat{\pi}(N)\theta^H = \mu\theta^H + \mu\theta^H \frac{m}{m+\varphi} \frac{e^{-\varphi T^+} - e^{-\varphi T^-} e^{mT^+} \frac{\varphi}{\varphi + m(1-e^{-\varphi T^-})}}{\frac{\mu}{1-\mu} + e^{mT^+} \frac{\varphi}{\varphi + m(1-e^{-\varphi T^-})}} \geq 1. \tag{41}$$

It remains to show that (41) is satisfied only for $T^+ \leq T(z)$. If $\mu\theta^H \leq 1$ it is easy to see, in this case for (41) to hold it must be that $e^{-\varphi T^+} - e^{-\varphi T^-} e^{mT^+} \frac{\varphi}{\varphi + m(1-e^{-\varphi T^-})} > 0$, and the $\hat{\pi}(N)\theta^H$ is decreasing with T^+. If $T^+ \to \infty$ then $\hat{\pi}(N)\theta^H \to \mu\theta^H - \mu\theta^H \frac{m}{m+\varphi} e^{-\varphi T^-} < 1$, i.e. for (41) to hold, T^+ must be lower than some threshold $T(z)$ which we characterize later.

Consider $\mu\theta^H \geq 1$ and rewrite (41) as follows

$$\frac{1-\mu}{\mu} \frac{\varphi}{\varphi + m(1-e^{-\varphi T^-})} \leq \theta^H \left[\mu e^{-mT^+} + (1-\mu) \frac{\varphi + m e^{-(m+\varphi)T^+}}{m+\varphi} \right] - e^{-mT^+}. \tag{42}$$

Denote by $\zeta(T^+)$ the term on the right-hand side of (42), take the derivative with respect to T^+ and obtain

$$\frac{\partial \zeta(T^+)}{\partial T^+} = -m e^{-mT^+} [\mu\theta^H - 1 + (1-\mu)\theta^H e^{-\varphi T^+}] < 0,$$

if $\mu\theta^H \geq 1$, i.e. the right-hand side of (42) is decreasing with T^+. Again, if $T^+ \to \infty$ then (41) and, consequently, (42) are violated. In other words (42) holds for $T^+ \leq T(z)$.

Substituting $z = \mu - \mu e^{-\varphi T^-}$ and putting (42) to equality gives (17) and determines $T(z)$.

QED.

Proof of Proposition 3. Consider $\mu\theta^H < 1$ and suppose the stationary equilibrium with trade can be implemented, i.e. conditions in Theorem 1 are satisfied:

$$\underline{z} = \frac{\varphi}{m}\frac{1-\mu\theta^H}{\theta^H - 1} < \mu,$$

First, let's show that $T^- \in [\underline{T}, \infty)$ in a stationary equilibrium with trade. By Theorem 1 only thresholds $z \in [\underline{z}, \mu)$ implement stationary equilibria with trade. By Theorem 3 the equivalent memory policies are given by (16), therefore they must lie in the corresponding range $T^- \in [\underline{T}, \infty)$, where $\underline{T}$ is obtained from (16): $\underline{z} = \mu - \mu e^{-\varphi \underline{T}}$

For a limited records policy T^-, T^+ to be compatible with a stationary equilibrium with trade, sellers with an N record must be able to sell. In the proof of Theorem 3 it was shown that for this to be the case condition (41) must hold, which in the case $\mu\theta^H < 1$ in turn requires $e^{-\varphi T^+} - e^{-\varphi T^-} e^{mT^+} \frac{\varphi}{\varphi + m(1-e^{-\varphi T^-})} > 0$. Let's us show that the latter condition is violated for $T^+ \geq T^-$. To do so suppose $T^+ \geq T^-$, then

$$e^{-\varphi T^+} - e^{-\varphi T^-} e^{mT^+} \frac{\varphi}{\varphi + m(1-e^{-\varphi T^-})} \leq e^{-\varphi T^-} - e^{-\varphi T^-} e^{mT^-} \frac{\varphi}{\varphi + m(1-e^{-\varphi T^-})} \leq 0,$$

because $\varphi + m(1-e^{-\varphi T^-}) - \varphi e^{mT^-} \leq 0$, for any $T^- \geq 0$. Indeed for $T^- = 0$ the condition holds as equality, but for any $T^- > 0$ its derivative is negative:

$$\frac{\partial\left(\varphi + m(1-e^{-\varphi T^-}) - \varphi e^{mT^-}\right)}{\partial T^-} = m\varphi e^{-\varphi T^-} - m\varphi e^{mT^-} < 0.$$

That is, if $T^+ \geq T^-$ then condition (41) is violated and sellers with an N record can't trade and the stationary equilibrium with trade can't be implemented. Therefore only policies with $T^+ < T^-$ can implement stationary equilibrium with trade when $\mu\theta^H < 1$.

QED.

Proof of Proposition 4. Here we prove that both types of sellers follow the recommendations of the intermediary under certain conditions, i.e. sell only if $\pi > z$ and $r = 1$. As in Theorem 1, we consider $z < \mu$ if $\mu\theta^H < 1$, since with $z \geq \mu$ there wouldn't be trade in the long run, and if $\mu\theta^H \geq 1$ we consider $z < 1/\theta^H$ because higher thresholds are not optimal. We first assume that sellers follow the recommendation, then compute the resulting value function, and finally check under which conditions it is optimal for sellers to follow recommendations.

Let us first derive Bellman equations for both types of the seller. Consider a high type seller and denote $V_H(\pi_t) = V_{\theta^H}(\pi_t)$ for brevity, then for a small dt we can express

$$\begin{aligned} &E_t[V_{\theta_{t+dt}}(\pi_{t+dt})|\theta^H, \pi_t, s_t] = s_t m dt V_H(1) + \\ &(1 - s_t m dt)\left[(1-\varphi(1-\mu)dt)V_H(\pi_t) + \varphi(1-\mu)dt V_L(\pi_t) + \frac{dV_H(\pi_t)}{d\pi_t}\varphi(\mu - \pi_t)dt\right]. \end{aligned} \tag{43}$$

In a stationary Markov Equilibrium $F_t = \hat{F}_t = F$ and $p_t(r,s) = p(r,s)$, so that the value function for each type only depends on π. Using (43) and (21) we obtain the Bellman equation for the high type seller

$$\begin{aligned} &\rho V_H(\pi) = \max_{s\in\{0,1\}} sm[p(r,s) - 1 + V_H(1) - V_H(\pi)] + \\ &\frac{dV_H(\pi)}{d\pi}\varphi(\mu - \pi) - \varphi(1-\mu)(V_H(\pi) - V_L(\pi)). \end{aligned} \tag{44}$$

Analogous Bellman equation for the low type seller is

$$\begin{aligned} &\rho V_L(\pi) = \max_{s\in\{0,1\}} sm[p(r,s) - 1 + V_L(0) - V_L(\pi)] + \\ &\frac{dV_L(\pi)}{d\pi}\varphi(\mu - \pi) + \varphi\mu(V_H(\pi) - V_L(\pi)). \end{aligned} \tag{45}$$

Consider $\pi > z$ and suppose sellers follow the recommendation and sell. Substituting $s_t = 1$ and subtracting (45) from (44) one obtains

$$\begin{aligned} &\rho(V_H(\pi) - V_L(\pi)) = m(V_H(1) - V_L(0) + V_L(\pi) - V_H(\pi)) + \\ &\frac{dV_H(\pi) - dV_L(\pi)}{d\pi}\varphi(\mu - \pi) - \varphi(V_H(\pi) - V_L(\pi)). \end{aligned} \tag{46}$$

Denote $\Delta(\pi) = V(\theta^H, \pi) - V(\theta^L, \pi)$ and rewrite (46) as

$$\Delta(\pi)(\rho + m + \varphi) = m(V_H(1) - V_L(0)) + \frac{d\Delta(\pi)}{d\pi}\varphi(\mu - \pi). \tag{47}$$

We first prove two useful lemmas, the first one shows that seller's value functions are constant for $\pi > z$, and the second one proves that the value functions are continuous at z.

Lemma 6. *For $\pi > z$ value functions for both types of sellers do not depend on π:*

$$V_H(\pi) = V_H(1) = \frac{m[p(1,1) - 1] - \varphi(1-\mu)[V_H(1) - V_L(1)]}{\rho}, \tag{48}$$

$$V_L(\pi) = V_L(1) = \frac{m[p(1,1) - 1 + V_L(0)] + \varphi\mu[V_H(1) - V_L(1)]}{\rho + m}. \tag{49}$$

Proof of Lemma 6. The solution to the differential equation (47) for $\pi > z$ can have two parts, the homogeneous part in the form $\Delta_{Homogeneous}(\pi) = A_1(\mu - \pi)^{-\frac{\rho+m+\varphi}{\varphi}}$ with some constant A_1, and the particular part $\Delta_{Particular}(\pi) = \frac{m(V_H(1) - V_L(0))}{\rho+m+\varphi}$. First, note that $z < \mu$ because of (10). Second, value functions $V_H(\pi)$ and $V_L(\pi)$ are bounded, hence the difference $\Delta(\pi)$ is also bounded, which in turn implies that $A_1 = 0$ in the general solution. Indeed, if $A_1 \neq 0$ then for $\pi \to \mu$ the difference is not bounded, a contradiction. Therefore for $\pi > z$, $\Delta(\pi) = \frac{m(V_H(1) - V_L(0))}{\rho+m+\varphi}$ is a constant. Substituting for $\Delta(\pi)$ in (44) and (45) one obtains differential equations for $V_H(\pi)$ and $V_L(\pi)$. By the same argument as for $\Delta(\pi)$ solutions to these equations are given by particular parts, i.e. for $\pi > z$ value functions $V_H(\pi)$ and $V_L(\pi)$ do not depend on π. It is easy to show that the solutions are given by (48) and (49).

QED.

Lemma 7. *Sellers' value functions $V_H(\pi)$ and $V_L(\pi)$ are continuous at z.*

Proof of Lemma 7. To prove the continuity of value functions, consider a prior $z^- = z - d\pi$, where $d\pi = \varphi(\mu - z)dt$. Compute $V_H(z^-)$ and $V_L(z^-)$. In equilibrium there should be no trade between two moments t and $t + dt$, because for $\pi < z$ there is a negative recommendation (later on we check that it is indeed the case), and we can write the value function for a high type seller at z^- as follows

$$V_H(z^-) = e^{-\rho dt}[V_H(z)(1 - (1-\mu)\varphi dt) + V_L(z)(1-\mu)\varphi dt] \tag{50}$$

Going to the limit $dt \to 0$, we obtain $V_H(z - 0) = V_H(z)$, i.e. $V_H(\pi)$ is continuous at z; the same property holds for $V_L(z)$.

QED.

Now we can characterize sellers' value functions for $\pi < z$. Substitute $s_t = 0$ in (44) and (45), and obtain

$$\rho V_H(\pi) = \frac{dV_H(\pi)}{d\pi}\varphi(\mu - \pi_t) - \varphi(1-\mu)\Delta(\pi). \tag{51}$$

$$\rho V_L(\pi) = \frac{dV_L(\pi)}{d\pi}\varphi(\mu - \pi_t) + \varphi\mu\Delta(\pi). \tag{52}$$

$$(\rho + \varphi)\Delta(\pi) = \frac{d\Delta(\pi)}{d\pi}\varphi(\mu - \pi_t). \tag{53}$$

The solution is

$$\Delta(\pi) = A(\mu - \pi)^{-\frac{\rho+\varphi}{\varphi}} = \Delta(1)\left(\frac{\mu - z}{\mu - \pi}\right)^{\frac{\rho+\varphi}{\varphi}}, \tag{54}$$

here we used the continuity of Δ: $\Delta(1) = \Delta(z) = A(\mu - z)^{-\frac{\rho+\varphi}{\varphi}}$. Rewrite (51) and (52) as

$$\rho V_H(\pi) = \frac{dV_H(\pi)}{d\pi}\varphi(\mu - \pi_t) - \varphi(1 - \mu)\Delta(1)\left(\frac{\mu - z}{\mu - \pi}\right)^{\frac{\rho+\varphi}{\varphi}}. \tag{55}$$

$$\rho V_L(\pi) = \frac{dV_L(\pi)}{d\pi}\varphi(\mu - \pi_t) + \varphi\mu\Delta(1)\left(\frac{\mu - z}{\mu - \pi}\right)^{\frac{\rho+\varphi}{\varphi}}. \tag{56}$$

Let's look for a solution in the following form $V_L(\pi) = (a\pi + b)\left(\frac{\mu-z}{\mu-\pi}\right)^{\frac{\rho+\varphi}{\varphi}}$, where a and b are arbitrary constants. Substituting in (56) one gets

$$\begin{aligned} \rho(a\pi + b)\left(\frac{\mu - z}{\mu - \pi}\right)^{\frac{\rho+\varphi}{\varphi}} &= (\rho + \varphi)(a\pi + b)\left(\frac{\mu - z}{\mu - \pi}\right)^{\frac{\rho+\varphi}{\varphi}} + \\ a\varphi(\mu - \pi)\left(\frac{\mu - z}{\mu - \pi}\right)^{\frac{\rho+\varphi}{\varphi}} &+ \varphi\mu\Delta(1)\left(\frac{\mu - z}{\mu - \pi}\right)^{\frac{\rho+\varphi}{\varphi}}. \end{aligned} \tag{57}$$

Which requires

$$\rho(a\pi + b) = (\rho + \varphi)(a\pi + b) + a\varphi(\mu - \pi) + \varphi\mu\Delta(1), \tag{58}$$

and we can express $b = -\mu(a + \Delta(1))$, and obtain

$$V_L(\pi) = (a(\pi - \mu) - \mu\Delta(1))\left(\frac{\mu - z}{\mu - \pi}\right)^{\frac{\rho+\varphi}{\varphi}}. \tag{59}$$

Combining (49) and (48) one obtains

$$V_L(1) - V_L(0) = \frac{\rho + \varphi}{m}\Delta(1), \tag{60}$$

which together with (49) in turn implies

$$V_L(1) = \frac{m[p(1,1)-1] - (\rho + \varphi(1-\mu))\Delta(1)}{\rho}. \tag{61}$$

Substituting for $V_L(\pi)$ in (60) we get

$$(a(z-\mu) - \mu\Delta(1)) + \mu(a + \Delta(1)) \left(\frac{\mu - z}{\mu}\right)^{\frac{\rho+\varphi}{\varphi}} = \frac{\rho + \varphi}{m}\Delta(1),$$

which allows to express

$$a = \Delta(1) \frac{\frac{\rho+\varphi}{m} + \mu\left(1 - \left(\frac{\mu-z}{\mu}\right)^{\frac{\rho+\varphi}{\varphi}}\right)}{z - \mu\left(1 - \left(\frac{\mu-z}{\mu}\right)^{\frac{\rho+\varphi}{\varphi}}\right)} \tag{62}$$

Note that $V_L(1) = V_L(z)$ and we can express

$$V_L(1) = a(z-\mu) - \mu\Delta(1) = \Delta(1) \frac{\frac{\rho+\varphi}{m} + \mu - (\mu - z)\left(\frac{\mu-z}{\mu}\right)^{\frac{\rho}{\varphi}}}{1 - \left(\frac{\mu-z}{\mu}\right)^{\frac{\rho}{\varphi}}} - \mu\Delta(1), \tag{63}$$

which together with (61) gives

$$\Delta(1) \frac{\frac{\rho+\varphi}{m} + \mu - (\mu - z)\left(\frac{\mu-z}{\mu}\right)^{\frac{\rho}{\varphi}}}{1 - \left(\frac{\mu-z}{\mu}\right)^{\frac{\rho}{\varphi}}} - \mu\Delta(1) = \frac{m}{\rho}[p(1,1)-1] - (1 + \frac{\varphi}{\rho}(1-\mu))\Delta(1),$$

which in turn allows to express

$$\Delta(1) = \frac{m[p(1,1)-1]}{\rho \frac{\frac{\rho+\varphi}{m} + z\left(\frac{\mu-z}{\mu}\right)^{\frac{\rho}{\varphi}}}{1 - \left(1 - \frac{z}{\mu}\right)^{\frac{\rho}{\varphi}}} + \rho + \varphi(1-\mu)}. \tag{64}$$

Now we consider the limit when $\rho \to 0$, in this case

$$\lim_{\rho \to 0} \Delta(1) = \frac{m}{\varphi} \frac{p(1,1) - 1}{\frac{\mu\varphi}{zm} + 1} \tag{65}$$

It is easy to see that if $p(1,1) \geq 1$, then $\Delta(1) \geq 0$.

Using (38) one can express

$$p(1,1) = E_F[\pi | \pi \geq z]\theta^H = \left(1 - \frac{\varphi(1-\mu)}{\varphi + mz}\right)\theta^H \tag{66}$$

It is easy to check that $z \geq \underline{z}$ implies

$$p(1,1) \geq 1,$$

and, therefore $\Delta(1) \geq 0$.

Equilibrium strategies

Now we are ready to check if both seller's types follow the intermediary's recommendations, i.e. $s = r = 1$ for $\pi > z$, and $s = r = 0$ for $\pi \leq z$.

Low type. The low type seller's Bellman equation is given by (45).

Consider first $\pi < z$, so that the intermediary does not recommend the seller to trade, i.e. $r = 0$. If the seller deviates from the recommendation and sells, beliefs of buyers' and the intermediary about his type drop to zero, i.e. $\hat{\pi}(0,1) = \pi(0,1) = 0$. The low type seller optimally chooses $s = 0$ for $\pi \leq z$ if

$$p(0,1) - 1 + V_L(0) \leq V_L(\pi).$$

Since $p(0,1) = 0$ it is sufficient to prove that $V_L(0) \leq V_L(\pi)$ for $\pi < z$, which is indeed the case as we show in Lemma 8 below. Hence, the low type seller does not sell when the sale is not recommended $s = r = 0$ if $\pi < z$.

Lemma 8. $V_L(0) \leq V_L(\pi)$ *for* $\pi \in [0, z]$.

Proof of Lemma 8. Consider a bad seller with a posterior $\pi \in [0, z]$, in equilibrium he does not trade until his posterior reaches the threshold z. His posterior reaches z after time t which is determined by the equation

$$\mu - (\mu - \pi)e^{-\varphi t'} = z.$$

Note that $\mu > z \geq \pi$ and t' decreases with π. The value function of a bad seller with posterior $\pi < z$ at time t can be written as

$$V_L(\pi) = e^{-\rho t'} \left(\pi' V_H(z) + (1 - \pi')V_L(z) \right),$$

where $\pi' = Pr[\theta_{t+t'} = \theta^H | \theta_t = \theta^L] = \mu(1 - e^{-\varphi t'})$ is the probability the low type seller becomes a high type by the date $t + t'$.

In order to prove that $V_L(0) \leq V_L(\pi)$ we check the sign of $\frac{dV_L(\pi)}{d\pi}$ which is the opposite of

$\frac{dV_L(\pi)}{dt'}$ because t' decreases with π. Note that, $V_L(z) = V_L(1)$, $V_H(z) - V_L(z) = \Delta(z) = \Delta(1)$ and we can express

$$\frac{dV_H(\pi)}{dt'} = -\rho e^{-\rho t'} \left(V_L(1) + \pi' \Delta(1) \right) + e^{-\rho t'} \Delta(1) \mu \varphi e^{-\varphi t'},$$

which after substitutions can be written as

$$\frac{dV_L(\pi)}{dt'} e^{\rho t'} = -\rho \left(V_L(1) + \mu \Delta(1) \right) + \Delta(1) \mu (\varphi + \rho) e^{-\varphi t'}.$$

If this derivative is not positive for any t', then $V_L(\pi)$ is non decreasing in π. It is enough to check that the derivative is negative for $t' = 0$, and substituting for $V_L(1)$ from (63) we get sufficient a condition for $V_L(\pi) \geq V_L(0)$:

$$-\rho \Delta(1) \frac{\frac{\rho+\varphi}{m} + \mu - (\mu - z) \left(\frac{\mu - z}{\mu} \right)^{\frac{\rho}{\varphi}}}{1 - \left(\frac{\mu - z}{\mu} \right)^{\frac{\rho}{\varphi}}} + \Delta(1) \mu (\varphi + \rho) \leq 0. \tag{67}$$

It is immediate to see that for $\rho \to 0$ this condition is equivalent to

$$-\frac{\mu}{z} \frac{\varphi^2}{m} \Delta(1) \leq 0.$$

The above condition holds, therefore $V_L(\pi) \geq V_L(0)$ for $\pi \in [0, z]$. QED.

Consider a bad seller with $\pi > z$ and a positive recommendation $r_t = 1$. If the seller becomes active, follows recommendation and trades $s_t = 1$ the feedback reveals the seller's low type and the intermediary's belief drops to $\pi_t = 0$. The bad seller optimally chooses $s = 1$ for $\pi > z$ if

$$p(1, 1) - 1 + V_L(0) \geq V_L(\pi).$$

Note that for $\pi > z$ $V_L(\pi) = V_L(1)$ and using (60) we obtain an equivalent condition

$$p(1, 1) - 1 \geq \frac{\rho + \varphi}{m} \Delta(1).$$

Using $\rho \to 0$ and (65) we obtain

$$\frac{\mu \varphi}{m z} + 1 \geq 1,$$

which always holds, i.e. the low type seller follows the intermediary's recommendation $s_t(\theta^L, \pi) = r_t$ for any $\pi \in [0, 1]$.

High type. Consider a high type seller with a posterior $\pi > z$ and a recommendation to trade $r = 1$. If the seller sells he gets a profit $p(1,1) - 1 \geq 0$, he gets a feedback from the buyer and his high type is revealed ($\pi = 1$). If he deviates and does not trade the posterior about him stays unchanged. The seller chooses to trade because

$$p(1,1) - 1 + V_H(1) \geq V_H(\pi).$$

Because $p(1,1) \geq 1$ and for $\pi \geq z$ the value function is constant $V_H(\pi) = V_H(1)$.

Consider a high type seller with a posterior $\pi \leq z$ and recommendation not to trade $r = 0$. If the seller does not trade $s = 0$, his posterior remains π. If he deviates and sells ($s = 1$), buyers' belief about his type drop to zero and the price would be $p(0,1) = 0$. The intermediary's beliefs at first also drop to zero, but then then buyers leave feedback and the posterior about the seller jumps to $\pi = 1$. The seller will choose $s = 0$ for $\pi \in [0, z)$ if

$$V_H(\pi) \geq -1 + V_H(1). \tag{68}$$

$V_H(\pi)$ is non decreasing in π, and it is sufficient to check that (68) holds for $\pi = 0$, i.e

$$V_H(1) - V_H(0) \leq 1.$$

Rewrite $V_H(1) - V_H(0) = V_L(1) + \Delta(1) - V_L(0) - \Delta(0)$, and using (54) and (60) we obtain

$$V_H(1) - V_H(0) = \Delta(1)\left(\frac{\rho + \varphi}{m} + 1 - \left(\frac{\mu - z}{\mu}\right)^{\frac{\rho+\varphi}{\varphi}}\right).$$

Let $\rho \to 0$, use (65) for $\Delta(1)$ to obtain a necessary and sufficient condition for $s = 0$ to be chosen by the high type seller for any $\pi \in [0, z)$

$$\frac{mz}{\mu\varphi}\left(p(1,1) - 1\right) \leq 1$$

Substituting for $p(1,1)$ from (66) we get an equivalent condition

$$\frac{\varphi\mu + mz}{\varphi + mz}\theta^H \leq \frac{\mu\varphi}{mz} + 1, \tag{69}$$

which further simplifies to

$$z \leq \frac{\varphi}{m} \frac{1}{\theta^H - 1}. \tag{70}$$

This condition is necessary and sufficient for the high type seller to follow the intermediary's recommendations when $\pi \in [0, z)$. Since low type seller always follows the recommendations, and so does the high type seller for $\pi \geq z$. We conclude that condition (70) is necessary and sufficient for a recommendation policy z to be incentive compatible when sellers are patient and informed.

QED.

Proof of Proposition 5. To prove that the seller always exerts high effort $a^* = \mu$, we first assume this is so, compute the corresponding value functions, and then check under which condition the seller optimally chooses to exert high effort.

The value functions for both types of the seller must satisfy differential equations similar to (44) and (45) derived in the previous analysis:

$$\begin{aligned} \rho V_H(\pi) &= sm[p(r_t, s, \hat{F}_t) - 1 + V_H(1) - V_H(\pi)] - ca^* \\ &+ \frac{dV_H(\pi)}{d\pi}\varphi(a^* - \pi) - \varphi(1 - \mu^*)(V_H(\pi) - V_L(\pi)). \end{aligned} \tag{71}$$

$$\begin{aligned} \rho V_L(\pi) &= sm[p(r_t, s, \hat{F}_t) - 1 + V_L(0) - V_L(\pi)] - ca^* + \\ &\frac{dV_L(\pi)}{d\pi}\varphi(a^* - \pi) + \varphi\mu^*(V_H(\pi) - V_L(\pi)). \end{aligned} \tag{72}$$

The above differential equations differ from (44) and (45) only because of the effort cost ca^*. Therefore, the corresponding value function differ from the ones derived before by the net present value of cost a^*c equal to a constant ca^*/ρ. This fact allows us to use the same formula as before for the difference between high and low value functions $\Delta(\pi) = V_H(\pi) - V_L(\pi)$, which can be expressed as follows[22]:

$$\Delta(\pi) = \Delta(1) \left(\frac{\mu - z}{\mu - \pi} \right)^{\frac{\rho + \varphi}{\varphi}},$$

Note that $\Delta(\pi)$ is increasing in π, and in order to guarantee that the high effort is chosen by the seller for any reputation level, it is enough to check that (24) holds for $\pi = 0$, i.e. $c \leq \varphi\Delta(0)$. Using $\rho \to 0$ and (65) we obtain

[22]Full derivations are in the Appendix, see formula (54) for a reference.

$$c \leq m\frac{\mu - z}{\mu}\frac{p(1,1) - 1}{\frac{\mu\varphi}{z\lambda m} + 1}.$$

Substituting for $p(1)$ from (66) and rearranging we get (25)

$$c \leq \bar{c} = \frac{z(\mu - z)m^2}{\mu}\left(\frac{\theta^H}{\varphi + mz} - \frac{1}{\mu\varphi mz}\right).$$

If this condition is satisfied sellers exert effort $a = \mu$ in equilibrium, if it is violated some sellers would deviate and exert $a \neq \mu$.

QED.

Proof of Proposition 6. Under a limited records policy the set of possible records for sellers consists of default records (D, τ), $\tau \in [0, T^-(z)]$ and of the record N. From Theorem 3 we know that policies z and $T^-(z)$, $T^+ = 0$ generate the same information for the intermediary, i.e. result in the same stationary $F(\pi)$, if sellers with record N trade and sellers with default records do not. We need to check that this is indeed the case. Also, we need to check that all sellers choose to exert effort $a^* = \mu$. If sellers' incentives are satisfied, then the limited record policy generates the same trades and the same welfare as the threshold policy.

Selling decisions. Bellman equations for high and low quality sellers can be obtained from analogous equations (44) and (45) if one substitutes record $r = 1$ with record N (they are equivalent), and record $r = 0$ with corresponding records (D, τ), $\tau \in [0, T^-(z)]$. Basically, record (D, τ) reveals to buyers the intermediary's information about the seller and allows them to learn the seller's posterior $\pi = \mu - \mu e^{-\varphi\tau} \in [0, z]$. Hence, the Bellman equation for a high quality seller with a (D, τ) record can be written as

$$\begin{aligned} \rho V_H(\pi) &= sm[p(\pi, s, \hat{F}_t) - 1 + V_H(1) - V_H(\pi)] - ca^* \\ &+ \frac{dV_H(\pi)}{d\pi}\varphi(a^* - \pi) - \varphi(1 - \mu^*)(V_H(\pi) - V_L(\pi)). \end{aligned} \tag{73}$$

If both types of the seller trade only with record N, i.e. when $\pi > z$ then their value function have the same values as in the proof of Proposition 5. Since record N and $r = 1$ are equivalent, and seller would sell with record $r = 1$, we conclude that he also optimally chooses to sell when his record is N.

Now we need to check that sellers with a (D, τ) record do not sell. Recall that under the threshold policy we assumed that buyers had pessimistic beliefs: they believed that any seller trying to sell with record $r = 0$ to be of low type. We assume the same here, if buyers see a seller who tries to sell with default record they believe the seller to be of low type, hence they are ready to pay a price $p = 0$. This makes Bellman equations under a threshold policy z and a limited records policy identical, which in turn implies that seller's optimal selling decision are the same under the two policies: sellers do not sell when they have a record (D, τ), $\tau \in [0, T^-(z)]$.

Effort choice. It remains to check that sellers choose to exert effort under the limited records policy. By the above arguments seller's value function $V_\theta(\pi)$ is the same under two policies, hence if condition (24) holds under a threshold policy z, it also holds under the limited records policy $T^-(z)$, $T^+ = 0$.

Since both policies provide the same incentives for the seller and threshold policy z implements a stationary equilibrium with trade, the corresponding limited record policy also implements an equivalent stationary equilibrium with trade.

QED.

Online Appendix

Buyers do not always leave feedback.

Here we relax the assumption that buyers always leave feedback after any transaction, and show that our results hold when buyers leave feedback only with a certain probability $\lambda \in (0, 1]$. This extension of our results is important for online market platforms such as Ebay and Amazon, where the probability of a buyer leaving feedback after a transaction can be as low as few percent. We assume that a buyer leaves feedback only with a certain probability $\lambda \in (0, 1]$.

Assumption 5. *The incidence of a buyer leaving feedback after purchasing from a seller i is a random variable $q_i^t = \{1, 0\}$ which is equal to one if the buyer leaves feedback, and* $\Pr[q_i^t = 1] = \lambda \in (0, 1]$.

Note that the joint probability of a seller trading and getting feedback is now $m\lambda$ instead of m. It turns out that our results do not change qualitatively in this case. For instance, it is easy to check that the stationary equilibrium with trade can be implemented if the modified condition of Theorem 1 holds

$$\underline{z}(\lambda) \equiv \frac{\varphi}{m\lambda} \frac{1 - \mu\theta^H}{\theta^H - 1} < \mu. \tag{74}$$

Proposition 7. *If $\mu\theta^H < 1$ stationary equilibrium with trade can be implemented if*

$$\lambda > \lambda^* = \frac{\varphi}{m\mu} \frac{1 - \mu\theta^H}{\theta^H - 1}, \tag{75}$$

and $\underline{z} \leq z < \mu$.

If $\mu\theta^H \geq 1$ stationary equilibrium with trade can be implemented with any $z \leq 1/\theta^H$.

We omit the proof because it can be easily obtained from the proofs of Lemma 5 and Theorem 1 by replacing m with $m\lambda$. Condition $\underline{z} < \mu$ is equivalent to (75) and implies that in markets with low average quality of sellers ($\mu\theta^H < 1$) the probability of feedback λ must be above the threshold λ^*, to implement a stationary equilibrium with trade. In markets with high average quality of sellers ($\mu\theta^H > 1$), stationary equilibrium with trade can be implemented for arbitrary small $\lambda > 0$.

Intuitively, when the average quality of sellers is high, it is easy to sustain trade, because even without any feedback the market would function. When the average quality of sellers is low, so that there would be no trade without an information intermediary according to Proposition 1 it is critical that buyers leave feedback with high enough probability. This is because with time the information available to the intermediary about sellers gets obsolete due to the Markovian sellers' types, in order to compensate this natural decay of information the intermediary needs to get enough new information about sellers through feedback generated by buyers. When λ is too low, the inflow of new information is not enough to compensate for this decay, and trade can not be sustained in the long run.

Similarly, it is easy to check that the optimal threshold policy can be obtained from the expression given in Theorem 2, by replacing m with $m\lambda$

$$\ln \frac{\mu}{\mu - z^*} = \frac{1}{\mu - z^*} \left(z^* + \frac{\varphi}{m\lambda} \frac{z^* \theta^H - 1}{\theta^H - 1} \right). \tag{76}$$

To sum up, our results can be easily generalized to situations when buyers do not always leave feedback, and do so with certain probability λ. Most of our results are qualitatively the same, and do not depend critically on λ, except for the case when the average quality of sellers is low ($\mu\theta^H < 1$). In this case, stationary equilibrium with trade can only be implemented when $\lambda > \lambda^*$.

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