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HOW NOISE AFFECTS EFFORT IN TOURNAMENTS

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Abstract

It is commonly understood that making a tournament ranking process more noisy leads to a reduction in effort exerted by players in the tournament. But what exactly does it mean to have "more noise?" We address this question and show that the level of *risk*, as measured by the variance or the second-order stochastic dominance order, is not the answer, in general. For rank-order tournaments with arbitrary prizes, equilibrium effort decreases as noise becomes more dispersed, in the sense of the *dispersive order*. For winner-take-all tournaments, we identify a weaker version of the dispersive order we call *quantile stochastic dominance*, as well as other orders and entropy measures linking equilibrium effort and noise.

Keywords: tournament, noise, dispersive order, quantile stochastic dominance, entropy.

JEL codes: C72, D72, D82.

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1 Introduction

A *tournament* takes place whenever agents make irreversible investments of effort or other resources trying to win a valuable prize. Examples include R&D competition, research grant applications, competition for promotion and bonuses in organizations, rent-seeking, and political campaigns. In many cases, the relationship between the agents' investments and tournament outcomes is *noisy*, in the sense that agent i investing more than agent j does not guarantee that i will be ranked above j as a result. Noise comes from many sources: Researchers experience random arrival of ideas, salespeople are subject to demand shocks, grant applications are evaluated by random reviewers, and money buys a lot but certainly not everything in politics. The effect of noise is especially critical in environments where agents are symmetric or nearly symmetric in their ability – in these settings, in equilibrium, noise is *the only* determinant of success.

In this paper, we study how the amount, or intensity, of noise in the ranking process affects effort in tournaments. We use a variant of the classic Lazear and Rosen (1981) rank-order tournament model with symmetric players. The players' output is given by effort distorted by additive noise, and the players receive fixed prizes based on the ranking of their output levels.[1] There is a common understanding that noise has a negative impact on incentives: The more uncertainty there is in the mapping between one's investment and relative position, i.e., the more rewards are determined by pure luck, the less sense it makes to invest in the tournament. But what does it mean to have "more noise?" Noise intensity is typically associated with the level of *risk*, which is determined by variance or, more generally, the second-order stochastic dominance (SOSD) order. Indeed, in several easy-to-compute examples such as when noise has the normal or uniform distribution, equilibrium effort is decreasing in the variance. The original Tullock (1980) paper on rent-seeking contests shows that a lower parameter r – the "discriminatory power" of the contest – leads to a lower equilibrium effort. A lower r also corresponds to a higher variance of the underlying Gumbel distribution of noise.

Consider, however, the example in Figure 1. The triangular distribution (left) has a lower variance than the absolute value distribution (center), and dominates it in the sense of the SOSD order; yet, the two distributions result in *the same* symmetric equilibrium

[1] The widely used contest model of Tullock (1980) is a special case (Fu and Lu, 2012; Jia, Skaperdas and Vaidya, 2013; Ryvkin and Drugov, 2020), with the noise following the extreme value type-I distribution (also known as the Gumbel distribution). The discriminatory power r in the Tullock (1980) contest success function (CSF) $p_i = \frac{e_i^r}{\sum_{j=1} e_j^r}$ is a parameter of this distribution. For the analysis of multi-prize Tullock contests, see Clark and Riis (1996), Fu and Lu (2012) and Schweinzer and Segev (2012).

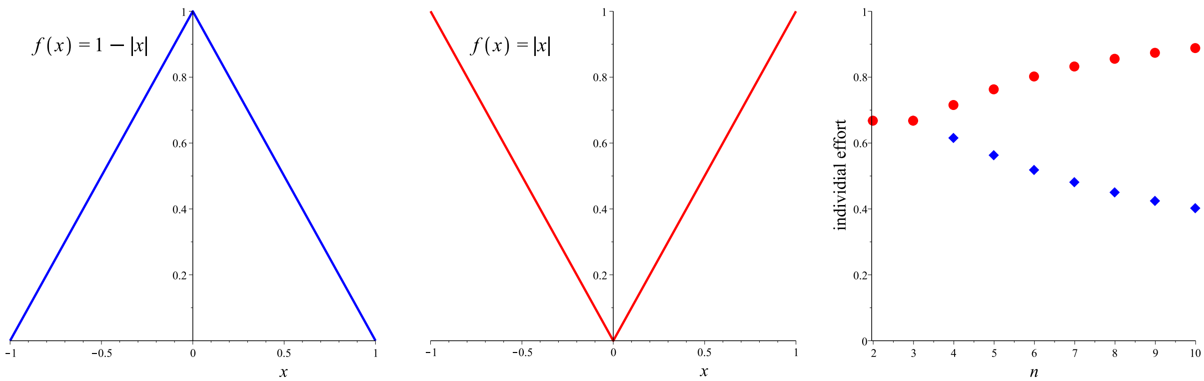

Figure 1: *Left*: The triangular distribution with pdf $f(x) = 1 - |x|$ on $[-1, 1]$. *Center*: The absolute value distribution with pdf $f(x) = |x|$ on $[-1, 1]$. *Right*: Equilibrium efforts for winner-take-all tournaments of n players, with prize normalized to one and cost of effort $\frac{e^2}{2}$, for the triangular distribution (blue diamonds) and the absolute value distribution (red circles). For $n = 2$ and 3, the efforts are the same for the two distributions.

efforts in a winner-take-all (WTA) tournament of two or three players. Even more strikingly, the triangular distribution generates *lower* efforts in WTA tournaments with $n \geq 4$ players despite having a lower variance.[2]

We show that, in general, the relevant stochastic order for the effect of noise on effort in tournaments is the *dispersive order* (Lewis and Thompson, 1981). It is stronger than the SOSD order and, in particular, is preserved by order statistics. Our main result is that the dispersive order is both necessary and sufficient to unambiguously rank noise distributions in terms of the equilibrium efforts they generate. In other words, noise X generates higher equilibrium efforts than noise Y for any number of players and any prize structure (subject to restrictions on the equilibrium existence) if and only if X dominates Y in the dispersive order.

To get a rough intuition for the role of the dispersive order, consider a "large contest" version of the model where a fraction α of best-performing players win a prize normalized to 1.[3] Let $y_i = e_i + X_i$ denote the output of player i, where e_i is the player's effort, and X_i is the noise term that is i.i.d. across players with a cdf $F(\cdot)$ and pdf $f(\cdot)$. In a symmetric equilibrium, the probability of winning is equal to α for all the players, and $\alpha = 1 - F(\theta - e^*)$, where θ is the endogenous output threshold needed to win, and e^* is the symmetric equilibrium effort. The marginal return to effort in the equilibrium is

[2] We come back to this example in Section 4.

[3] In a large contest, an individual player's choice of effort does not affect other players' probability of winning. We are grateful to an anonymous referee for this intuition. For the analysis of large contests, see, e.g., Olszewski and Siegel (2016).

then $f(\theta - e^*) = f(F^{-1}(1-\alpha))$, where $F^{-1}(\cdot)$ is the quantile function of noise. Function $f(F^{-1}(z))$ is known as the inverse quantile density (Parzen, 1979), and it is ranked by the dispersive order for all $z \in (0,1)$ (see, e.g., Shaked and Shanthikumar, 2007). Hence, in large contests the dispersive order ranks equilibrium efforts for all α. We show that this result continues to hold for any (finite) number of players and any prize structure in the tournament.

For a fixed prize schedule – such as WTA – the necessity part does not apply. For WTA tournaments, we provide a number of results involving weaker stochastic orders and *entropy* characterizations. In particular, we show that equilibrium effort is determined by the entropy of $X_{\frac{n}{2}:\frac{n}{2}}$ – the highest order statistic of noise from the sample of size $\frac{n}{2}$, i.e., half the original number of players. A similar result holds more generally for tournaments involving two distinct prizes. The entropy in question is H_2 – the Rényi entropy of order 2, also known as "collision entropy" (Rényi, 1961).[4] In the case of two players, it is the entropy of the original noise distribution, which explains why the two distributions in the example above generate the same effort when $n = 2$. Indeed, entropy is invariant to reshuffling of realizations, and the two distributions in Figure 1 are each other's re-arrangements. We also introduce a new *spread order* which is linked to the ranking of H_2, and hence to the ranking of equilibrium effort.

As the number of players increases, the ranking of distributions can change since the pdf of $X_{\frac{n}{2}:\frac{n}{2}}$ may no longer be a re-arrangement of the pdf of $Y_{\frac{n}{2}:\frac{n}{2}}$ even if the pdf of X is a re-arrangement of the pdf of Y. The presence of order statistics makes various parts of the support of the distribution matter differently. We identify a new class of progressively weaker *quantile stochastic dominance* (QSD) orders that are similar to the standard FOSD, SOSD, and higher orders of stochastic dominance but apply to inverse quantiles densities. In particular, first-order QSD can be interpreted as the *upper-tail conditional entropy* order that ranks entropy H_2 in the upper tail of the distribution of noise. The concept of QSD also helps explain the diverging effects of risk and entropy on equilibrium effort, especially as n increases. Risk, or the standard SOSD order, measures variability relative to the mean; whereas entropy is a measure of variability over the entire support of the distribution. In the symmetric equilibrium, player i wins a WTA tournament if her noise realization, X_i, surpasses $X_{n-1:n-1}$ – the largest of the noise realizations of the other $n-1$ players. For $n = 2$, this is equivalent to simply surpassing X; hence, the effect is determined by entropy H_2 over the entire range of noise. As n

[4] The more standard entropy in information theory is Shannon entropy, which is the Rényi entropy of order 1.

increases, the distribution of $X_{n-1:n-1}$ shifts to the upper tail, and entropy over the upper tail of the support becomes dominant.

As we discuss below in the literature review section, the majority of imperfectly discriminating contest models in the literature rely, sometimes implicitly, on a noise distribution that has a scale parameter, such as the uniform, normal or Gumbel distribution. Variation in scale leads to the dispersive order, but also to the SOSD order and associated changes in the variance. The latter may have created an impression in the literature that it is the level of *risk* that determines equilibrium effort. However, we show that effort is instead determined by *informational* properties of noise. These characteristics are distinct from risk and may or may not change in the same direction.

Relation to prior literature We focus here on "imperfectly discriminating" models of tournaments – most notably, those originating from the seminal contributions of Tullock (1980) and Lazear and Rosen (1981).[5] In a generic model of this sort, a player's output can be represented as $y_i = \varphi(e_i, X_i)$, where e_i is player i's costly investment (or effort), X_i is an idiosyncratic random shock, and φ is a "production function" increasing in both arguments. The most common functional forms for φ are additive and multiplicative. The additive version gives rise to the Lazear and Rosen (1981) model, while the multiplicative one can be reduced to it by taking logs and appropriately transforming the distribution of noise and the cost function.

There are relatively few papers focusing specifically on the effect of noise on equilibrium effort in tournaments. The most basic result follows already from the original Tullock (1980) paper showing that equilibrium effort in a symmetric contest with the probability of player i winning given by the contest success function (CSF) $p_i = \frac{e_i^r}{\sum_{j=1} e_j^r}$ increases in r – the discriminatory power of the contest. For contests with endogenous entry, with the same CSF, Fu, Jiao and Lu (2015) show that an increase in r may lead to a reduction in the number of entrants, thus introducing a trade-off between the intensive and extensive margins of effort provision. As a result, there is an interior optimum level of discriminatory power maximizing aggregate effort. Both of these papers only consider scenarios with pure strategy bidding, and thus their results apply for r not too large. Wang (2010) analyzes two-player Tullock contests of heterogeneous players and arbitrary r, including the range of mixed equilibria, and shows that an interior value of r maximizes aggregate

[5] For a recent review see, e.g., Konrad (2009), Congleton, Hillman and Konrad (2008), Corchón (2007), Connelly et al. (2014). Parallel to this literature there is a branch studying "perfectly discriminating" contests or all-pay auctions without noise (e.g., Hillman and Riley, 1989; Baye, Kovenock and De Vries, 1996; Moldovanu and Sela, 2001; Siegel, 2009).

effort, and more noise (a lower r) is optimal as the players become more heterogeneous.[6] Finally, Morgan, Tumlinson and Vardy (2018) consider a more general Lazear-Rosen tournament model with the pdf of noise parameterized as $f(x, \sigma) = \frac{1}{\sigma} f(\frac{x}{\sigma}, 1)$, where the scaling parameter σ is a measure of noise intensity. Extending the analysis also to mixed equilibria, they show that aggregate equilibrium effort is single-peaked in σ, and the optimal (positive) σ corresponds to the smallest possible noise that still allows for the symmetric pure strategy equilibrium with full participation.[7] A common feature of these studies is that, effectively, they all consider variation in noise intensity in terms of a single parameter – the scale of the distribution of noise, as in Morgan, Tumlinson and Vardy (2018). Indeed, discriminatory power r in Tullock contests is the inverse scale parameter of the corresponding type I extreme value (or Gumbel) distribution. Incidentally, an increase in the scale parameter also implies an increase in variance and a decrease in the SOSD order and the dispersive order.

In this paper, we restrict attention to settings where noise is "large enough" so that the symmetric pure strategy equilibrium in pure strategies exists.[8] In contrast to prior studies, we consider the effects of more general changes in the distribution of noise, including changes in its *shape*. We identify a number of uncertainty orders for the distribution of noise that lead to a ranking of equilibrium effort. Gerchak and He (2003) provide an important first step in this direction, noting that the in two-player tournaments the equilibrium effort is determined by the collision entropy of noise. However, their result is restricted to two-player tournaments, and they do not relate it to any uncertainty orders.

Finally, while we are unaware of the use of the dispersive order in the contest literature, there are several recent applications in the auction theory literature (see, e.g., Ganuza and Penalva, 2010; Kirkegaard, 2012).

The rest of the paper is organized as follows. Section 2 sets up the model. Section 3 provides results for general prize schedules. Section 4 looks at winner-take-all tournaments. Section 5 discusses several extensions. Section 6 concludes. Missing proofs are contained in the Appendix.

[6] The existence and properties of mixed-strategy equilibria arising in contests with small noise (or large discriminatory power) have been studied more comprehensively by Ewerhart (2015) for Tullock contest and by Ewerhart (2017) for general imperfectly discriminating contests.

[7] Strictly speaking, this result is only demonstrated for Lazear-Rosen tournaments with $n = 2$ players and Tullock contests with any number of players. Otherwise, Morgan, Tumlinson and Vardy (2018) use the "large contest" approximation where each player's effort does not affect the ranking of others.

[8] When there is too little noise in the tournament, the symmetric pure strategy equilibrium no longer exists and effort goes down due to players dropping out (Morgan, Tumlinson and Vardy, 2018). We focus on the effect of adding noise to a tournament with full participation.

2 Model setup

We consider a tournament of $n \geq 2$ identical players indexed by $i = 1 \ldots, n$. Each player i chooses effort $e_i \in \mathbb{R}_+$ at a cost $c(e_i)$. Function $c(\cdot)$ is continuous on $[0, \bar{e}]$; and strictly increasing, strictly convex and C^2 on $(0, \bar{e}]$, where $\bar{e} = c^{-1}(1)$ exists and is finite. Player i's output is $y_i = e_i + X_i$, where X_i are random shocks, i.i.d. across players, with $\mathrm{E}(X_i) = 0$, support $\mathcal{X} = [\underline{x}, \overline{x}]$ (finite or infinite), absolutely continuous cdf $F(\cdot)$ and continuous, differentiable a.e., and square-integrable pdf $f(\cdot)$. The players are awarded with prizes based on the ranking of their output; that is, the player whose output is ranked r (where $r = 1$ corresponds to the highest output, $r = 2$ to the second highest, etc.) receives a prize with utility V_r.[9] For brevity, we will refer to V_r as "prizes" in what follows. Ties occur with probability zero. Prizes are nonconstant, decreasing in rank and normalized so that $V_1 \geq V_2 \geq \ldots \geq V_n \geq 0$, $V_1 > V_n$, and $\sum_{r=1}^n V_r = 1$.

Assuming all players other than i choose effort e^*, the expected utility of player i from effort e_i is

$$U^{(i)}(e_i, e^*) = \sum_{r=1}^{n} p^{(i,r)}(e_i, e^*) V_r - c(e_i), \tag{1}$$

where $p^{(i,r)}(e_i, e^*)$ – the probability for player i's output to be ranked r – is

$$p^{(i,r)}(e_i, e^*) = \binom{n-1}{r-1} \int F(e_i - e^* + x)^{n-r}[1 - F(e_i - e^* + x)]^{r-1} dF(x).$$

Here and below, integration over $\mathcal{X}$ is implied unless noted otherwise. The symmetric first-order condition, $U^{(i)}_{e_i}(e^*, e^*) = 0$, takes the form

$$c'(e^*) = \sum_{r=1}^{n} \beta_{r,n} V_r, \tag{2}$$

where coefficients $\beta_{r,n} \equiv p^{(i,r)}_{e_i}(e^*, e^*)$ are given by

$$\beta_{r,n} = \binom{n-1}{r-1} \int F(x)^{n-r-1}[1 - F(x)]^{r-2}[n - r - (n-1)F(x)] f(x) dF(x). \tag{3}$$

The solution of Eq. (2) (which is unique if it exists) is the only candidate for the symmetric equilibrium effort level e^*. Sufficient conditions on the primitives of the model that

[9] Thus, we allow for risk-averse players with utility separable in prizes and effort costs. See Drugov and Ryvkin (2019) for the analysis of optimal prize allocation in this case.

guarantee the equilibrium existence are discussed in detail by Drugov and Ryvkin (2018). From this point on, we assume that those conditions are satisfied, and the e^* solving (2) is indeed the equilibrium.[10]

For a fixed prize schedule, equilibrium effort e^* is affected by noise through coefficients $\beta_{r,n}$, Eq. (3). These coefficients can be positive or negative, and have the property $\sum_{r=1}^{n}\beta_{r,n} = 0$. It is, therefore, convenient to introduce cumulative coefficients $B_{r,n} = \sum_{k=1}^{r}\beta_k$ and utility differentials $D_r = V_r - V_{r+1} \geq 0$ (for $r = 1, \ldots, n-1$) and apply "summation by parts" to the right-hand side of (2). Equation (2) then becomes

$$c'(e^*) = \sum_{r=1}^{n-1} B_{r,n} D_r. \tag{4}$$

Coefficients $B_{r,n}$ are given by

$$B_{r,n} = \frac{(n-1)!}{(n-r-1)!(r-1)!}\int F(x)^{n-r-1}[1-F(x)]^{r-1}f(x)dF(x), \tag{5}$$

with $B_{n,n} = 0$ and $B_{r,n} > 0$ for all $r = 1, \ldots, n-1$. Three alternative representations of coefficients $B_{r,n}$ are useful in what follows.

Inverse quantile density representation Let $F^{-1}(z) = \inf\{x : F(x) \geq z\}$ denote the left-continuous quantile function of noise, and let $m(z) = f(F^{-1}(z))$ denote the inverse quantile density function. Through the probability integral transformation, $z = F(x)$, coefficients $B_{r,n}$ can be written as

$$B_{r,n} = \frac{(n-1)!}{(n-r-1)!(r-1)!}\int_0^1 z^{n-r-1}(1-z)^{r-1}m(z)dz. \tag{6}$$

Order statistics representation Let $f_{\mu:\nu}(\cdot)$, $0 < \mu \leq \nu$, denote the pdf

$$f_{\mu:\nu}(x) = \frac{\Gamma(\nu+1)}{\Gamma(\mu)\Gamma(\nu-\mu+1)}F(x)^{\mu-1}[1-F(x)]^{\nu-\mu}f(x). \tag{7}$$

Here, $\Gamma(\mu) = \int_0^\infty t^{\mu-1}\exp(-t)dt$ is the gamma function defined for $\mu > 0$ and equal to $(\mu-1)!$ when μ is a positive integer. For positive integer μ and ν, Eq. (7) gives the pdf

[10]The key restrictions on the primitives are as follows: (i) function $c(\cdot)$ has a second derivative bounded from below; that is, there exists a $c_0 > 0$ such that $c''(e) \geq c_0$ on $[0, \bar{e}]$; (ii) pdf $f(\cdot)$ and its derivative, $|f'(\cdot)|$, are bounded by $f_m, f'_m < \infty$, respectively; and (iii) c_0 is "large enough" compared to f_m and f'_m. For details, see Drugov and Ryvkin (2018).

of the μ-th order statistic in a sample of ν i.i.d. draws of X. However, (7) is a valid pdf also when μ and ν are not integer. Equations (5) and (6) then give

$$B_{r,n} = \int f_{n-r:n-1}(x)dF(x) = \int_0^1 f^B(z; n-r, r)m(z)dz, \tag{8}$$

where $f^B(z;\mu,\nu)$ is the pdf of the beta distribution with parameters (μ,ν), which is also the pdf of order statistic $(\mu:\nu)$ of the uniform distribution on $[0,1]$.

Entropy representation For a random variable with square-integrable pdf $f(\cdot)$, the Rényi entropy of order 2, also known as "collision entropy," is defined as $H_2[f] = -\log\left[\int f(x)^2dx\right]$ (Rényi, 1961).[11] For brevity, we will refer to H_2 simply as "entropy" in what follows. Equation (5) can then be written in the form

$$B_{r,n} = A_{r,n}\int f_{\frac{n-r+1}{2}:\frac{n}{2}}(x)^2dx = A_{r,n}\exp\left(-H_2\left[f_{\frac{n-r+1}{2}:\frac{n}{2}}\right]\right), \tag{9}$$

$$A_{r,n} = \frac{(n-1)!}{(n-r-1)!(r-1)!}\left[\frac{\Gamma(\frac{r+1}{2})\Gamma(\frac{n-r+1}{2})}{\Gamma(\frac{n}{2}+1)}\right]^2.$$

3 General prize schedules

In this section we consider arbitrary prize schedules. We start by introducing the dispersive order, and then formulate our main result: The dispersive order is both necessary and sufficient to unambiguously rank noise distributions in terms of the equilibrium efforts they generate (Proposition 1). We then consider two common special cases of the dispersive order – stretching and scaling transformations, the latter being hitherto the primary case discussed in the literature. We finish the section by discussing two-prize schedules, which emerge as optimal prize schedules for risk-neutral players.

We will use subscripts such as X or Y to denote objects pertaining to different random variables representing the noise. The *dispersive order* (Lewis and Thompson, 1981) is defined as follows.

Definition 1 *X is more dispersed than Y in the dispersive order if for all $z, z' \in [0,1]$ such that $z' > z$*

$$F_X^{-1}(z') - F_X^{-1}(z) \geq F_Y^{-1}(z') - F_Y^{-1}(z),$$

and the inequality is strict in some subset of $[0,1]$ of positive measure.

[11] The general expression for the Rényi entropy of order α is $H_\alpha[f] = \frac{1}{1-\alpha}\log\left[\int f(x)^\alpha dx\right]$.

The definition is rather intuitive: X is more dispersed than Y if the distance between any two quantiles of X is at least as large as the distance between the same quantiles of Y. As discussed by Shaked and Shanthikumar (2007), whenever X is more dispersed than Y, $\mathrm{Var}(X) \geq \mathrm{Var}(Y)$; the converse, however, is not true. Similarly, the dispersive order for variables with equal means implies SOSD, but the converse is not true.

The following proposition shows that the dispersive order is a necessary and sufficient condition for the ranking of equilibrium efforts in general.[12]

Proposition 1 *$e^*_X < e^*_Y$ for all admissible prize schedules and tournament sizes n if and only if X is more dispersed than Y in the dispersive order.*

The "if" part of Proposition 1 follows directly from representation (6) of coefficients $B_{r,n}$ and the fact that X being more dispersed than Y in the dispersive order is equivalent to $m_X(z) \leq m_Y(z)$ for all $z \in [0, 1]$ (Shaked and Shanthikumar, 2007). The "only if" part also uses representation (6). In the proof, we show that for any two random variables X and Y such that $m_X(z) > m_Y(z)$ in some interval, a prize scheme can be constructed such that for a sufficiently large n this interval plays the dominant role in determining equilibrium effort.

An important special case which satisfies the dispersive order, allows for an explicit characterization of equilibrium effort, and incorporates several important examples is when additional dispersion is generated by a "stretching" transformation.

Definition 2 *Function $\phi : \mathcal{X} \to \mathbb{R}$ is a stretching transformation if it is strictly monotone and for any $x, x' \in \mathcal{X}$ such that $x' > x$*

$$|\phi(x') - \phi(x)| \geq x' - x,$$

with strict inequality in some subset of $\mathcal{X}$ of positive measure.

For differentiable functions, Definition 2 is equivalent to the requirement that $|\phi'(x)| \geq 1$ for all $x \in \mathcal{X}$, with strict inequality in some subset of $\mathcal{X}$ of positive measure.

Proposition 2 *Suppose $X = \phi(Y)$ where ϕ is a stretching transformation. Then X is more dispersed than Y and hence $e^*_X < e^*_Y$.*

[12] Proposition 1 also applies to tournaments with a stochastic number of players, see Section 5.1.

For an explicit characterization (and a proof) for a differentiable ϕ, notice that $F_X(x) = F_Y(\phi^{-1}(x))$, which implies $f_X(x) = \frac{1}{|\phi'(\phi^{-1}(x))|} f_Y(\phi^{-1}(x))$ and $F_X^{-1}(z) = \phi(F_Y^{-1}(z))$; therefore,

$$m_X(z) = f_X(F_X^{-1}(z)) = \frac{1}{|\phi'(F_Y^{-1}(z))|} m_Y(z) \leq m_Y(z),$$

with strict inequality in some subset of $[0,1]$ of positive measure. The result follows immediately from representation (6).

A straightforward example of stretching is a scaling transformation, $X = \sigma Y$ with $\sigma > 1$ (cf. Theorem 3.B.4 in Shaked and Shanthikumar, 2007). A parameterized cdf $F(x, \sigma)$ is said to have a scale parameter σ if it satisfies $F(x, \sigma) = F(\frac{x}{\sigma}, 1)$. The corresponding scaled pdf is $f(x, \sigma) = \frac{1}{\sigma} f(\frac{x}{\sigma}, 1)$. For example, the standard deviation of a zero-mean normal distribution, the length of the support of a uniform distribution, the expected value of an exponential distribution and the scale of the Gumbel distribution (and hence $\frac{1}{r}$, where r is the discriminatory power of the Tullock contest) are scale parameters. Explicitly, in the case of scaling $\phi'(x) = \sigma$, which gives $m_X(z) = \frac{1}{\sigma} m_Y(z)$; therefore, equilibrium effort is decreasing in σ.

Let us finish this section by discussing two-prize schedules, $V_1 = \ldots = V_s > V_{s+1} = \ldots = V_n$ for some s; that is, ranks $r = 1, \ldots, s$ receive the same high prize V_1 and ranks $r > s$ receive the same low prize V_{s+1}. Such prize schedules are optimal for risk-neutral players, with the location of s determined by the properties of the failure (or hazard) rate of the distribution of noise (Drugov and Ryvkin, 2018). Two-prize schedules may also be optimal for risk-averse players under some conditions. For example, the extreme punishment schedule, with $s = n - 1$, is optimal when the distribution of noise is DFR (has a decreasing failure rate) (Drugov and Ryvkin, 2019). The winner-take-all (WTA) prize schedule, with $s = 1$, is optimal under a wide range of conditions as well.

Let $V^{(s,n)}$ denote a two-prize schedule in a tournament of n players with s high prizes. In this case, $D_s > 0$ and $D_r = 0$ for all $r \neq s$; therefore, the equilibrium effort is determined by a single coefficient $B_{s,n}$, cf. (4), and hence by a single entropy of order statistic $X_{\frac{n-s+1}{2}:\frac{n}{2}}$, cf. (9).[13]

Consider a sequence of tournaments with increasing n and two-prize schedules $V^{(s_n,n)}$

[13]When support $[\underline{x}, \overline{x}]$ is finite, the entropy reaches its maximum for the uniform distribution. Hence, the effort-*minimizing* distribution of noise is the one with $X_{\frac{n-s+1}{2}:\frac{n}{2}}$ uniform; that is, its cdf $F_{\min}(x)$ satisfies $F^B\left(F_{\min}(x); \frac{n-s+1}{2}, \frac{s+1}{2}\right) = \frac{x-\underline{x}}{\overline{x}-\underline{x}}$, where $F^B(z; \mu, \nu)$ is the regularized incomplete beta function, or the cdf of the beta distribution with parameters (μ, ν) (Paris, 2010). For example, for WTA tournaments $(s = 1)$, $F_{\min}(x) = \left(\frac{x-\underline{x}}{\overline{x}-\underline{x}}\right)^{\frac{2}{n}}$.

such that $\frac{s_n}{n} = \alpha \in (0,1)$. Under these two-prize schedules, a fixed fraction α of top performers, or players in the $(1-\alpha)$-th quantile, receive a high prize. Representation (8) shows that $B_{s_n,n} = \mathrm{E}(m(Z_{n-s_n:n-1}))$, where $Z_{n-s_n:n-1}$ is the order statistic of the uniform distribution on $[0,1]$. For $s_n = \alpha n$, these order statistics are asymptotically normally distributed with mean $1-\alpha$ and variance $\frac{(1-\alpha)\alpha}{n}$ (Arnold, Balakrishnan and Nagaraja, 1992); therefore, $B_{s_n,n} \to m(1-\alpha)$. Thus, in large two-prize tournaments the equilibrium effort is determined by the inverse quantile density of noise at $1-\alpha$.

4 Winner-take-all tournaments

In this section we study the effect of noise on effort in winner-take-all tournaments. A restriction to a particular prize schedule allows for conditions weaker than the dispersive order to guarantee a ranking of equilibrium effort. We start by showing that the entropy representation (9) has an intuitively appealing interpretation. Then, noting that the ranking of the relevant entropy may change with the number of players, we introduce *quantile stochastic dominance* – new stochastic orders which rank equilibrium effort for any number of players. Finally, we introduce yet another stochastic order for the case of two players.

Winner-take-all (WTA) tournaments are characterized by two-prize schedules $V^{(1,n)}$, with only the best performer receiving the high prize. WTA schedules are optimal in small tournaments with symmetrically distributed noise (Krishna and Morgan, 1998), as well as in tournaments of arbitrary sizes where noise is sufficiently light-tailed (Drugov and Ryvkin, 2018, 2019). WTA schedules also emerge naturally in many environments where prize sharing is impossible for institutional reasons; for example, there may be only one job vacancy or one managerial position at a certain level in an organization, or only one contractor is needed to complete a competitively allocated project.

In WTA tournaments, equilibrium effort is determined by coefficient $B_{1,n}$, cf. Eq. (4). We will sometimes use $B_{1,n}[f]$ to explicitly denote the coefficient $B_{1,n}$ corresponding to a noise distribution with pdf $f(\cdot)$. From (9), the entropy representation of $B_{1,n}$ is

$$\begin{aligned} B_{1,n}[f] &= \frac{4(n-1)}{n^2} \int \left[\frac{n}{2} F(x)^{\frac{n}{2}-1} f(x)\right]^2 dx = \frac{4(n-1)}{n^2} B_{1,2}[f_{\frac{n}{2}:\frac{n}{2}}] \\ &= \frac{4(n-1)}{n^2} \exp(-H_2[f_{\frac{n}{2}:\frac{n}{2}}]). \end{aligned} \tag{10}$$

Thus, coefficient $B_{1,n}$ in a tournament of n players can be represented as an appropriately rescaled coefficient $B_{1,2}$ in a tournament of two symmetric players each having the cdf of noise $F_{\frac{n}{2}:\frac{n}{2}}(x)$. The latter coefficient can then be expressed through the entropy of pdf $f_{\frac{n}{2}:\frac{n}{2}}$.

Lemma 1 *In a WTA tournament of n players, equilibrium effort decreases in the entropy of a distribution with pdf $f_{\frac{n}{2}:\frac{n}{2}}$.*

Representation (10) and Lemma 1 have an appealing interpretation when n is even. Instead of the original n-player tournament, consider a tournament of two players where each player has access to $\frac{n}{2}$ independent draws from the original noise distribution and selects the highest draw.[14] Another, though less precise, interpretation is that the n players are split arbitrarily into two equal subgroups with $\frac{n}{2}$ players each. Then $f_{\frac{n}{2}:\frac{n}{2}}$ is the pdf of noise of the two players whose shocks are the largest in each subgroup, and the player with a larger shock between these two wins the tournament.

While intuitively appealing, Lemma 1 is not very useful in practice because entropy ordering is not preserved by order statistics. That is, if $n = 2$ and $H_2[f_X] > H_2[f_Y]$ (and hence $e^*_X < e^*_Y$ for $n = 2$), it does not imply that $H_2[f_{X_{\frac{n}{2}:\frac{n}{2}}}] > H_2[f_{Y_{\frac{n}{2}:\frac{n}{2}}}]$ (and $e^*_X < e^*_Y$) for $n > 2$.

While the sufficiency part of Proposition 1 still applies,[15] the dispersive order is relatively strong, and in many cases of interest it does not rank distributions. For example, two (different) distributions cannot be ranked by the dispersive order when they have the same finite support (Theorem 3.B.14. in Shaked and Shanthikumar, 2007). However, the restriction to WTA prizes schedules also allows for the development of other, weaker methods of comparing the entropy of order statistics.

4.1 Quantile stochastic dominance orders

The standard FOSD and SOSD orders represent the first two levels in the hierarchy of progressively weaker stochastic orders of random variables (Marshall, Olkin and Arnold, 2011). These orders can be defined through inequalities for cdfs and their integrals. Here, we introduce a similar hierarchy of *quantile stochastic dominance* orders that progressively relax the dispersive order and are based on inequalities for inverse quantile densities and their integrals.

[14]This is the case in some Olympic sports where participants have several attempts and choose the best result, such as discus throw, shot put, javelin throw, long jump, triple jump, etc.

[15]The necessity part of Proposition 1 does not apply anymore because the prize structure is fixed.

For a random variable X, let $\bar{M}_X^{(0)}(z) = m_X(z)$, and recursively define functions $\bar{M}_X^{(k)}(z) = \int_z^1 \bar{M}_X^{(k-1)}(t)dt$. Function $\bar{M}_X^{(1)}(z) = \int_z^1 m_X(t)dt$ is the quantile analog of survival function $\bar{F}_X(x) = 1 - F_X(x)$; however, since the inverse quantile density $m_X(z)$ is not normalized, the value of $\bar{M}_X^{(1)}(0) = \exp(-H_2[f_X])$ is determined by the entropy and can differ across random variables.

Definition 3 *Y dominates X in quantile stochastic dominance of order k (QSD(k)) if $\bar{M}_Y^{(k)}(z) \geq \bar{M}_X^{(k)}(z)$ for all $z \in [0,1]$.*

QSD(0) is the dispersive order; QSD(1) and QSD(2) are similar to the FOSD and SOSD orders, respectively. It is easy to see that the orders are progressively weaker: If Y dominates X in QSD(k) than it also dominates X in QSD(k') for all $k' > k$.

Proposition 3 *Suppose there exists a $k \geq 1$ such that*
(a) $\bar{M}_Y^{(l)}(0) \geq \bar{M}_X^{(l)}(0)$ for all $l = 1, \ldots, k-1$;
(b) Y dominates X in QSD(k).
Then $e_Y^* \geq e_X^*$ in WTA tournaments for any $n \geq 2$.

To understand how Proposition 3 works, suppose first that $n = 2$. Then we have $B_{1,2}[f_Y] - B_{1,2}[f_X] = \int_0^1 [m_Y(z) - m_X(z)]dz$. The dispersive order requires that $\Delta m(z) = m_Y(z) - m_X(z) \geq 0$ for all z, but in fact $e_Y^* \geq e_X^*$ will hold even if $\Delta m(z)$ changes sign multiple times, as long as $\int_0^1 \Delta m(z)dz \geq 0$. For a general $n \geq 2$, we have $B_{1,n}[f_X] = (n-1)\int_0^1 m_X(z)z^{n-2}dz$, cf. (6). Thus, $e_Y^* \geq e_X^*$ for all $n \geq 2$ if and only if

$$\int_0^1 m_Y(z)z^{n-2}dz \geq \int_0^1 m_X(z)z^{n-2}dz \quad \forall n \geq 2. \tag{11}$$

Inequality (11) has the form of an inequality between two expectations of z^{n-2} with different unnormalized densities, m_X and m_Y. Note that $m_X(z) = -\bar{M}_X^{(1)\prime}(z)$; integrating (11) by parts, obtain

$$[\bar{M}_Y^{(1)}(0) - \bar{M}_X^{(1)}(0)]\mathbb{1}_{n=2} + (n-2)\int_0^1 [\bar{M}_Y^{(1)}(z) - \bar{M}_X^{(1)}(z)]z^{n-3}dz \geq 0 \quad \forall n \geq 2. \tag{12}$$

As seen from (12), QSD(1) is sufficient for any $n \geq 2$. Moreover, for $n > 2$ the first term in (12) is zero, and the second term has exactly the same structure as (11), with m replaced by $\bar{M}^{(1)}$. Integrating by parts again will produce a similar structure involving $\bar{M}^{(2)}$, etc., while terms in condition (a) of Proposition 3 will arise due to boundary conditions.

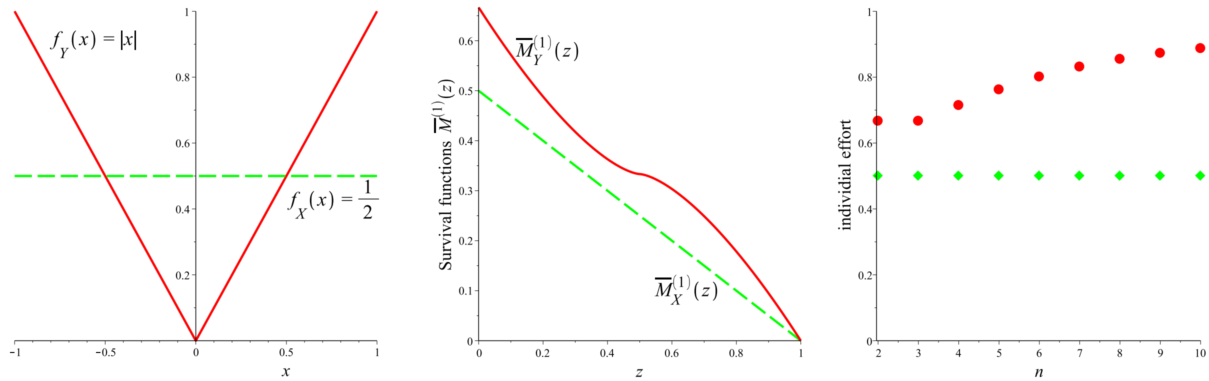

Figure 2: Example of QSD(1). *Left*: The pdfs of the absolute value (solid red line) and uniform distributions (dashed green line). *Center*: Quantile survival functions. *Right*: Equilibrium efforts for winner-take-all tournaments of n players, with prize normalized to one and cost of effort $\frac{e^2}{2}$, for the uniform distribution (green diamonds) and the absolute value distribution (red circles).

Proposition 3 provides a sufficient condition for the ranking of equilibrium efforts for any $n \geq 2$. For a given n, a weaker condition can be formulated.

Corollary 1 *Suppose any one of the following conditions holds:*
(a) $\bar{M}_Y^{(n-1)}(0) \geq \bar{M}_X^{(n-1)}(0)$*;*
(b) Y *dominates* X *in QSD(*k*) for some* $k \leq n-1$*.*
Then $e_Y^* \geq e_X^*$ *in WTA tournaments with* n *players.*

Condition (a) in Corollary 1 can be obtained by integrating (11) by parts $n-2$ times, whereas condition (b) is stronger and sufficient for (a).

Example of QSD(1) Consider a uniform distribution, $f_X(x) = \frac{1}{2}\mathbb{1}_{[-1,1]}(x)$, and the absolute value distribution, $f_Y(x) = |x|\mathbb{1}_{[-1,1]}(x)$, see Figure 2. Since the two distributions have the same finite support, the dispersive order does not rank them (the inverse quantile density functions, $m_X(z) = \frac{1}{2}$ and $m_Y(z) = \sqrt{|2z-1|}$, intersect). However, Y dominates X in QSD(1). Indeed, computing the survival functions yields

$$\bar{M}_X^{(1)}(z) = \frac{1-z}{2}, \quad \bar{M}_Y^{(1)}(z) = \begin{cases} \frac{1+(1-2z)^{2/3}}{3}, & z \leq \frac{1}{2} \\ \frac{1-(2z-1)^{2/3}}{3}, & z > \frac{1}{2} \end{cases}$$

and $\bar{M}_Y^{(1)}(z) \geq \bar{M}_X^{(1)}(z)$ for all $z \in [0,1]$. With QSD(1) in place, condition (a) of Proposition 3 is void; hence, $e_Y^* \geq e_X^*$ in WTA tournaments for any $n \geq 2$, cf. Figure 2.

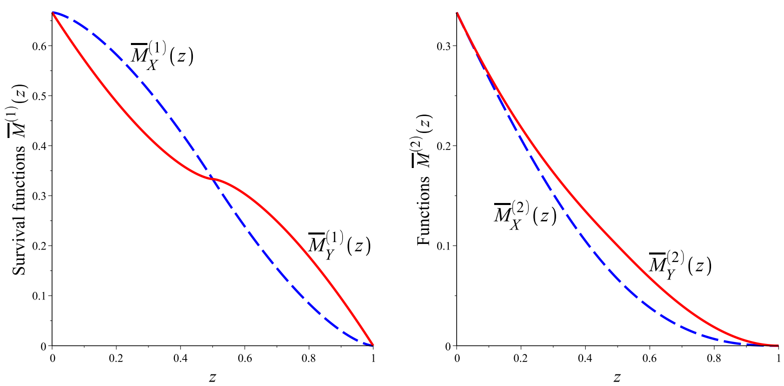

Figure 3: Example of QSD(2): the triangular distribution with pdf $f_X(x) = 1 - |x|$ on $[-1, 1]$ (dashed blue line) and the absolute value distribution with pdf $f_Y(x) = |x|$ on $[-1, 1]$ (solid red line). *Left*: Survival functions $\bar{M}^{(1)}(z)$. *Right*: Functions $\bar{M}^{(2)}(z)$.

Example of QSD(2) Consider the example in the Introduction, with the triangular distribution, $f_X(x) = (1 - |x|)\mathbb{1}_{[-1,1]}(x)$, and the absolute value distribution, $f_Y(x) = |x|\mathbb{1}_{[-1,1]}(x)$, shown in Figure 1. The two distributions are not ranked according to the QSD(0) (the dispersive order) or QSD(1). Indeed, the survival functions

$$\bar{M}_X^{(1)}(z) = \begin{cases} \frac{2-(2z)^{3/2}}{3}, & z \le \frac{1}{2} \\ \frac{(2(1-z))^{3/2}}{3}, & z > \frac{1}{2} \end{cases}, \quad \bar{M}_Y^{(1)}(z) = \begin{cases} \frac{1+(1-2z)^{3/2}}{3}, & z \le \frac{1}{2} \\ \frac{1-(2z-1)^{3/2}}{3}, & z > \frac{1}{2} \end{cases}$$

intersect at $z = \frac{1}{2}$, cf. Figure 3. Hence, compute functions $\bar{M}^{(2)}(z)$:

$$\bar{M}_X^{(2)}(z) = \begin{cases} \frac{1-2z}{3} + \frac{(2z)^{5/2}}{15}, & z \le \frac{1}{2} \\ \frac{(2(1-z))^{5/2}}{15}, & z > \frac{1}{2} \end{cases}, \quad \bar{M}_Y^{(2)}(z) = \frac{|1-2z|^{5/2}}{15} + \frac{4-5z}{15}.$$

As seen from Figure 3, $\bar{M}_Y^{(2)}(z) \ge \bar{M}_X^{(2)}(z)$. It also shows that $\bar{M}_Y^{(1)}(0) = \bar{M}_X^{(1)}(0)$, which is equivalent to the two distributions having the same entropy.[16] Hence, both conditions (a) and (b) of Proposition 3 are satisfied. Then, $e_Y^* \ge e_X^*$ in WTA tournaments for any $n \ge 2$ as we saw in Figure 1.

QSD(1) and upper-tail conditional entropy Order QSD(1) is an appropriately modified version of first-order stochastic dominance that has an interpretation through conditional entropy.

[16] This means that $e_X^* = e_Y^*$ for $n = 2$. As shown by Ryvkin and Drugov (2020), e^* is the same for $n = 2$ and $n = 3$ when the distribution of noise is symmetric, see Figure 1.

Definition 4 *X is more dispersed than Y in the upper-tail conditional entropy order if $H_2[f_{X|X \geq F_X^{-1}(z)}] \geq H_2[f_{Y|Y \geq F_Y^{-1}(z)}]$ for all quantiles $z \in [0, 1]$.*

Lemma 2 *The upper-tail conditional entropy order is equivalent to QSD(1).*

Indeed, the inequality in Definition 4 can be written in the form

$$\bar{M}_Y^{(1)}(z) \geq \bar{M}_X^{(1)}(z) \quad \forall z \in [0, 1], \tag{13}$$

As discussed above, QSD(1) (or the upper-tail conditional entropy order) is weaker than the dispersive order as it allows for multiple sign changes of $\Delta m(z)$. It is necessary, however, that the last sign change be $-+$, because otherwise condition (13) will not hold for z sufficiently close to 1. It is easy to see that when $\Delta m(z)$ is single-crossing $-+$, condition (13) is equivalent to $\int_0^1 \Delta m(z)dz \geq 0$, i.e., to the requirement that $B_{1,2}[f_Y] \geq B_{1,2}[f_X]$.

Corollary 2 *Suppose that $\Delta m(z)$ is single-crossing $-+$ and $B_{1,2}[f_Y] \geq B_{1,2}[f_X]$. Then $B_{1,n}[f_Y] \geq B_{1,n}[f_X]$, and hence $e_Y^* \geq e_X^*$, for any $n \geq 2$.*

The dispersive order condition, $\Delta m(z) \geq 0$, can be thought of as a requirement that conditional entropy over any interval of quantiles is ranked. In contrast, condition (13) only requires that the upper-tail entropies are ranked. QSD of higher orders places even more weight on the upper tail. This is consistent with the intuition that, as n increases, the upper tail of the distribution of noise plays an increasingly important role in determining the equilibrium effort (Ryvkin and Drugov, 2020).

The role of the upper tail can also be understood by considering the marginal benefit of effort in the symmetric equilibrium, $B_{1,n} = \int f_{n-1:n-1}(x)f(x)dx$, cf. Eq. (8). It is equal to the probability density of $X_i - X_{n-1:n-1}$ at zero. Indeed, player i wins the tournament when her effort surpasses $X_{n-1:n-1}$ – the largest shock among the remaining $n-1$ players. As n increases, $X_{n-1:n-1}$ FOSD-shifts towards the upper tail, and dispersion in the upper tail of the distribution determines $B_{1,n}$.

4.2 Spread order for $n = 2$

When there are two players, the equilibrium effort is determined by

$$B_{1,2}[f] = \int f(x)^2 dx = \exp(-H_2[f]). \tag{14}$$

Making the pdf "flatter" in some way should increase its entropy and hence reduce equilibrium effort. Having in mind how, say, the normal distribution changes as its variance changes, consider a partial order that we call *spread*.

Definition 5 *X is a spread of Y (or f_X is a spread of f_Y) if*
(a) f_X and f_Y are unimodal;
(b) there exist $x_1 < x_2$ such that f_X crosses f_Y from above at x_1, then from below at x_2;
(c) the modes of both f_X and f_Y are between x_1 and x_2.

Condition (b) implies that the cdfs F_X and F_Y cross once. Hence, the spread is a special case of single-crossing cdfs that have been studied by Diamond and Stiglitz (1974), Hammond (1974) and Johnson and Myatt (2006), among others. While it may seem intuitive that the spread implies the ranking of entropies, further restrictions are needed for this to be the case in general (see the counterexample in the Appendix). The following Lemma provides a sufficient condition. Denote by $\hat{x}$ the intersection point of the two cdfs where $F_X(\hat{x}) = F_Y(\hat{x})$. It is easy to see that $x_1 \leq \hat{x} \leq x_2$.

Lemma 3 *Suppose that f_X is a spread of f_Y and*

$$f_X(\hat{x}) + f_Y(\hat{x}) \geq 2\max\{f_X(x_1), f_X(x_2)\}. \tag{15}$$

Then $B_{1,2}[f_X] \leq B_{1,2}[f_Y]$ (and $e^*_X \leq e^*_Y$ for $n = 2$).

To understand the intuition for condition (15) note that f_X and f_Y are FOSD-ranked on each side of $\hat{x}$. If both modes coincide with $\hat{x}$, then the two pdfs are monotone on each side of $\hat{x}$, and f_X has a higher entropy than f_Y.[17] Since the modes may not coincide, there are non-monotone parts of the densities, and condition (15) effectively guarantees that they do not contribute enough to reverse the entropy ordering. It is satisfied in several easy-to-check situations, such as when f_X has its mode at $\hat{x}$ or when f_X and f_Y intersect at the same level, $f_X(x_1) = f_X(x_2)$.

[17]Indeed, since f_X and f_Y are increasing and Y FOSD X, for any increasing function $u(x)$ we have $\int_{\underline{x}}^{\overline{x}} f_Y(x)u(x)dx \geq \int_{\underline{x}}^{\overline{x}} f_X(x)u(x)dx$. Using $u(x) = f_Y(x)$, obtain $\int_{\underline{x}}^{\overline{x}} f_Y(x)^2 dx \geq \int_{\underline{x}}^{\overline{x}} f_X(x)f_Y(x)dx$; using $u(x) = f_X(x)$, obtain $\int_{\underline{x}}^{\overline{x}} f_Y(x)f_X(x)dx \geq \int_{\underline{x}}^{\overline{x}} f_X(x)^2 dx$. Combining the two inequalities, obtain the result. This was shown first by Gerchak and He (2003). For symmetric distributions this implies that the *peakedness order* (X is smaller than Y in the peakedness order if $|Y - \hat{x}|$ FOSD $|X - \hat{x}|$, Birnbaum (1948)) leads to the entropy ranking.

5 Extensions

In this Section we briefly discuss three extensions of the initial model: When the number of players is stochastic (Section 5.1), when players are asymmetric (Section 5.2), and when the number of players is endogenous due to free entry (Section 5.3).

5.1 Stochastic number of players

Suppose that the number of players is stochastic, following an arbitrary distribution. Indeed, in many situations the players do not know how many other players are competing. This is the case in open innovation contests such as those conducted under the Longitude Act, the Orteig Prize, or the XPRIZE Foundation. In promotion tournaments, workers may not know how many of their colleagues are considered for the same promotion.

For WTA tournaments, this setting is considered in detail in Ryvkin and Drugov (2020). For our purposes here, suppose that the number of tournament participants, K, is stochastic. The principal commits to a prize allocation rule V contingent on the realized number of participants, $K = k$. That is,

$$V = ((V_{1,1}), (V_{1,2}, V_{2,2}), \dots, (V_{1,n}, \dots, V_{n,n})),$$

where $V_{r,k}$ is the prize allocated to a player ranked r when there are k participants. The usual monotonicity and budget constraints apply, with $V_{1,k} \geq \dots \geq V_{k,k} \geq 0$ and $\sum_{r=1}^{k} V_{r,k} \leq 1$ for all k. Note that a prize schedule independent of the number of participants is a special case of this more general allocation rule. Equation (2) then becomes

$$c'(e^*) = \tilde{\mathrm{E}}_K \sum_{r=1}^{K} \beta_{r,K} V_{r,K}, \tag{16}$$

where $\tilde{\mathrm{E}}_K$ denotes expectation over the number of players from a viewpoint of a participating player.[18] Since each $\beta_{r,k} = B_{r,k} - B_{r-1,k}$ is linear in the inverse quantile density $m(z)$, cf. Eq. (6), the expectation in (16) is affected by the dispersive order in the same way as the right-hand side of (4), i.e., Proposition 1 is still valid.

[18] This distribution is different from the original distribution of K because a participating player knows about her or his own participation (cf., e.g., Harstad, Kagel and Levin, 1990).

5.2 Asymmetric players

In tournament models with additive noise à la Lazear and Rosen (1981), heterogeneous players are typically introduced as having an ability a_i which adds to their effort, so that player i's output is $y_i = e_i + a_i + X_i$. Consider the case of two players and one prize normalized to 1, and let $\Delta a = a_1 - a_2$ denote the difference in abilities. The equilibrium effort is the same for both players, $e_1^* = e_2^* = e^*$, with e^* given by the first-order condition

$$c'(e^*) = \int f(x + \Delta a) f(x) dx = g(\Delta a),$$

where $g(\cdot)$ is the pdf of $X_1 - X_2$, which is symmetric around zero and has a peak at zero.

As an example, suppose that $X_i \sim N(0, \frac{\sigma^2}{2})$ and hence $g(\cdot, \sigma)$ – the corresponding pdf of $X_1 - X_2$ – is $N(0, \sigma^2)$. Let $e^*(\sigma, \Delta a)$ denote the resulting equilibrium effort. For $\sigma' > \sigma$, we have $e^*(\sigma', 0) < e^*(\sigma, 0)$ because an increase in σ leads to the dispersive order, which is sufficient for the symmetric case. It is straightforward to show, however, that $e^*(\sigma', \Delta a) < (>) e^*(\sigma, \Delta a)$ if $(\frac{1}{\sigma^2} - \frac{1}{\sigma'^2})\Delta a^2 < (>) \log \frac{\sigma'}{\sigma}$. In other words, the equilibrium effort increases in σ when players are sufficiently asymmetric; hence, in general, Proposition 1 does not hold for asymmetric players.

This example is generic, in the following sense. For any discrete change in a parameter of the distribution of noise, by continuity $g(\Delta a)$ will move in the same direction as $g(0)$ for a sufficiently small Δa; therefore, the effect will be similar to the case of symmetric players. However, if this change in the parameter does not affect the support of $g(\cdot)$, then necessarily $g(\cdot)$ will change in the other direction in some intervals of its support, i.e., the effect will have the opposite direction for some values of Δa, to keep the total mass equal to 1.

5.3 Endogenous number of players

Let $e^*_{X,n}$ denote the symmetric equilibrium effort in a tournament with total prize money normalized to 1, n players, and noise X. We will assume that the equilibrium exists and the participation constraint is satisfied, $\pi^*_{X,n} = \frac{1}{n} - c(e^*_{X,n}) \geq 0$, for all $n \leq N$. Here, $N \geq 2$ is the number of potential tournament participants. Suppose also that $\pi^*_{X,n}$ is decreasing in n for $n \leq N$.

Consider a two-stage game where at the first stage the N potential players simultaneously and independently decide whether to enter the tournament, or to stay out and receive an outside option $\omega > 0$. At the second stage, all entrants observe how many others have entered and choose effort. We will consider a pure strategy equilibrium where some number of player n_X enter and exert effort e^*_{X,n_X}, and $N - n_X$ players stay out. The equilibrium number of entrants is determined by conditions[19]

$$\frac{1}{n_X} - c(e^*_{X,n_X}) \geq \omega, \quad \frac{1}{n_X + 1} - c(e^*_{X,n_X+1}) < \omega. \tag{17}$$

Suppose Y is a different noise such that $e^*_{X,n} \geq e^*_{Y,n}$ for all $n \leq N$. For example, Y can be dominated by X in the dispersive order. We will explore under what conditions $e^*_{X,n_X} \geq e^*_{Y,n_Y}$; that is, effort is ranked in the same way in the equilibrium with endogenous participation.

First, we show that $n_X \leq n_Y$. By contradiction, suppose $n_X > n_Y$; then $n_X \geq n_Y + 1$ and hence

$$\frac{1}{n_Y + 1} - c(e^*_{Y,n_Y+1}) \geq \frac{1}{n_X} - c(e^*_{Y,n_X}) \geq \frac{1}{n_X} - c(e^*_{X,n_X}). \tag{18}$$

The first inequality holds because $\pi^*_{Y,n}$ is decreasing in n, and the second one holds by our assumption about noise Y. However, from (17), the initial expression in (18) is strictly below ω, whereas the final expression is weakly above ω, which is impossible.

Second, we compare $c(e^*_{X,n_X})$ to $c(e^*_{Y,n_Y})$. When $n_X = n_Y$, the comparison is trivial; suppose, therefore, that $n_X < n_Y$ and hence $n_X + 1 \leq n_Y$. From (17),

$$c(e^*_{X,n_X+1}) > \frac{1}{n_X + 1} - \omega \geq \frac{1}{n_Y} - \omega \geq c(e^*_{Y,n_Y}).$$

Thus, we can show that either $e^*_{X,n_X} \geq e^*_{Y,n_Y}$ or $e^*_{X,n_X+1} > e^*_{Y,n_Y}$. When effort is decreasing in the number of players, this implies the result $e^*_{X,n_X} \geq e^*_{Y,n_Y}$ holds in general. Also, in the "large contest" approximation the two inequalities are identical. But when effort is increasing or nonmonotone in n *and* the number of players is small, the discreteness of n plays a role.

Let us now consider the total cost of effort. Define total equilibrium cost of effort in a tournament of n players as $C_{X,n} = nc(e^*_{X,n})$. To compare total equilibrium costs in the case of endogenous participation, C_{X,n_X} and C_{Y,n_Y}, rewrite parts of (17) for X and Y as

$$(n_X + 1)c(e^*_{X,n_X+1}) > 1 - (n_X + 1)\omega, \quad n_Y c(e^*_{Y,n_Y}) \leq 1 - n_Y\omega.$$

[19]For concreteness, we assume that indifference is resolved in favor of entry.

If $n_X = n_Y$, then $C_{X,n_X} \geq C_{Y,n_Y}$ holds. Suppose that $n_X < n_Y$, i.e., $n_X + 1 \leq n_Y$. Then

$$(n_X + 1)c(e^*_{X,n_X+1}) > 1 - (n_X + 1)\omega \geq 1 - n_Y\omega \geq n_Y c(e^*_{Y,n_Y}).$$

Thus, we have shown that either $C_{X,n_X} \geq C_{Y,n_Y}$ or $C_{X,n_X+1} > C_{Y,n_Y}$. In the "large contest" approximation, the two inequalities are identical. Also, the second inequality implies the first provided $C_{X,n}$ is decreasing in n. That is, the total cost of effort decreases with noise in large contests or when $C_{X,n}$ is decreasing in n. For a small n, and without restrictions imposed on $C_{X,n}$, there is a correction for discreteness similar to the comparison of equilibrium efforts above.

6 Concluding remarks

It has been the common wisdom since the beginning of the literature that when more noise is injected into a tournament, players' effort is reduced, at least under full participation. However, it has remained unknown which details of noise are responsible for this effect. In this paper, we show that the key factor is the *informational* content of noise, rather than the associated level of risk. Information, characterized by the dispersive order and various forms of entropy, plays a role due to strategic interactions in tournaments: Marginal incentives are determined by the probability density of *differences* between shocks, as opposed to deviations of shocks from a deterministic threshold.

We show that the dispersive order is both necessary and sufficient to rank equilibrium effort across tournaments with arbitrary prize schedules. For two-prize tournaments, we show that effort is determined by an appropriately defined Rényi entropy of order statistics of noise. We also introduce new *quantile stochastic dominance* (QSD) orders that are similar to the standard stochastic dominance defined for inverse quantile densities. The simplest of these orders – QSD(1) – is related to upper-tail conditional entropy. These orders are weaker than the dispersive order and allow for the ranking of equilibrium effort in winner-take-all tournaments.

Our results have empirical implications in settings where the distribution of fluctuations does not follow the standard unimodal pattern and hence risk and entropy can move in opposite directions when parameters of the distribution change. Important examples are settings characterized by bimodal distributions. In the presence of bimodality (or, more generally, multimodality), risk increases while entropy remains unchanged or may go down when the distance between the peaks of the distribution increases and the peaks become more narrow. For example, recent trends in growing political polarization (Dixit and Weibull, 2007; Fiorina and Abrams, 2008) and polarization of skills and wages in the labor market (Autor, Katz and Kearney, 2006) point at the bimodality in various dimensions of preferences and income of consumers. Bimodal fluctuations have also been identified in macroeconomic variables due to feedback loops arising from interactions of the economy with the financial sector (Brunnermeier and Sannikov, 2014).[20] Examples of bimodality outside economics include fluctuations in medical costs (Patterson, 2011), mortality times (Gunst et al., 2010), time intervals in human communication (Wu et al., 2010), and failure times of various devices (Fischer et al., 2000).

Our results extend to tournaments with stochastic and endogenous participation, with caveats regarding the effects of discreteness in the equilibrium number of entrants in the latter case. For heterogeneous players, the ranking of equilibrium effort by the dispersive order and related entropy orders survives under small asymmetries in ability, but it can be reversed when players are sufficiently asymmetric.

More broadly, the results of this paper show that tournament incentives can be affected in nontrivial ways by the properties of noise. However, the existing empirical literature on tournaments and contests using natural data (e.g., Ehrenberg and Bognanno, 1990; Knoeber and Thurman, 1994; Eriksson, 1999) or experiments (for a review, see Dechenaux, Kovenock and Sheremeta, 2015) provides virtually no guidance on such effects. One exception we are aware of is the experiment of List et al. (2014) who study effort in tournaments with different noise distributions, focusing on the effects of changes in the number of agents. We hope that our results will lead to more research linking effort in tournaments to the properties of noise.

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Appendix

Proof of Proposition 1 The sufficiency of the dispersive order follows directly from representation (6) for coefficients $B_{r,n}$ and the fact that X being more dispersed than Y is equivalent to $m_X(z) \leq m_Y(z)$ (with strict inequality in some interval) (Shaked and Shanthikumar, 2007). To see this, note that Definition 1 implies $F_X^{-1}(z) - F_Y^{-1}(z)$ is increasing in z, and the result follows by differentiation.

For necessity, consider X and Y such that $\Delta m(z) = m_Y(z) - m_X(z) < 0$ in some interval $(z_1, z_2) \subset [0,1]$. Restricting attention to two-prize schedules, with $D_s > 0$ for some $s \in \{1, \dots, n-1\}$ and $D_r = 0$ for $r \neq s$, it is sufficient to show that there exist (s, n) such that $B_{X,s,n} > B_{Y,s,n}$. From (8),

$$B_{X,s,n} - B_{Y,s,n} = -\int_0^1 f^B(z; n-s, s)\Delta m(z)dz. \tag{19}$$

Function $f^B(z; n-s, s)$ is single-peaked, with a unique maximum at $z^* = \frac{n-s-1}{n-2}$. For a sufficiently large n, $s = s_n$ can be chosen such that z^* is arbitrarily close to the middle of the interval (z_1, z_2). With s_n chosen this way, we have $f^B(z; n-s_n, s_n) \to f^B(z; 1 + (n-2)z^*, (n-1) - (n-2)z^*)$, which is the distribution of the z^*-th sample quantile of the uniform distribution on $[0,1]$. Since $z^* \in (0,1)$, this is a central order statistic that is asymptotically $N(z^*, \frac{z^*(1-z^*)}{n-1})$ (Arnold, Balakrishnan and Nagaraja, 1992). Therefore, $B_{X,s,n} - B_{Y,s,n} \to -\Delta m(z^*) > 0$. ■

Proof of Proposition 3 Consider a WTA tournament with $n \geq 2$ players and noise X. The equilibrium effort in it is determined by $B_{1,n}[f_X] = (n-1)\int_0^1 m_X(z) z^{n-2} dz$. For $k = 0$, the result holds due to Proposition 1; therefore, suppose $k \geq 1$. Note that $m_X(z) = -\bar{M}_X^{(1)\prime}(z)$, $\bar{M}_X^{(1)}(z) = -\bar{M}_X^{(2)\prime}(z)$, $\dots$, $\bar{M}_X^{(k-1)}(z) = -\bar{M}_X^{(k)\prime}(z)$. Integrating by parts k times, obtain

$$
\begin{aligned}
B_{1,n}[f_X] &= -(n-1)\bar{M}_X^{(1)}(z)z^{n-2}|_0^1 + (n-1)(n-2)\int_0^1 \bar{M}_X^{(1)}(z)z^{n-3}dz \\
&= \bar{M}_X^{(1)}(0)\mathbb{1}_{n=2} + \frac{(n-1)!}{(n-3)!}\int_0^1 \bar{M}_X^{(1)}(z)z^{n-3}dz \\
&= \bar{M}_X^{(1)}(0)\mathbb{1}_{n=2} - \frac{(n-1)!}{(n-3)!}\bar{M}_X^{(2)}(z)z^{n-3}|_0^1 + \frac{(n-1)!}{(n-4)!}\int_0^1 \bar{M}_X^{(2)}(z)z^{n-4}dz \\
&= \bar{M}_X^{(1)}(0)\mathbb{1}_{n=2} + 2!\bar{M}_X^{(2)}(0)\mathbb{1}_{n=3} + \frac{(n-1)!}{(n-4)!}\int_0^1 \bar{M}_X^{(2)}(z)z^{n-4}dz \\
&\dots \\
&= \sum_{l=1}^{k} l!\bar{M}_X^{(l)}(0)\mathbb{1}_{n=l+1} + \frac{(n-1)!}{(n-2-k)!}\int_0^1 \bar{M}_X^{(k)}(z)z^{n-2-k}dz \\
&= (n-1)!\bar{M}_X^{(n-1)}(0)\mathbb{1}_{k\geq n-1} + \frac{(n-1)!}{(n-2-k)!}\int_0^1 \bar{M}_X^{(k)}(z)z^{n-2-k}dz.
\end{aligned}
$$

The result follows directly from condition (a) and Definition 3. ■

Proof of Lemma 2 We need to show that condition $\bar{M}_Y^{(1)}(z) \geq \bar{M}_X^{(1)}(z)$ is equivalent to the upper-tail conditional entropy order. Indeed,

$$
\bar{M}_X^{(1)}(z) = \int_z^1 m_X(t)dt = \int_{F_X^{-1}(z)}^{\overline{x}} f_X(x)^2 dx = (1-z)^2 \int_{F_X^{-1}(z)}^{\overline{x}} f_{X|X\geq F_X^{-1}(z)}(x)^2 dx;
$$

therefore, $\bar{M}_Y^{(1)}(z) \geq \bar{M}_X^{(1)}(z)$ is equivalent to

$$
\int_{F_Y^{-1}(z)}^{\overline{y}} f_{Y|Y\geq F^{-1}(z)}(x)^2 dx \geq \int_{F_X^{-1}(z)}^{\overline{x}} f_{X|X\geq X\geq F_X^{-1}(z)}(x)^2 dx,
$$

which is equivalent to $H[f_{X|X\geq F_X^{-1}(z)}] \geq H[f_{Y|Y\geq F_Y^{-1}(z)}]$. ■

Proof of Lemma 3 Let $[\underline{x}, \overline{x}]$ denote the union of supports of X and Y. Define

$$
\Delta B_{1,2} = \int_{\underline{x}}^{\overline{x}} [f_Y^2(x) - f_X^2(x)]dx = \int_{\underline{x}}^{\overline{x}} f_+(x)f_-(x)dx,
$$

where $f_\pm(x) = f_Y(x) \pm f_X(x)$. Note that $f_-(x) \leq 0$ for $x \in [\underline{x}, x_1] \cup [x_2, \overline{x}]$ and $f_-(x) \geq 0$

for $x \in [x_1, x_2]$. Thus, we can write

$$\Delta B_{1,2} = -\int_{\underline{x}}^{x_1} f_+(x)|f_-(x)|dx + \int_{x_1}^{\hat{x}} f_+(x)f_-(x)dx + \int_{\hat{x}}^{x_2} f_+(x)f_-(x)dx - \int_{x_2}^{\overline{x}} f_+(x)|f_-(x)|dx.$$

By the mean-value theorem for definite integrals, there exist $x_1^* \in (\underline{x}, x_1)$, $x_2^* \in (x_1, \hat{x})$, $x_3^* \in (\hat{x}, x_2)$ and $x_4^* \in (x_2, \overline{x})$ such that

$$\Delta B_{1,2} = -f_+(x_1^*)\int_{\underline{x}}^{x_1} |f_-(x)|dx + f_+(x_2^*)\int_{x_1}^{\hat{x}} f_-(x)dx + f_+(x_3^*)\int_{\hat{x}}^{x_2} f_-(x)dx - f_+(x_4^*)\int_{x_2}^{\overline{x}} |f_-(x)|dx.$$

Recall that $F_X(\hat{x}) = F_Y(\hat{x})$, which implies $\int_{\underline{x}}^{\hat{x}} f_X(x)dx = \int_{\underline{x}}^{\hat{x}} f_Y(x)dx$, and hence $\int_{\underline{x}}^{\hat{x}} f_-(x)dx = 0$ and $\int_{x_1}^{\hat{x}} f_-(x)dx = \int_{\underline{x}}^{x_1} |f_-(x)|dx$. Similarly, $\int_{\hat{x}}^{x_2} f_-(x)dx = \int_{x_2}^{\overline{x}} |f_-(x)|dx$. This gives

$$\Delta B_{1,2} = [f_+(x_2^*) - f_+(x_1^*)]\int_{\underline{x}}^{x_1} |f_-(x)|dx + [f_+(x_3^*) - f_+(x_4^*)]\int_{x_2}^{\overline{x}} |f_-(x)|dx.$$

It follows from the condition $f_+(\hat{x}) \geq 2\max\{f_X(x_1), f_X(x_2)\}$ that $f_+(\hat{x}) \geq f_+(x_1)$, which implies $f_+(x_2^*) \geq f_+(x_1)$ and hence $f_+(x_2^*) \geq f_+(x_1^*)$. Similarly, $f_+(x_3^*) \geq f_+(x_4^*)$, which implies $\Delta B_{1,2} \geq 0$. ■

Counterexample for the spread order The spread order alone (e.g., without condition (15)), is not sufficient to generate unambiguous entropy rankings, as seen from the following example.

In this example we use piece-wise constant functions f_X and f_Y, which are discontinuous. This is not a problem, because any such function can be approximated arbitrarily closely by a smooth and continuous function. Also, the statement of Lemma 3 is true for discontinuous functions as long as they are L^2.

Define functions f_X and f_Y on support $[0, 1]$ as follows:

$$f_X(x) = \begin{cases} 1.1, & x \in [0, 0.25) \\ 1.7, & x \in [0.25, 0.5) \\ 0.6, & x \in [0.5, 1] \end{cases}, \quad f_Y(x) = \begin{cases} 0.95, & x \in [0, 0.125) \\ 1.15, & x \in [0.125, 0.25) \\ 1.7, & x \in [0.25, 0.5) \\ 0.7, & x \in [0.5, 0.75) \\ 0.55, & x \in [0.75, 1] \end{cases}.$$

It is easy to check that both functions have the same integral equal to one and that f_X is

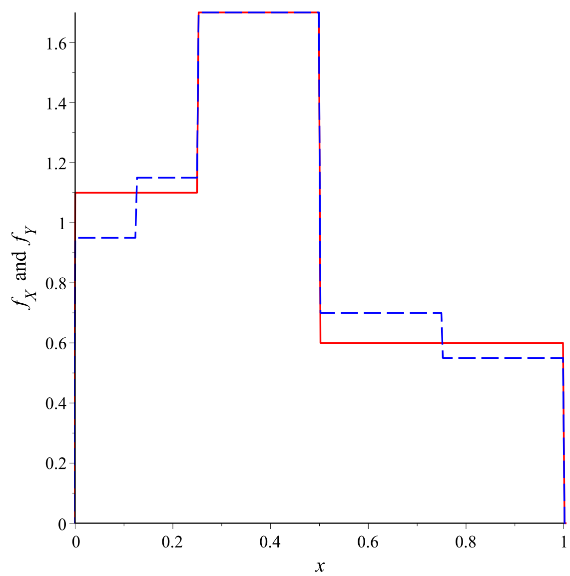

Figure 4: Functions f_X (red solid line) and f_Y (blue dashed line). f_X is a spread of f_Y but generates higher efforts. Condition (15) of Lemma 3 is not satisfied.

a spread of f_Y, see Figure 4. In particular, $x_1 = 0.125$ and $x_2 = 0.75$.

Note that condition (15) of Lemma 3 is not satisfied since the two CDFs intersect at $\hat{x} = 0.625$ and $f_X(\hat{x}) + f_Y(\hat{x}) = 1.3 < 2\max\{f_X(x_1), f_X(x_2)\} = 2f_X(x_1) = 2.2$.

Function f_X has a lower entropy than f_Y because $\int_0^4 f_X(x)^2dx = 1.205 > \int_0^4 f_Y(x)^2dx = 1.19875$. Hence, a spread alone does not guarantee that the entropy increases and hence, efforts go down.