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Mikhail Drugov
Dmitry Ryvkin

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Mikhail Drugov* Dmitry Ryvkin†

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Abstract

Heavy-tailed fluctuations are common in many environments, such as sales of creative and innovative products or the financial sector. We study how the presence of heavy tails in the distribution of shocks affects the optimal allocation of prizes in rank-order tournaments. While a winner-take-all prize schedule maximizes aggregate effort for light-tailed shocks, prize sharing becomes optimal when shocks acquire heavy tails, increasingly so following a skewness order. Extreme prize sharing – rewarding all ranks but the very last – is optimal when shocks have a decreasing failure rate, such as power laws. Hence, under heavy-tailed uncertainty, typically associated with strong inequality in the distribution of gains, providing incentives and reducing inequality go hand in hand.

Keywords: heavy tails, power law, tournament, optimal allocation of prizes, failure rate

JEL codes: C72, D86, M52

*New Economic School and CEPR, mdrugov@nes.ru.

†Department of Economics, Florida State University, Tallahassee, FL 32306-2180, USA, dryvkin@fsu.edu.

1 Introduction

We don't live in a normal world; we live under a power law.
Peter Thiel, "Zero to One"

A series of *power laws*, established empirically and explained theoretically, is arguably the unique case in economics of a quantitative law that is both nontrivial and true (Gabaix, 2016). Starting with Pareto (1896), who described the distribution of income in a society with the help of the eponymous distribution, power laws have been identified for a variety of economic variables such as city sizes (Auerbach, 1913), firm sizes (Axtell, 2001), stock market movements (Mandelbrot, 1963), CEO compensation (Roberts, 1956), demand for books at Amazon (Chevalier and Goolsbee, 2003) or movie ratings in Netflix (Bimpikis and Markakis, 2016).[1] The key feature of a power law distribution is its *heavy tail*, implying that realizations far away from the mean are quite likely. This is in contrast to light-tailed distributions, such as the normal or Gumbel distribution, whose realizations are mostly confined within few standard deviations from the mean.

In this paper, we study how the shape of the distribution of noise, and especially the presence of heavy tails, affects the optimal allocation of prizes in rank-order tournaments which are ubiquitous in hiring, the assignment of bonuses, merit raises and promotions in organizations (Prendergast, 1999; Bognanno, 2001; Waldman, 2013). In these incentive schemes, first analyzed by Lazear and Rosen (1981), agents expending effort or other resources are rewarded based on the rank of their output, which is random but stochastically increasing in effort. For example, salespeople experience random demand shocks; workers in the service sector face random arrival of clients; stock traders are subject to market fluctuations; whereas researchers and creative artists randomly get ideas and inspiration. Given that rank-based rewards and heavy-tailed fluctuations coexist in many areas, such as the financial sector or sales of creative products, what are the optimal tournament incentives in the presence of heavy tails?

We fully characterize the optimal structure of prizes for symmetric risk-neutral agents when the principal allocates a fixed budget. We show that the presence of heavy tails has drastic consequences for the structure of optimal tournament contracts, specifically for *prize sharing*. We find that the optimal number of top prizes is determined by the

[1] Power laws also describe many patterns outside economics such as the frequency of words in natural languages (Zipf, 1949), the intensity of earthquakes (Christensen et al., 2002) or popularity in social networks (Barabási and Albert, 1999).

failure (or hazard) rate of the distribution of noise. For noise distributions with increasing failure rate (IFR), the *winner-take-all* (WTA) contract rewarding only the top performer is optimal. In contrast, for distributions with decreasing failure rate (DFR), a contract rewarding everyone but the very bottom performer is optimal.[2] Decreasing or unimodal failure rates are a common, albeit not defining, feature of heavy-tailed distributions, including power laws; whereas IFR is more typical for light-tailed distributions. This suggests that organizations facing heavy-tailed uncertainty may benefit from restructuring their rank-based compensation in the direction of prize sharing, as opposed to high-powered rewards at the top that are optimal in the "normal" world.

The intuition for our results is as follows. In the symmetric equilibrium, ranking is determined entirely by luck; that is, the player with the largest realization of noise receives the first prize, the second largest realization – the second prize, etc. Therefore, qualitatively, the incentives are related to spacings – distances between adjacent order statistics of noise. The smaller they are, the more likely a small increase in effort will affect the ranking and hence, the incentives are higher. The growth of spacings with rank is determined by the failure rate. When the distribution of noise is light-tailed (IFR), the spacings are increasing slowly, if at all, which means competition at the top is the fiercest and WTA is optimal. In contrast, when the distribution of noise is heavy-tailed, spacings at the top are increasing fast. In order to maximize incentives in this case the principal has to give prizes to multiple ranks, more so the faster the spacings grow, i.e., the thicker the tail. In the extreme case when spacings are increasing fast for all ranks (DFR), all ranks except the very last need to be incentivized.

Our results can be interpreted in a broader sense as providing a neo-classical justification for prize sharing as an alternative to winner-take-all incentives. The latter have become prevalent in many sectors, exacerbating the rise in inequality and fall in social mobility (Frank and Cook, 1996), which can have negative consequences for long-term economic growth, the stability of democratic institutions and the likelihood of civil conflict (e.g., Alesina and Rodrik, 1994; Stiglitz, 2012; Huber and Maoyral, 2018). The justification stems from the changing nature of fluctuations people in the modern, post-industrial world face. Indeed, while disasters and catastrophes we face are heavy-tailed "black swans" (Taleb, 2007), the winners of the world's tournaments in various domains rely increasingly on the heavy-tailed "white swans" of percolating innovation and cre-

[2]For noise distributions with unimodal failure rates, intermediate contracts can be optimal, with a monotone transition between the two extremes following the convex transform order, or increasing skewness (Shaked and Shanthikumar, 2007).

ative ideas, and more and more markets are characterized by the economics of superstars (Rosen, 1981). It is then suggested by our results that prize sharing among winners is a win-win approach because it *simultaneously* provides stronger incentives and reduces inequality.

We also apply our results to a few different but related settings. In particular, we identify a natural ordering of general rank-based prize schedules – the majorization order – that leads to the ordering of equilibrium effort. We also study tournaments for status, i.e., tournaments in which, following Moldovanu, Sela and Shi (2007), prizes are proportional to the difference between the number of players ranked strictly below and above a given player. While in their model the top category always contains a single – the best – player, in our model the top category may include multiple players when shocks are heavy-tailed and includes all players but the very last in the extreme case when shocks are DFR.

On a technical side, an important contribution of this paper is in providing general sufficient conditions for the validity of the first-order approach and the existence of a unique symmetric equilibrium in rank-order tournaments with arbitrary monotone prize schedules. While it has been generally understood since Nalebuff and Stiglitz (1983) that a symmetric pure strategy equilibrium may fail to exist in tournament models when noise dispersion is small or the number of players is large, the available existence results are very limited, as explained below.

Interestingly, as long as noise is sufficiently dispersed, there are no restrictions for the first-order approach in tournaments in terms of the *shape* of the distribution of noise. This is in contrast to the standard individual moral hazard problem with effort distorted by additive noise, which does not have a known solution in the presence of heavy-tailed shocks. Indeed, the monotone likelihood ratio property necessary for the first-order approach in the moral hazard problem (Rogerson, 1985; Jewitt, 1988) means in this case a log-concave distribution of noise; that is, the approach is restricted to light-tailed distributions. While there are some recent advances in weakening the conditions for the first-order approach or dispensing with it altogether (Kadan and Swinkels, 2013; Kirkegaard, 2017), the optimal contract in the moral hazard model with heavy-tailed shocks is still unknown. This provides yet another independent reason to study tournament contracts with heavy-tailed shocks.

Related literature

Starting with the groundbreaking paper by Lazear and Rosen (1981), multiple studies have explored optimal tournament contracts in an organizational context, comparing their efficiency to other incentive schemes, such as piece rates (Nalebuff and Stiglitz, 1983; Green and Stokey, 1983), and identifying their various properties pertaining to the number, risk-aversion and heterogeneity of agents (Krishna and Morgan, 1998; Akerlof and Holden, 2012; Balafoutas et al., 2017).[3] Yes, relatively little is known about the general properties of these incentive schemes pertaining to the shape of the distribution of noise. Ryvkin and Drugov (2017) recently provided a comprehensive analysis of comparative statics for WTA tournaments.

Our setting is similar to the one studied by Krishna and Morgan (1998) and Moldovanu and Sela (2001), and different from the more traditional setting of Lazear and Rosen (1981) and most later studies in that it does not impose a binding zero-profit constraint on the principal or a participation constraint on the agents. The model allows us to obtain a unique optimal allocation of prizes, unlike the more traditional settings where a continuum of prize allocations can be optimal under risk-neutrality.

The paper that is closest to ours is Krishna and Morgan (1998). They have risk-averse agents but restrict the analysis to tournaments with $n \leq 4$ agents and symmetric unimodal noise distributions, and do not relate the structure of optimal contracts to the properties of noise. In contrast, we focus on risk-neutral agents but provide very general results regarding the effects of noise. In particular, we generalize the (risk-neutral version of the) "winner-take-all principle" formulated by Krishna and Morgan (1998) for small tournaments.[4] We show that it is the symmetry, and not unimodality, of the pdf of noise that leads to the optimality of WTA when $n \leq 4$, and the IFR property provides an independent sufficient condition for the optimality of WTA that holds for any $n \geq 2$. Schweinzer and Segev (2012) demonstrate the optimality of the WTA schedule for multi-prize Tullock contests.[5]

There is a parallel literature on the optimal allocation of prizes in all-pay auctions of complete (Barut and Kovenock, 1998; Fang, Noe and Strack, 2018) and incomplete information (Moldovanu and Sela, 2001). While the private information all-pay auction model of Moldovanu and Sela (2001) is quite different from ours, there are some interesting

[3]For a review of the earlier literature see, e.g., McLaughlin (1988), Lazear (1999), Prendergast (1999), Sisak (2009), Connelly et al. (2014).

[4]Under their assumptions, Krishna and Morgan (1998) demonstrate the WTA principle for $n \leq 3$ for risk-averse agents and for $n \leq 4$ for risk-neutral agents. Both bounds are tight.

[5]The symmetric equilibrium in the Tullock (1980) contest model is the same as in an appropriately transformed Lazear-Rosen tournament with additive noise following the Gumbel (extreme value type-I) distribution; thus, in equilibrium the Tullock contest model can be thought of as a special case of the Lazear-Rosen model, even though generally the two games are different (Jia, Skaperdas and Vaidya, 2013; Ryvkin and Drugov, 2017). The Gumbel distribution is IFR.

parallels. In their model, agents are endowed with i.i.d. private types and in equilibrium bid according to a symmetric monotone bidding function of type that provides a one-to-one assortative matching between types and prizes. In our model, the equilibrium effort is symmetric and prizes are assigned according to realizations of noise. Therefore, some properties of aggregate equilibrium effort, such as the role of order statistics, are similar between the two models, but cardinal effects related to the distribution of noise/types and the cost function of effort are quite different. Specifically, a WTA contest is optimal in Moldovanu and Sela (2001) if the cost function is linear or concave, regardless of the distribution of types; when costs are convex, multiple prizes may be optimal, but there is no obvious relationship between the structure of optimal contracts and properties of the distribution of types. In Section 4.2, we present an application of our model to tournaments for status – another setting previously analyzed using the private value all-pay auction approach (Moldovanu, Sela and Shi, 2007) – that further highlights similarities and differences between the two models.

A few papers provide different explanations for why only punishing the bottom performer is optimal. In the complete information all-pay auction model with convex costs, Fang, Noe and Strack (2018) show that this is always the optimal contract. Letina, Liu and Netzer (2018) show that this contract is optimal when a manager has to delegate incentive provision to an intermediate reviewer who cares about workers' well-being. Behaviorally, extreme punishment contracts generate extra effort due to "last-place aversion" (Kuziemko et al., 2014; Dutcher et al., 2015; Gill et al., 2018). In contrast, we show precisely in which cases such contracts are optimal for selfish, payoff-maximizing agents.

Finally, we are only aware of two papers addressing equilibrium existence in multi-prize tournament models: Schweinzer and Segev (2012) provide equilibrium existence results for multi-prize Tullock contests and Akerlof and Holden (2012) derive local second-order conditions, but not global conditions for the existence of best responses, for rank-order tournaments. For WTA tournaments, Ewerhart (2016) demonstrates the limits of the first-order approach and develops an alternative method for equilibrium characterization using an envelope function for incentive compatibility constraints. In contrast, our existence conditions are sufficient for the optimality of the first-order approach.

The rest of the paper is organized as follows. Section 2 sets up the model. The optimal prize allocation is characterized in Section 3. Section 4 provides comparative statics results and looks at some extensions and applications. Section 5 contains a technical result on equilibrium existence that is separated for the ease of exposition. Section 6 concludes.

All missing proofs are collected in the Appendix.

2 Model setup

2.1 Preliminaries

There are $n \geq 2$ risk-neutral players indexed by $i = 1, \ldots, n$. The players simultaneously and independently choose effort levels $e_i \in \mathbb{R}_+$. The cost of effort e_i to player i is $c(e_i)$, where $c(\cdot)$ is strictly increasing, strictly convex and twice differentiable on $(0, c^{-1}(1)]$, and continuous and differentiable at zero with $c(0) = c'(0) = 0$. The output of player i is stochastic and given by $y_i = e_i + X_i$,[6] where shocks X_i are i.i.d. across players, with a cumulative density function (cdf) $F(\cdot)$ and probability density function (pdf) $f(\cdot)$ defined on an interval support $U = [\underline{x}, \overline{x}]$ (finite or infinite). The cdf $F(\cdot)$ is continuously differentiable, and the pdf $f(\cdot)$ is atomless, continuous, differentiable almost everywhere and square-integrable in U. Let $F^{-1}(z) = \inf\{x \in U : F(x) \geq z\}$ denote the corresponding quantile function, and $m(\cdot) = f(F^{-1}(\cdot)) : [0, 1] \to \mathbb{R}_+$ denote the inverse quantile density (Parzen, 1979). Function $m(\cdot)$, inheriting its properties from $f(\cdot)$ and $F(\cdot)$, is continuous, differentiable almost everywhere and integrable on $[0, 1]$.

A risk-neutral principal observes the ranking of outputs and allocates nonnegative rank-dependent prizes $v = (v_1, \ldots, v_n)$ to the n players. Specifically, a player whose output is ranked r (where $r = 1$ corresponds to the highest output, $r = 2$ to the second highest, etc.) receives a prize $v_r \geq 0$.[7] Prizes are decreasing[8] in rank, $v_1 \geq v_2 \geq \ldots \geq v_n$. By default, throughout the paper it is assumed that prizes satisfy the budget constraint $\sum_{r=1}^{n} v_r = 1$, but in Section 4 we discuss situations where this assumption is relaxed.

2.2 Equilibrium characterization

Consider a symmetric pure-strategy Nash equilibrium where all players exert some effort $e^* > 0$. Following the symmetric opponents form approach (SOFA) (Hefti, 2017), assume that all players other than one indicative player choose effort e^*, and write the payoff of

[6] Via a change of variables, this model can also accommodate tournaments with multiplicative noise, with $y_i = e_i X_i$ (see Jia, 2008; Jia, Skaperdas and Vaidya, 2013; Ryvkin and Drugov, 2017).

[7] Ties in the ranking are broken randomly, but occur with zero probability for an atomless $f(\cdot)$.

[8] Throughout this paper, "increasing" means nondecreasing and "decreasing" means nonincreasing. Whenever the distinction is important, we use the terms "strictly increasing" and "strictly decreasing."

the indicative player from a deviation effort e as

$$u(e, e^*) = \sum_{r=1}^{n} \pi^{(r)}(e, e^*) v_r - c(e). \tag{1}$$

Here $\pi^{(r)}(e, e^*)$, the probability that the indicative player's output is ranked r, is given by

$$\pi^{(r)}(e, e^*) = \binom{n-1}{r-1} \int_U F(e - e^* + x)^{n-r} [1 - F(e - e^* + x)]^{r-1} dF(x). \tag{2}$$

Indeed, in order to be ranked r, the indicative player's output must be higher than the output of exactly $n - r$ other players, and there are $\binom{n-1}{r-1}$ ways to choose those players. We will use subscript e to denote partial derivatives of functions with respect to e. The symmetric first-order condition, $u_e(e^*, e^*) = 0$, produces the equation

$$\sum_{r=1}^{n} \beta_{r,n} v_r = c'(e^*), \tag{3}$$

where coefficients $\beta_{r,n} \equiv \pi_e^{(r)}(e^*, e^*)$, the marginal probabilities of reaching rank r, are given by

$$\beta_{r,n} = \binom{n-1}{r-1} \int_U F(x)^{n-r-1} [1 - F(x)]^{r-2} [n - r - (n-1)F(x)] f(x) dF(x). \tag{4}$$

Coefficients $\beta_{r,n}$, Eq. (4), are determined entirely by the distribution of noise. The following properties follow immediately from Eq. (4) (cf. also Akerlof and Holden, 2012): (i) $\sum_{r=1}^{n} \beta_{r,n} = 0$, $\beta_{1,n} > 0$, $\beta_{n,n} < 0$; (ii) if pdf f is symmetric then $\beta_{r,n} = -\beta_{n-r+1,n}$ for all r.

The e^* given by Eq. (3) is a natural candidate for the symmetric equilibrium effort level. Proposition 9 in Section 5 provides appropriate sufficient conditions, which we assume from this point on are satisfied. These conditions guarantee that there is a unique e^* solving (3) and it is the symmetric equilibrium for *all* monotone prize schedules v. Consistent with intuition, the sufficient conditions impose restrictions on the dispersion of noise and the magnitude and convexity of the cost function. Crucial for our purposes, they impose minimal restrictions on the *shape* of the distribution of noise. This allows us, in what follows, to explore optimal prize allocations and, more generally, the comparative statics of equilibrium effort, by relying entirely on the properties of the left-hand side of (3).

3 Optimal prize allocations

In this section, we characterize the optimal allocation of prizes. In Section 3.1, we rewrite the problem in a more convenient form and identify the general structure of the solution. In Section 3.2, we relate the properties of the solution to the shape of the distribution of noise.

3.1 Restating the problem

We consider a principal whose objective is to maximize total equilibrium effort, ne^*. It follows immediately from (3), and the assumption that $c(\cdot)$ is strictly convex, that the principal's prize allocation problem has the form

$$\max_{v_1,\ldots,v_r} \sum_{r=1}^{n} \beta_{r,n} v_r \quad \text{s.t. } v_1 \geq \ldots \geq v_n \geq 0, \quad \sum_{r=1}^{n} v_r = 1. \tag{5}$$

Problem (5) is a linear programming problem. If prize schedules were not restricted to be monotone, its solution would simply be to allocate the entire prize to the rank r that maximizes $\beta_{r,n}$. Therefore, if $\beta_{r,n}$ is maximized at $r = 1$, the optimal prize allocation is a *winner-take-all* (WTA) tournament with $v = (1, 0, \ldots, 0)$, for which the monotonicity constraint is satisfied automatically. If, however, $\beta_{r,n}$ is (strictly) maximized at some $r > 1$, it is impossible to allocate a high prize to rank r without also allocating at least the same prizes to ranks $1, \ldots, r-1$. The larger r is, the smaller these prizes will be, thereby diminishing incentives to compete for higher ranks. A different prize allocation, therefore, may be optimal in this case.

After the probability integral change of variable, $z = F(x)$, Eq. (4) becomes

$$\beta_{r,n} = \binom{n-1}{r-1} \int_0^1 z^{n-r-1}(1-z)^{r-2}[n-r-(n-1)z]m(z)dz. \tag{6}$$

Representation (6) is useful in that it separates the effects of the distribution of noise, contained entirely in the inverse quantile density $m(\cdot)$, from the effects of the number of players, n, and performance rank, r. Note that $\beta_{1,n} = (n-1)\int_0^1 z^{n-2}m(z)dz = \mathrm{E}(m(Z_{(n-1:n-1)}))$; that is, in a WTA tournament with $v = (1, 0, \ldots, 0)$ the equilibrium effort is determined by the expectation of $m(\cdot)$ over the highest order statistic among $n-1$ i.i.d. draws from the uniform distribution on $[0, 1]$. As we now show, more generally, for an arbitrary prize schedule v the left-hand side of (3) can be written as a linear combination of expectations of $m(\cdot)$ over the set of uniform order statistics $Z_{(n-r:n-1)}$, $r = 1, \ldots, n-1$.

Recall that $\beta_{n,n} < 0$. This implies that it is never optimal to assign a positive prize to rank $r = n$. That is, $v_n = 0$ in any optimal prize schedule. It is convenient to introduce nonnegative variables $d_r = v_r - v_{r+1}$ for $r = 1, \ldots, n-1$, from which the original prizes can be recovered as $v_r = \sum_{k=r}^{n-1} d_k$. Further, let $B_{r,n} = \sum_{k=1}^{r} \beta_{k,n}$ denote the cumulative version of coefficients $\beta_{r,n}$. Using (6), it is straightforward to verify that

$$B_{r,n} = \frac{(n-1)!}{(n-r-1)!(r-1)!} \int_0^1 z^{n-r-1}(1-z)^{r-1} m(z) dz. \tag{7}$$

Note that $B_{r,n} > 0$ for all $r = 1, \ldots, n-1$ and $B_{n,n} = 0$. Using summation by parts, rewrite the objective function of problem (5) as $\sum_{r=1}^{n} \beta_{r,n} v_r = \sum_{r=1}^{n-1} B_{r,n} d_r$. Taking into account that $v_n = 0$, we can also rewrite the budget constraint in the new variables as $\sum_{r=1}^{n} v_r = \sum_{r=1}^{n-1} r d_r = 1$. Thus, problem (5) reduces to

$$\max_{d_1,\ldots,d_{n-1}} \sum_{r=1}^{n-1} B_{r,n} d_r \quad \text{s.t. } d_1, \ldots, d_{n-1} \geq 0, \quad \sum_{r=1}^{n-1} r d_r = 1. \tag{8}$$

Representation (8) can be understood intuitively as follows. Recall that $\beta_{r,n}$ is the marginal probability of reaching rank r in the symmetric equilibrium, cf. Eqs. (3) and (4). Therefore, coefficient $B_{r,n} = \sum_{k=1}^{r} \beta_{k,n}$ can be interpreted as the marginal probability of reaching the rank of *at least* r. At the same time, prize differential d_r is the premium for reaching the rank of at least r (as compared to reaching the rank of at least $r+1$). Indeed, a player who reached the rank of at least r earns at least $d_{n-1} + \ldots + d_r = v_r$. Since exactly r players reach the rank of at least r, the budget constraint takes the form as in (8).

Problem (8) is a standard linear utility maximization problem in $\mathbb{R}_+^{n-1}$ with a budget constraint where the r-th "commodity" has price r. Commodities in this problem are the differentials $d_r = v_r - v_{r+1}$ between adjacent prizes. Such commodities indeed become more expensive for the principal as r increases because a positive prize differential once introduced for some r has to be carried over to all lower r to preserve the monotonicity of prizes. Coefficients $B_{r,n}$ play the role of (constant) marginal utilities of these commodities. A generic solution to problem (8) is a vertex solution where the entire budget is allocated to one commodity that yields the highest marginal utility per dollar spent. We, therefore, arrive at the following result.

Proposition 1 *The optimal prize allocation is a two-prize schedule, with* $v_1 = \ldots = v_{r^*} = \frac{1}{r^*}$ *and* $v_{r^*+1} = \ldots = v_n = 0$*, where the location* r^* *of a positive prize differential is given by*

$$r^* \in \arg\max_{r \in \{1,\ldots,n-1\}} \frac{B_{r,n}}{r}. \tag{9}$$

The optimal prize schedule described by Proposition 1 allocates equal prizes to the top r^* players. From Eq. (3), the resulting equilibrium effort is given by the equation $\frac{B_{r^*,n}}{r^*} = c'(e^*)$. The location of r^* is determined entirely by coefficients $B_{r,n}$, Eq. (7), i.e., by the properties of the distribution of noise. Generically, the optimal prize schedule is unique.

It should be noted that the simple two-prize structure of optimal prize schedules arises due to risk-neutrality. More sophisticated prize schedules involving several levels of prizes may be optimal for risk-averse agents.[9] However, the two-prize structure is sufficient to address our main research questions: Under what conditions are WTA tournaments optimal ($r^* = 1$), and what are the consequences of heavy tails in the distribution of noise for the optimality of WTA? In other words, we are interested in the effects of noise on r^*. An increase in r^* in this model constitutes movement away from WTA tournaments emphasizing rewards at the top to tournaments with prize sharing and, in extreme cases, to tournaments emphasizing punishment at the bottom.

The optimal location r^* of the prize differential, Eq. (9), maximizes the objective $\bar{\beta}_{r,n} = \frac{B_{r,n}}{r} = \frac{1}{r}\sum_{k=1}^{r} \beta_{k,n}$, which is the running *average* of coefficients $(\beta_{1,n}, \ldots, \beta_{r,n})$. As discussed previously, if $\beta_{r,n}$ is maximized at $r = 1$, this implies immediately that $r^* = 1$. Similarly, as long as $\beta_{r,n}$ is increasing, $\bar{\beta}_{r,n}$ is also increasing. Thus, if $\beta_{r,n}$ is increasing for all $r = 1, \ldots, n-1$, we have $r^* = n-1$. However, if $\beta_{r,n}$ is interior unimodal, r^* is located at or to the right of the maximum of $\beta_{r,n}$. The exact location of r^* for a nonmonotone $\beta_{r,n}$ depends on the details of the distribution of noise. This dependence is studied in the next section.

3.2 A characterization of optimal prize allocations

We turn to the analysis of $\bar{\beta}_{r,n} = \frac{B_{r,n}}{r}$, the objective of problem (9) that determines the location r^* of the positive prize differential in the optimal allocation of prizes. Using (7),

[9] A complete characterization of optimal prizes for risk-averse agents is rather complex but can be obtained using similar methods. Details are available upon request.

we write it in the form

$$\bar{\beta}_{r,n} = \frac{1}{n}\int_0^1 \frac{m(z)}{1-z} dF^B(z; n-r, r+1) = \frac{1}{n}\mathrm{E}(h(Z_{(n-r:n)})). \tag{10}$$

Here, $F^B(z; x, y)$ is the cdf of the beta distribution with parameters (x, y),[10] and $Z_{(n-r:n)}$ is the $(n-r)$-th order statistic among n i.i.d. draws from the uniform distribution on $[0, 1]$. The hazard quantile function $h(z) = \frac{m(z)}{1-z}$ (Nair, Sankaran and Balakrishnan, 2013) is a quantile representation of failure (or hazard) rate $\frac{f(x)}{1-F(x)}$. We will refer to distributions as having increasing failure rate (IFR) if $h(z)$ is increasing and decreasing failure rate (DFR) if $h(z)$ is decreasing. The exponential distribution with pdf $f(x) = \lambda\exp(-\lambda x)$ is the only distribution with a constant failure rate (equal λ). Heavy-tailed distributions are typically DFR or have an (interior) unimodal failure rate.

Representation (10) leads to the following properties of $\bar{\beta}_{r,n}$.

Lemma 1 *(i) If $f(\cdot)$ is IFR (DFR) then $\bar{\beta}_{r,n}$ is decreasing (increasing) in r for $r = 1, \ldots, n-1$.*
(ii) If $f(\cdot)$ has a unimodal (U-shaped) failure rate then $\bar{\beta}_{r,n}$ is unimodal (U-shaped) in r for $r = 1, \ldots, n-1$.

Part (i) of Lemma 1 follows directly from representation (10) since order statistics $Z_{n-r:n}$ are FOSD-decreasing in r. Part (ii), proved in the Appendix, is a result of the preservation of unimodality under uncertainty. It is closely related to the results of Athey (2002) on the preservation of single-crossing under uncertainty. Indeed, a unimodal function has a single-crossing derivative, and the specific structure of the densities of order statistics satisfies the needed log-supermodularity condition on the parameterization of uncertainty. Part (i) further leads to the following proposition, which is the first major result of this section.

Proposition 2 *(i) If $f(\cdot)$ is IFR then $r^* = 1$, i.e., the WTA tournament is optimal.*
(ii) If $f(\cdot)$ is DFR then $r^* = n-1$, i.e., it is optimal to award prize $\frac{1}{n-1}$ to all but the very last player.
(iii) If $f(\cdot)$ is an exponential distribution, any allocation of prises with $v_n = 0$ is optimal.

Many standard distributions fall into one of the monotone failure rate classes covered by Proposition 2. In particular, distributions with increasing or log-concave pdfs are IFR, and distributions with log-convex pdfs and $f(\overline{x}) = 0$ are DFR.

[10] Function $F^B(z; x, y) = \frac{1}{\mathcal{B}(x,y)}\int_0^z t^{x-1}(1-t)^{y-1}dt$, where $\mathcal{B}(x, y)$ is the beta function, is also known as the regularized incomplete beta function (see, e.g., Paris, 2010).

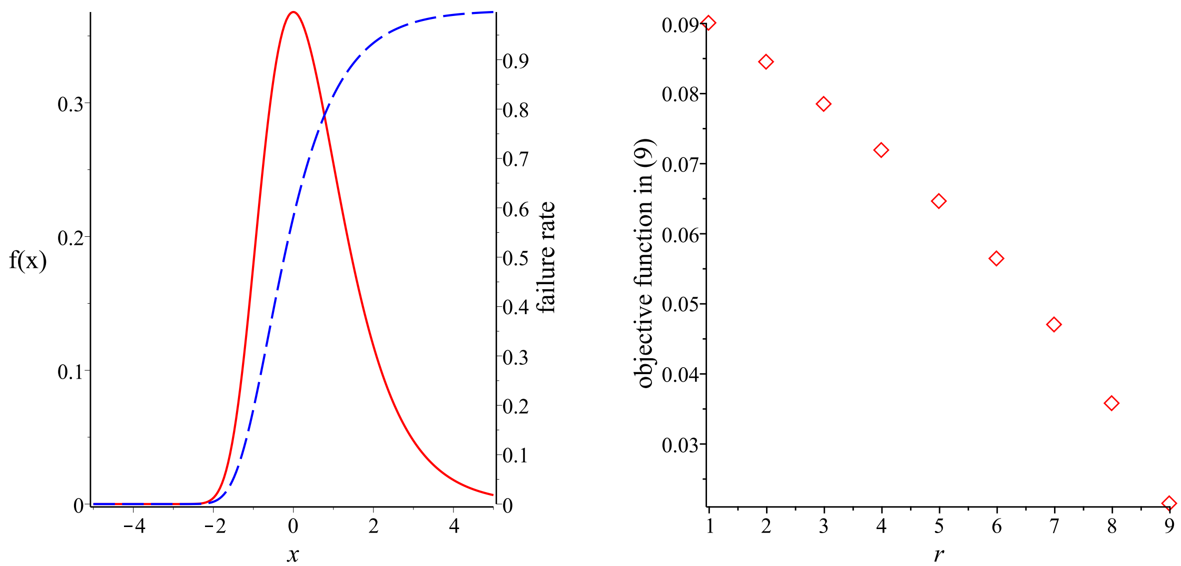

Figure 1: Gumbel distribution with parameter 1. *Left*: Pdf $f(x)$ (solid line, left scale) and failure rate $\frac{f(x)}{1-F(x)}$ (dashed line, right scale). *Right*: The objective $\bar{\beta}_{r,n}$ for $n = 10$ is maximized at $r^* = 1$.

The role of failure rate in the optimal allocation of prizes can be understood intuitively from the following arguments. Eq. (10) can be rewritten as $\bar{\beta}_{r,n} = \frac{1}{n} \int_U f(x|X \geq x) f_{(n-r:n)}(x)dx$, where the failure rate $\frac{f(x)}{1-F(x)}$ is written as the density at x of variable X conditional on $X \geq x$. Thus, $\bar{\beta}_{r,n}$ is determined by the density at zero of the difference between X and $X_{(n-r:n)}$ conditional on $X \geq X_{(n-r:n)}$. Indeed, the probability of reaching a rank of at least r can be expressed as the probability of surpassing the r-th highest noise realizations out of n conditional on X being among the top r realizations, multiplied by the probability that X is in the top r (equal $\frac{r}{n}$). This explains why, as r increases, $\bar{\beta}_{r,n} = \frac{1}{r} B_{r,n}$ changes according to the shape of the failure rate.

As an example of an IFR distribution, consider the Gumbel distribution with parameter 1, whose pdf $f(x) = \exp[-x - \exp(-x)]$ is shown in the left panel of Figure 1. This distribution generates the contest success function (CSF) of the Tullock contest (Jia, Skaperdas and Vaidya, 2013), for which the symmetric equilibrium with multiple prizes was identified by Clark and Riis (1996) and Fu and Lu (2012). The right panel in Figure 1 shows the objective $\bar{\beta}_{r,n}$ of problem (9) as a function of r. As seen from the figure, $r = 1$ is indeed optimal. The optimality of the WTA prize schedule for Tullock contests was demonstrated by Schweinzer and Segev (2012). Part (i) of Proposition 2 generalizes this result to arbitrary tournaments with IFR distributions of noise.

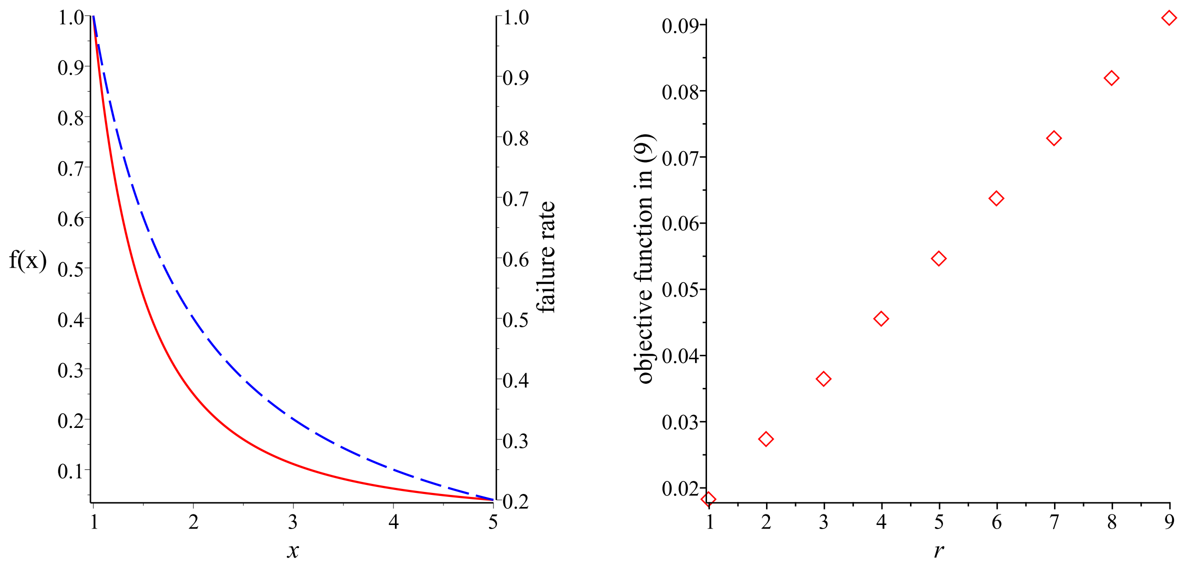

Figure 2: The Pareto distribution with parameters $(1, 1)$. *Left*: Pdf $f(x)$ (solid line, left scale) and failure rate $\frac{f(x)}{1-F(x)}$ (dashed line, right scale). *Right*: The objective $\bar{\beta}_{r,n}$ for $n = 10$ is maximized at $r^* = 9$.

For an example of a DFR distribution, consider the Pareto distribution with parameters $(1, 1)$, with pdf $f(x) = \frac{1}{x^2}\mathbb{1}_{x\geq 1}$, for which $m(z) = (1-z)^2$ and $\bar{\beta}_{r,n} = \frac{r+1}{n(n+1)}$ is increasing in r. Hence, it is optimal to reward every player but the very last, see Figure 2.

Finally, for an example of a distribution with an interior unimodal failure rate consider the Burr distribution with parameters $(2, 1)$ and $(3, 1)$ which have pdfs $f_{Burr(2,1)}(x) = \frac{2x}{(x^2+1)^2}\mathbb{1}_{x\geq 0}$ and $f_{Burr(3,1)}(x) = \frac{3x^2}{(x^3+1)^2}\mathbb{1}_{x\geq 0}$, and the inverse quantile densities $m_{Burr(2,1)}(z) = 2z^{\frac{1}{2}}(1-z)^{\frac{3}{2}}$ and $m_{Burr(3,1)}(z) = 3z^{\frac{2}{3}}(1-z)^{\frac{4}{3}}$. Then, a two-prize schedule with an interior r^* is optimal in both cases, see Figure 3.

In this section we have established that noise distributions with monotone failure rates produce one of the two extreme prize schedules – WTA or extreme prize sharing punishing the bottom performer. In the next section we bridge the gap between the IFR and DFR cases by considering distributions with unimodal failure rates, which lead to "intermediate" prize schedules being optimal.

3.3 Intermediate prize allocations

As we show in this section, the optimal number of winners r^* depends mainly on the skewness of the noise distribution. Hence, we start with the simplest case of a non-

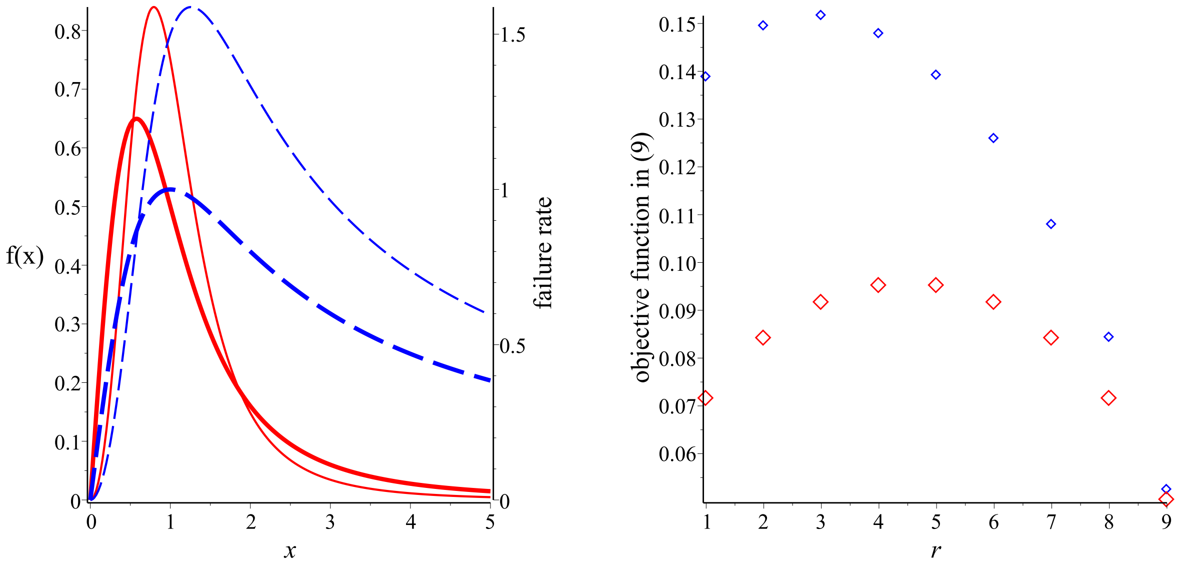

Figure 3: The Burr distribution. *Left*: Pdfs $f(x)$ (solid lines, left scale) and failure rates $\frac{f(x)}{1-F(x)}$ (dashed lines, right scale), thick lines for $Burr(2,1)$ and thin ones for $Burr(3,1)$. *Right*: The objective $\bar{\beta}_{r,n}$ for $n = 10$ is maximized at $r^* = 4$ and 5 for $Burr(2,1)$ (large diamonds) and at $r^* = 3$ for $Burr(3,1)$ (small diamonds).

skewed, that is, a symmetric distribution. In this case, r^* can be restricted as described in following proposition.

Proposition 3 *If $f(\cdot)$ is symmetric then $r^* < \frac{n}{2}$.*

Note that $f(\cdot)$ in Proposition 3 is not required to be unimodal or to have a well-behaved failure rate. Propositions 2 and 3 can be compared to the results of Krishna and Morgan (1998), who explored tournaments of $n \leq 4$ players assuming that the pdf of noise is symmetric and unimodal. For risk-neutral players, Krishna and Morgan (1998) show that the WTA schedule is optimal for $n = 2, 3$ and 4. These results follow directly from our Proposition 3, since for $n = 4$ fewer than $\frac{n}{2} = 2$ prizes are to be awarded. Note that the unimodality of $f(\cdot)$ is not needed.[11] Additionally, Proposition 2 shows the optimality of the WTA schedule for arbitrary n when the distribution of noise is IFR (such distributions are not necessarily unimodal or symmetric).

Prize schedules with $r^* < \frac{n}{2}$ can be characterized as "rewards tournaments" in that they award a prize to relatively few top performers. Similarly, prize schedules with $r^* > \frac{n}{2}$

[11]It can be shown that $n = 4$ is a tight bound for when the symmetry of $f(\cdot)$ leads to the optimality of WTA. That is, already for $n = 5$ a symmetric (and unimodal) distribution of noise can be found such that the WTA schedule is not optimal. For example, $r^* = 2$ for the t-distribution with 0.5 degrees of freedom.

can be thought of as "punishment tournaments" as they single out relatively few bottom performers receiving zero prize. These concepts of reward and punishment should be understood in a relative sense (e.g., in terms of status, see Section 4.2). Proposition 3 then implies that *skewness* in the distribution of noise is necessary for the optimality of punishment contracts.

We turn next to a general ranking of skewness of distributions which is provided by the *convex transform order*, defined by the property that $F_Y^{-1} \circ F_X$ is convex (Shaked and Shanthikumar, 2007). The convex transform order is characterized as "less skewed" and "more IFR" (Marshall and Olkin, 2007); that is, X is less positively skewed and is closer to IFR than Y (if Y is not IFR to begin with). An equivalent condition for X being smaller than Y in the convex transform order is an increasing ratio $\frac{m_X(z)}{m_Y(z)}$, where $m_X(z)$ and $m_Y(z)$ denote the inverse quantile density corresponding to random variables X and Y, respectively. Similarly, denote the optimal number of winners as r_X^* and r_Y^*.

Proposition 4 *Suppose Y has a unimodal failure rate. Consider another random variable X such that $\frac{m_X(z)}{m_Y(z)}$ is increasing. Then $r_X^* \le r_Y^*$.*

Proposition 4 states conditions under which the location of the optimal prize differential shifts to higher ranks. When Y is DFR, we have $r_Y^* = n-1$ and the result holds automatically. For the case when Y is IFR or has an interior unimodal failure rate, it can be shown that if $\bar{\beta}_{r,n}$ decreases in r for some r when the noise is Y then it also decreases in r when the noise is X. This implies, since $\bar{\beta}_{r,n}$ is unimodal for Y due to Lemma 1, that the maximum of $\bar{\beta}_{r,n}$ for X is (weakly) to the left of the maximum for Y. In particular, under the conditions of Proposition 4, when Y is IFR and the WTA tournament is optimal for Y then it is also optimal for X. The example with two Burr distributions above illustrates Proposition 4. Indeed, the ratio of quantile densities $\frac{m_{Burr(3,1)}(z)}{m_{Burr(2,1)}(z)} = \frac{3}{2}(\frac{z}{1-z})^{\frac{1}{6}}$ is increasing. Then, $r^*_{Burr(3,1)} = 3$ while $r^*_{Burr(2,1)}$ is 4 and 5, see Figure 3.

The convex transform order, therefore, provides a connection between the two extreme cases (IFR and DFR) described in Proposition 2. It is also related to Proposition 3. As the distribution of noise becomes more DFR or more positively skewed, the optimal allocation of prizes becomes more spread out.[12]

[12] For regularly varying distributions, there is a simple connection between the convex transform order and tail thickness measured by the tail index. A distribution $F(\cdot)$ is regularly varying if $1 - F(x) \equiv \bar{F}(x) = x^{-\frac{1}{\xi}} L(x)$, where $L(\cdot)$ is a slowly varying function, i.e., such that $\lim_{t\to\infty} \frac{L(xt)}{L(t)} = 1$ for all $x > 0$. Parameter $\xi > 0$ is the tail index, where a larger ξ corresponds to a heavier tail. It can be shown that if X is smaller than Y in the convex transform order and both X and Y are regularly varying then $\xi_Y \ge \xi_X$.

When failure rate is interior unimodal, part (ii) of Lemma 1 establishes that several prizes ($r^* > 1$) *may* be optimal, but does not provide a sufficient condition. The following proposition shows that if failure rate is interior unimodal then $r^* > 1$ is always optimal in sufficiently large tournaments.

Proposition 5 *If $f(\cdot)$ has an interior unimodal failure rate then $r^* > 1$ for n sufficiently large.*

Finally, the following result establishes comparative statics of r^* with respect to the number of agents for unimodal failure rates.

Proposition 6 *If $f(\cdot)$ has an interior unimodal failure rate then r^* is increasing in n.*

Recall that heavy-tailed distributions are characterized by decreasing or interior unimodal failure rates. Taken together, the results of this section show that the presence of heavy tails shifts the optimal allocation of prizes away from WTA, at least in large tournaments, and the effect becomes stronger with tournament size. DFR distributions, such as power laws, produce optimal contracts with extreme punishment.

4 Extensions and applications

In this section we apply the results from Section 3 to different but related settings. In Section 4.1, we identify a natural ordering of general prize schedules (i.e., not necessarily two-prize ones) that leads to the ordering of equilibrium effort. In Section 4.2 we apply our findings to tournaments for status, i.e., tournaments in which, following Moldovanu, Sela and Shi (2007), prizes are proportional to the difference between the number of players ranked strictly below and above a given player.

4.1 General prize schedules

As we showed in Section 3, two-prize schedules of the form $v = (\frac{1}{r}, \ldots, \frac{1}{r}, 0, \ldots, 0)$ are optimal for a principal maximizing aggregate effort. It may be, however, that two-prize schedules are not feasible for legal, institutional or political reasons. For example, in tournaments for status the number of status categories can be predetermined by an organizational or social structure. More generally, the principal may be restricted to only using prize schedules from a certain class, $\Gamma = \{v(\theta)\}_{\theta \in \Theta}$, where $v(\theta) = (v_1(\theta), \ldots, v_n(\theta))$

denotes a prize schedule, and parameter θ can be continuous or discrete and takes values in some set $\Theta \subseteq \mathbb{R}$. For example, Γ may be all linearly decreasing schedules such that $v_{r+1} = v_r - \theta$, or exponentially decreasing schedules such that $v_{r+1} = \frac{v_r}{1+\theta}$. Indeed, bonuses and pay raises in organizations may be structured as linear or percentage increases between ranks.

In this section, we explore how individual (and hence aggregate) equilibrium effort is affected by changes in a general prize schedule, i.e., how e^* given by Eq. (3) responds to changes in θ. We maintain the budget constraint and monotonicity assumptions, $\sum_{r=1}^{n} v_r(\theta) = 1$ and $v_1(\theta) \geq \ldots \geq v_n(\theta) \geq 0$, for all $\theta \in \Theta$.

Under our assumptions, a vector of prizes $v(\theta)$ can be thought of as a probability mass function (pmf) of a random variable K_θ taking values $1, \ldots, n$ such that $\Pr(K_\theta = r) = v_r(\theta)$. Let $V_r(\theta) = \sum_{k=1}^{r} v_k(\theta)$ denote the corresponding cumulative mass function (cmf), which is increasing in r and satisfies $V_n(\theta) = 1$. Equation (3) then can be written as $c'(e^*) = \mathrm{E}(\beta_{K_\theta,n})$, where the expectation is taken with respect to random variable K_θ.

Comparing general monotone prize schedules requires stronger conditions on the distributions than in Section 3. In particular, monotonicity of $\beta_{r,n}$ implies the monotonicity of $\bar{\beta}_{r,n}$ but the converse is not true. From parts (iii) and (iv) of Lemma 3 in the Appendix, $\beta_{r,n} = B_{r,n} - B_{r-1,n}$ is decreasing (increasing for $r = 1, \ldots, n-1$) if $f(\cdot)$ is log-concave (log-convex with $f(\overline{x}) = 0$). The following result is a direct application of first-order stochastic dominance.

Proposition 7 *Suppose $V_r(\theta)$ is increasing in θ for all $r = 1, \ldots, n$.*
(i) If $f(\cdot)$ is log-concave then e^* is increasing in θ.
(ii) If $f(\cdot)$ is log-convex, $f(\overline{x}) = 0$ and $v_n(\theta)$ is independent of θ, then e^* is decreasing in θ.

Indeed, consider some $\theta' > \theta$ and suppose $V_r(\theta') \geq V_r(\theta)$ for all $r = 1, \ldots, n$. This implies that K_θ dominates $K_{\theta'}$ in the FOSD order and, therefore, $\mathrm{E}(\beta_{K_{\theta'},n}) \geq \mathrm{E}(\beta_{K_\theta,n})$ when $\beta_{r,n}$ is decreasing in r. This result is rather intuitive. We already know from part (i) of Proposition 2 that the WTA tournament maximizes effort for log-concave distributions since they are IFR. The (downward) FOSD shift in K_θ produces a more sharply decaying prize schedule with larger differentials at the top, making it more similar to the WTA schedule and thus leading to a higher equilibrium effort.

A useful alternative interpretation of Proposition 7 can be obtained using the notions of majorization and Schur convexity (Marshall, Olkin and Arnold, 2011). For two vectors $v, v' \in \mathbb{R}^n_+$ whose components are arranged in descending order and sum up to one, v

majorizes v' if $\sum_{k=1}^{r} v_r \geq \sum_{k=1}^{r} v'_r$ for all $r = 1, \ldots, n$. When applied to prize schedules, such *majorization order* is a natural ordering of prize schedules by how closely they approximate WTA. A function $g(v)$ is called *Schur-convex* if it preserves majorization order; that is, $g(v) \geq g(v')$ whenever v majorizes v'. Proposition 7 then implies that the equilibrium effort is Schur-convex in the prizes when $f(\cdot)$ is log-concave. Similarly, it is Schur-concave for prize schedules with a fixed v_n when $f(\cdot)$ is log-convex and $f(\overline{x}) = 0$.

To illustrate Proposition 7, consider a class of linearly decreasing prize schedules, $v_r(\theta) = \frac{1}{n} + \theta\left(\frac{n+1}{2} - r\right)$, where $\theta \in \Theta = \left(0, \frac{2}{n(n-1)}\right]$. A higher θ makes the decline in prizes steeper, the normalization $\sum_{r=1}^{n} v_r(\theta) = 1$ is maintained, and the restriction $\theta \leq \frac{2}{n(n-1)}$ ensures that $v_n \geq 0$. The cmf for this prize schedule, $V_r(\theta) = \frac{r}{n} + \frac{\theta r(n-r)}{2}$, is increasing in θ, implying that an increase in θ leads to a downward FOSD shift in the distribution of prizes across ranks. It then follows from Proposition 7 that, assuming $f(\cdot)$ is log-concave, e^* is increasing in θ. The results are similar for a class of exponentially decreasing prize schedules, with $v_r(\theta) = \frac{\theta(1+\theta)^{n-r}}{(1+\theta)^n - 1}$, where cmf $V_r(\theta) = \frac{(1+\theta)^n - (1+\theta)^{n-r}}{(1+\theta)^n - 1}$ is also increasing in θ.

4.2 Tournaments for status

Consider tournaments for status where a player's reward is modeled as the difference between the number of players ranked below and above her (Moldovanu, Sela and Shi, 2007; Dubey and Geanakoplos, 2010). Following the setup of Moldovanu, Sela and Shi (2007) (henceforth, MSS07), let $P = (C_1, \ldots, C_k)$ denote a partition of the sequence of ranks $(1, \ldots, n)$ into $k \geq 2$ status categories $C_i = \{r_{i-1} + 1, \ldots, r_i\}$. Rank r_i serves as the upper bound of category i, with $r_0 = 0$ and $r_k = n$. The status prize of a player in category i is given by[13]

$$w_i = (n - r_i) - r_{i-1}.$$

These status prizes produce the complete set of rank-based prizes $v_r = w_i$ such that $r \in C_i$. The corresponding prize differentials $d_r = v_r - v_{r+1}$ are zero for $r \neq r_i$, whereas

$$d_{r_i} = w_i - w_{i+1} = r_{i+1} - r_{i-1}.$$

[13]The definition of MSS07 has the opposite sign because in their formulation prizes are increasing in rank and C_k is the top category.

For a given partition P, the principal's objective then becomes

$$T_P = \sum_{r=1}^{n-1} B_{r,n} d_r = \sum_{i=1}^{k-1} B_{r_i,n}(r_{i+1} - r_{i-1}). \tag{11}$$

The following proposition summarizes our results.

Proposition 8 *In tournaments for status,*
(i) If $f(\cdot)$ is IFR then the top status category containing a unique element is optimal.
(ii) If $f(\cdot)$ is log-concave then the finest partition with one element in each category is optimal.
(iii) If $f(\cdot)$ is DFR then the top status category containing $n-1$ elements is optimal.
(iv) If $f(\cdot)$ has an interior unimodal failure rate then the top status category containing more than one element is optimal for n sufficiently large; moreover, for n sufficiently large an arbitrary partition can be improved by combining its top two categories.

It is instructive to compare Proposition 8 to the results of MSS07. First, their Theorem 3 states that the top status category (C_1 in our formulation) always contains a unique element. In our case, the top category can contain multiple elements in the presence of heavy tails, all the way up to $n-1$ players when the distribution of noise is DFR. In order to understand the source of the difference, rewrite objective (11) using (10) as

$$T_P = \sum_{i=1}^{k-1} \bar{\beta}_{r_i,n} r_i (r_{i+1} - r_{i-1}) = \frac{1}{n} \sum_{i=1}^{k-1} (r_{i+1} - r_{i-1}) r_i \mathrm{E}(h(Z_{(n-r_i:n)})).$$

This representation is similar to the one in Theorem 1 of MSS07, except in their model the expectation $\mathrm{E}(h(Z_{(n-r_i:n)}))$ is replaced by $\mathrm{E}(A_{(n-r_i:n)})$, the expectation of the $(n-r_i)$-th order statistic of ability. Randomly assigned abilities in MSS07 are similar to noise in our model and lead to the same assortative matching to prizes in equilibrium; however, the information structure of the game and equilibrium configuration are different. As a result, the objective in our case is determined by the expectations of the hazard rate of order statistics.

The second major result (Theorem 4) of MSS07 is that the finest partition, with each category containing one element, is optimal for IFR distributions of abilities. This is based on a result of Barlow and Proschan (1966) about the stochastic ordering of *spacings* – distances between adjacent order statistics. In fact, the MSS07 result relies only on a property that restricts the growth rate of expectations of spacings with rank.

In our model, incentives are determined by "quasi-spacings" $\beta_{r,n} = B_{r,n} - B_{r-1,n}$, which are distances between expectations of random variable $m(\cdot)$ over adjacent order statistics $Z_{(n-r:n-1)}$ and $Z_{(n-r+1:n-1)}$. The IFR property alone is not sufficient in our case because the rate of growth of quasi-spacings may be too high when $m(\cdot)$ is convex. The log-concavity of $f(\cdot)$ ensures that $m(\cdot)$ is concave and hence quasi-spacings behave similar to spacings in MSS07.

5 Equilibrium existence

General equilibrium existence and uniqueness conditions have so far eluded the literature on Lazear-Rosen tournaments. Qualitatively, the existence of a pure-strategy equilibrium requires that the distribution of noise be sufficiently dispersed and the number of players not too large (Nalebuff and Stiglitz, 1983). Our conditions below formalize these requirements. The focus of this paper is on the effects of noise on the properties of optimal prize allocations; hence, we impose minimal restrictions on the *shape* of the distribution of noise but substantially restrict its dispersion and the cost function $c(e)$. By construction, all effort levels above $c^{-1}(1)$ are strictly dominated; therefore, we only have to restrict the behavior of $c(e)$ on the interval $[0, c^{-1}(1)]$.

Let $f_m = \sup\{f(x) : x \in U\}$, $f'_{\max} = \sup\{f'(x) : x \in U\}$ and $f'_{\min} = \inf\{f'(x) : x \in U\}$ denote the tight, possibly infinite, bounds of pdf $f(\cdot)$ and its derivative $f'(\cdot)$ on U. We impose the following restrictions on the pdf of noise.

Assumption 1 *(a) $f(\cdot)$ is uniformly bounded; that is, $f_m < \infty$.*
(b) $f'(\cdot)$ is uniformly bounded above or below or both; that is, either $f'_{\max} < \infty$ or $f'_{\min} > -\infty$ or both.

Further, introduce the bounds

$$D_- = \binom{n-1}{\lfloor \frac{n}{2} \rfloor}(f_m^2 - f'_{\min}), \quad D_+ = \binom{n-1}{\lfloor \frac{n}{2} \rfloor}[(n-1)f_m^2 + f'_{\max}]. \tag{12}$$

Under Assumption 1, at least one of these bounds is finite. The following proposition provides the equilibrium existence result.

Proposition 9 *Suppose Assumption 1 is satisfied and*
(a) There exists a $c_0 > 0$ such that $c''(e) \geq c_0$ on $[0, c^{-1}(1)]$.
(b) $c_0 > D \equiv \min\{D_-, D_+\}$.

(c) $nc(c'^{-1}(f_m)) \le 1$.

Then e^* *given by (3) is the unique symmetric equilibrium effort in the tournament.*

Let us comment on the conditions in Proposition 9 in reverse order. Condition (c) ensures that Eq. (3) has a solution and the equilibrium payoff $u(e^*, e^*)$ is nonnegative. For the solution to exist, we show that the left-hand side of (3) is bounded by f_m; hence, $c(c'^{-1}(f_m)) \le 1$ is sufficient. Factor n on the left-hand side makes condition (c) even stronger in order to guarantee that $u(e^*, e^*) = \frac{1}{n} - c(e^*) \ge 0$.

Condition (b) ensures the strict concavity of $u(e, e^*)$ in e for $e \in [0, c^{-1}(1)]$. With the curvature of the cost function restricted from below by c_0 (condition (a)), constant D serves as an upper bound on the curvature of the revenue part of payoff (1), $R(e) = \sum_{r=1}^{n} \pi^{(r)}(e, e^*) v_r$. As we show, both D_- and D_+ defined in (12) can serve as bounds for $R''(e)$, and hence only one of them has to be finite and D is defined as the minimum of the two.

The two bounds for $R''(e)$ are due to two different representations for the marginal revenue function $R'(e)$ (cf. Eqs. (22) and (21) in the proof). The first representation is a linear combination of expectations of the form $\mathrm{E}(f(X_{n-r:n-1} - e + e^*))$, where $X_{n-r:n-1}$ denotes the $(n-r)$-th order statistic in the sample of $n-1$ i.i.d. draws of noise X. These expectations represent the densities of differences $X_{n-r:n-1} - X$ evaluated at $e - e^*$, and curvature $R''(e)$ is a linear combination of slopes of these densities with respect to e. The corresponding upper bound on $R''(e)$, therefore, involves an upper bound of $-f'(\cdot)$, and hence the presence of $-f'_{\min}$ in D_-. But the same densities can also be written as expectations $\mathrm{E}(f_{n-r:n-1}(X + e - e^*))$, where $f_{n-r:n-1}(\cdot)$ is the pdf of order statistic $X_{n-r:n-1}$. In this case, an upper bound on $R''(e)$ involves an upper bound of $f'(\cdot)$, through $f'_{n-r:n-1}(\cdot)$, and hence the presence of $f'_{\max}$ and an extra factor on f_m^2 in D_+.

The binomial coefficient $\binom{n-1}{\lfloor \frac{n}{2} \rfloor}$ arises in the bounds (12) for condition (b) because of the requirement that the equilibrium must exist for all monotone prize schedules v. For any specific prize schedule, such as WTA, condition (b) can be further relaxed. When the cost function is quadratic, condition (b) implies condition (c).

Conditions (b) and (c) combine all the ingredients previously identified in the literature as critical for the existence of pure-strategy equilibria in various tournament models. The bounds f_m, $-f'_{\min}$ and $f'_{\max}$ decrease as noise becomes more dispersed. Both conditions (b) and (c) become more restrictive as the number of players, n, increases. This is in line with the formal results in Tullock contests where the unique symmetric equilibrium seizes to exist when the discriminatory power parameter measuring the dispersion of noise becomes too high or when there are too many players (Schweinzer and Segev, 2012).

The boundedness conditions in Assumption 1 are satisfied for many widely used distributions such as Normal, Gumbel, Laplace, Cauchy, Pareto, exponential and uniform distributions. They also allow for unbounded $f'(\cdot)$ from above or below, which is the case for Beta and lognormal distributions for some parameter values.

Condition (a) defines a class of cost functions with a second derivative bounded away from zero. Quadratic costs are, obviously, in this class. More generally, such functions can be written in the form $c(e) = \frac{c_0}{2}e^2 + \kappa(e)$, where $\kappa(\cdot)$ is a convex function on $[0, c^{-1}(1)]$, with $\kappa(0) = \kappa'(0) = 0$. This class of functions is related but not equivalent to functions that are convex transforms of e^2.[14] Generalizing the observation above, condition (b) implies condition (c) when $\kappa(\cdot)$ is positive and increasing.

6 Discussion and conclusions

In this paper, we study how a tournament prize budget should be allocated in order to maximize aggregate effort. Our focus is on settings where, in addition to effort, agents' performance is affected by noise – fluctuations beyond their control. The existing models of noisy tournaments, going back to Lazear and Rosen (1981), treat noise mostly as a nuisance. We argue, however, that the properties of noise – most notably, the behavior of its upper tail – can have drastic consequences for the structure of optimal tournament contracts. Specifically, the presence of a heavy upper tail shifts optimal contracts from winner-take-all to more equitable prize sharing, more so the heavier the tail, in a well-defined sense (of the convex transform order or increasing skewness).

Prize sharing, whereby tournament prize money is divided among multiple winners, is common in many areas. For example, the US federal government uses multiple awards in its indefinite-delivery/indefinite-quantity (IDIQ) procurement processes.[15] In R&D competition, the XPRIZE Foundation offers multi-prize innovation tournaments that "...

[14] It can be shown that function $c(e) = \frac{c_0}{2}e^2 + \kappa(e)$ is a convex transform of e^2 if and only if $\kappa(e)$ is a convex transform of e^2. For example, $c(e) = \frac{c_0}{2}e^2 + c_1 e^{\frac{3}{2}}$ has a bounded second derivative but is not a convex transform of e^2, whereas $c(e) = c_1 e^4$ is a convex transform of e^2 but does not have a bounded second derivative. More generally, consider functions $c(e) = c_1 e^{\xi}$ with $\xi > 1$, $c_1 > 0$. Then $c''(e) = c_1\xi(\xi-1)e^{\xi-2} \geq c_0$ for all $e \in [0, c_1^{-\frac{1}{\xi}}]$ if $\xi \in (1, 2]$ and $c_0 \leq c_1^{\frac{2}{\xi}}\xi(\xi-1)$. Thus, a power function can only have bounded second derivative when it is a *concave* transform of e^2.

[15] Halac, Kartik and Liu (2017) provide a few examples. For a discussion of multiple-award procurement contracts, see https://www.govexec.com/federal-news/2011/03/sba-considering-mandating-set-asides-on-multiple-award-contracts/33644/.

[push] the limits of what's possible to change the world for the better."[16] TopCoder – a crowdsourcing platform using coding contests to deliver designs and applications to interested buyers[17] – provides a variety of multi-prize incentives for competitive coders.

Many Internet companies, including Microsoft, Amazon, Facebook and Google, conduct large-scale online experiments with users exploring the effectiveness of various strategies and innovations using a technique termed "A/B testing" (Kohavi and Thomke, 2017). The problem of optimal design of A/B testing is essentially a multi-armed bandit problem where a fixed resource – the number of tests – must be allocated optimally among multiple levers – potentially profitable innovations with uncertain returns. Recent data from Microsoft Bing's EXP platform[18] indicate the presence of heavy tails in these returns (Azevedo et al., 2018). Similar to our results, Azevedo et al. (2018) show that the optimal A/B testing strategy changes dramatically in the presence of heavy tails; specifically, while relatively few high-powered tests are optimal for light-tailed distributions, a larger number of smaller-scale experiments – a form of resource sharing – becomes optimal with heavy tails.

Another setting where tournament incentives are common is sales contests. Lim, Ahearne and Ham (2009) provide a number of examples of such contests across industries, using both WTA (Avis Rent A Car) and prize sharing (Phillips Foods, Webb Furniture and NBC) reward schemes. The authors also present the results of two field experiments with groups of 15 professional salespeople, comparing their performance under various prize schedules. The first experiment compares a WTA schedule with a top prize of \$300 to a prize sharing scheme awarding \$60 to top five employees. These schedules are ranked by the majorization order, and the second scheme produced more than 50% higher sales. Our results then suggest that the distribution of noise the salespeople faced was heavy-tailed. The second experiment compares two prize sharing schemes – $(40, 40, 40, 40, 40, 0, \ldots, 0)$ and $(70, 51, 36, 23, 13, 7, 0, \ldots, 0)$ – that are not ranked by majorization, and our predictions have no bite. Consistently, no significant difference in average sales is registered in the second experiment.

Heavy-tailed distributions – the Pareto distribution being the primary example – have been found to describe a wide range of phenomena in economics, finance and other domains, such as distributions of income, wealth, firm sizes, insurance losses (Kleiber and Kotz, 2003), or the sizes of sand particles and oil fields (Reed and Jorgensen, 2004).

[16] See https://www.xprize.org/.
[17] See https://www.topcoder.com.
[18] See https://exp-platform.com/.

Moreover, heavy-tailed shocks are becoming more prominent in the new, post-industrial world of innovation, crowdsourcing, and Internet-based delivery of goods and services (e.g., Fleming, 2007). Our results, therefore, imply more broadly that in a world where individual success is shaped to a large extent by luck (Frank, 2016), and luck is heavy-tailed, prize sharing not only serves as a form of insurance against the pitfalls of a winner-take-all society, but is actually efficient in terms of productivity.

The model we use is rather stylized. First, we assume that players are risk-neutral. As a consequence, optimal prize schedules only include two distinct prizes. Thus, the *number* of top prizes serves as a natural measure of prize sharing in our setting. This is a simplification as compared to gradually decaying prizes obtained in all-pay auctions with convex costs (Moldovanu and Sela, 2001) or tournaments with risk-averse players (Krishna and Morgan, 1998; Akerlof and Holden, 2012). However, results for risk-averse players in our setting can be derived using essentially the same techniques, and, although they are not as clear-cut or explicit, the qualitative effects of noise are very similar. Importantly, for DFR distributions the extreme punishment contract with $n-1$ equal prizes is still optimal regardless of risk-aversion. Second, we assume that players are symmetric. This assumption is helpful in that marginal incentives in equilibrium, and hence the structure of optimal contract, are determined entirely by the properties of noise. The same will hold approximately for weakly asymmetric players as long as participation constraints are not binding (cf. Balafoutas et al., 2017). When players are strongly heterogeneous, the influence of noise is diminished and equilibrium characterization is difficult. That said, due to the standard efficiency argument (Lazear and Rosen, 1981), settings with symmetric (and weakly asymmetric) players are the most relevant applications for tournament contracts.

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Appendix

Several results below rely on Lemma 2 that we state without proof and refer the reader to Ryvkin and Drugov (2017) for details. Essentially, Lemma 2 can be obtained via integration by parts from the result of Athey (2002) on the preservation of single-crossing under uncertainty. The following definition is used in the lemma.

Definition 1 *Function $\psi : S_1 \times S_2 \to \mathbb{R}$, where $S_1, S_2 \subseteq \mathbb{R}$, is log-supermodular if for any $x_1, x_1' \in S_1$, $x_2, x_2' \in S_2$ such that $x_1' > x_1$, $x_2' > x_2$,*

$$\psi(x_1, x_2')\psi(x_1', x_2) \leq \psi(x_1, x_2)\psi(x_1', x_2'),$$

or, equivalently, the ratio $\frac{\psi(x_1', x_2)}{\psi(x_1, x_2)}$ is increasing in x_2.

Suppose function $\phi(z) : [0, 1] \to \mathbb{R}$ is integrable, continuous, differentiable almost everywhere; and $H(z, \theta)$ is a cdf of a random variable $Z|\theta$ defined on $[0, 1]$ and parameterized by $\theta \in \Theta \subseteq \mathbb{R}$. Suppose $Z|\theta$ increases in θ in the usual stochastic order (FOSD); that is, $H(z, \theta)$ is decreasing in θ. Let $H_\theta(z, \theta) \leq 0$ denote the derivative of $H(z, \theta)$ with respect to θ if θ is a continuous parameter (in which case we assume that $H(z, \theta)$ is differentiable) or the first difference, $H(z, \theta + \Delta) - H(z, \theta)$, if θ is a discrete index with step size $\Delta > 0$. The following lemma is a simplified version (sufficiency part only) of Lemma 1 from Ryvkin and Drugov (2017).

Lemma 2 *If $\phi(z)$ is unimodal and $|H_\theta(z,\theta)|$ is log-supermodular then $\gamma(\theta) = \int_0^1 u(z)dH(z,\theta)$ is unimodal.*

Lemma 3 *For any $1 \le r \le n-1$,*
(i) If $f(\cdot)$ is increasing (decreasing) then $B_{r,n}$ is decreasing (increasing) in r and increasing (decreasing) in n.
(ii) If $f(\cdot)$ is unimodal (U-shaped) then $B_{r,n}$ is unimodal (U-shaped) in r and in n.
(iii) If $f(\cdot)$ is log-concave then $B_{r,n}$ is concave in r.
(iv) If $f(\cdot)$ is log-convex and $f(\overline{x}) = 0$ then $B_{r,n}$ is convex in r for $r = 1,\dots,n-1$.
(v) If $f(\cdot)$ is symmetric then $B_{r,n}$ is symmetric in r, with $B_{r,n} = B_{n-r,n}$.

Proof of Lemma 3 From Eq. (7), coefficients $B_{r,n}$ can be written in the form

$$B_{r,n} = \int_0^1 m(z)dF^B(z; n-r, r) = \mathrm{E}(m(Z_{(n-r:n-1)})), \tag{13}$$

where $F^B(z;x,y)$ is the cdf of the beta distribution with parameters (x,y) (regularized incomplete beta function) and $Z_{(n-r:n-1)}$ is the $(n-r)$-th order statistic among $n-1$ i.i.d. draws from the uniform distribution on $[0,1]$. We will use the following properties of the regularized incomplete beta function (Paris, 2010):

$$F^B(z; x+1, y) = F^B(z;x,y) - \frac{z^x(1-z)^y}{x\mathcal{B}(x,y)} \tag{14}$$

$$F^B(z; x, y+1) = F^B(z;x,y) + \frac{z^x(1-z)^y}{y\mathcal{B}(x,y)}. \tag{15}$$

Here, $\mathcal{B}(x,y)$ is the beta function.

By construction, order statistics $Z_{n-r:n-1}$ are FOSD-decreasing in r. Property (14) implies that they are also FOSD-increasing in n. Part (i) then follows directly from (13).

For part (ii), we will first show that $F^B_r(z; n-r, r) = F^B(z; n-r-1, r+1) - F^B(z; n-$

$r,r)$ is log-supermodular in $(-r,z)$. Using properties (14) and (15), obtain

$$\begin{aligned}
&F^B(z;n-r-1,r+1)-F^B(z;n-r,r)\\
&=F^B(z;n-r-1,r+1)-F^B(z;n-r,r+1)+F^B(z;n-r,r+1)-F^B(z;n-r,r)\\
&=\frac{z^{n-r-1}(1-z)^{r+1}}{(n-r-1)\mathcal{B}(n-r-1,r+1)}+\frac{z^{n-r}(1-z)^{r}}{r\mathcal{B}(n-r,r)}\\
&=\frac{z^{n-r-1}(1-z)^{k+1}(n-1)!}{(n-r-1)(n-r-2)!r!}+\frac{z^{n-r}(1-z)^{r}(n-1)!}{r(n-r-1)!(r-1)!\mathcal{B}(n-r,r)}\\
&=\binom{n-1}{r}z^{n-r-1}(1-z)^r.
\end{aligned}$$

Thus, for any $r'<r$ the ratio $\frac{F_k^B(z;n-r',r')}{F_k^B(z;n-r,r)}\propto\left(\frac{z}{1-z}\right)^{r-r'}$ is increasing in z; therefore, $F_r^B(z;n-r,r)$ is log-supermodular in $(-r,z)$.

Second, we will show that $|F_n^B(z;n-r,r)|=|F^B(z;n+1-r,r)-F^B(z;n-r,r)|$ is log-supermodular in (n,z). Indeed, from (14) $|F_n^B(z;n-r,r)|=\frac{z^{n-r}(1-z)^k}{(n-r)\mathcal{B}(n-r,r)}$, which gives, for some $n'>n$,

$$\frac{F_n^B(z;n'-r,r)}{F_n^B(z;n-r,r)}=\frac{(n'-r)\mathcal{B}(n'-r,r)z^{n'-n}}{(n-r)\mathcal{B}(n-r,r)}.$$

The above ratio is increasing in z, i.e., indeed $|F_n^B(z;n-r,r)|$ is log-supermodular in (n,z). The results then follow from Lemma 2.

For part (iii), note that if $f(\cdot)$ is log-concave then $m(\cdot)$ is concave. We will show that in this case $\beta_{r,n}=B_{r,n}-B_{r-1,n}$ is decreasing in r. Integrating Eq. (6) by parts, obtain

$$\begin{aligned}
\beta_{r,n}&=\binom{n-1}{r-1}\int_0^1 m(z)z^{n-r-1}(1-z)^{r-2}[n-r-(n-1)z]dz \qquad (16)\\
&=\binom{n-1}{r-1}\int_0^1 m(z)d[z^{n-r}(1-z)^{r-1}]\\
&=\binom{n-1}{r-1}\left[m(z)z^{n-r}(1-z)^{r-1}\Big|_0^1-\int_0^1 z^{n-r}(1-z)^{r-1}m'(z)dz\right]\\
&=\binom{n-1}{r-1}[m(1)\mathbb{1}_{r=1}-m(0)\mathbb{1}_{r=n}]-\frac{1}{n}\frac{n!}{(n-r)!(r-1)!}\int_0^1 z^{n-r}(1-z)^{r-1}m'(z)dz\\
&=m(1)\mathbb{1}_{r=1}-m(0)\mathbb{1}_{r=n}-\frac{1}{n}\int_0^1 m'(z)dF^B(z;n-r+1,r).
\end{aligned}$$

Here, $F^B(z;n-r+1,r)$ is the cdf of order statistic $Z_{n+1-r:n}$. These order statistics are FOSD-decreasing in r; therefore, given that $m'(z)$ is decreasing, the integral is increasing in r. The first term in the expression above is equal to $m(1)$ for $r=1$, $-m(0)$ for $r=n$

and 0 otherwise; hence, it is decreasing in r. Thus, combined we have a sequence that is decreasing in r.

For part (iv), note that if $f(\cdot)$ is log-convex then $m(\cdot)$ is convex, and $f(\overline{x}) = 0$ implies $m(1) = 0$. Eq. (16) then gives a sequence that is increasing in r for $r = 1, \ldots, n-1$.

The symmetry of $f(\cdot)$ implies that $m(1-z) = m(z)$. Part (v) then follows directly from (13). ■

Proof of Lemma 1 Part (i) follows immediately from representation (10). Part (ii) is immediate from the proof of part (ii) of Lemma 3 above. ■

Proof of Proposition 3 For a symmetric $f(\cdot)$, $B_{r,n} = B_{n-r,n}$ by Lemma 3(v). This implies $\bar{\beta}_{r,n} = \frac{B_{r,n}}{r} > \frac{B_{n-r,n}}{n-r} = \bar{\beta}_{n-r,n}$ for $r < \frac{n}{2}$. To rule out the case $r^* = \frac{n}{2}$ when n is even and $n \geq 4$, we will show that $\bar{\beta}_{\frac{n}{2}-1,n} > \bar{\beta}_{\frac{n}{2},n}$. Indeed, using (13),

$$\begin{aligned}
&\bar{\beta}_{\frac{n}{2}-1,n} - \bar{\beta}_{\frac{n}{2},n} = \frac{B_{\frac{n}{2}-1,n}}{\frac{n}{2}-1} - \frac{B_{\frac{n}{2},n}}{\frac{n}{2}} \\
&= \binom{n-1}{\frac{n}{2}-1} \int_0^1 z^{n-(\frac{n}{2}-1)-1}(1-z)^{\frac{n}{2}-1-1} m(z)dz - \binom{n-1}{\frac{n}{2}} \int_0^1 z^{n-\frac{n}{2}-1}(1-z)^{\frac{n}{2}-1} m(z)dz \\
&= \binom{n-1}{\frac{n}{2}} \int_0^1 z^{\frac{n}{2}-1}(1-z)^{\frac{n}{2}-1} \frac{2z-1}{1-z} m(z)dz.
\end{aligned}$$

We now show that the last integral is positive, by writing it as

$$\begin{aligned}
&-\int_0^{\frac{1}{2}} z^{\frac{n}{2}-1}(1-z)^{\frac{n}{2}-1} \frac{1-2z}{1-z} m(z)dz + \int_{\frac{1}{2}}^1 z^{\frac{n}{2}-1}(1-z)^{\frac{n}{2}-1} \frac{2z-1}{1-z} m(z)dz \\
&> -\int_0^{\frac{1}{2}} z^{\frac{n}{2}-1}(1-z)^{\frac{n}{2}-1} \frac{1-2z}{\frac{1}{2}} m(z)dz + \int_{\frac{1}{2}}^1 z^{\frac{n}{2}-1}(1-z)^{\frac{n}{2}-1} \frac{2z-1}{\frac{1}{2}} m(z)dz = 0.
\end{aligned}$$

The inequality follows by replacing the term $\frac{1}{1-z}$ by its maximum value in the first integral and minimum value in the second integral. The resulting expression is equal to zero due to the symmetry of $m(z)$ around $z = \frac{1}{2}$. ■

Proof of Proposition 4 If Y is DFR, $r_Y^* = n-1$ and the result holds automatically. Suppose Y is IFR or with an interior unimodal failure rate. It is sufficient to show that if $\bar{\beta}_{r,n,Y}$ is decreasing for some r then $\bar{\beta}_{r,n,X}$ is decreasing for that same r; that is, we will show that if $\frac{B_{r-1,n,Y}}{r-1} \geq \frac{B_{r,n,Y}}{r}$ then $\frac{B_{r-1,n,X}}{r-1} \geq \frac{B_{r,n,X}}{r}$.

Suppose $\frac{B_{r-1,n,Y}}{r-1} \geq \frac{B_{r,n,Y}}{r}$, then $\frac{B_{r-1,n,Y}}{B_{r,n,Y}} \geq \frac{r-1}{r}$. It is sufficient to show that $\frac{B_{r-1,n,X}}{B_{r,n,X}} \geq$

$\frac{B_{r-1,n,Y}}{B_{r,n,Y}}$, or, equivalently (denoting $\rho(z) = \frac{m_X(z)}{m_Y(z)}$),

$$\frac{\int_0^1 z^{n-r}(1-z)^{r-2}\rho(z)m_Y(z)dz}{\int_0^1 z^{n-r-1}(1-z)^{r-1}\rho(z)m_Y(z)dz} \geq \frac{\int_0^1 z^{n-r}(1-z)^{r-2}m_Y(z)dz}{\int_0^1 z^{n-r-1}(1-z)^{r-1}m_Y(z)dz}.$$

Define function $a(z) = z^{n-r-1}(1-z)^{r-1}m_Y(z)$. The inequality above then can be rewritten as

$$\frac{\int_0^1 \frac{z}{1-z}a(z)\rho(z)dz}{\int_0^1 a(z)\rho(z)dz} \geq \frac{\int_0^1 \frac{z}{1-z}a(z)dz}{\int_0^1 a(z)dz}.$$

Note that $\frac{z}{1-z}$ is an increasing function; therefore, the inequality will hold if the distribution with pdf $\frac{a(z)\rho(z)}{\int_0^1 a(z')\rho(z')dz'}$ FOSD the distribution with pdf $\frac{a(z)}{\int_0^1 a(z')dz'}$. Thus, we need to show that for any $z \in [0,1]$ we have

$$\frac{\int_0^z a(z')\rho(z')dz'}{\int_0^1 a(z')\rho(z')dz'} \leq \frac{\int_0^z a(z')dz'}{\int_0^1 a(z')dz'}$$

or

$$\int_0^z a(z')\rho(z')dz' \int_0^1 a(z')dz' \leq \int_0^z a(z')dz' \int_0^1 a(z')\rho(z')dz'.$$

Subtracting $\int_0^z a(z')\rho(z')dz' \int_0^z a(z')dz'$ from both sides, obtain

$$\int_0^z a(z')\rho(z')dz' \int_z^1 a(z')dz' \leq \int_0^z a(z')dz' \int_z^1 a(z')\rho(z')dz'.$$

But $\rho(z)$ is an increasing function; therefore, the left-hand side of this inequality does not exceed $\rho(z)\int_0^z a(z')dz' \int_z^1 a(z')dz'$, whereas the right-hand side is not less than the same number, and hence the inequality holds. ■

Proof of Proposition 5 It suffices to show that $\bar{\beta}_{2,n} > \bar{\beta}_{1,n}$ for a large enough n. From (10),

$$\bar{\beta}_{r+1,n} - \bar{\beta}_{r,n} = \frac{(n-1)!}{(n-2-r)!(r+1)!}\int_0^1 z^{n-2-r}(1-z)^{r+1}h(z)dz$$
$$- \frac{(n-1)!}{(n-1-r)!r!}\int_0^1 z^{n-1-r}(1-z)^r h(z)dz$$
$$= \frac{(n-1)!}{(n-1-r)!(r+1)!}\int_0^1 z^{n-2-r}(1-z)^r[(n-1-r)(1-z)-(r+1)z]h(z)dz.$$

Let $s_{r,n}$ denote the the integral in the expression above. Integrating by parts, obtain

$$s_{r,n} = \int_0^1 h(z) d[z^{n-1-r}(1-z)^{r+1}] = -\int_0^1 z^{n-1-r}(1-z)^{r+1} h'(z) dz. \tag{17}$$

Here, we used the fact that $h(z)$ is interior unimodal and hence $h(0)$ and $h(1)$ are finite. We also assumed that $r < n-1$.

Let $\hat{z}$ denote a mode of $h(z)$ such that $h(z)$ is increasing (decreasing) for $z \in [0, \hat{z}]$ ($z \in [\hat{z}, 1]$). Splitting the integral in (17), obtain,

$$\begin{aligned}
s_{r,n} &= -\int_0^{\hat{z}} z^{n-1-r}(1-z)^{r+1} h'(z) dz + \int_{\hat{z}}^1 z^{n-1-r}(1-z)^{r+1} |h'(z)| dz \\
&\geq -M_1 \int_0^{\hat{z}} z^{n-1-r} dz + \hat{z}^{n-1-r} \int_{\hat{z}}^1 (1-z)^{r+1} |h'(z)| dz \\
&= -\frac{M_1 \hat{z}^{n-r}}{n-r} + \hat{z}^{n-1-r} M_2 = \hat{z}^{n-1-r} \left(M_2 - \frac{M_1 \hat{z}}{n-r} \right).
\end{aligned}$$

Here, $M_1 = \sup_{z \in [0,\hat{z}]} (1-z)^{r+1} h'(z)$ and M_2 – the integral in the second line above – are positive constants independent of n. The resulting expression becomes positive for n sufficiently large. The result follows by setting $r = 1$. ■

Proof of Proposition 6 We know from Lemma 1 that $\bar{\beta}_{r,n}$ is unimodal for a unimodal failure rate. It is, therefore, sufficient to show that if $\bar{\beta}_{r+1,n} \geq \bar{\beta}_{r,n}$ for some (r, n) then $\bar{\beta}_{r+1,n+1} \geq \bar{\beta}_{r,n+1}$.

Suppose $\bar{\beta}_{r+1,n} \geq \bar{\beta}_{r,n}$. Then, using the notation introduced in the proof of Proposition 5, $s_{r,n} \geq 0$ and we need to show that $s_{r,n+1} \geq 0$. With $\hat{z}$ denoting a mode of $h(z)$, Eq. (17) gives

$$\begin{aligned}
s_{r,n+1} &= -\int_0^1 z^{n-r}(1-z)^{r+1} h'(z) dz \\
&= -\int_0^{\hat{z}} z^{n-r}(1-z)^{r+1} h'(z) dz + \int_{\hat{z}}^1 z^{n-r}(1-z)^{r+1} |h'(z)| dz \\
&> -\hat{z} \int_0^{\hat{z}} z^{n-r-1}(1-z)^{r+1} h'(z) dz + \hat{z} \int_{\hat{z}}^1 z^{n-r-1}(1-z)^{r+1} |h'(z)| dz \\
&= -\hat{z} \int_0^1 z^{n-r-1}(1-z)^{r+1} h'(z) dz = \hat{z} s_{r,n} \geq 0.
\end{aligned}$$

The first inequality is strict because $h(z)$ is nonconstant. ■

Proof of Proposition 8 The proof follows the proofs of Theorems 3 and 4 of Moldovanu, Sela and Shi (2007) with appropriate modifications. For parts (i) and (iii), consider a partition P such that $|C_1| > 1$ and create another partition, P', by splitting the top category into two. Let r_d be such that $1 \le r_d < r_1$, and $P' = (C_1', C_2', C_2, \ldots, C_k)$ where $C_1' = \{1, \ldots, r_d\}$, $C_2' = \{r_d + 1, \ldots, r_1\}$ and $C_2, \ldots, C_k$ are the same as in partition P. This gives

$$T_{P'} - T_P = r_1 r_d \bar{\beta}_{r_d,n} + (r_2 - r_d) r_1 \bar{\beta}_{r_1,n} + \sum_{i=2}^{k} (r_{i+1} - r_{i-1}) r_i \bar{\beta}_{r_i,n}$$

$$- r_2 r_1 \bar{\beta}_{r_1,n} - \sum_{i=2}^{k} (r_{i+1} - r_{i-1}) r_i \bar{\beta}_{r_i,n} = r_1 r_d (\bar{\beta}_{r_d,n} - \bar{\beta}_{r_1,n}).$$

For IFR distributions, this expression is positive due to (10); therefore, it is optimal to split the top category. For DFR distributions, the expression is negative; therefore, it is optimal to combine the top two categories into one. Parts (i) and (iii) then follow by iterating these results.

For part (iv), suppose the failure rate is interior unimodal and consider the same two partitions P and P' as above. As seen from the proof of Proposition 5, for any fixed r, and n sufficiently large, we have $\bar{\beta}_{k,n}$ increasing in k for $k \le r$. This gives $T_{P'} - T_P \le 0$, and hence combining the two categories at the top of P' leads to an increase in aggregate effort for a sufficiently large n.

For part (ii), consider partition $P = (C_1, \ldots, C_k)$ such that $|C_i| > 1$ for some category $i > 1$. Here, we generate another partition, P', by splitting C_i into $C_i' = \{r_{i-1} + 1\}$ and

$C'_{i+1} = \{r_{i-1}+2, \ldots, r_i\}$ and keeping all other categories intact. This gives

$$
\begin{aligned}
T_{P'} - T_P = {} & \sum_{j=1}^{i-2} (r_{j+1} - r_{j-1}) r_j \bar{\beta}_{r_j,n} + (r_{i-1} + 1 - r_{i-2}) r_{i-1} \bar{\beta}_{r_{i-1},n} \\
& + (r_i - r_{i-1})(r_{i-1}+1)\bar{\beta}_{r_{i-1}+1,n} + (r_{i+1} - r_{i-1} - 1) r_i \bar{\beta}_{r_i,n} + \sum_{j=i+1}^{k-1} (r_{j+1} - r_{j-1}) r_j \bar{\beta}_{r_j,n} \\
& - \sum_{j=1}^{i-2} (r_{j+1} - r_{j-1}) r_j \bar{\beta}_{r_j,n} - (r_i - r_{i-2}) r_{i-1} \bar{\beta}_{r_{i-1},n} - (r_{i+1} - r_{i-1}) r_i \bar{\beta}_{r_i,n} \\
& - \sum_{j=i+1}^{k-1} (r_{j+1} - r_{j-1}) r_j \bar{\beta}_{r_j,n} \\
= {} & (r_i - r_{i-1})(r_{i-1}+1)\bar{\beta}_{r_{i-1}+1,n} - (r_i - r_{i-1} - 1) r_{i-1} \bar{\beta}_{r_{i-1},n} - r_i \bar{\beta}_{r_i,n} \\
= {} & (r_i - r_{i-1}) B_{r_{i-1}+1,n} - (r_i - r_{i-1} - 1) B_{r_{i-1},n} - B_{r_i,n} \\
= {} & (r_i - r_{i-1} - 1)(B_{r_{i-1}+1,n} - B_{r_{i-1},n}) - (B_{r_i,n} - B_{r_{i-1}+1,n}) \\
= {} & (r_i - r_{i-1} - 1)\beta_{r_{i-1}+1,n} - (\beta_{r_i,n} + \beta_{r_i-1,n} + \ldots + \beta_{r_{i-1}+2,n}) \\
= {} & (\beta_{r_{i-1}+1,n} - \beta_{r_i,n}) + (\beta_{r_{i-1}+1,n} - \beta_{r_i-1,n}) + \ldots + (\beta_{r_{i-1}+1,n} - \beta_{r_{i-1}+2,n}).
\end{aligned}
$$

From Lemma 3(iii), $\beta_{r,n}$ is decreasing in r for a log-concave $f(\cdot)$, and hence the expression above is positive. ■

Proof of Proposition 9 For an arbitrary $e \geq 0$, Eq. (1) gives

$$u_e(e, e^*) = \sum_{r=1}^{n} \pi_e^{(r)}(e, e^*) v_r - c'(e). \tag{18}$$

It is convenient to introduce nonnegative prize differentials $d_r = v_r - v_{r+1}$ for $r = 1, \ldots, n-1$, from which the original prizes can be recovered as $v_r = \sum_{k=r}^{n-1} d_k$. The budget constraint takes the form $\sum_{r=1}^{n-1} r d_r + v_n = 1$. Further, let $\Pi^{(r)}(e, e^*) = \sum_{k=1}^{r} \pi^{(k)}(e, e^*)$ denote the cumulative version of probabilities $\pi^{(r)}(e, e^*)$; that is, $\Pi^{(r)}(e, e^*)$ represents the probability for the indicative player's output to be ranked r *or higher*. By construction, $\Pi^{(n)}(e, e^*) = 1$, and hence $\Pi_e^{(n)}(e, e^*) = 0$, for all e. Using summation by parts, Eq. (18) can be rewritten as

$$u_e(e, e^*) = \sum_{r=1}^{n-1} \Pi_e^{(r)}(e, e^*) d_r - c'(e). \tag{19}$$

Let $\Delta e = e - e^*$. From Eq. (2),

$$\pi_e^{(r)}(e, e^*) = \binom{n-1}{r-1} \int_U F(\Delta e + x)^{n-r-1}[1 - F(\Delta e + x)]^{r-2} \tag{20}$$
$$\times[(n-r)(1 - F(\Delta e + x)) - (r-1)F(\Delta e + x)]f(\Delta e + x)dF(x).$$

It can be directly verified that

$$\Pi_e^{(r)}(e, e^*) = r\binom{n-1}{r} \int_U F(\Delta e + x)^{n-r-1}[1 - F(\Delta e + x)]^{r-1} f(\Delta e + x)dF(x). \tag{21}$$

Indeed, (20) and (21) coincide for $r = 1$. It is, therefore, sufficient to show that (20) and (21) satisfy $\pi_e^{(r)}(e, e^*) = \Pi_e^{(r)}(e, e^*) - \Pi_e^{(r-1)}(e, e^*)$ for $r = 2, \dots, n$. From (21), we have

$$\Pi_e^{(r)}(e, e^*) - \Pi_e^{(r-1)}(e, e^*) = \int_U F(\Delta e + x)^{n-r-1}[1 - F(\Delta e + x)]^{r-2}$$
$$\times \left[r\binom{n-1}{r}(1 - F(\Delta e + x)) - (r-1)\binom{n-1}{r-1}F(\Delta e + x)\right] f(\Delta e + x)dF(x).$$

Equation (20) obtains by noting that $r\binom{n-1}{r} = (n-r)\binom{n-1}{r-1}$.

Via a change of variable $\Delta e + x \to x$, Eq. (21) can be written as

$$\Pi_e^{(r)}(e, e^*) = r\binom{n-1}{r} \int_U F(x)^{n-r-1}[1 - F(x)]^{r-1} f(x - \Delta e)dF(x). \tag{22}$$

Next, we differentiate $\Pi_e^{(r)}(e, e^*)$ with respect to e one more time. This needs to be done carefully because $f(x - \Delta e)$ is not necessarily continuous in e for all $x \in U$. To preserve the continuity of the integrand in (22), the interval of integration must be changed to $[\underline{x} + \Delta e, \overline{x}]$ for $\Delta e > 0$ and to $[\underline{x}, \overline{x} + \Delta e]$ for $\Delta e < 0$, and the case of $\Delta e = 0$ must be treated separately. This gives

$$\Pi_{ee}^{(r)}(e, e^*) = r\binom{n-1}{r} \left[-\int_U F(x)^{n-r-1}[1 - F(x)]^{r-1} f'(x - \Delta e)dF(x)\right.$$
$$- F(\underline{x} + \Delta e)^{n-r-1}[1 - F(\underline{x} + \Delta e)]^{r-1} f(\underline{x})f(\underline{x} + \Delta e)\mathbb{1}_{\Delta e>0}$$
$$+ F(\overline{x} + \Delta e)^{n-r-1}[1 - F(\overline{x} + \Delta e)]^{r-1} f(\overline{x})f(\overline{x} + \Delta e)\mathbb{1}_{\Delta e<0}$$
$$\left. - f(\underline{x})^2 \mathbb{1}_{r=n-1}\mathbb{1}_{\Delta e=0} + f(\overline{x})^2 \mathbb{1}_{r=1}\mathbb{1}_{\Delta e=0}\right]$$
$$\le r\binom{n-1}{r}(f_m^2 - f'_{\min}). \tag{23}$$

Here, $\mathbb{1}_S$ is the indicator equal to one if S is true and zero otherwise. The inequality in (23) follows from the following considerations: (i) the term with the integral including $f'(x-\Delta e)$ does not exceed $-f'_{\min}$; (ii) the negative terms can be ignored; (iii) the remaining two positive terms cannot be nonzero simultaneously, and each of them does not exceed f_m^2.

Let $R(e) = \sum_{r=1}^{n} \pi^{(r)}(e, e^*)v_r$ denote the revenue part of payoff (1). Equation (19) then gives $R'(e) = \sum_{r=1}^{n-1} \Pi_e^{(r)}(e, e^*)d_r$, and we obtain un upper bound on the second derivative of the revenue,

$$R''(e) = \sum_{r=1}^{n-1} \Pi_{ee}^{(r)}(e, e^*)d_r \leq (f_m^2 - f'_{\min}) \sum_{r=1}^{n-1} \binom{n-1}{r} r d_r.$$

From the budget constraint, $\sum_{r=1}^{n-1} r d_r \leq 1$ for any monotone prize schedule; therefore,

$$\sum_{r=1}^{n-1} \binom{n-1}{r} r d_r \leq \max_{1 \leq r \leq n-1} \binom{n-1}{r} = \binom{n-1}{\lfloor \frac{n}{2} \rfloor},$$

which produces the first bound, $R''(e) \leq D_- \equiv (f_m^2 - f'_{\min})\binom{n-1}{\lfloor \frac{n}{2} \rfloor}$.

An alternative upper bound on $\Pi_{ee}^{(r)}(e, e^*)$ can be obtained by differentiating (21) directly. Taking into account the variable limits of integration similar to (23), this gives

$$\begin{aligned}
&\Pi_{ee}^{(r)}(e, e^*) = r\binom{n-1}{r} \Bigg[\int_U F(\Delta e + x)^{n-r-2}[1 - F(\Delta e + x)]^{r-2} \\
&\times [(n-r-1)(1 - F(\Delta e + x)) - (r-1)F(\Delta e + x)] f(\Delta e + x)^2 dF(x) \\
&+ \int_U F(\Delta e + x)^{n-r-1}[1 - F(\Delta e + x)]^{r-1} f'(\Delta e + x) dF(x) \\
&- F(\Delta e + \underline{x})^{n-r-1}[1 - F(\Delta e + \underline{x})]^{r-1} f(\Delta e + \underline{x}) f(\underline{x}) \mathbb{1}_{\Delta e > 0} \\
&+ F(\Delta e + \overline{x})^{n-r-1}[1 - F(\Delta e + \overline{x})]^{r-1} f(\Delta e + \overline{x}) f(\overline{x}) \mathbb{1}_{\Delta e < 0} \\
&- f(\underline{x})^2 \mathbb{1}_{r=n-1} \mathbb{1}_{\Delta e = 0} + f(\overline{x})^2 \mathbb{1}_{r=1} \mathbb{1}_{\Delta e = 0} \Bigg] \\
&\leq r\binom{n-1}{r} [(n-r-1) f_m^2 + f'_{\max} + f_m^2] = r\binom{n-1}{r} [(n-r) f_m^2 + f'_{\max}].
\end{aligned} \tag{24}$$

Using the budget constraint,

$$\sum_{r=1}^{n-1} (n-r)\binom{n-1}{r} r d_r \leq \max_{1 \leq r \leq n-1} (n-r)\binom{n-1}{r} \leq (n-1)\binom{n-1}{\lfloor \frac{n}{2} \rfloor},$$

which produces the second bound, $R''(e) \le D_+ \equiv [(n-1)f_m^2 + f'_{\max}]\binom{n-1}{\lfloor \frac{n}{2} \rfloor}$.

Both bounds are sufficient, and hence $R''(e) \le \min\{D_-, D_+\}$. Recall that $c''(e) \ge c_0$; therefore, condition $c_0 > \min\{D_-, D_+\}$ guarantees that $u(e, e^*)$ is strictly concave in e for $e \in [0, c^{-1}(1)]$.

Setting $e = e^*$ in (19), and defining $B_{r,n} \equiv \Pi_e^{(r)}(e^*, e^*)$, we obtain the first-order condition (3) in the form

$$\sum_{r=1}^{n-1} B_{r,n} d_r = c'(e^*),$$

where the left-hand side is nonnegative for any monotone prize schedule v. In order to show that a unique $e^* \in [0, c^{-1}(1))$ solving (3) exists, it is sufficient to show that $\sum_{r=1}^{n-1} B_{r,n} d_r < c'(c^{-1}(1))$. According to Proposition 1, the maximum of $\sum_{r=1}^{n-1} B_{r,n} d_r$ is equal to $\frac{B_{r,n}}{r}$ for some r; therefore, this expression does not exceed $\max_r B_{r,n}$. Further, representation (13) for $B_{r,n}$ shows that $B_{r,n} \le f_m$ for any r. Therefore, $f_m < c'(c^{-1}(1))$, which is equivalent to $c(c'^{-1}(f_m)) < 1$ and is implied by (c), is sufficient for the solution to exist.

Finally, we show that $u(e^*, e^*) \ge 0$. Using the upper bound for the left-hand side of (3), symmetry and condition (c),

$$u(e^*, e^*) = \frac{1}{n} - c(e^*) = \frac{1}{n} - c\left(c'^{-1}\left(\sum_{r=1}^{n-1} B_{r,n} d_r\right)\right) \ge \frac{1}{n} - c(c'^{-1}(f_m)) \ge 0.$$

■