Centre for
Economic
and Financial
Research
at
New Economic
School

January 2016

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Albert S. Kyle
Anna A. Obizhaeva
Yajun Wang

Working Paper No 226

CEFIR / NES Working Paper series

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Albert S. Kyle, Anna A. Obizhaeva, and Yajun Wang*

First Draft: June 14, 2012

This Draft: January 7, 2016

We describe a symmetric continuous-time model of trading among relatively overconfident, oligopolistic informed traders with exponential utility. Traders agree to disagree about the precisions of their continuous flows of Gaussian private information. The price depends on a trader's inventory (permanent price impact) and the derivative of a trader's inventory (temporary price impact). More disagreement makes the market more liquid; without enough disagreement, there is no trade. Target inventories mean-revert at the same rate as private signals. Actual inventories smoothly adjust toward target inventories at an endogenous rate which increases with disagreement. Faster-than-equilibrium trading generates "flash crashes" by increasing temporary price impact. A "Keynesian beauty contest" dampens price fluctuations.

JEL: D8, D43, D47, G02, G14

Keywords: market microstructure, price impact, liquidity, transaction costs, double auctions, information aggregation, rational expectations, agreement-to-disagree, imperfect competition, Keynesian beauty contest, overconfidence, strategic trading, dynamic trading, flash crash.

* Kyle: Robert H. Smith School of Business, University of Maryland, College Park, MD 20742 USA, akyle@rhsmith.umd.edu. Obizhaeva: New Economic School, Moscow, Russia, aobizhaeva@nes.ru. Wang: Robert H. Smith School of Business, University of Maryland, College Park, MD 20742 USA, ywang22@rhsmith.umd.edu. We thank Bradyn Breon-Drish, Vincent Fardeau, Thierry Foucault, Jungsuk Han, Jeongmin Lee, Wei Li, Hong Liu, Mark Loewenstein, Natalie Popovic, Ioanid Roşu, Marzena Rostek, Dimitri Vayanos, Bin Wei, Liyan Yang, and three anonymous referees for their comments.

When large traders in financial markets seek to profit from perishable private information while keeping transaction costs low, they face a fundamental trade-off. On the one hand, they want to trade slowly, to reduce their own temporary price impact costs resulting from adverse selection. On the other hand, they want to trade quickly, before the permanent price impact of competitors trading on similar information makes temporary profit opportunities go away. To illustrate this trade-off, we use a stationary model of continuous information-based trading among oligopolistic traders who agree to disagree about the precisions of private signals. The derived equilibrium with smooth trading reveals important insights about dynamic properties of inventories, prices, and liquidity in financial markets.

The model combines the following assumptions: (1) There is one type of trader, a strategic informed trader; there are no noise traders or market makers. (2) Each trader has a flow of private information about the same underlying fundamental value; the "noise" in their signals is uncorrelated. (3) Traders are "relatively over-confident," in that each trader believes his private information is more precise than other traders believe it to be. (4) Given his beliefs about the precision of his own signals and the signals of others, each trader applies Bayes law correctly; in doing so, he infers from prices the economically relevant aggregation of other traders' information. (5) Traders trade strategically, correctly taking into account how the permanent and temporary price impact of their trades affects prices. (6) Random variables are jointly normally distributed, and traders have additive exponential utility functions. (7) Traders are "symmetric" in that they have the same utility functions and symmetrically different beliefs about the information structure. (8) All model state variables are stationary.

While Vayanos (1999) and Du and Zhu (2015) also describe dynamic symmetric models of strategic informed trading with gradual adjustment of inventories, our approach is different. In our model, prices, inventories, and expected returns have stationary distributions; in their models, these variables are non-stationary. Vayanos (1999) motivates trade from shocks to inventories while Du and Zhu (2015) motivate trade from shocks to private values, both in a common prior setting. In our model, trade is motivated by disagreement about the precision of stationary private signals which decay as other traders acquire substitute information. We also consider an alternate model with a common prior and trading based on stationary privately-observed shocks to both cash-flow information and private values.

The one-period version of our model is an equilibrium in demand curves, like Kyle (1989) and Rostek and Weretka (2012). An equilibrium with linear trading strategies and positive trading volume exists if and only if each trader believes that his signal is slightly more than twice as accurate as other traders' signals. The equilibrium has a simple closed-form solution. As disagreement falls, liquidity dries up and trade vanishes.

The continuous-time model implements a continuous auction in which traders continuously submit demand schedules. An "almost-closed-form" steady-state equilibrium is characterized by six endogenous parameters which solve a set of six polynomial equations. Numerical calculations indicate that the same existence condition holds in the continuous-time model as in the one-period model.

Although none of our individual assumptions is new to the financial economics literature, combining all of the assumptions together into one model leads to new results which can explain realistic features of inventories, liquidity, and prices in speculative markets.

1. Inventories. Our stationary model provides a realistic description of trading by large asset managers who seek out risks to exploit private information about individual stocks. Each trader calculates a "target inventory" based on how his own estimate of the long-term dividend growth rate differs from the estimates of other traders. Since the market offers no instantaneous liquidity for block trades, each trader only partially adjusts his inventory linearly in the direction of a target inventory. Traders "shred orders" so that actual inventories are differentiable or "smooth" functions of time.[1] We prove analytically that the half-life of traders' target inventories matches the half-life of private information; both decay at a rate equal to the sum of the mean reversion rate of dividend growth and the total precision of all information in the market.

We obtain additional robust results numerically. The endogenous speed with which actual inventories move toward target inventories is faster when signals decay faster and when there is more disagreement which makes markets more liquid. When traders have very precise signals and there is a great deal of disagreement, then markets are very liquid, traders trade aggressively, inventories mean-revert rapidly, and the price is very informative about long-term fundamental value. This contradicts the common intuition that high trading volume and short holding periods indicate a myopic focus on quarterly earnings announcements rather than long-term value.

These results depend on imperfect competition and on an absence of noise trading. With perfect competition, traders adjust holdings to target inventories infinitely fast; imperfect competition induces traders to slow down inventory adjustment, like in Vayanos (1999). In models with noise trading such as Wang (1993), holding periods depend on both the arrival rate of information and the decay rate of noise traders' inventories.

Inventories follow a partial adjustment process with coefficients implied by the model's deep parameters. Specifically, we show analytically that when traders' beliefs are "correct on average," a more liquid market tends to be associated with lower autocorrelation of actual inventories but a higher contemporaneous correlation of actual inventories with target inventories. In an extensive literature on institutional trading, Atkyn and Dyl (1997) study turnover rates; Chakrabarty, Moulton and Trzcinka (2015) study holding periods; Chan and Lakonishok (1995)

[1] The market clears in time-derivatives of inventories, not inventories themselves. Our informal use of the term "smooth trading" is different from the mathematical usage, which implies derivatives of all orders exist. Since the first derivatives of traders' inventories follow diffusions, higher order derivatives do not exist.

study the length of trading packages; Cremers and Pareek (2014) study stock duration; Bae et al. (2014) study the number of buy-sell switching points; Cremers and Petajisto (2009) study the size of active shares; and Puckett and Yan (2011) study the amount of short-term trading. Our model generates specific stylized facts—with testable implications—for this empirical literature.

Although our model has no separate intermediaries, we expect our predictions to apply to the empirical market-making literature, including Hasbrouck and Sofianos (1993), Madhavan and Smidt (1993), and Menkveld and Hendershott (2014). These papers find that intermediaries' inventories adjust rapidly toward time-varying targets and tend to have higher autocorrelations and lower mean-reversion rates in smaller and less-frequently-traded stocks.

2. Liquidity. Our model endogenously generates a clean distinction between permanent and temporary price impact. From a trader's perspective, the level of prices is a linear function of his level of inventories and the derivative of his inventories. Changes in prices therefore depend on two liquidity parameters: (1) a permanent price impact parameter, denoted λ as in Kyle (1985), measuring the price impact of a change in the level of inventories, and (2) a temporary price impact parameter, denoted κ, measuring the price impact of a change in the derivative of inventories. The temporary component of price impact makes trading a given quantity over a shorter horizon more expensive than trading the same quantity over a longer horizon; the market offers no instantaneous liquidity for block trades. The speed with which actual inventories move toward target inventories results from a trade-off between temporary price impact costs and the speed with which information decays. Modeling this important trade-off requires a stationary model.

Our continuous-time approach makes the distinction between permanent and temporary price impact intuitively and mathematically clear. In the discrete-time set-up of Vayanos (1999), an analogous distinction between permanent and temporary price impact could be derived by carefully taking the limit as the interval between rounds of trading goes to zero.

In our model, trading scales down with the traders' risk aversion parameter. Since both permanent and temporary price impact are proportional to risk aversion, inventories and trading volume are inversely proportional to it. In Vayanos (1999), by contrast, greater risk aversion is associated with less liquidity and more trading. In addition, we show numerically that increasing disagreement makes markets more liquid and increases the speed of trading. The smooth trading model therefore realistically predicts that high volume markets will be highly liquid.

Black (1971) describes liquidity using the concepts of tightness, depth, and resiliency. In our continuous-time model, the market has no instantaneous depth, tightness is related to temporary price impact, and resiliency depends on the aggregate rate of information production. These concepts of liquidity play out differently from Kyle (1985), in which the equilibrium would break if noise traders—like the informed trader—were also allowed to smooth their trading; when all traders smooth their trading, the nature of liquidity changes.

In models with "impatient" noise traders—such as Chau and Vayanos (2008), Foster and Viswanathan (1994), Caldentey and Stacchetti (2010), and Holden and Subrahmanyam (1992)—a discrete-time setting is needed to prevent traders from trading infinitely fast. Back, Cao and Willard (2000) are able to implement the discrete-time model of Foster and Viswanathan (1996) in continuous time, because declining permanent price impact over time deters infinitely aggressive trading immediately after trading begins.

Our use of the terms "temporary" and "permanent" price impact differs from that of empirical researchers who think of temporary impact as short-term mean reversion in prices arising from dealer spreads ("bid-ask bounce") and permanent impact as persistent price changes arising from private information being impounded into market prices. Consistent with Black (1982), it is impossible to infer price impact from price fluctuations which result from optimal trading. As a result of traders' optimizing behavior, higher trading costs show up indirectly as smoother inventories, not as more short-term mean reversion in prices. In principle, price impact can be inferred from abnormally fast "out-of-equilibrium" execution of a bet, which leads to a price spike resembling a "flash crash."

Our *endogenously* derived price impact model is similar to transaction costs models used by practitioners such as Grinold and Kahn (1995), Almgren and Chriss (2000), and Obizhaeva and Wang (2013). The model also provides a theoretical explanation for robust empirical findings that the speed of trading affects transaction costs and often relates to the size of temporary price changes, as documented by Holthausen, Leftwich and Mayers (1990), Chan and Lakonishok (1995), Keim and Madhavan (1997), and Dufour and Engle (2000). Numerous papers examine the economic implications of fast trading given *exogenously* specified price impact functions depending on the speed of trading; examples include Brunnermeier and Pedersen (2005), Carlin, Lobo and Viswanathan (2007), and Longstaff (2001).

3. Prices, Keynesian Beauty Contest, and Private Values. Although traders adjust inventories slowly, prices immediately reflect all of the information in the market, both public and private. In the absence of noise trading, each trader can infer the average valuation of other traders from the price.

Prices reflect the "beauty contest" described by Keynes (1936), in the sense that traders forecast how the expectations of other traders will evolve in the future and trade to take advantage of these forecasts. We obtain numerically the interesting result that prices are dampened due to this beauty contest. The growth-rate component of prices is a weighted average of the growth-rate expectations of each trader; "dampening" means that the weights sum to a constant less than one. Here is the intuition: When prices are high and a trader believes that the high prices reflect fundamental value, the trader forecasts that the other overconfident traders will revise their forecasts down so quickly that it is temporarily profitable to sell ahead of such revisions in the short run. This dampens price fluctuations and leads to momentum in returns. We find that this dampening is more pronounced when disagreement is larger and markets are more liquid. These predictions explain the otherwise puzzling empirical finding that momentum is more pronounced in high volume and liquid securities, as documented by Lee and Swaminathan (2000), Moskowitz, Ooi and Pedersen (2012), and Cremers and Pareek (2014).

We characterize equilibrium in an otherwise similar model of perfect competition in which traders immediately adjust inventories to target levels, as in Kyle and Lin (2001). Consistent with the intuition that low trading costs amplify the economic importance of the dampening effect, we find that perfect competition leads to more pronounced dampening than imperfect competition.

We also examine an otherwise similar model with privately observed shocks to private values and a common prior. For analytical tractability, we assume that shocks to private values mean revert at the same rate as private information. This model has properties analogous to our preferred model of overconfidence in all respects except that price dampening goes away: prices are an average of traders' private valuations, adjusted for private values. Similarly, the model of Du and Zhu (2015), with non-stationary private values, has no price dampening either.

In noisy rational expectations models such as Wang (1993), Wang (1994), and He and Wang (1995), noise affects the weights placed on signals but the sum of weights on all valuations is equal to one; there is no price dampening. There is also no price dampening in Vayanos (1999). Banerjee and Kremer (2010) is a myopic model in which price dampening goes away because of myopia.

We infer from these results that price dampening in the Keynesian beauty contest results from a combination of overconfidence and substantial market liquidity, not from models of noise trading or private values with a common prior. Harsanyi (1976) conjectures that a model without a common prior can be mapped into an isomorphic model with a common prior. Our price dampening result shows that such a mapping is not straightforward; it would likely require complicated *ad hoc* assumptions involving "externalities" related to the cross-correlation structure of private values. By contrast, our parsimonious model of disagreement provides a natural micro-founded explanation for trading while satisfying Ockham's razor.

Grinold and Kahn (1995), an influential book for quantitative asset managers, describes a simplistic partial-equilibrium model of trading with decaying private information, risk aversion, and temporary transaction costs. They pose as an important open research question (p. 580) how to set up a proper trading model with a finite half-life for information, risk aversion, and a model of transaction costs which captures the components of tightness, depth, and resiliency. Our model not only solves an appropriate optimization problem for all asset managers simultaneously but also derives endogenously a realistic transaction cost model with stationary dynamics for inventories, prices, and expected returns.

This paper is structured as follows. Section 1 presents a one-period model. Section 2 presents the continuous-time model. Section 3 examines properties of the smooth-trading equilibrium. Section 4 concludes. Proofs are in Appendix A. Appendix B presents a similar model of competitive trading. Appendix C presents a similar model in which private values and a common prior replace overconfidence.

1. One-period Model

The following one-period model has a simple closed-form solution illustrating the interaction between overconfidence and market power.

A risky asset with random liquidation value $v \sim N(0, 1/\tau_v)$ is traded for a safe numeraire asset. It is common knowledge that the asset is in zero net supply. Trader n is endowed with a privately observed inventory S_n, with $\sum_{n=1}^{N} S_n = 0$. While non-zero initial inventories play no significant role in this one-period model, they are useful for mapping results into the continuous-time model below. Traders observe signals about the normalized liquidation value $\tau_v^{1/2} v \sim N(0, 1)$. All traders observe a public signal $i_0 := \tau_0^{1/2} \, (\tau_v^{1/2} v) + e_0$ with $e_0 \sim N(0, 1)$. Each trader n observes a private signal $i_n := \tau_n^{1/2} \, (\tau_v^{1/2} v) + e_n$ with $e_n \sim N(0, 1)$. The asset payoff v, the public signal error e_0, and N private signal errors $e_1, \ldots, e_N$ are independently distributed.

Traders agree about the precision of the public signal τ_0 and agree to disagree about the precisions of private signals τ_n. Each trader is "relatively overconfident," believing his own signal has a high precision $\tau_n = \tau_H$ and other traders' signals have low precision $\tau_m = \tau_L$ for $m \neq n$, with $\tau_H > \tau_L \geq 0$.

Each trader believes other traders are like noise traders who over-trade on their information. Unlike Grossman and Stiglitz (1980) or Kyle (1985), there are no explicit noise traders or market makers. The model is like Treynor (1995), who discusses "transactors acting on information which they believe has not yet been fully discounted in the market price but which in fact has." Similarly, Black (1986) defines noise trading as "trading on noise as if it were information."

As in Kyle (1989) and Rostek and Weretka (2012), each trader submits a demand schedule $X_n(p) := X_n(i_0, i_n, S_n, p)$ to a single-price auction. An auctioneer clears the market at price $p := p[X_1, \ldots, X_N]$. Trader n's terminal wealth is

$$W_n := v \, (S_n + X_n(p)) - p \, X_n(p). \tag{1}$$

Each trader n maximizes the same expected exponential utility function of wealth $\mathrm{E}^n\{-\mathrm{e}^{-A \, W_n}\}$ using his own beliefs about τ_H and τ_L to calculate the expectation.

An *equilibrium* is a set of trading strategies $X_1, \ldots, X_N$ such that each trader's strategy maximizes his expected utility, taking as given the trading strategies of other traders. Except for the assumption that traders do not share a common prior, this is equivalent to a Bayesian Nash equilibrium. As imperfect competitors, traders take into account how the price p depends on the quantities they trade.

1.1. Linear Strategies and Bayesian Updating

Let $i_{-n} := \frac{1}{N-1} \sum_{m \neq n} i_m$ denote the average of other traders' signals. When trader n conjectures that other traders submit symmetric linear demand schedules

$$X_m(i_0, i_m, S_m, p) = \alpha \, i_0 + \beta \, i_m - \gamma \, p - \delta \, S_m, \quad m = 1, \ldots, N, \quad m \neq n, \tag{2}$$

he infers from the market-clearing condition

$$x_n + \sum_{m \neq n} (\alpha\, i_0 + \beta\, i_m - \gamma\, p - \delta\, S_m) = 0 \tag{3}$$

that his residual supply schedule $P(x_n)$ is a function of his quantity x_n given by

$$P(x_n) = \frac{\alpha}{\gamma}\, i_0 + \frac{\beta}{\gamma}\, i_{-n} + \frac{\delta}{(N-1)\gamma}\, S_n + \frac{1}{(N-1)\gamma}\, x_n. \tag{4}$$

Since trader n observes the public signal i_0, his own inventory S_n, and the quantity he trades himself x_n, he can infer the average of other traders' signals i_{-n} from observing the intercept of his residual supply schedule.

Let $\mathrm{E}^n\{\ldots\}$ and $\mathrm{Var}^n\{\ldots\}$ denote trader n's expectation and variance operators conditional on all signals $i_0, i_1, \ldots, i_N$. Define "total precision" τ by

$$\tau := (\mathrm{Var}^n\{v\})^{-1} = \tau_v\ (1 + \tau_0 + \tau_H + (N-1)\,\tau_L)\,. \tag{5}$$

The projection theorem for jointly normally distributed random variables implies

$$\mathrm{E}^n\{v\} = \frac{\tau_v^{1/2}}{\tau}\ \Big(\tau_0^{1/2}\, i_0 + \tau_H^{1/2}\, i_n + (N-1)\,\tau_L^{1/2}\, i_{-n}\Big)\,. \tag{6}$$

1.2. Utility Maximization with Market Power

Conditional on all information, trader n's terminal wealth W_n is a normally distributed random variable with mean and variance given by

$$\mathrm{E}^n\{W_n\} = \mathrm{E}^n\{v\}\,(S_n + x_n) - P(x_n)\,x_n, \qquad \mathrm{Var}^n\{W_n\} = (S_n + x_n)^2\ \mathrm{Var}^n\{v\}. \tag{7}$$

Normal distributions imply that expected utility is given by

$$\mathrm{E}^n\{-\,\mathrm{e}^{-A\,W_n}\} = -\exp\Big(-A\ \mathrm{E}^n\{W_n\} + \tfrac{1}{2}A^2\ \mathrm{Var}^n\{W_n\}\Big). \tag{8}$$

Maximizing this function is equivalent to maximizing the simpler function $\mathrm{E}^n\{W_n\} - \frac{1}{2}A\ \mathrm{Var}^n\{W_n\}$. Plugging equations (5), (6), and (7) into equation (8), trader n solves the maximization problem

$$\max_{x_n} \left(\frac{\tau_v^{1/2}}{\tau} \Big(\tau_0^{1/2}\, i_0 + \tau_H^{1/2}\, i_n + (N-1)\tau_L^{1/2}\, i_{-n}\Big)\ (S_n + x_n) - P(x_n)\, x_n - \frac{A}{2\tau}\,(S_n + x_n)^2 \right). \tag{9}$$

Oligopolistic trader n exercises market power by taking into account how his chosen quantity x_n affects the price $P(x_n)$ on his residual supply schedule (4).

1.3. Equilibrium with Linear Demand Schedules

There always exists a no-trade equilibrium in which each trader submits a no-trade schedule $X_n(.) \equiv 0$ and the auctioneer cannot establish a meaningful price.

An equilibrium with trade may also exist. Appendix section A.1 proves the following theorem using the "no-regret" approach: Each trader observes his residual linear supply schedule, infers the average of other traders' signals from its intercept, picks the optimal quantity x_n, and implements this choice with a demand schedule $x_n = X_n(i_0, i_n, S_n, p)$, without observing the residual supply schedule itself.[2]

Let τ_H/τ_L measure "disagreement." Define the exogenous quantity Δ_H by

$$\Delta_H := \frac{\tau_H^{1/2}}{\tau_L^{1/2}} - 2 - \frac{2}{(N-2)}. \tag{10}$$

THEOREM 1: ***Characterization of Equilibrium in the One-Period Model with Overconfidence and Imperfect Competition.*** *There exists a unique symmetric equilibrium with linear trading strategies and non-zero trade if and only if the second order condition $\Delta_H > 0$ holds. This equilibrium has the following properties:*

1. *Trader n chooses the quantity x_n^* given by*

$$x_n^* = \frac{(N-2)\, \tau_L^{1/2}\, \Delta_H}{A\, N}\, \tau_v^{1/2}\, (i_n - i_{-n}) - \delta\, S_n. \tag{11}$$

2. *The price p^* is the average of traders' valuations based on all information:*

$$p^* = \frac{1}{N} \sum_{n=1}^{N} \mathrm{E}^n\{v\} = \frac{\tau_v^{1/2}}{\tau} \left(\tau_0^{1/2}\, i_0 + \frac{\tau_H^{1/2} + (N-1)\tau_L^{1/2}}{N} \sum_{n=1}^{N} i_n \right). \tag{12}$$

3. *The parameters $\alpha > 0$, $\beta > 0$, $\gamma > 0$, and $\delta > 0$ defining the linear trading strategies in equation (2) have unique closed-form solutions, defined in (A-6).*

For an equilibrium with positive trading volume to exist, there must be "enough" disagreement so that $\Delta_H > 0$. This requires $N \geq 3$ and requires $\tau_H^{1/2}$ to be sufficiently more than twice as large as $\tau_L^{1/2}$. Each trader trades in the direction of his private signal i_n, trades against the average of other traders' signals i_{-n}, and hedges a fraction δ of his initial inventory. Trading volume increases in disagreement and decreases in risk aversion.

[2]Substituting equation (4) into equation (9) to find his optimal demand given in (A-1). Solving for i_{-n} in the market-clearing condition (3), substituting this solution into equation (A-1), and then solving for x_n, yields a demand schedule $X_n(i_0, i_n, S_n, p)$ for trader n as a function of price p. In a symmetric linear equilibrium, the strategy chosen by trader n must be the same as the linear strategy (2) conjectured for the other traders. Equating the corresponding coefficients of the variables i_0, i_n, p, and S_n yields a system of four equations in terms of the four unknowns α, β, γ, and δ. The unique solution is given in (A-6). Substituting (A-6) into (A-3) yields trader n's optimal demand. Then using the market-clearing condition yields the equilibrium price.

1.4. Equilibrium Properties

Like Kyle (1989) and Rostek and Weretka (2012), each trader exercises market power by "shading" the quantity traded relative to the quantity a perfect competitor would trade. Define a trader's "target inventory" S_n^{TI} as the inventory such that he would not want to trade ($x_n^* = 0$), given from equation (11) by

$$(13) \qquad S_n^{TI} = \frac{1}{A}\left(1 - \frac{1}{N}\right) \tau_v^{1/2} \left(\tau_H^{1/2} - \tau_L^{1/2}\right) \left(i_n - i_{-n}\right).$$

Then trader n's optimal quantity traded can be written

$$(14) \quad x_n^* = \delta(S_n^{TI} - S_n), \qquad \text{where} \qquad 0 < \delta = \frac{(N-2)\,\tau_H^{1/2} - 2\,(N-1)\,\tau_L^{1/2}}{(N-1)(\tau_H^{1/2} - \tau_L^{1/2})} < 1.$$

The parameter δ, calculated from equations (A-6), is the fraction by which imperfectly competitive traders adjust positions toward target levels. As a function of disagreement τ_H/τ_L, δ increases monotonically from a lower bound of zero when the existence condition $\tau_H^{1/2}/\tau_L^{1/2} - 2 - 2/(N-2) > 0$ is barely satisfied toward an upper bound of $(N-2)/(N-1)$ as $\tau_H^{1/2}/\tau_L^{1/2} \to \infty$. If there is not enough disagreement to sustain an equilibrium with trade, each trader would want to shade his bid more than the others, and this breaks the equilibrium.[3]

In an otherwise equivalent one-period model with perfect competition, traders trade the quantity which brings their inventories to target levels, equivalent to $\delta = 1$. Prices in the competitive equilibrium are the same as with imperfect competition. Appendix section B.1 proves the following:

THEOREM 2: ***Characterization of Competitive Equilibrium in the One-Period Model.*** *There exists a unique symmetric equilibrium with linear trading strategies and non-zero trade if and only if $\tau_H > \tau_L$. In this equilibrium:*

1. *Trader n chooses the quantity $x_n^* = S_n^{TI} - S_n$ (equation (14) with $\delta = 1$).*

[3]When there is not enough disagreement to sustain an equilibrium with pure strategies, one might imagine that it is possible to have an equilibrium with mixed strategies. For mixed strategies to be an equilibrium, the trader must be indifferent across the various randomized choices of quantities he trades. For example, if we add normally distributed noise to quantities traded, symmetrically across all traders, a mixed strategy equilibrium requires the second order condition to be exactly zero. This means that the quadratic objective function reduces to a linear function, *i.e.*, the denominator of equation (A-1) is zero. Since the trader has to be indifferent across various randomizations, this further implies that the linear function must be a constant, distributed independently from the quantity traded. This assumption cannot hold, because a trader with a positive value of i_n would always want to buy unlimited quantities, and a trader with a negative i_n would always want to sell unlimited quantities. Thus, an equilibrium with symmetric normally distributed noise cannot exist. When noise is not normally distributed or the equilibrium is not symmetric, the objective function is not quadratic any more, but it will still be difficult to find a mixed strategy equilibrium given that the sensitivity of utility to a trader's own private information must be well-defined.

2. The price p^* is the same as in the model with imperfect competition (equation (12)).

3. The parameters $\alpha > 0$, $\beta > 0$, $\gamma > 0$, and $\delta = 1$, which define the linear trading strategies in equation (B-4), have unique closed-form solutions defined in (B-7).

With perfect competition, traders take the equilibrium price as given. With imperfect competition, traders realize that both their inventories and the quantity they trade affect prices.

From the perspective of trader n, equation (4) implies that with imperfect competition, price impact can be written as a function of both x_n and S_n,

$$P(x_n, S_n) := p_{0,n} + \lambda\, S_n + \kappa\, x_n, \tag{15}$$

where $p_{0,n}$ is a linear combination of random variables i_0 and i_{-n}, and equations (A-5) and (A-6) imply that constants λ and κ are given by

$$\lambda := \frac{\delta}{(N-1)\gamma} = \frac{A}{\tau}\, \frac{\tau_H^{1/2} + (N-1)\, \tau_L^{1/2}}{(N-1)(\tau_H^{1/2} - \tau_L^{1/2})} \tag{16}$$

and

$$\kappa := \frac{\lambda}{\delta} = \frac{1}{(N-1)\gamma} = \frac{A}{\tau}\, \frac{\tau_H^{1/2} + (N-1)\, \tau_L^{1/2}}{(N-2)\, \tau_L^{1/2}\, \Delta_H}. \tag{17}$$

The price impact parameters λ and κ increase in risk aversion A and decrease in disagreement τ_H/τ_L; these results are consistent with the continuous-time model considered next. In the continuous-time model, the first component $\lambda\, S_n$ is analogous to permanent price impact as in Kyle (1985). The second component $\kappa\, x_n$ is analogous to temporary price impact determined by the speed of trading, with x_n replaced by the derivative of the trader's inventory dS_n/dt.

2. Continuous-Time Model

The continuous-time model has the following structure. There are N risk-averse oligopolistic traders who trade at price $P(t)$ a risky asset in zero net supply against a risk-free asset which earns constant risk-free rate $r > 0$.

The risky asset pays out dividends at continuous rate $D(t)$. Dividends follow a stochastic process with mean-reverting stochastic growth rate $G^*(t)$, constant instantaneous volatility $\sigma_D > 0$, and constant rate of mean reversion $\alpha_D > 0$:

$$dD(t) := -\alpha_D\, D(t)\, dt + G^*(t)\, dt + \sigma_D\, dB_D(t). \tag{18}$$

The dividend $D(t)$ is publicly observable, but the growth rate $G^*(t)$ is not observed by any trader. The growth rate $G^*(t)$ follows an AR-1 process with mean reversion α_G and volatility σ_G:

$$dG^*(t) := -\alpha_G\, G^*(t)\, dt + \sigma_G\, dB_G(t). \tag{19}$$

Each trader n observes a continuous stream of private information $I_n(t)$ defined by the stochastic process

$$(20) \qquad dI_n(t) := \tau_n^{1/2} \frac{G^*(t)}{\sigma_G \, \Omega^{1/2}} \, dt + dB_n(t), \quad n = 1, \ldots, N.$$

Since its drift is proportional to $G^*(t)$, each increment $dI_n(t)$ in the process $I_n(t)$ is a noisy observation of the unobserved growth rate $G^*(t)$. The denominator $\sigma_G \, \Omega^{1/2}$ scales $G^*(t)$ so that its conditional variance is one; this simplifies the intuitive interpretation of the model. The "precision" parameter τ_n measures the informativeness of the signal $dI_n(t)$ as a signal-to-noise ratio describing how fast new information flows into the market. The parameter Ω measures the steady-state error variance of the trader's estimate of $G^*(t)$ in units of time; it is defined algebraically below (see equation (24)).[4]

Analogous to the one-period model, each trader is absolutely certain that his own private information $I_n(t)$ has "high" precision $\tau_n = \tau_H$ and the other traders' private information has "low" precision $\tau_m = \tau_L$ for $m \neq n$, with $\tau_H > \tau_L \geq 0$. Traders do not update their dogmatic beliefs about τ_H and τ_L over time; for plausible parameter values, it would take a long time for a trader to learn that his beliefs are incorrect. Since "relatively overconfident" traders "agree to disagree" about the precisions of their private signals, they do not share a common prior even though their beliefs are common knowledge.[5] Agreement-to-disagree is a simple assumption with realistic implications: it can break no-trade results and naturally generate trading volume. It is important to distinguish between the common prior assumption (which we do not make) and the traditional economists' assumption of rationality as consistently applying Bayes law when maximizing expected utility with respect to some probability distribution (which we do make). Morris (1995) further discusses why "dropping the common prior assumption from otherwise rational behavior" is an important research agenda.

Each trader's information set at time t, denoted $\mathcal{F}_n(t)$, consists of the histories of (1) the dividend process $D(s)$, (2) the trader's own private information $I_n(s)$, and (3) the market price $P(s)$, $s \in (-\infty, t]$. All traders process information rationally; they apply Bayes law correctly given their possibly incorrect beliefs.

Let $S_n(t)$ denote the inventory of trader n at time t. Zero net supply implies $\sum_{n=1}^{N} S_n(t) = 0$.

This section considers "smooth trading" equilibria in which inventories $S_n(t)$

[4] Since the innovation variance of the signal $dI_n(t)$ can be estimated arbitrarily precisely by observing past signals continuously, it is common knowledge that the innovation variance of the signal is one. Scaling the innovation variance of $I_n(t)$ in equation (20) to make it equal to one is therefore a normalization without loss of generality.

[5] We call this belief structure "relative overconfidence" to distinguish it from a belief structure with "absolute overconfidence" in which traders believe the precisions of their signals are greater than empirically true precisions. Empirically true precisions do not affect the equilibrium strategies investigated in this paper but do affect empirical predictions about asset returns (see section 3.6).

are differentiable functions of time. Therefore, trading strategies and the market-clearing condition are specified using rates of trading, not shares traded. Each trader's trading strategy X_n is a mapping from his information set $\mathcal{F}_n(t)$ at time t into a "flow-demand schedule" which defines the derivative of his inventory $x_n(t) := X_n(t, P(t); \mathcal{F}_n(t))$ ("trading intensity") as a function of the market-clearing price $P(t)$. An auctioneer continuously calculates the market-clearing price $P(t) := P[X_1, \ldots, X_N](t)$ such that the market-clearing condition $\sum_{n=1}^{N} x_n(t) = 0$ is satisfied. Each trader explicitly takes into account the effect of his trading intensity on market prices.

Each trader has the same time preference parameter ρ and the same time-additively-separable exponential utility function $U(c_n(s)) := -\,\mathrm{e}^{-A\, c_n(s)}$ with constant-absolute-risk-aversion parameter A. Trader n's consumption strategy C_n defines a consumption rate $c_n(t) := C_n(t; \mathcal{F}_n(t))$ for all $t > -\infty$. Let $\mathrm{E}_t^n\{\ldots\}$ denote the conditional expectations operator $\mathrm{E}\{\ldots|\mathcal{F}_n(t)\}$ based on trader n's beliefs.

Define an *equilibrium* as a set of trading strategies $X_1^*, \ldots, X_N^*$ and consumption strategies $C_1^*, \ldots, C_N^*$ such that, for $n = 1, \ldots, N$, trader n's optimal consumption and trading strategies $X_n = X_n^*$ and $C_n = C_n^*$ solve his maximization problem taking as given the optimal strategies of the other traders. For all dates $t > -\infty$, the optimal strategies X_n^* and C_n^* solve trader n's maximization problem

$$\max_{\{C_n, X_n\}} \quad \mathrm{E}_t^n \left\{ \int_{s=t}^{\infty} \mathrm{e}^{-\rho(s-t)} \;\, U(c_n(s))\, ds \right\}, \tag{21}$$

where inventories follow the process $dS_n(t) = x_n(t)\, dt$ and money holdings $M_n(t)$ follow the process

$$dM_n(t) = (r\, M_n(t) + S_n(t)\, D(t) - c_n(t) - P(t)\, x_n(t))\;\, dt. \tag{22}$$

When solving the maximization problem, trader n takes as given the trading strategies X_m, $m \neq n$, for the other $N - 1$ traders; in doing so, he exercises market power by taking into account how his own trading strategy affects equilibrium prices $P(t)$ and future trading opportunities. Except for the assumption that traders do not share a common prior (since $\tau_H \neq \tau_L$), the equilibrium is a perfect Bayesian equilibrium; traders follow dynamically consistent trading strategies, taking as given the strategies of other traders. This generalizes the Bayesian Nash equilibrium concept used in the one-period model in a natural way.

The values of all ten exogenous parameters α_D, σ_D, α_G, σ_G, τ_H, τ_L, N, r, A, and ρ are common knowledge. It is also common knowledge that each trader believes that $dB_D(t)$, $dB_G(t)$, $dB_1(t), \ldots, dB_N(t)$ are independently-distributed Brownian motions, given traders' beliefs. Note that since traders disagree about whether a signal has precision τ_H or τ_L, they also disagree about how to construct the Brownian motions $dB_n(t)$ from the signals $dI_n(t)$. Symmetry of parameter values prevents the number of state variables from exploding, avoiding the forecasting-the-forecasts-of-others problem described by Townsend (1983).

We show that if disagreement is large enough—*i.e.*, if τ_H is sufficiently larger than τ_L—there will be trade based on private information. The degree of disagreement τ_H/τ_L affects the equilibrium prices and quantities traded. Without overconfidence—*e.g.*, in a model of rational expectations with a common prior—there would be no trade except for possible unwinding of initial suboptimal endowments.

Infinitely fast portfolio updating cannot be an equilibrium. Temporary price impact is intuitively necessary to prevent a breakdown in equilibrium which would occur with infinitely fast updating toward target inventories. With temporary price impact, infinitely fast trading toward a target inventory is infinitely expensive because the price is an unboundedly increasing function of the derivative of a trader's inventory. If there were no temporary price impact—and the price were only an increasing function of the level of a trader's inventory—then a trader would reduce price impact costs by moving continuously but very quickly along his residual demand schedule, trading at increasingly less favorable prices like a perfectly discriminating monopolist. This could not be a symmetric equilibrium, however, because the counter-parties would require compensation, in the form of temporary price impact costs, to compensate for losses from being "picked off" by the discriminating monopolist. To reduce transaction costs, each trader would try to slow his trading relative to others, and the equilibrium would break.

The continuous equilibrium of Kyle (1985) is different. While the informed trader optimally smooths out his trading so that his inventory is a continuous function of time, the noise traders are assumed to trade sub-optimally. In response to a shock to desired inventories ΔU, the noise traders immediately trade the quantity ΔU all at once, incurring price impact cost $\lambda \Delta U$. If the noise traders were instead to trade smoothly at rate $\Delta U/\Delta t$ over some small time interval Δt, moving quickly but continuously along their residual demand schedule like a perfectly discriminating monopolist, then they would incur approximately only one-half the price impact cost, $\frac{1}{2}\lambda\, \Delta U$. Such optimized smooth trading by noise traders would break the equilibrium of Kyle (1985) because the market makers on the other side of this smooth trading would suffer losses.

2.1. Bayesian Updating by Traders in the Model

In addition to private information, traders also use the history of the dividend process $D(t)$ to forecast the unobserved dividend growth rate $G^*(t)$. To simplify Kalman filtering formulas, the information content of the publicly observable dividend $D(t)$ can be expressed in a form analogous to the notation for private information $I_n(t)$ in equation (20). Define $dI_0(t) := \left[\alpha_D\, D(t)\, dt + dD(t)\right]/\sigma_D$ and $dB_0 := dB_D$. Then the public information $I_0(t)$ in the divided stream (18) can be written

$$dI_0(t) := \tau_0^{1/2}\, \frac{G^*(t)}{\sigma_G\, \Omega^{1/2}}\, dt + dB_0(t), \quad \text{where} \quad \tau_0 := \frac{\Omega\, \sigma_G^2}{\sigma_D^2}. \tag{23}$$

The process $I_0(t)$ is informationally equivalent to the dividend process $D(t)$. The quantity τ_0 measures the precision of the dividend process in units analogous to the units of precision for private information.

Consider next how traders update their estimates of the unobserved growth rate. In a symmetric equilibrium with no noise trading, each trader infers from prices a sufficient statistic for other traders' private information. Thus, all traders update estimates of the unobserved growth rate $G^*(t)$ as if fully informed about all information $I_0(s) \equiv D(s)$, $I_1(s), \ldots, I_N(s)$, $s \in (-\infty, t]$, including the private information of other traders. Let $G_n(t) := \mathrm{E}_t^n\{G^*(t)\}$ denote trader n's estimate of the unobserved growth rate $G^*(t)$ conditional on all information. The superscript n indicates that conditional distributions of growth rates are calculated by trader n based on his belief that his own signal has high precision τ_H and other traders' signals have low precision τ_L. The subscript t denotes conditioning on the history of all information at date t. Similarly, let $\mathrm{Var}_t^n\{G^*(t)\}$ denote trader n's conditional variance using all information at date t.

Appendix section A.2 presents Stratonovich-Kalman-Bucy filtering formulas for calculating estimates of the unobserved growth rate $G^*(t)$ from information of arbitrary precision $\tau_0, \tau_1, \ldots, \tau_N$. For traders' specific beliefs τ_H and τ_L, these results show how traders correctly use Bayes law to update their estimates $G_n(t)$.

Equations (A-8) and (A-9) imply that, for the beliefs of any trader n, "total precision" τ and non-time-varying "scaled error variance" Ω are given by

$$\tau := \tau_0 + \tau_H + (N-1)\,\tau_L, \qquad \Omega^{-1} := \left(\frac{\mathrm{Var}_t^n\{G^*(t)\}}{\sigma_G^2}\right)^{-1} = 2\,\alpha_G + \tau. \tag{24}$$

Although traders disagree about which signal has high precision τ_H, it is common knowledge that they use the same values of τ and Ω.

From the history of each "raw information" process $I_n(s)$, $s \in (-\infty, t]$, define a "signal" $H_n(t)$, $n = 0, \ldots, N$, by plugging τ and Ω into equation (A-13); the resulting exponentially weighted average of past innovations, given by

$$H_n(t) := \int_{u=-\infty}^{t} \mathrm{e}^{-(\alpha_G+\tau)\,(t-u)}\; dI_n(u), \quad n = 0, 1, \ldots, N, \tag{25}$$

is a sufficient statistic for the information in the history of the raw information process $I_n(s)$. Equation (25) reveals that the information content of an innovation in the raw information process $dI_n(t)$ decays at rate $\alpha_G + \tau$, where τ measures the rate at which new information is being produced. Let $H_{-n}(t)$ denote the average of the other traders' signals:

$$H_{-n}(t) := \tfrac{1}{N-1} \sum_{m=1,\ldots,N;\, m\neq n} H_m(t). \tag{26}$$

Equation (A-15) implies that trader n's estimate of the growth rate $G_n(t)$ is a linear combination of $H_0(t)$, $H_n(t)$, and $H_{-n}(t)$ given by

$$G_n(t) := \sigma_G\, \Omega^{1/2}\, \Big(\tau_0^{1/2}\, H_0(t) + \tau_H^{1/2}\, H_n(t) + \tau_L^{1/2}\, (N-1)\, H_{-n}(t)\Big). \tag{27}$$

This equation has a simple intuition. All traders place the same weight $\tau_0^{1/2}$ on the dividend-information signal $H_0(t)$. Because they disagree about the precisions, each trader assigns a larger weight $\tau_H^{1/2}$ to his own signal and a lower weight $\tau_L^{1/2}$ to each of the other $N-1$ traders' signals.

As discussed next, trader n's optimal trading strategy depends on both the average of other traders' estimates of the unobserved growth rate $G^*(t)$, defined as $G_{-n}(t) := \frac{1}{N-1} \sum_{m \neq n} G_m(t)$, and his own beliefs about the dynamic statistical relationship between $G^*(t)$ and the sufficient statistics $H_0(t)$, $H_n(t)$, and $H_{-n}(t)$.

2.2. Linear Conjectured Strategies

We seek to find a symmetric, steady-state equilibrium in which traders use simple Markovian linear strategies. To reduce the number of state variables, it is convenient to replace the three state variables $H_0(t)$, $H_n(t)$, $H_{-n}(t)$ with two composite state variables $\hat{H}_n$ and $\hat{H}_{-n}$ defined using a constant $\hat{A}$ by

$$\hat{H}_n(t) := H_n(t) + \hat{A} H_0(t), \quad \hat{H}_{-n}(t) := H_{-n}(t) + \hat{A} H_0(t), \quad \hat{A} := \frac{\tau_0^{1/2}}{\tau_H^{1/2} + (N-1)\tau_L^{1/2}}. \tag{28}$$

Let $x_n(t) = dS_n(t)/dt = X_n(t, P[X_1, \ldots, X_N](t); \mathcal{F}_n(t))$ denote the "flow-quantity" traded by any trader n. Trader n conjectures that for other traders m, four constant "γ-parameters" γ_D, γ_H, γ_S, γ_P define symmetric linear flow-schedules

$$x_m(t) = dS_n(t)/dt = \gamma_D\, D(t) + \gamma_H\, \hat{H}_m(t) - \gamma_S\, S_m(t) - \gamma_P\, P(t). \tag{29}$$

The market-clearing condition and this linear conjecture imply that trader n's own flow-demand schedule satisfies

$$x_n(t) + \sum_{m=1,\ldots,N;\, m\neq n} \Big(\gamma_D\, D(t) + \gamma_H\, \hat{H}_m(t) - \gamma_S\, S_m(t) - \gamma_P\, P(t)\Big) = 0. \tag{30}$$

This equation can be solved for $P(t)$ as a function of $x_n(t)$. Zero net supply $\sum_{m=1}^{N} S_m(t) = 0$ and stationarity yield trader n's conjectured price impact function

$$P(x_n(t)) = \frac{\gamma_D}{\gamma_P}\, D(t) + \frac{\gamma_H}{\gamma_P}\, \hat{H}_{-n}(t) + \frac{\gamma_S}{(N-1)\gamma_P}\, S_n(t) + \frac{1}{(N-1)\gamma_P}\, x_n(t). \tag{31}$$

Equation (31) is analogous to equation (4) from the one-period model, with the interpretation of $x_n(t)$ changed from quantity traded to time-derivative of quantity traded. The intercept of the residual supply schedule depends on dividends $D(t)$

and the information of other traders $\hat{H}_{-n}(t)$. Call the term linear in $S_n(t)$ "permanent impact" and the term linear in $x_n(t)$ "temporary impact." By analogy with equations (16) and (17) for the one-period model, equation (31) defines coefficients of permanent impact λ and temporary impact κ:

$$\lambda := \frac{\gamma_S}{(N-1)\,\gamma_p}, \qquad \kappa := \frac{1}{(N-1)\,\gamma_p}. \tag{32}$$

Imperfect competition requires trader n to take into account both his permanent and temporary price impact in choosing how fast to change his inventory. Trader n exercises monopoly power in choosing how fast to demand liquidity from other traders to profit from information. He also exercises monopoly power in choosing how fast to provide liquidity to the other $N-1$ traders who, according to trader n's beliefs, trade with overconfidence and therefore make supplying liquidity to them profitable. Intuitively, the symmetry of equilibrium trading strategies requires traders to believe they are being adequately compensated for both supplying and demanding liquidity in a manner consistent with market clearing.

2.3. Equilibrium with Linear Trading Strategies

Define a steady-state equilibrium with symmetric, linear flow-strategies as a Bayesian perfect equilibrium in which traders maximize expected utility by choosing flow-strategies of the form (29) with constant γ-parameters (as functions of time). The Bayesian perfect equilibrium concept requires strategies to be dynamically consistent. A steady-state equilibrium also requires inventories to have a non-stochastic, finite variance which does not vary over time.

Appendix section A.3 uses the "no-regret" approach in the same way as the proof of theorem 1 for the one-period model. Trader n solves for his optimal consumption and trading strategy by plugging the price impact function (31) into his dynamic optimization problem. Trader n infers the value of $H_{-n}(t)$ by observing his residual flow-supply schedule, picks the optimal point on this residual flow-supply schedule, and implements this optimal point with a linear demand schedule. Linear conjectured strategies for other traders $m \neq n$ make the optimization problem quadratic in trading intensity $x_n(t)$; thus, the optimal flow-demand $x_n^*(t)$ is the solution to a linear equation. This linear solution generates higher profits than any non-linear demand schedule.

The proof in Appendix section A.3 conjectures an exponential value function whose exponent is a specific quadratic function of the state variables $M_n(t)$, $D(t)$, $\hat{H}_n(t)$, $\hat{H}_{-n}(t)$, and $S_n(t)$, defined in terms of nine "ψ-parameters"; obtains first-order necessary conditions from the Hamilton-Jacobi-Bellman equation; equates coefficients in the conjectured linear solution; and then combines the resulting nine ψ-equations with four γ-equations, imposing symmetry on the solution. The proof shows that these thirteen equations can be reduced to six polynomial equations (A-57)–(A-62) in six unknowns, whose solution determines the nine ψ-parameters defining the value function in equation (A-37) and the four γ-parameters defining trading strategies in equation (29). The thirteen endogenous parameters are functions of the ten exogenous parameters r, ρ, A, α_D, σ_D, α_G, σ_G, N, τ_H, and τ_L (in terms of which the quasi-exogenous parameters τ_0, τ, Ω, and $\hat{A}$ are also defined).[6]

There always exists a no-trade equilibrium $X_n \equiv 0$, with no well-defined price.

THEOREM 3: ***Characterization of Equilibrium in the Continuous-Time Model with Overconfidence and Imperfect Competition.*** *There exists a steady-state, Bayesian-perfect equilibrium with symmetric, linear flow-strategies and positive trading volume if and only if the six polynomial equations (A-57)–(A-62) have a solution satisfying the second-order condition $\gamma_P > 0$ and the stationarity condition $\gamma_S > 0$. Such an equilibrium has the following properties:*

1) *There is an endogenously determined constant $C_L > 0$, defined in equation (A-49), such that trader n's optimal flow-strategy $x_n^*(t)$ makes time-differentiable inventories $S_n(t)$ change at rate*

$$x_n^*(t) = \frac{dS_n(t)}{dt} = \gamma_S \Big(C_L \left(\hat{H}_n(t) - \hat{H}_{-n}(t) \right) - S_n(t) \Big). \tag{33}$$

2) *There is an endogenously determined constant $C_G > 0$, defined in equation (A-49), such that the equilibrium price is*

$$P^*(t) = \frac{D(t)}{r + \alpha_D} + C_G \, \frac{\bar{G}(t)}{(r + \alpha_D)(r + \alpha_G)}, \tag{34}$$

where $\bar{G}(t) := \frac{1}{N} \sum_{n=1}^{N} G_n(t)$ denotes the average expected growth rate.

The equilibrium has a surprisingly simple structure. Equations (33) and (34) are similar to equations (11) and (12) in the one-period model.

The price immediately reveals the average of all signals, responding instantaneously to innovations in each trader's private information. This occurs despite the fact that, to reduce price impact costs resulting from adverse selection, each trader intentionally slows down his trading.

If C_G were equal to one, equation (34) would imply that the equilibrium price would be the average of traders' risk-neutral buy-and-hold valuations, consistent

[6]The six polynomial equations are implications of first-order conditions under the assumptions that trading strategies have the conjectured linear form and the equilibrium is symmetric. For a solution to the six polynomial equations to define a stationary equilibrium, it is sufficient for the solution to satisfy (1) a second-order condition implying $\gamma_P > 0$, (2) a stationarity condition implying $\gamma_S > 0$, (3) a transversality condition requiring $r > 0$, and (4) a budget constraint ruling out Ponzi schemes (implied by $r > 0$ and stationarity of inventories). The second-order condition $\gamma_P > 0$ requires temporary price impact to be positive; if temporary price impact were negative, traders could achieve infinite utility by first buying and then selling at extremely fast rates over short periods of time. The stationarity condition $\gamma_S > 0$ requires permanent price impact to be positive; inventories would blow up over time if permanent price impact were negative, and this would be inconsistent with stationarity. Mathematical details supporting this intuition are provided at the end of Appendix section A.3.

with Gordon's growth formula and the one-period model. As discussed in section 3.3, a "Keynesian beauty contest" makes the multiplier C_G less than one.

An important difference between our model and other models concerns the scaling of trading with risk aversion. We have the following analytical result:

THEOREM 4: ***Comparative Statics for Risk Aversion.*** *If risk aversion A is scaled by a factor of F to A/F, then C_L changes to $C_L\,F$, λ changes to λ/F, κ changes to κ/F, but γ_S and C_G remain the same.*

Theorem 4 implies that when risk tolerance $1/A$ increases by a factor of $F > 1$, traders increase target inventories and quantities traded proportionally in response to proportional reductions in temporary and permanent price impact. Risk aversion affects only quantities. The speed of trading and equilibrium prices remain the same. Section 3 discusses related empirical implications.

2.4. An Existence Condition

Calculation of an analytical solution for the equilibrium in theorem 3 requires solving the six polynomial equations (A-57)–(A-62). While these equations do not admit an obvious analytical solution, they can be solved numerically. Extensive numerical calculations lead us to conjecture that the existence condition for the continuous-time model is exactly the same as the existence condition for the one-period model:

CONJECTURE 1: ***Existence Condition.*** *A steady-state, Bayesian-perfect equilibrium with symmetric, linear flow-strategies exists if and only if*

$$\Delta_H := \frac{\tau_H^{1/2}}{\tau_L^{1/2}} - 2 - \frac{2}{N-2} > 0. \tag{35}$$

We have examined numerical solutions to the six equations (A-57)–(A-62) for a large number of exogenous parameter values. When existence condition (35) is satisfied, the numerical algorithm always finds precisely one solution satisfying the second-order condition $\gamma_P > 0$, and this solution also satisfies the stationarity condition $\gamma_S > 0$. When existence condition (35) is reversed, the numerical algorithm sometimes finds solutions satisfying the second-order condition $\gamma_P > 0$, but these solutions do not satisfy the stationarity condition $\gamma_S > 0$.

Similar to the one-period model, we expect equilibrium with trade to exist only if there is enough disagreement. With continuous trading, each trader tries to exercise his monopoly power by smoothly walking up the residual supply schedule rather than by trading a block at one market-clearing price. If ΔP denotes the total price impact of trading some quantity smoothly by walking up a linear residual supply schedule, then the average transaction price incorporates a realized price impact cost of approximately $\Delta P/2$. To be willing to take the other side of such smooth trades of their competitors, traders must believe that their competitors' signals are only about "half as precise" as their competitors believe them to be. This intuition is consistent with the existence condition $\Delta_H > 0$, which is equivalent to $\tau_H^{1/2}/2 > \tau_L^{1/2}\,(1 + 1/(N-2))$. In this context, "half-as-precise" means $\tau_H^{1/2}/2 \approx \tau_L^{1/2}$; the term $1/(N-2)$ is due to market power.

A closed-form solution exists for the limiting case when market liquidity vanishes (*i.e.*, when $\Delta_H \to 0$ implies $\kappa \to \infty$ or $\gamma_P \to 0$). Consistent with the intuition above, the existence condition (35) holds exactly in this limit. A closed-form solution also exists for the special case when traders believe other traders have no information ($\tau_L = 0$) and the number of traders N is large. These two closed-form solutions are discussed in more detail in sections 3.4 and 3.5.

2.5. A Competitive Model as Benchmark

To understand how imperfect competition affects the equilibrium, Appendix section B.2 characterizes the equilibrium of the analogous continuous-time model in which the assumption of perfect competition replaces imperfect competition.

Conceptually, the model with perfect competition differs from the model with imperfect competition in two ways. First, when traders construct their strategies $(c_n(t), S_n(t))$, they do not take into account the effect of their trades on prices, and this simplifies their wealth dynamics (B-12). Second, since it is not necessary for a trader to consider separately money holdings M_n and a stock holdings S_n in the case of perfect competition, the value function conjectured in (B-14) is a quadratic exponential function of only three state variables, wealth W_n and two information variables $\hat{H}_n$ and $\hat{H}_{-n}$. This reduces the number of parameters in the value function. The results are summarized in the following theorem:

THEOREM 5: ***Characterization of Competitive Equilibrium for the Continuous-Time Model.*** *There exists a steady-state, Bayesian-perfect equilibrium with symmetric, linear strategies with positive trading volume if and only if the three polynomial equations (B-23)–(B-25) have a solution satisfying $\gamma_P > 0$. Such an equilibrium has the following properties:*

1) *There is an endogenously determined constant $C_L > 0$, defined in equation (B-20), such that trader n's optimal inventories $S_n^*(t)$ are*

$$S_n^*(t) = C_L\,(\hat{H}_n(t) - \hat{H}_{-n}(t)). \tag{36}$$

2) *There is an endogenously determined constant $C_G > 0$, defined in equation (B-18), such that the equilibrium price is*

$$P^*(t) = \frac{D(t)}{r + \alpha_D} + C_G\,\frac{\bar{G}(t)}{(r + \alpha_D)(r + \alpha_G)}, \tag{37}$$

where $\bar{G}(t) := \frac{1}{N}\sum_{n=1}^{N} G_n(t)$ denotes the average expected growth rate.

Equations (36) and (37) are similar to corresponding equations (33) and (34) in theorem 3, but the values of C_L and C_G are different. The equilibrium has a simple structure, which is very different from the model of smooth trading. The most important difference is that traders do not smooth their trading. Each trader immediately adjusts actual inventories to target levels; since inventories are diffusions, trading volume is infinite. With imperfect competition, traders strategically smooth their trades out over time to exploit their market power in the presence of adverse selection; since trading is "smooth," trading volume is finite.

In the model with perfect competition, the price dampening factor C_G continues to be less than one. As discussed in section 3.3 below, competition leads to greater price dampening, making the value of C_G even smaller than with imperfect competition.

2.6. Interpretation As A Model with Private Values

Instead of motivating trade using a model based on agreement-to-disagree (and no common prior), trade can instead be motivated by an alternative model based on private values with a common prior. The alternative model is identical to the model of disagreement except for two important differences: (1) Instead of agreeing to disagree, all private signals have the same precision τ_I. (2) In addition to the common cash dividend $D(t)$, each trader receives an orthogonal private-value or "convenience yield" $\pi_J\, H_n^J(t)$ which follows an AR-1 process. This structure is common knowledge; traders share a common prior.

Making the two models as similar as possible requires making the assumption that the exogenous mean-reversion rate of the convenience yield is the same as the endogenous mean-reversion rate of private information; otherwise, the number of state variables explodes.

To compare these two approaches, Appendix C examines such a model with private values. All of the equations in section 2 and Appendix A map nicely into corresponding equations in Appendix C.

The noise due to private values lowers by some endogenous factor the precision of other traders' signals inferred from prices. To make the models as similar as possible, the parameter τ_I can be chosen to equal the parameter τ_H, and the level of innovation variance in shocks to private values can be chosen so that the endogenous lower precision inferred from prices is equal to τ_L.

When the parameters of the private-values processes are chosen to map the model with private values into the model with disagreement as closely as possible, it is straightforward to see that all equations are similar except for one specific, intriguing difference. In the model with private values, traders agree that they have different valuations in the present, and they furthermore agree that these different valuations will mean revert toward the same unconditional common mean consistent with a common prior. In the model with disagreement, traders also agree that they have different valuations in the present, but—in contrast to the model with private values—they furthermore agree to disagree about the stochastic process their different current valuations will follow in the future. Specifically, each trader believes that the other traders' valuations will converge to his own valuation in the long run but deviate in the short run; because they have different beliefs about valuation dynamics as a result of not sharing a common prior, they disagree in the present about how their expectations will differ in the future. Algebraically, this effect shows up in equations (C-33) and (C-34); the discussion following these equations clarifies the intuition further.

As discussed in detail below, disagreement about the dynamics of valuations leads to a Keynesian beauty contest with dampening of prices ($0 < C_G < 1$). With private values, it can be shown analytically that no such dampening occurs ($C_G = 1$). The private-values model is simpler than the model with disagreement because traders disagree about the present only; they do not disagree about the future.

This inconsistency between the two models leads to a practical insight about the conjecture of Harsanyi (1976) that any model with different priors is isomorphic to a model with a common prior, therefore making models with different priors unnecessary. Harsanyi's conjecture potentially simplifies economic modeling by allowing game theorists to employ machinery developed specifically for common prior models.

To generate an isomorphic model with a Keynesian beauty contest, it would be necessary to make very specific *ad hoc* complicated assumptions about the evolution of private values and correlations between them in the future. These assumptions would need to be designed specifically to mimic artificially the natural dynamics of Bayes law in the context of agreement-to-disagree. Thus, as a practical matter, successfully implementing Harsanyi's conjecture is unlikely to simplify the analysis of the resulting common-values model. Ockham's razor supports modeling trading based on disagreement, not based on a common prior assumption.

Both Vayanos (1999) and Du and Zhu (2015) obtain no price dampening ($C_G = 1$) in models with private values, inventories, and expected returns following random walks. Our smooth trading model with disagreement, by contrast, is designed to make sharp predictions about individual traders' asset holdings. As discussed further below, we believe it is empirically realistic that both higher information flow (increased τ) and increased liquidity resulting from more disagreement (higher τ_H/τ_L) shorten the holding period of individual traders.

3. Implications of the Continuous-Time Model

This section discusses implications of the continuous-time model for (1) trading strategies, (2) market liquidity, and (3) prices. In what follows, we suppress for simplicity the superscript "$*$" on equilibrium prices and strategies.

3.1. Trading Strategies: A Partial Adjustment Process

The equilibrium trading strategies have a simple form similar to the one-period model. Define trader n's "target inventory," denoted $S_n^{TI}(t)$, as the inventory level such that trader n chooses not to trade ($x_n(t) = 0$). From equation (33), the target inventory is given by

$$S_n^{TI}(t) = C_L \left(\hat{H}_n(t) - \hat{H}_{-n}(t)\right). \tag{38}$$

Trader n targets a long position if his own signal $\hat{H}_n(t)$ is greater than the average signal of other traders $\hat{H}_{-n}(t)$ and a short position if his own signal is less than the average signal of others. The endogenously determined proportionality constant C_L measures the sensitivity of target inventories to this difference.

Trader n follows a partial adjustment strategy such that his inventory $S_n(t)$ gradually converges toward its target level $S_n^{TI}(t)$ at an endogenously determined rate γ_S:

$$x_n(t) = \frac{dS_n(t)}{dt} = \gamma_S \left(S_n^{TI}(t) - S_n(t)\right). \tag{39}$$

While sample paths for the target inventory level $S_n^{TI}(t)$ and the trading intensity $x_n(t)$ follow diffusions (of order $dt^{1/2}$) whose innovations respond to the arrival of new information, the sample path of inventories $S_n(t)$ is a differentiable function of time (of order dt), not a diffusion. In this sense, trader n trades smoothly.

The smooth trading model captures in an intuitive and realistic manner—consistent with Grinold and Kahn (1995)—the inventory behavior of equity asset managers who use public and private information to forecast stock returns. Partial adjustment toward mean-reverting target inventories provides a realistic structural benchmark. When information changes, an asset manager updates his estimate of the asset's value, recalculates his target inventory, and adjusts trading to move inventories in the direction of the new target. Since moving large blocks over short periods of time is expensive, an asset manager builds positions gradually, taking into account both price impact and the speed with which information decays.

Trader n believes that the information variables $\hat{H}_n(t)$ and $\hat{H}_{-n}(t)$ in equation (38) follow a bivariate vector auto-regression process. Traders disagree about drift rates. Trader n believes that $\hat{H}_n(t) - \hat{H}_{-n}(t)$ mean-reverts at rate $\alpha_G + \tau$ but also drifts in a direction proportional to $(\tau_H^{1/2} - \tau_L^{1/2})\, G_n(t)$ (see equation (A-36)).

Intuition suggests that more disagreement will make markets more liquid, and this additional liquidity will be associated with more rapid adjustment of actual inventories toward target levels. Although we do not have formal analytical results proving that this intuition is correct, it is consistent with results we have obtained numerically.

Figure 1 shows that as disagreement τ_H/τ_L increases, the speed of inventory adjustment γ_S increases (first panel) while the size of target inventories C_L increases when disagreement is low and then decreases before leveling off at high levels of disagreement. Intuitively, when disagreement increases, it becomes less costly for a trader to trade toward his target inventory more rapidly because other trades are more willing to provide liquidity. Figure 1 shows that the speed with which traders' inventories converge to target levels also increases when the decay rate of their signals $\alpha_G + \tau$ increases. Intuitively, when a signal decays faster, a trader trades faster.[7]

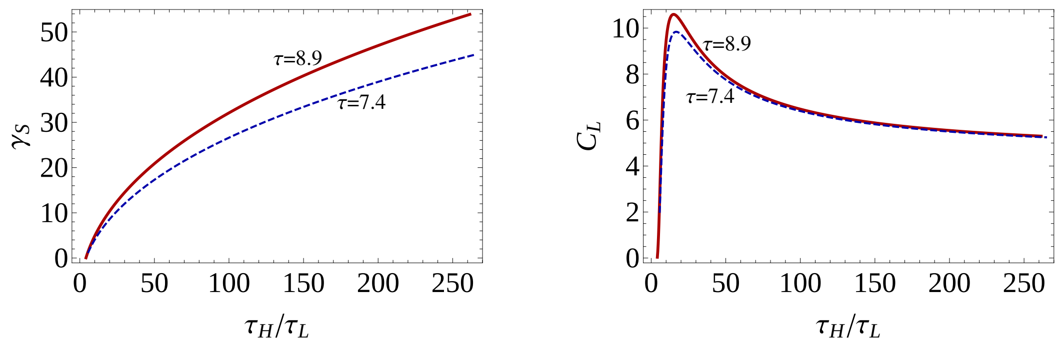


FIGURE 1. COEFFICIENTS γ_S AND C_L AGAINST τ_H/τ_L WHILE FIXING $\tau = 7.4$ AND $\tau = 8.9$.

The left panel of figure 2 shows that as the number of traders N increases, the speed of inventory adjustment γ_S increases steadily; the intuition is that more competition makes trading less costly. The right panel shows that the size of target inventories C_L increases toward a constant level when N is large; the intuition is that risk aversion limits the maximum size of inventories when more competition makes trading costs fall.[8]

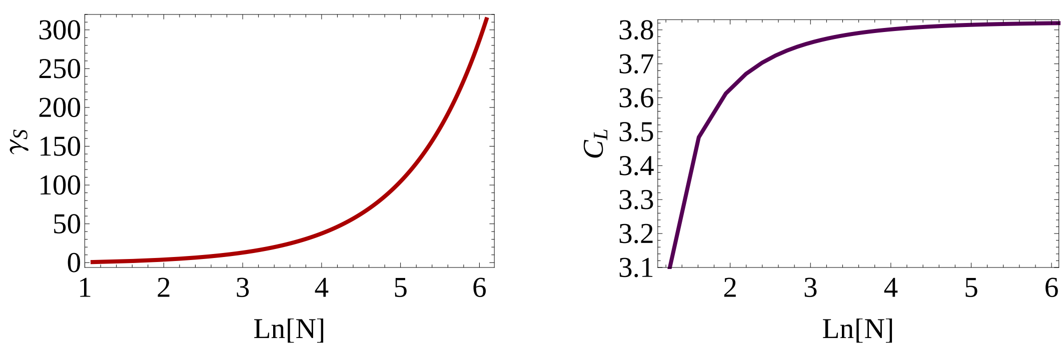


FIGURE 2. COEFFICIENTS γ_S AND C_L AGAINST N WHILE FIXING $\tau = 1.4$ AND $\tau_L = 0$.

[7]Numerical calculations in figure 1, figure 4, and panel (a) of figure 8 are based on exogenous parameter values $\tau = 7.4$ (or $\tau = 8.9$), $r = 0.01$, $A = 1$, $\alpha_D = 0.1$, $\alpha_G = 0.02$, $\sigma_D = 0.5$, $\sigma_G = 0.1$, $\tau_0 = \Omega\sigma_G^2/\sigma_D^2 = 0.0054$, and $N = 100$.

[8]Numerical calculations in figure 2, figure 5, and panel (b) of figure 8 are based on the exogenous parameter values $\tau_L = 0$, $\tau = 1.4$, $r = 0.01$, $A = 1$, $\alpha_D = 0.1$, $\alpha_G = 0.02$, $\sigma_D = 0.5$, $\sigma_G = 0.1$ and $\tau_0 = \Omega\sigma_G^2/\sigma_D^2 = 0.0279$.

Immediate Price Adjustment, Gradual Inventory Change. In the smooth trading model, prices adjust instantaneously but quantities adjust slowly. As soon as trader n changes his trading intensity $x_n(t)$ due to arrival of new information, the price instantaneously moves to a new equilibrium level, even though the trader has not yet traded a single share. Even if signals $\hat{H}_n(t)$ and $\hat{H}_{-n}(t)$ were to remain constant over some period of time and the price did not change, trader n would continue to trade based on the level of his "past" disagreement with the market. The integral representation of the inventory dynamics in equations (38) and (39),

$$(40) \quad S_n(t+s) = \mathrm{e}^{-\gamma_S\, s} \; \Big(S_n(t) + \int_{u=t}^{t+s} \mathrm{e}^{-\gamma_S\, (t-u)} \;\, \gamma_S \, C_L \, (\hat{H}_n(u) - \hat{H}_{-n}(u)) \, du\Big),$$

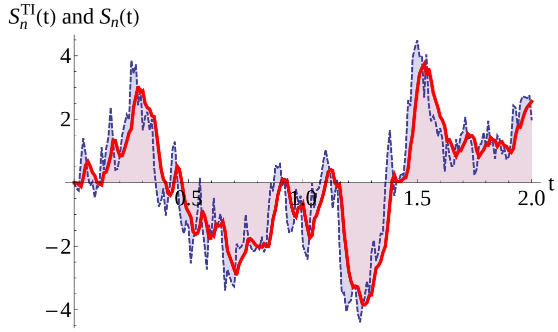
shows that traders add to existing inventories based on current differences in signals $\hat{H}_n(t) - \hat{H}_{-n}(t)$ and liquidate their existing inventories accumulated based on past signal differences at rate γ_S.

Simulated Inventory Paths. Figure 3 presents three simulated paths of target inventories (dashed lines) and actual inventories (solid line).[9] Panel (a) shows that when disagreement is larger—and the market is more liquid—actual inventories closely track target inventories. Panel (b) shows that when disagreement is smaller—and the market is less liquid—actual inventories deviate significantly from target inventories since traders restrict their speed of trading. To illustrate that the speed at which traders' inventories converge to target levels also depends on the decay rate of their signals, panel (c) plots actual and target inventories using the same level of disagreement as in panel (a) but a lower decay rate $\alpha_G + \tau$ of traders' signals; actual inventories track target inventories less closely than in panel (a), in line with figure 1. Note that target and actual inventories coincide in the competitive model.

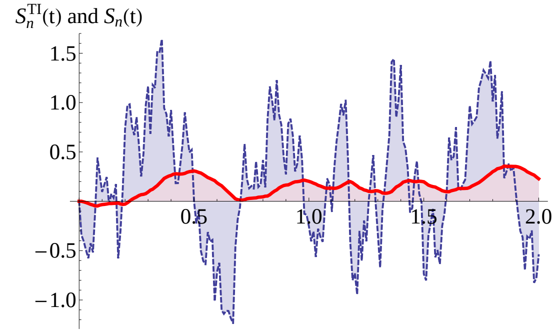
Literature. Inventories in the discrete-time model of Vayanos (1999) also follow partial adjustment processes toward randomly changing target levels. While smooth trading in that model is also a consequence of imperfect competition, the economic forces governing target and actual inventories are different, and this leads to different empirical implications.

Like Grossman and Miller (1988), Vayanos (1999) assumes shocks to endowments affect target inventories. Risk aversion motivates demand for immediacy. Higher risk aversion makes the speed of inventory adjustment faster and price impact higher. As risk aversion varies across markets, the model of Vayanos (1999) therefore implies that illiquid markets are characterized by short holding periods and high trading volume. Vayanos (1999) also assumes that both inventories and expected returns are non-stationary, diverging to infinity over time.

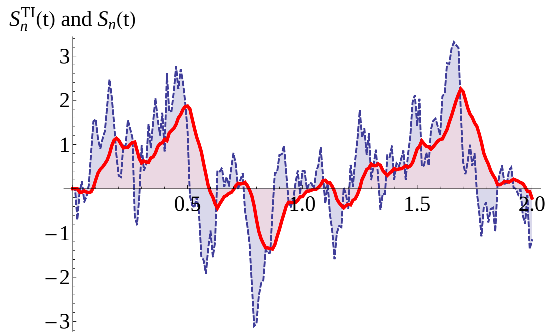
[9]The paths are generated using equations (38), (39), (40), (A-34), and (A-35), which describe the dynamics of $H_n(t), H_{-n}(t), S_n(t)$, and $S_n^{TI}(t)$. Numerical calculations in figure 3 are based on the exogenous parameter values $\alpha_D = 0.1$, $\alpha_G = 0.02$, $\sigma_D = 0.5$, $\sigma_G = 0.1$, and $N = 100$, with $\tau = 8.9$ and $\tau_0 = \Omega \sigma_G^2 / \sigma_D^2 = 0.0045$ in both (a) and (b); $\tau_H = 4.46$ and $\tau_L = 0.045$ in (a); $\tau_H = 0.5$ and $\tau_L = 0.085$ in (b); and $\tau = 3.15$, $\tau_H = 1.56$, $\tau_L = 0.016$, and $\tau_0 = 0.0126$ in (c),.



(a) With high disagreement



(b) With low disagreement



(c) With low decay rate and high disagreement

FIGURE 3. SIMULATED PATHS OF $S_n^{TI}(t)$ AND $S_n(t)$.

In our model, risk tolerance $1/A$ as an empirical proxy for assets under management. Theorem 4 implies that higher risk aversion does not affect the speed of inventory adjustment; instead, it scales down target inventories and the number of shares traded and increases price impact coefficients λ and κ (theorem 4). As risk aversion goes to infinity, our model makes the realistic prediction that trading volume goes to zero while Vayanos (1999) makes the unrealistic prediction that trading volume goes to infinity.

Our model explains how asset managers try to outperform benchmarks by trading securities they perceive to be undervalued or overvalued. Stationary, mean-reverting target inventories and perceived expected returns are endogenous consequences of the simultaneous solutions to optimization problems based on public and private information flow, total precision of information in the market, disagreement among traders, and traders' risk bearing capacity. If actively traded stocks have faster information flow (larger $\alpha_G + \tau$), then our model predicts more rapidly mean-reverting target inventories in more active markets. Our model also predicts that the speed of inventory adjustment γ_S tends to increase with faster information decay (increasing $\alpha_G + \tau$) or more disagreement (increasing τ_H/τ_L); markets with high trading volume are therefore more liquid and have shorter holding periods.

These predictions are consistent with empirical patterns in inventories of market makers documented by Hasbrouck and Sofianos (1993), Madhavan and Smidt (1993), and Menkveld and Hendershott (2014). The inventories of market makers tend to be stationary with lower mean-reversion rates in less liquid securities. The half-life of inventories of intermediaries is usually shorter in larger stocks.

These predictions are also relevant for the empirical literature which studies trades and holdings of institutional traders. Atkyn and Dyl (1997) study turnover rates, Chakrabarty, Moulton and Trzcinka (2015) study institutional holding periods, Chan and Lakonishok (1995) study the length of trading packages, Cremers and Pareek (2014) study stock duration, Bae et al. (2014) study the number of buy-sell switching points, and Cremers and Petajisto (2009) study "active shares." Monthly and quarterly data on institutional holdings, such as 13-F filings, suggest complicated patterns of long-term trading. Using more granular proprietary databases, Puckett and Yan (2011) and Chakrabarty, Moulton and Trzcinka (2015) find that institutional investors also engage in intensive short-term trading.

Patterns of simultaneous long- and short-term trading are consistent with partial adjustment toward fluctuating target inventories. One of the main contributions of our paper is the empirical hypothesis that long-term trading results from slow information flow and high trading costs in low-volume markets while short-term trading results from fast information flow and low trading costs in high-volume markets.

3.2. Temporary and Permanent Price Impact

The concepts of permanent and temporary price impact are crucially important for the practical management of transaction costs. Our model generates insights by linking both permanent and temporary impact to deep parameters such as the precision of information flow and the magnitude of disagreement. The models of Vayanos (1999) and Du and Zhu (2015) also have permanent and temporary price impact, influenced by risk aversion and either the size of inventory shocks or the size of shocks to private values, respectively. In these two papers, permanent and temporary price impact are more difficult to distinguish because these papers are set in discrete time.[10]

We do not use the terms "temporary price impact" and "permanent price impact" like these terms are usually used in the empirical market microstructure literature. In this literature, temporary and permanent price impact are time series properties of market prices. Temporary price impact is associated with negative first-order autocorrelation in price changes (bid-ask bounce), and permanent price impact

[10]Describing these effects is more complicated in the discrete-time models of Vayanos (1999) and Du and Zhu (2015). If time intervals between rounds of trading in these discrete-time models were to become infinitely short, then price impact would equal a product of an infinitely large price impact coefficient and an infinitely small quantity traded. Continuous-time gracefully deals with this "infinity-times-zero" problem, crystalizing how the speed of trading affects the equilibrium. Since an infinitely small quantity traded can be presented as a finite time-derivative of inventory multiplied by an infinitely small time interval, price impact naturally decomposes into two components. One component, which we call permanent price impact, is linear in the level of inventories (stocks). The other component, which we call temporary price impact, is linear in the time derivative of inventories (flows).

is associated with persistent correlations between price changes and order flow. Instead, like sophisticated traders in the asset management industry, we think of temporary and permanent price impact as components of transaction costs, which are explicitly optimized in the construction of trading strategies. Traders correctly understand that faster execution leads to larger temporary price impact but has no effect on permanent price impact.

Combining equations (31) and (32), the price can be expressed as a linear combination of (1) a weighted average $p_{0,n}(t)$ of other traders' signals, (2) the trader's own inventory level $S_n(t)$, and (3) the intensity of his trading measured by the time-derivative of his inventory $x_n(t)$:

$$P(S_n(t), x_n(t)) := p_{0,n}(t) + \lambda\, S_n(t) + \kappa\, x_n(t). \tag{41}$$

The intercept $p_{0,n}(t)$ is the random variable $p_{0,n}(t) = \gamma_D/\gamma_P\, D(t) + \gamma_H/\gamma_P\, \hat{H}_{-n}(t)$. The "permanent price impact" parameter λ and the "temporary price impact" parameter κ are defined in equation (32). A trader correctly believes that the price changes when the level of his inventory changes or when the intensity of his trading changes. If trader n suddenly stops trading, then the price will immediately reverse by $\kappa\, x_n(t)$ as his temporary price impact disappears, but his permanent price impact $\lambda\, S_n(t)$ will remain.

Since our model is symmetric across traders, market clearing implies that temporary and permanent price impact cannot show up as correlations between prices and quantities. The equilibrium price process resembles a Brownian motion. It is therefore theoretically difficult to learn about price impact from price and inventory dynamics when traders trade optimally. Black (1982) makes a similar point when he writes that we can only hope to learn about causal relationships between variables from studying mistakes that firms make when they act sub-optimally or from performing experiments.

Notwithstanding the empirical difficulties of detecting price resiliency suggested by our model, the empirical literature has found evidence of temporary price impact, which can be explained either as sub-optimally fast order execution or as intermediation costs imposed by dealers. Keim and Madhavan (1997) find that more aggressive trades of index funds and technical traders have larger costs than trades of more patient value investors. Dufour and Engle (2000) find that the price impact of trades increases when duration between transactions decreases. Chan and Lakonishok (1995) document that high demand for immediacy tends to be associated with larger price impact. Holthausen, Leftwich and Mayers (1990) measure temporary and permanent price effects associated with block trades and find that most of the adjustment occurs during the very first trade; the immediate price response to changes in target inventories is a property of our model as well.

Asset management practitioners have long recognized the importance of managing both permanent and temporary price impact costs. The practitioner-oriented model of Almgren and Chriss (2000) is essentially a non-linear generalization of our equation (41). One difference is that our intercept $p_{0,n}(t)$ changes over time in a manner which trader n believes he can predict, whereas the intercept in the Almgren-Chriss model follows a random walk. Obizhaeva and Wang (2013) propose an alternative model in which—rather than decaying instantaneously when a trader stops trading—temporary price impact decays gradually at an exponential rate; the paper derives an optimal way to manage temporary price impact costs in the context of a slightly different dynamic model of price resilience.

Our information-based price impact model is significantly different from models with linear permanent price impact but no temporary price impact. Consider the continuous-time model of Kyle (1985). The informed trader correctly conjectures that the price is given by $P(t) = P(0) + \lambda\,(\sigma_U\, B_U(t) + S_n(t))$, where $\sigma_U\, B_U(t)$ is the inventory of noise traders and $S_n(t)$ is the inventory of the informed trader. This formula is similar to our equation (41), except there is no temporary price impact term. The informed trader only optimizes the permanent impact of his trades. If the informed trader buys Q shares over a fixed period of time T, then he "walks up the demand schedule" and expects to incur a price impact cost of $\frac{1}{2}\lambda\, Q$ per share, while the price gradually increases to $\lambda\, Q$. There is no temporary impact as long as the informed trader's inventory is a differentiable function of time.

In our model, by contrast, temporary price impact makes trading costs depend on the execution speed. Suppose a trader buys Q shares at a constant rate over an interval of time T. For simplicity, assume $P(t) = \hat{H}_n(t) = \hat{H}_{-n}(t) = 0$. The average execution price is $(\frac{1}{2}\lambda + \kappa/T)\, Q$ (obtained by integrating over equation (41)). The first term $\frac{1}{2}\lambda\, Q$ is the permanent price impact cost and the second term $(\kappa/T)\, Q$ is the additional temporary price impact cost proportional to the execution speed $1/T$. When the trader initiates order execution, the price immediately jumps from zero to $\kappa\, Q/T$, then gradually rises to $(\lambda + \kappa/T)\, Q$ over the time interval T, and finally drops back to $\lambda\, Q$ when the trader stops buying.

In our model, traders provide liquidity to one another; there are no dealers acting as intermediaries. Figure 4 shows that as disagreement τ_H/τ_L increases, permanent depth $1/\lambda$ increases monotonically and temporary depth $1/\kappa$ increases almost linearly. In addition, both permanent price impact and temporary price impact decrease as the total precision increases. Figure 5 shows that, as N increases, permanent depth $1/\lambda$ and temporary depth $1/\kappa$ increase as well. Our numerical results are consistent with the intuition that more disagreement or a greater number of traders decreases transaction costs by making traders more willing to provide liquidity to one another.

Suboptimal γ_S. To illustrate further how suboptimal order execution can be detected empirically, consider the following off-equilibrium scenario. Suppose trader n silently decides to deviate from his equilibrium strategy by trading toward his target inventory at some rate $\bar{\gamma}_S$, which is arbitrarily faster or slower than the equilibrium rate γ_S. To fix ideas, suppose he thinks about implementing the following strategy $\bar{x}_n(t)$:

$$\bar{x}_n(t) = \bar{\gamma}_S\, (S_n^{TI}(t) - \bar{S}_n(t)), \tag{42}$$

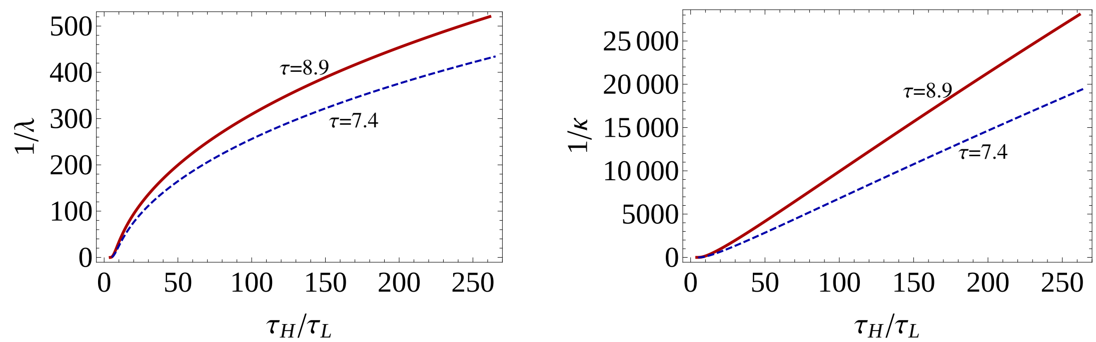


FIGURE 4. COEFFICIENTS $1/\lambda$ AND $1/\kappa$ AGAINST τ_H/τ_L WHILE FIXING $\tau = 7.4$ AND $\tau = 8.9$.

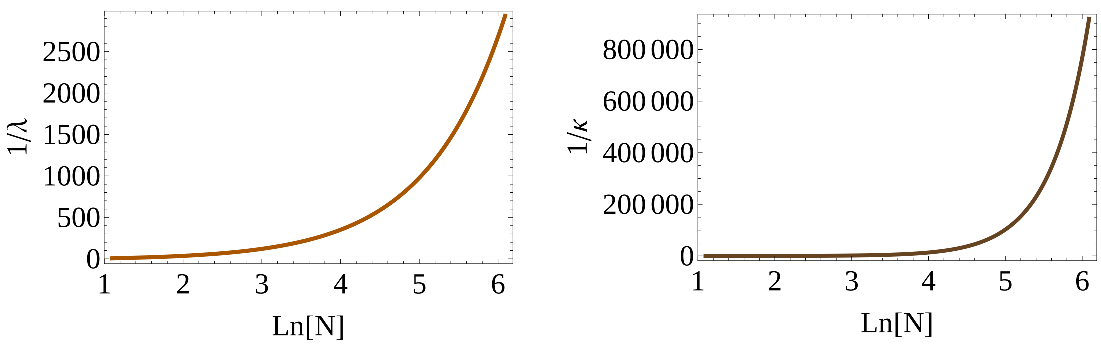


FIGURE 5. COEFFICIENTS $1/\lambda$ AND $1/\kappa$ AGAINST N WHILE FIXING $\tau = 1.4$ AND $\tau_L = 0$.

at each point t after date 0. When $\bar{\gamma}_S = \gamma_S$, this equation coincides with the equilibrium strategy in equation (39); when $\bar{\gamma}_S > \gamma_S$, the trader moves to his target inventory $S_n^{TI}(t)$ more aggressively, and when $\bar{\gamma}_S < \gamma_S$ the trader is more patient. After date $t = 0$, the off-equilibrium inventory level $\bar{S}_n(t)$ is given by

$$\bar{S}_n(t) = \mathrm{e}^{-\bar{\gamma}_S\, t} \left(S_n(0) + \int_{u=0}^{t} \mathrm{e}^{\bar{\gamma}_S\, u}\; \bar{\gamma}_S\, C_L\, (\hat{H}_n(u) - \hat{H}_{-n}(u))\, du \right). \tag{43}$$

For simplicity, suppose trader n holds a positive target inventory at time $t = 0$, with other traders' signals at their long-term mean $\hat{H}_{-n}(0) = 0$, implying

$$S_n(0) = S_n^{TI}(0) = C_L\, \hat{H}_n(0) > 0, \tag{44}$$

$$P(0) = \frac{\gamma_S}{(N-1)\gamma_P}\, S_n(0) > 0. \tag{45}$$

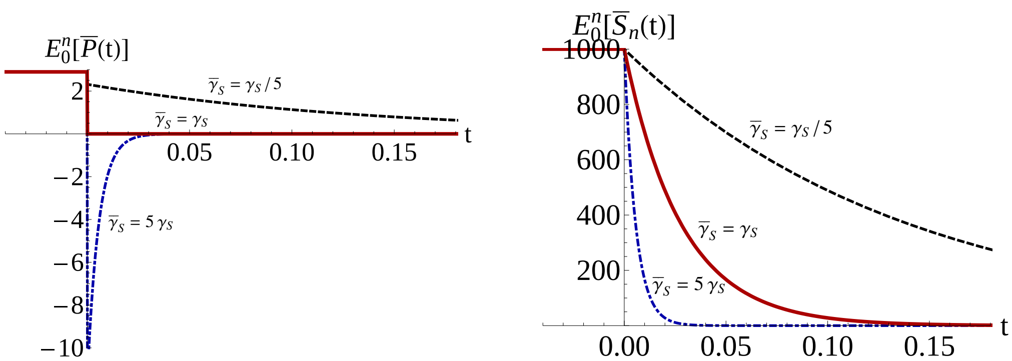
Next, assume that at time $t = 0^+$, trader n's sufficient statistic $\hat{H}_n(0)$ suddenly drops to zero, reducing both his target inventory and the price to zero. Since $\hat{H}_n(0^+) = \hat{H}_{-n}(0^+) = 0$, the implied equilibrium price $\mathrm{E}_0^n\{P(t)\} = 0$. Equations (43) and (41) imply that trader n's expectation of future inventories and price—the impulse-response functions from the perspective of trader n—are given by

$$\text{(46)} \qquad \mathrm{E}_0^n\{\bar{S}_n(t)\} = \mathrm{e}^{-\bar{\gamma}_S\, t}\; S_n(0),$$

$$\text{(47)} \qquad \mathrm{E}_0^n\{\bar{P}(t)\} = -\frac{\bar{\gamma}_S - \gamma_S}{(N-1)\gamma_P}\; \mathrm{e}^{-\bar{\gamma}_S t}\; S_n(0).$$

In figure 6, panel (a) shows expected paths of future prices based on equation (47) and panel (b) shows paths of future inventories based on equation (46). As shown by the solid red lines, if trader n liquidates his inventory at an equilibrium rate $\bar{\gamma}_S = \gamma_S$, then the price immediately drops to zero, but the trader continues to trade out of his inventories over time. Since his equilibrium trading is "expected," it has no additional effect on prices after time 0^+; the initial temporary price impact gradually turns into permanent impact at a pace that keeps price changes relatively unpredictable.

Figure 6 also illustrates two off-equilibrium cases:[11]



(a) The Dynamics of Expected Prices (b) The Dynamics of Inventories

Figure 6. The Dynamics of Expected Prices and Inventories.

When trader n sells at a rate five times slower than the equilibrium rate, $\bar{\gamma}_S = \gamma_S/5$, the immediate price drop is only 1/5 as large as in equilibrium. The slow rate is not optimal because the higher profits on the early trades at initially better prices are more than offset by lower profits on later trades, when information is being incorporated into prices through the trading of others.

When trader n sells at a rate five times faster than the equilibrium rate, $\bar{\gamma}_S = 5\,\gamma_S$, the price is expected to drop sharply initially, by five times as much as in equilibrium. Speeding up execution exacerbates temporary price impact initially and elevates transaction costs overall. As the price comes back, the price path exhibits a distinct V-shaped pattern.

[11]We assume $S_n(0) = 1,000$ shares, $\tau = 9.95$ with $\tau_0 = 0.004$, $\tau_L = 0.05$, and $\tau_H = 5.00$, implying equilibrium price $P(0) = 2.896$ and equilibrium $\gamma_S = 35.8$. The other exogenous parameter assumptions are $r = 0.01, A = 1, \alpha_D = 0.1, \alpha_G = 0.02, \sigma_D = 0.5, \sigma_G = 0.1$, $N = 100$, and $D(0^+) = 0$.

Flash Crash. The price response from trading too fast is very similar to the price patterns observed during the "flash crash" of May 6, 2010. On that day, the E-mini S&P 500 futures price plunged by 5% over a 13-minute period, triggered a five-second trading halt, and then rose by 6% over the next 23 minutes. The Staffs of the CFTC and SEC (2010a,b) report that the flash crash was triggered by an automated execution algorithm that sold S&P 500 E-mini futures worth approximately $4 billion. Kyle and Obizhaeva (2013) note that market microstructure invariance would imply a price impact of less than one percent and attribute the difference between predicted and realized price dynamics to unusually fast execution of the sales. Indeed, the entire $4 billion quantity was executed over a 36-minute period; orders of similar magnitude would normally be executed over several hours. Our model does not explain why some trader chose to execute an order so quickly, but it does predict how market prices would respond to a gigantic order, executed much faster than the market expects orders of such size to be executed.

Our explanation for flash crashes is different from explanations based on other existing models. For example, sharp price changes may occur in the continuous-time model of Kyle (1985) in response to large trades by noise traders, but the size of price declines in models with linear price impact depends only on quantities sold, not on the speed of selling; furthermore, such price declines are corrected only slowly by subsequent trading by the informed trader, who pushes the price back to fundamental value.

Foucault and Dugast (2014) suggest that similar V-shaped price patterns may occur in response to false signals. While the flash crash patterns in figure 6 represent the perspective of a trader who contemplates speeding up or slowing down his trading, the market responds to such selling essentially as if it were a false signal that is quickly corrected.

Duffie (2010) suggests that flash crashes may occur because capital moves too slowly. Our model shows why traders endogenously choose to move their capital slowly due to adverse selection.

Menkveld and Yueshen (2013) conclude that the flash crash could not have been caused by one large sell order because most of the selling took place after the market had crashed and while prices were recovering. This observed pattern is, however, reasonably consistent with our model, which predicts that price crash is triggered by changes in speed of trading and the selling itself occurs *after* prices crash and while the market recovers. It is not necessary for this selling to trigger additional selling by others.

Of course, flash crashes do not happen in equilibrium in our model. A rare event, perhaps unintended, the flash crash of May 2010 was like an experiment from which something about price impact can be learned.

3.3. Prices: A Keynesian Beauty Contest

Define trader n's estimate of the "fundamental value" of the risky asset $F_n(t)$ as the expected present value of all future dividends based on all information, discounted at the risk-free rate r, calculated using the beliefs of trader n. Gordon's growth formula implies that $F_n(t)$ is a function of trader n's expected growth rate $G_n(t)$:

$$F_n(t) = \frac{D(t)}{r + \alpha_D} + \frac{G_n(t)}{(r + \alpha_D)(r + \alpha_G)}. \tag{48}$$

Since the risky asset is in zero net supply, intuition might suggest that the equilibrium price is the average estimate of fundamental value $\sum_n F_n(t)/N$ obtained by replacing $G_n(t)$ with $\bar{G}(t)$ in equation (48). This intuition is precisely consistent with the one-period model. Surprisingly, in the model with continuous trading, this intuition turns out to be wrong!

A comparison of equations (34) and (48) reveals that the equilibrium price equals the average estimate of fundamental $\sum_n F_n(t)/N$ value if and only if $C_G = 1$. Since, in our numerical calculations, we always find that C_G is less than one, we conjecture that $0 < C_G < 1$ in any equilibrium with trade. Even if all N traders unanimously agree on the same expected growth rate $G_n(t) = \bar{G}(t)$, the equilibrium price will reflect a dampened implied growth rate $C_G\,\bar{G}(t)$, not $\bar{G}(t)$ itself.

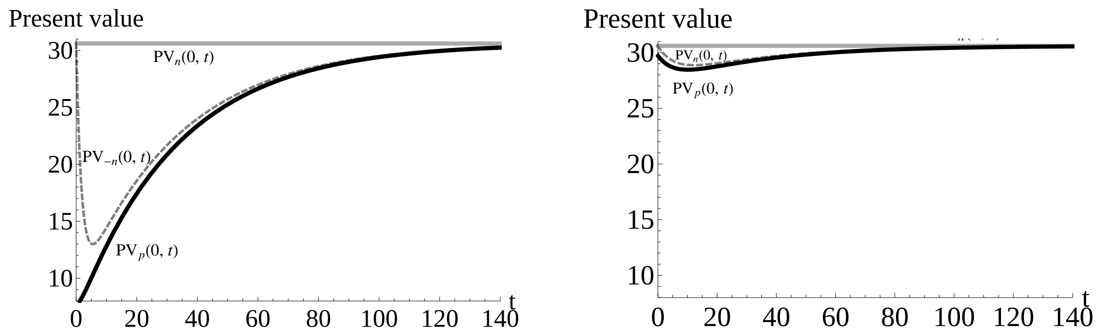
Since $C_G = 1$ in our one-period model and $C_G \to 1$ in the limit as liquidity vanishes in our continuous-time model, the intuitive explanation for why $0 < C_G < 1$ must relate to having multiple rounds of trading. The explanation is not based on imperfect competition because we find a similar dampening result in our competitive model, as in Kyle and Lin (2001).

Price dampening does not occur when disagreement is replaced by the assumption of private values with a common prior, such as in our private-values model or in Du and Zhu (2015). Price dampening also does not occur in noisy rational expectations models, such as Wang (1993), Wang (1994), and He and Wang (1995).

Figure 7 illustrates the intuition behind the dampening effect.[12] For simplicity of exposition, assume that the buy-and-hold valuations of all N traders coincide at time 0, and these estimates are positive, *i.e.*, for all n, assume $G_n(0) = G_{-n}(0) = \bar{G}(0) > 0$. For negative values, figure 7 will be symmetric.

Each panel of figure 7 depicts a graph of three functions, with time t on the horizontal axis and the results of three different present value calculations on the vertical axis. Each of these functions represents the expected discounted payoff to trader n resulting from collecting dividends on one share of stock between dates 0 and t, depositing the dividends into a money market account between dates 0 and

[12] Numerical calculations are based on the parameter values $r = 0.01$, $A = 1$, $\alpha_D = 0.1$, $\alpha_G = 0.02$, $\sigma_D = 0.5$, $\sigma_G = 0.01$, $N = 10$, $G_n(0) = 0.08$, and $D(0) = 0.7$. In addition, $\tau_H = 0.524$, $\tau_L = 0.0009$, and $\tau_0 = 0.0007$ in Panel (a), and $\tau_H = 0.054$, $\tau_L = 0.0098$, and $\tau_0 = 0.0022$ in Panel (b).



(a) With high disagreement (b) With low disagreement

FIGURE 7. PRESENT VALUE OF DIVIDENDS AND LIQUIDATION VALUE FROM THE PERSPECTIVE OF A TRADER.

t, selling the asset based on an assumed valuation at date t, then discounting the resulting cash flows back to purchase date 0 at the risk-free rate. Each of the three graphs corresponds to a different valuation assumption. Details for the present value calculations are given in Appendix section A.7, equations (A-81)–(A-91). By assumption, these three calculations are done using trader n's beliefs, but they are identical for all traders.

The horizontal light solid line is based on the assumption that trader n liquidates the asset at date t at a valuation equal to his own estimate of its fundamental value $F_n(t)$. Let $PV_n(0, t)$ denote the result of this present value calculation. Since trader n applies Bayes law correctly given his beliefs, the martingale property of his valuation (law of iterated expectations) makes the present value $PV_n(0, t)$ a constant function for $t \geq 0$; its graph is a horizontal line.

The light dashed curve is based on the assumption that trader n liquidates the asset at a valuation equal to the average estimate of fundamental value of the other $N-1$ traders, $\sum_{m \neq n} F_m(t)/(N-1)$. Let $PV_{-n}(0, t)$ denote the result of this present value calculation. The N traders' estimates of fundamental value are the same at date 0. Due to disagreement about signal precision, trader n believes that the other $N-1$ traders' estimates of the growth rate $G^*(t)$ will mean revert to zero at rate $\alpha_G + [\tau_H^{1/2} - \tau_L^{1/2}]^2$, which is faster than the mean reversion rate α_G he assumes for his own forecast, as shown in Appendix section A.7. As a result of the higher mean-reversion rate, trader n believes that $PV_{-n}(0, t)$ will fall in the short run. Since trader n believes that his own initial present value calculation is correct, trader n believes that $PV_{-n}(0, t)$ will rise back to his own estimate of the fundamental value in the long run. Thus, the graph depicted by the dashed line in figure 7 first falls below the horizontal line in the short run and then rises asymptotically back toward it in the long run. This pattern is analytically proved in Appendix section A.7.

The dark solid curve is based on the assumption that trader n liquidates the asset at a valuation equal to his estimate of the equilibrium market price $P(t)$. Let $PV_p(0, t)$ denote the result of this present value calculation. Consistent with the equilibrium result $0 < C_G < 1$, the initial price $P(0)$ is lower than the consensus fundamental value $F_n(0)$, even though all traders by assumption agree about this current fundamental value, agree about how it will evolve in the future, and know that they agree with the valuation dynamics. The dampening effect nevertheless arises due to interactions among expectations of traders in our model. If prices were equal to the consensus fundamental valuation $F_n(0)$, all traders would want to hold short positions because all of them would expect prices to fall below fundamental value in the short run as the others temporarily became more bearish. As a result, the price $P(0)$ is dampened relative to the average fundamental valuation in the market; yet this is consistent with each trader having a target inventory of zero at date 0.

As figure 7 illustrates, trader n may expect prices to increase monotonically from a dampened value toward his estimate of fundamental value (panel (a)) or decline first and then increase later (panel (b)). Appendix section A.7 proves that only those two patterns are possible. If C_G is less than some threshold $\hat{C}_G := (1 + (1 - 1/N)(\tau_H^{1/2} - \tau_L^{1/2})^2/(r + \alpha_G))^{-1}$, then $PV_p(0, t)$ increases monotonically over time, as in panel (a). If C_G is greater than the threshold $\hat{C}_G$, then $PV_p(0, t)$ first decreases over a time interval $\hat{t}$ defined in equation (A-93) and increases monotonically afterwards, as in panel (b).

The complicated dynamics of price-based present-value $PV_p(0, t)$ can be attributed to two factors. First, it tracks the average of $PV_n(0, t)$ and $PV_{-n}(0, t)$, where $PV_n(0, t)$ remains constant and $PV_{-n}(0, t)$ falls in the short run and then rises back in the long run, as discussed above. Second, there is also an additional effect related to the magnitude of C_G.

The above discussion implies that our model captures precisely the intuition of the beauty contest described by Keynes (1936):[13]

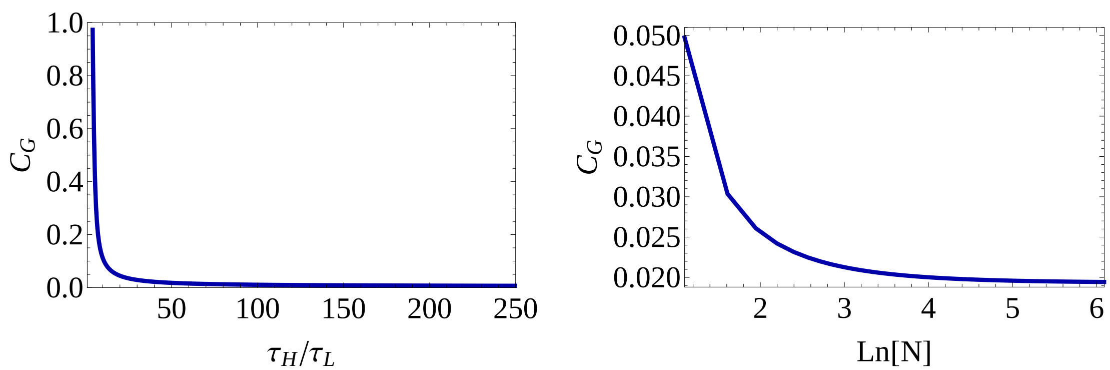
> "For most of these persons are, in fact, largely concerned, not with making superior long-term forecasts of the probable yield on an investment over its whole life, but with foreseeing changes in the conventional basis of valuation a short time ahead of the general public. They are concerned not with what an investment is really worth to a man who buys it 'for keeps,' but with what the market will value it at, under the

[13] "...Professional investment may be likened to those newspaper competitions in which the competitors have to pick out the six prettiest faces from a hundred photographs, the prize being awarded to the competitor whose choice most nearly corresponds to the average preferences of the competitors as a whole; so that each competitor has to pick, not those faces which he himself finds prettiest, but those which he thinks likeliest to catch the fancy of the other competitors, all of whom are looking at the problem from the same point of view. It is not a case of choosing those which, to the best of one's judgment, are really the prettiest, nor even those which average opinion genuinely thinks the prettiest. We have reached the third degree where we devote our intelligences to anticipating what average opinion expects the average opinion to be. And there are some, I believe, who practise the fourth, fifth and higher degrees." Our model implicitly assumes that all traders anticipate the expectations of other traders for arbitrarily higher degrees. Han and Kyle (2013) examine a one-period model in which, instead of agreeing to disagree, traders disagree about higher order beliefs.

> influence of mass psychology, three months or a year hence."

As in Keynes (1936), traders in our model use trading strategies which respond to short-term price dynamics. As Keynes puts it, "it is not sensible to pay 25 for an investment of which you believe the prospective yield to justify value of 30, if you also believe that the market will value it at 20 three months hence." The behavior of traders in our model is consistent with this intuition.

As a result of his belief that financial markets are dominated by short-term speculation rather than long-term enterprize, Keynes thought that financial markets are not too different from a casino and exhibit excessive volatility. In contrast to Keynes' intuition, short-term trading dynamics dampen price volatility in our model relative to the volatility of fundamentals. Furthermore, prices in our model are not "noisy;" the levels of current prices and dividends are sufficient statistics for inferring the average "true" valuations of traders.



(a) C_G against τ_H/τ_L (b) C_G against N

FIGURE 8. THE COMPARATIVE STATICS OF C_G.

The Keynesian price effect is related to liquidity. Consider the intuition of two special cases studied in detail in sections 3.4 and 3.5. When the market is very illiquid (the degree of disagreement τ_H/τ_L is close to the existence boundary $\Delta_H = 0$), it is costly for traders to implement short-term trading strategies due to high temporary price impact costs, the profit opportunities based on the beauty contest are therefore too costly to exploit, and therefore $C_G \to 1$. When the market is very liquid (the degree of disagreement τ_H/τ_L is large), short-term strategies are cheap, traders trade aggressively against one another's perceived mistakes, the dampening effect is substantial, and C_G decreases with the degree of disagreement. Both intuitions are illustrated in figure 8. Thus, one should expect to find more pronounced price dampening effects and therefore time-series momentum in more liquid markets. Indeed, the price reduction relative to fundamentals is greater in the left panel than the right panel of figure 7 because the degree of disagreement is larger and therefore the market is more liquid.

Price dampening may help to explain several empirical findings. Lee and Swaminathan (2000) document that price momentum is more pronounced among high volume stocks. Moskowitz, Ooi and Pedersen (2012) find significant time series momentum in equity index, currency, commodity, and bond futures markets; they show, in line with our predictions, that more liquid contracts tend to exhibit greater momentum profits. Similarly, Cremers and Pareek (2014) report that momentum profits increase with decreasing stock duration, a measure of how quickly institutions turnover their positions. Also, our results may help to explain a recent growth in assets of managed futures funds, which essentially implement trend-following strategies in liquid markets.

To summarize, each trader believes that equilibrium prices deviate from fundamental values and do not have a martingale property.

3.4. The Special Case of Vanishing Liquidity

Intuition suggests that market liquidity should depend on the exogenous degree of disagreement τ_H/τ_L. Market liquidity is larger when the endogenous parameter γ_P increases, since the quantity traders supply is more sensitive to price changes. When disagreement increases, intuition suggests that traders offer flatter flow-supply schedules because they believe adverse selection in the order flow of other traders decreases; flatter residual flow-supply schedules allow each trader to trade larger quantities based on his own private information. When disagreement decreases, intuition thus suggests that market liquidity γ_P should decrease as well, with γ_P becoming zero at exactly the point where there is not enough disagreement to support trade. The existence condition (Conjecture 1) suggests that this point is reached precisely when the exogenous value of Δ_H is zero.

COROLLARY 1: *Assume $\gamma_P = 0$. Then the six equations characterizing equilibrium (A-57)–(A-62) have a solution if and only if $\Delta_H = 0$. This solution has a closed form, presented in equations (A-70)–(A-71), which implies $\gamma_S = \gamma_H = \gamma_D = 0$ and $C_G = 1$.*

The proof is in Appendix section A.5.

Our numerical results suggest that the converse is also true: market liquidity falls to zero precisely when the existence condition $\Delta_H > 0$ fails to be satisfied. Since the solution to the six equations (A-57)–(A-62) is continuous in the exogenous parameters, this suggests that when Δ_H is a small positive number, there will be an equilibrium with low liquidity and modest trade. As the values of the exogenous parameters τ_H, τ_L, and N change so that $\Delta_H := \tau_H^{1/2}/\tau_L^{1/2} - 2 - 2/(N-2) \to 0$, this liquidity will vanish, the value of trading on private information will vanish, and trading volume will fall to zero. Note that liquidity vanishes when $\tau_H = \tau_L$ in the model with perfect competition, as shown in Appendix section B.2.

When trading volume falls to zero, traders not only cease to trade on private information, but they also cease to trade based on risk-sharing. Similarly, Vayanos (1999) presents a closed-form solution for a limiting case when the amount of risk in the economy to share converges to zero and traders optimally choose not to trade.

Figure 1, figure 4, and the left panel of figure 8 show effects of changes in the degree of overconfidence τ_H/τ_L on the endogenous parameters γ_S, C_L, $1/\lambda$, $1/\kappa$, and C_G. The horizontal axis plots the ratio τ_H/τ_L, calculated with τ_H increasing and τ_L decreasing so that the total precision τ and other exogenous parameters are fixed. As disagreement τ_H/τ_L decreases ($\gamma_P \to 0$), the values of γ_S, C_L, $1/\lambda$, and $1/\kappa$ converge to zero and C_G converges to one. The graphs only show values when equilibrium exists (τ_H/τ_L greater than about 4). When the existence condition $\Delta_H > 0$ fails to hold, the numerical algorithm for solving the system (A-57)–(A-62), as expected, fails to converge to a meaningful solution.

3.5. The Special Case of "Noise" Traders ($\tau_L = 0$)

Consider next the case $\tau_L = 0$, where each trader believes that the other traders observe a signal with no information. We assume that τ_0 is close to zero for tractability and allow the number of traders N to vary. An equilibrium always exists because there is unbounded disagreement. Increasing the number of traders N increases both risk bearing capacity and competition.

In the limit as $N \to \infty$, Appendix section A.6 shows that equations (A-57)–(A-62) have a closed form solution, presented in equations (A-73)–(A-77), in which the parameters γ_P and γ_S are proportional to N. In the spirit of Black (1986) and Treynor (1995), the limit $\tau_L \to 0$ implies each trader believes that all other traders trade on noise as if it were information. In this limit, permanent and temporary market impact parameters λ and κ converge to zero and inventory adjustment γ_S is infinitely fast. The parameter C_G converges to a constant limit which is less than one: $\lim_{N\to\infty} C_G = (r + \alpha_G)/(r + \alpha_G + \tau) < 1$.

Therefore, consistent with our extensive numerical results, the closed-form solutions to these two cases (sections 3.4 and 3.5) further help us to draw the conclusion that when disagreement increases, markets become more liquid and price dampening becomes more pronounced.

3.6. The Case when Traders Are "Correct on Average"

So far, this paper has focussed on understanding trading strategies, price impact, and price levels from the perspective of individual traders. Since traders have different beliefs about precision parameters τ_H and τ_L, traders calculate expected returns differently. Describing the properties of prices and quantities from the perspective of a non-trader, such as an economist or econometrician, requires also considering the unobserved "true values" of the parameters.

To illustrate, assume that traders' beliefs about the precisions of signals are the same and are "correct on average" in the sense that the correct aggregate precision matches the aggregate precision used by traders, *i.e.*, $\tau_n = (\tau_H + (N-1)\tau_L)/N$. For simplicity, assume that traders' beliefs about the parameters N, α_G, σ_G, α_D, and σ_D are correct.

THEOREM 6: ***Characterization of Equilibrium when Traders are Correct on Average.*** *When traders are correct on average, an equilibrium has the following properties:*

1) *There is an endogenously determined constant $C_G > 0$, defined in equation (A-49), such that the equilibrium price is given in (37). Under the correct beliefs, the fundamental price is*

$$F(t) = \frac{D(t)}{r + \alpha_D} + \frac{G_{true}(t)}{(r + \alpha_D)(r + \alpha_G)}, \tag{49}$$

where $G_{true}(t)$ defined in (54) is the estimate of the growth rate under the correct beliefs.

2) *Target inventories $S_n^{TI}(t)$ and actual inventories $S_n(t)$ follow the simple bivariate process*

$$\begin{pmatrix} dS_n^{TI}(t) \\ dS_n(t) \end{pmatrix} = \begin{pmatrix} -(\alpha_G + \tau) & 0 \\ \gamma_S & -\gamma_S \end{pmatrix} \begin{pmatrix} S_n^{TI}(t) \\ S_n(t) \end{pmatrix} dt + \begin{pmatrix} C_L \\ 0 \end{pmatrix} \Bigg(dB_n(t) - \tfrac{1}{N-1} \sum_{m=1, m \neq n}^{N} dB_m(t) \Bigg). \tag{50}$$

3) *The autocorrelation function for actual inventories and the correlation between actual inventory and target inventory are*

$$\mathit{Corr}\{S_n(t), S_n(t + \Delta t)\} = \frac{(\alpha_G + \tau)\ \mathrm{e}^{-\gamma_S\, \Delta t} - \gamma_S\ \mathrm{e}^{-(\alpha_G + \tau)\, \Delta t}}{\alpha_G + \tau - \gamma_S}, \tag{51}$$

$$\mathit{Corr}\{S_n(t), S_n^{TI}(t)\} = \Big(\frac{\gamma_S}{\alpha_G + \tau + \gamma_S}\Big)^{1/2}. \tag{52}$$

It can be easily shown that prices are dampened relative to fundamentals. From equation (27), we have

$$\bar{G}(t) = \frac{1}{N} \sum_{n=1}^{N} G_n(t) = \sigma_G\, \Omega^{1/2} \Bigg(\tau_0^{1/2}\, H_0(t) + \tfrac{\tau_H^{1/2} + (N-1)\tau_L^{1/2}}{N} \sum_{n=1}^{N} H_n(t) \Bigg), \tag{53}$$

while the empirically correct expected growth rate showing up in equation (49) assigns weights equal to the square root of the average of the precision parameters in equation (A-15),

$$G_{\mathrm{true}}(t) = \sigma_G\, \Omega^{1/2} \Bigg(\tau_0^{1/2}\, H_0(t) + \big(\tfrac{1}{N}\, \tau_H + \tfrac{N-1}{N}\, \tau_L\big)^{1/2} \sum_{n=1}^{N} H_n(t) \Bigg). \tag{54}$$

The market prices in equation (37) are less sensitive to information flow than fundamentals in equation (49), and therefore returns are predictable. Indeed, there are two effects. First, there is the price dampening effect implied by $C_G < 1$. Second, there is another price dampening due to averaging of expectations in equations (53) and (54), since Jensen's inequality implies that $\frac{1}{N}\,\tau_H^{1/2} + \frac{N-1}{N}\,\tau_L^{1/2} \leq \left(\frac{1}{N}\,\tau_H + \frac{N-1}{N}\,\tau_L\right)^{1/2}$. Section 3.3 relates this return predictability to a Keynesian beauty contest. Kyle, Obizhaeva and Wang (2014) provide a detailed analysis of return predictability in a competitive setting with a general specification for empirically correct beliefs.

The bivariate process of inventory dynamics is fully characterized by the three parameters $\alpha_G + \tau$, γ_S, and C_L. Target inventories $S_n^{TI}(t)$ follow a univariate AR-1 process. Symmetry implies that both $S_n(t)$ and $S_n^{TI}(t)$ are distributed independently from prices and have correlation $-1/(N-1)$ with one another. Since the model is Gaussian, the auto-correlation function completely describes the statistical properties of inventories. Equation (52) implies that the correlation between the actual inventory and target inventory increases when markets become more liquid (*i.e.*, γ_S increases).

Consistent with the empirical evidence discussed in the introduction, equation (51) implies that the auto-correlation of actual inventories tends to be smaller in more liquid markets. It can be easily shown that $\mathrm{Corr}\{S_n(t), S_n(t+\Delta t)\}$ decreases in γ_S if $\alpha_G + \tau < \gamma_S$ or if $\alpha_G + \tau > \gamma_S$ and $\Delta t > 1/(\alpha_G + \tau - \gamma_S)$. In more illiquid markets, actual inventories are closer to past actual inventories and further away from target inventories.

The auto-correlation function (51) has the following interesting symmetry: If the values of $\alpha_G + \tau$ and γ_S are interchanged, then the auto-correlation function is unchanged. The symmetry property of γ_S and $\alpha_G + \tau$ implies that the inventory process in a model with rapidly mean-reverting target inventories and slow convergence of actual inventories to target inventories (large $\alpha_G + \tau$ and small γ_S) is observationally equivalent to the inventory process in a different model with slowly mean-reverting target inventories and fast convergence of actual inventories to target inventories (small $\alpha_G + \tau$ and large γ_S). In both cases, actual inventories will change slowly and appear to be almost non-stationary.

Throughout this paper, we have emphasized that our model of disagreement provides a connection between the half-life of information in prices and the half-life of traders' target inventories. Empirically, it is more likely that actual inventories $S_n(t)$ are observed than target inventories $S_n^{TI}(t)$. If it is known whether $\alpha_G + \tau$ is greater or less than γ_S, then the values of both $\alpha_G + \tau$ and γ_S can be inferred from the autocorrelation of actual inventories $S_n(t)$. Interesting, the liquidity parameter $\gamma_S = \lambda/\kappa$ is not identified from the contemporaneous correlation of price changes and inventories (which are not correlated by symmetry, see section 3.2).

4. Conclusion

We have described a steady-state model of continuous trading, in which trading reflects both overconfidence and market power. This model provides a framework for thinking about how the dynamics of trading affects market liquidity, transaction costs, and market prices. It helps to analyze temporary and permanent price impact.

Our model of smooth order flow implements ideas about market liquidity described informally by Black (1995). Black envisioned a future frictionless market for exchanges as "an equilibrium in which traders use indexed limit orders at different levels of urgency but do not use market orders or conventional limit orders." In that equilibrium, there is no conventional liquidity available for market orders and conventional limit orders. Placement of indexed orders onto the market moves the price by an amount increasing in level of urgency. Algorithms for executing orders have been incorporating the idea of urgency for years. For example, in popular algorithms based on VWAP ("Volume Weighted Average Price"), a trader chooses a target number of shares to trade, a time frame (say one day), and a participation rate (say 5% of volume); the higher the participation rate, the greater the trader's impatience.

The idea that securities markets offer a flow equilibrium rather than a stock equilibrium may seem far-fetched at first glance. Yet, recent trends in the way liquidity is supplied and demanded in electronic markets are in many ways consistent with the way our model predicts liquidity to be supplied and demanded. For example, our model predicts vanishingly small market depth to be available at a given point in time; instead, market depth is made available only over time.

In today's markets, the actual level of market depth available at the "top of the book"—*i.e.*, at the best bid and the best offer—is influenced by tick size (the smallest units in which prices are allowed to fluctuate) and by rules for time and price priority. Since time priority mandates execution of the older resting limit orders before newer ones, time priority encourages traders to place bids and offers into the limit order book. Relative to our model, time priority creates an externality which results in more depth in the limit order book than would otherwise be present. This externality may be more important when minimum tick size is large. Today's markets probably have more instantaneous market depth available than our theory would imply, but they may have less instantaneous depth available than they would have if tick size were larger.

In the future, exchanges might change order matching rules to implement limit orders conforming to the intuition of our model. For example, exchanges might approximate our flow model by having frequent batch auctions, say once per second, consistent with Budish, Cramton and Shim (2013). Limit orders could easily be implemented with a time parameter. For example, a trader who might in today's market place a limit order to buy 10,000 shares at a price of $40 per share might instead enter an order to purchase one share per second at a price of $40 or better for the next 10,000 seconds. Such new order types would allow traders to implement smooth trading strategies without generating the heavy message traffic associated with submitting and canceling thousands of conventional limit orders.

REFERENCES

Almgren, Robert, and Neil Chriss. 2000. "Optimal Execution of Portfolio Transactions." *Journal of Risk*, 3: 5–39.

Atkyn, Allen B., and Edward A. Dyl. 1997. "Transactions costs and holding periods for common stocks." *Journal of Finance*, 52(1): 309325.

Back, Kerry, Henry C. Cao, and Gregory A. Willard. 2000. "Imperfect Competition among Informed Traders." *Journal of Finance*, 55(5): 2117–2155.

Bae, Kyoung-hun, Albert S. Kyle, Eun Jung Lee, and Anna Obizhaeva. 2014. "An Invariance Relationship in the Number of Buy-sell Switching Points." Working Paper.

Banerjee, Snehal, and Ilan Kremer. 2010. "Disagreement and Learning: Dynamic Patterns of Trade." *Journal of Finance*, 65(4): 1269–1302.

Black, Fischer. 1971. "Towards a Fully Automated Exchange, Part I." *Financial Analysts Journal*, 27: 29–34.

Black, Fischer. 1982. "The trouble with econometric models." *Financial Analysts Journal*, 38(2): 29–37.

Black, Fischer. 1986. "Noise." *Journal of Finance*, 41: 529–543.

Black, Fischer. 1995. "Equilibrium Exchanges." *Financial Analysts Journal*, 51: 23–29.

Brunnermeier, Markus K., and Lasse H. Pedersen. 2005. "Predatory Trading." *Journal of Finance*, 60(4): 1825–1863.

Budish, Eric, Peter Cramton, and John Shim. 2013. "The High-Frequency Trading Arms Race: Frequent Batch Auctions as a Market Design Response." Working Paper, University of Chicago.

Caldentey, René, and Ennio Stacchetti. 2010. "Insider Trading With a Random Deadline." *Econometrica*, 78(1): 245–283.

Carlin, Bruce, Miguel Lobo, and S. Viswanathan. 2007. "Episodic Liquidity Crises: Cooperative and Predatory Trading." *Journal of Finance*, 62(5): 2235–2274.

Chakrabarty, Bidisha, Pamela C. Moulton, and Charles Trzcinka. 2015. "Why Are Short-term Institutional Trades Losers?" Working Paper.

Chan, Louis, and Josef Lakonishok. 1995. "The Behavior of Stock Prices Around Institutional Trades." *Journal of Finance*, 50(4): 1147–1174.

Chau, Minh, and Dimitri Vayanos. 2008. "Strong-Form Efficiency with Monopolistic Insiders." *Review of Financial Studies*, 21(5): 2275–2306.

Cremers, Martijn, and Ankur Pareek. 2014. "Short-term trading and stock return anomalies: Momentum, reversal, and share issuance." *Review of Finance*, 19(4): 1649–1701.

Cremers, Martijn, and Antii Petajisto. 2009. "How Active is your Fund Manager? A New Measure that Predicts Performance." *Review of Financial Studies*, 22(9): 3329–3365.

Duffie, Darrell. 2010. "Presidential Address: Asset Price Dynamics with Slow-Moving Capital." *Journal of Finance*, 65(4): 1237–1267.

Dufour, Alfonso, and Robert Engle. 2000. "Time and the Price Impact of a Trade." *Journal of Finance*, 55(6): 2469–2498.

Du, Songzi, and Haoxiang Zhu. 2015. "Welfare and Optimal Trading Frequency in Dynamic Double Auctions." Working Paper, URL: http://ssrn.com/abstract=2040609.

Foster, Douglas F., and S. Viswanathan. 1996. "Strategic Trading When Agents Forecast the Forecasts of Others." *Journal of Finance*, 51(4): 1437–1478.

Foster, F Douglas, and S. Viswanathan. 1994. "Strategic Trading with Asymmetrically Informed Traders and Long-lived Information." *Journal of Financial and Quantitative Analysis*, 29(4): 499–518.

Foucault, Thierry, and Jérôme Dugast. 2014. "False News, Informational Efficiency, and Price Reversals." Working Paper.

Grinold, Richard C., and Ronald N. Kahn. 1995. *Active Portfolio Management: Quantitative Theory and Applictions.* Irwin Professional Publishing.

Grossman, Sanford J., and Joseph E. Stiglitz. 1980. "On the Impossibility of Informationally Efficient Markets." *American Economic Review*, 70(3): 393–408.

Grossman, Sanford J., and Merton H. Miller. 1988. "Liquidity and Market Structure." *Journal of Finance*, 43(3): 617–633.

Han, Jungsuk, and Albert S. Kyle. 2013. "Speculative Equilibrium with Differences in Higher Order Beliefs." Working Paper, Stockholm School of Economics.

Harsanyi, John C. 1976. "Games with Incomplete Information Played by 'Bayesian' Players. Part I: the Basic Model." *Management Science*, 14: 159–182.

Hasbrouck, Joel, and George Sofianos. 1993. "The Trades of Market Makers: An Empirical Analysis of NYSE Specialists." *Journal of Finance*, 48(5): 1565–1593.

He, Hua, and Jiang Wang. 1995. "Differential Information and Dynamic Behavior of Stock Trading Volume." *Review of Financial Studies*, 8(4): 919–972.

Holden, Craig W., and Avanidhar Subrahmanyam. 1992. "Long-Lived Private Information and Imperfect Competition." *Journal of Finance*, 47(1): 247–270.

Holthausen, Robert W., Richard W. Leftwich, and David Mayers. 1990. "Large-Block Transactions, the Speed of Response, and Temporary and Permanent Stock-Price Effects." *Journal of Finance*, 26(1): 71–95.

Keim, Donald B., and Ananth Madhavan. 1997. "Transaction Costs and Investment Style: An Inter-Exchange Analysis of Institutional Equity Trades." *Journal of Financial Economics*, 46(3): 265–292.

Keynes, John M. 1936. *The General Theory of Employment, Interest and Money.* Palgrave Macmillan, United Kingdom.

Kyle, Albert S. 1985. "Continuous Auctions and Insider Trading." *Econometrica*, 53(6): 1315–1335.

Kyle, Albert S. 1989. "Informed Speculation with Imperfect Competition." *Review of Economic Studies*, 56: 317–355.

Kyle, Albert S., and Anna A. Obizhaeva. 2013. "Market Microstructure Invariance: Theory and Empirical Tests." Working Paper, University of Maryland.

Kyle, Albert S., and Tao Lin. 2001. "Continuous Speculation with Overconfident Competitors." Working Paper, Duke University.

Kyle, Albert S., Anna A. Obizhaeva, and Yajun Wang. 2014. "A Market Microstructure Theory of the Term Structure of Asset Returns." Working Paper, University of Maryland.

Lee, Charles, and Bhaskaran Swaminathan. 2000. "Transactions costs and holding periods for common stocks." *Journal of Finance*, 55(5): 2017–2069.

Longstaff, Francis. 2001. "Optimal Portfolio Choice and the Valuation of Illiquid Securities." *Review of Financial Studies*, 14: 407–431.

Madhavan, Ananth, and Seymour Smidt. 1993. "An Analysis of Changes in Specialist Inventories and Quotations." *Journal of Finance*, 48(5): 1595–1628.

Menkveld, Albert, and Terrence Hendershott. 2014. "Price Pressures." *Journal of Financial Economics*, 114: 405–423.

Menkveld, Albert J., and Bart Z. Yueshen. 2013. "Anatomy of the Flash Crash." Working Paper.

Morris, Stephen. 1995. "The Common Prior Assumption in Economic Theory." *Economics and Philosophy*, 11: 227–253.

Moskowitz, Tobias J., Yao Hua Ooi, and Lasse H. Pedersen. 2012. "Time series momentum." *Journal of Financial Economics*, 104(2): 228–250.

Obizhaeva, Anna A., and Jiang Wang. 2013. "Optimal Trading Strategy and Supply/Demand Dynamics." *Journal of Financial Markets*, 16(1): 1–32.

Puckett, Andy, and Xuemin Yan. 2011. "The interim trading skills of institutional investors." *Journal of Finance*, 66(2): 601–633.

Rostek, Marzena, and Marek Weretka. 2012. "Price Inference in Small Markets." *Econometrica*, 80(2): 687–711.

Staffs of the CFTC and SEC. 2010a. *Preliminary Findings Regarding the Market Events of May 6, 2010.* Report of the Staffs of the CFTC and SEC to the Joint Advisory Committee on Emerging Regulatory Issues. May 18, 2010. Preliminary Report.

Staffs of the CFTC and SEC. 2010b. *Findings Regarding the Market Events of May 6, 2010.* Report of the Staffs of the CFTC and SEC to the Joint Advisory Committee on Emerging Regulatory Issues. September 30, 2010. Final Report.

Townsend, Robert M. 1983. "Forecasting the Forecasts of Others." *Journal of Political Economy*, 546–588.

Treynor, Jack. 1995. "The Only Game in Town." *Financial Analysts Journal*, 51(1): 81–83 (reprinted from *Financial Analysts Journal*, 1971(22):12–14, 22).

Vayanos, Dimitri. 1999. "Strategic Trading and Welfare in a Dynamic Market." *Review of Economic Studies*, 66(2): 219–254.

Wang, Jiang. 1993. "A Model of Intertemporal Asset Prices Under Asymmetric Information." *Review of Economic Studies*, 60(2): 249–282.

Wang, Jiang. 1994. "A Model of Competitive Stock Trading Volume." *Journal of Political Economy*, 102(1): 127–167.

A. Proofs

A.1. Proof of Theorem 1

Under the tentative assumption that trader n knows the value of i_{-n}, plug equation (4) into equation (9) and use the first-order condition to find his optimal demand:

(A-1)

$$x_n = \frac{\frac{\tau_v^{1/2}}{\tau}\left(\tau_0^{1/2}\, i_0 + \tau_H^{1/2}\, i_n + (N-1)\tau_L^{1/2}\, i_{-n}\right) - \left(\frac{\alpha}{\gamma}\, i_0 + \frac{\beta}{\gamma}\, i_{-n}\right) - \left(\frac{\delta}{(N-1)\gamma} + \frac{A}{\tau}\right) S_n}{\frac{2}{(N-1)\gamma} + \frac{A}{\tau}}.$$

In the numerator of this equation, the first term is trader n's expectation of the liquidation value, the second term is the market-clearing price when trader n trades a quantity of zero and has no inventory, and the last term is the adjustment for existing inventory. In the denominator, the first and second terms reflect how trader n restricts the quantity traded due to market power and risk aversion, respectively.

As in Kyle (1989), even though trader n does not observe i_{-n} explicitly, he is still able to implement this optimal strategy with a demand schedule which implicitly infers i_{-n} from the market-clearing price.

Define the constant

$$C := \frac{1}{(N-1)\,\gamma} + \frac{A}{\tau} + \frac{\tau_L^{1/2}\,\tau_v^{1/2}}{\tau\,\beta}. \tag{A-2}$$

Solving for i_{-n} instead of p in the market-clearing condition (3), substituting this solution into equation (A-1) above, and then solving for x_n, yields a demand schedule $X_n(i_0, i_n, S_n, p)$ for trader n as a function of price p:

$$\begin{aligned} X_n(i_0, i_n, S_n, p) = \; & \frac{1}{C}\Bigg[\frac{\tau_v^{1/2}}{\tau}\left(\tau_0^{1/2} - (N-1)\tau_L^{1/2}\,\frac{\alpha}{\beta}\right) i_0 + \frac{\tau_H^{1/2}}{\tau}\,\tau_v^{1/2}\, i_n \\ & + \left(\frac{(N-1)\tau_L^{1/2}\gamma\,\tau_v^{1/2}}{\tau\,\beta} - 1\right) p - \left(\frac{\delta\tau_L^{1/2}\tau_v^{1/2}}{\tau\,\beta} + \frac{A}{\tau}\right) S_n\Bigg]. \end{aligned} \tag{A-3}$$

In a symmetric linear equilibrium, the strategy chosen by trader n must be the same as the linear strategy (2) conjectured for the other traders. Equating the corresponding coefficients of the variables i_0, i_n, p, and S_n yields a system of four equations in terms of the four unknowns α, β, γ, and δ:

$$\alpha = \frac{\tau_v^{1/2}}{C}\left(\frac{\tau_0^{1/2}}{\tau} - \frac{(N-1)\tau_L^{1/2}}{\tau}\,\frac{\alpha}{\beta}\right), \qquad \beta = \frac{\tau_v^{1/2}}{C}\,\frac{\tau_H^{1/2}}{\tau}, \tag{A-4}$$

$$\gamma = -\frac{1}{C}\left(\frac{(N-1)\tau_L^{1/2}}{\tau}\,\frac{\gamma}{\beta}\,\tau_v^{1/2} - 1\right), \qquad \delta = \frac{1}{C}\left(\frac{\tau_L^{1/2}}{\tau}\,\frac{\delta}{\beta}\,\tau_v^{1/2} + \frac{A}{\tau}\right). \tag{A-5}$$

The unique solution is

$$\beta = \frac{(N-2)\tau_H^{1/2} - 2(N-1)\tau_L^{1/2}}{A\,(N-1)}\,\tau_v^{1/2}, \tag{A-6}$$

$$\alpha = \frac{\tau_0^{1/2}}{\tau_H^{1/2} + (N-1)\tau_L^{1/2}}\,\beta, \quad \gamma = \frac{\tau}{\tau_H^{1/2} + (N-1)\tau_L^{1/2}}\,\frac{\beta}{\tau_v^{1/2}}, \quad \delta = \frac{A}{\tau_H^{1/2} - \tau_L^{1/2}}\,\frac{\beta}{\tau_v^{1/2}}.$$

Substituting (A-6) into (A-3) yields trader n's optimal demand (11). Substituting (11) into the market-clearing condition $\sum_{m=1}^{N} X_m(i_0, i_m, S_m, p) = 0$ yields the equilibrium price (12).

The second order condition has the correct sign if and only if $\frac{2}{(N-1)\gamma} + \frac{A}{\tau} > 0$. Given the definition $\Delta_H := \tau_H^{1/2}/\tau_L^{1/2} - 2 - 2/(N-2)$, this is equivalent to

$$\frac{A}{\tau}\,\frac{N}{N-2}\,\frac{\tau_H^{1/2}}{\tau_L^{1/2}}\,\frac{1}{\Delta_H} > 0. \tag{A-7}$$

Therefore, assuming $N > 2$, the second order condition holds if and only if $\Delta_H > 0$.

A.2. Bayesian Updating with Signals of Arbitrary Precision

This section derives signal processing formulas for arbitrary "generic" beliefs $\bar{\tau}_0, \bar{\tau}_1, \ldots, \bar{\tau}_N$ about signal precisions.

Define $G(t) = \mathrm{E}_t\{G^*(t)\}$, where the subscript t denotes conditioning on the history of the signals $I_0(s), \ldots, I_N(s)$ for $s \in (-\infty, t]$. Without loss of generality, let $\bar{\Omega}$ denote the error variance $\bar{\Omega} := \mathrm{Var}\{(G^*(t) - G(t))/\sigma_G\}$. Assume a steady state in which $\bar{\Omega}$ is a constant which does not depend on time. Like a squared Sharpe ratio, $\bar{\Omega}$ measures the error variance in units of time. For example, if time is measured in years, $\bar{\Omega} = 4$ means that the estimate of $G^*(t)$ is "behind" the true value of $G^*(t)$ by an amount equivalent to four years of volatility unfolding at rate σ_G. There are simple and intuitive formulas for information processing:

LEMMA 1: *Given generic beliefs $\bar{\tau}_1, \ldots, \bar{\tau}_N$, let $\bar{\tau}$ denote the sum of precisions*

$$\bar{\tau} := \bar{\tau}_0 + \sum_{n=1}^{N} \bar{\tau}_n. \tag{A-8}$$

Then $\bar{\Omega}$ and $dG(t)$ satisfy

$$\bar{\Omega}^{-1} := \mathrm{Var}^{-1}\left\{\frac{G^*(t) - G(t)}{\sigma_G}\right\} = 2\,\alpha_G + \bar{\tau}, \tag{A-9}$$

$$dG(t) = -\left(\alpha_G + \bar{\tau}\right)\,G(t)\,dt + \sigma_G\,\bar{\Omega}^{1/2}\,\sum_{n=0}^{N} \bar{\tau}_n^{1/2}\,dI_n. \tag{A-10}$$

Proof. Apply the Stratonovich-Kalman-Bucy filter to the filtering problem summarized by equation (19) for signals and by equations (20) and (23) for observations. This yields the filtering estimate defined by the Itô differential equation

$$\text{(A-11)} \qquad dG(t) = -\alpha_G\, G(t)\, dt + \sum_{n=0}^{N} \sigma_G^2\, \bar{\Omega}\, \frac{\bar{\tau}_n^{1/2}}{\sigma_G\, \bar{\Omega}^{1/2}} \Big(dI_n(t) - G(t)\, \frac{\bar{\tau}_n^{1/2}}{\sigma_G\, \bar{\Omega}^{1/2}}\, dt \Big).$$

Rearranging terms yields equation (A-10). The mean-square filtering error of the estimate $G(t)$, denoted $\sigma_G^2\, \bar{\Omega}(t)$, is defined by the Riccati differential equation

$$\text{(A-12)} \qquad \sigma_G^2\, \frac{d\bar{\Omega}(t)}{dt} = -2\alpha_G\, \sigma_G^2\, \bar{\Omega}(t) + \sigma_G^2 - \sigma_G^4\, \bar{\Omega}(t)^2 \sum_{n=0}^{N} \Big(\frac{\bar{\tau}_n^{1/2}}{\sigma_G\, \bar{\Omega}(t)^{1/2}} \Big)^2.$$

Let $\bar{\Omega}$ denote the steady state of the function of time $\bar{\Omega}(t)$. Using the steady-state assumption $d\bar{\Omega}(t)/dt = 0$, solve the second equation for the steady state value $\bar{\Omega} = \bar{\Omega}(t)$ to obtain equation (A-9). *Q.E.D.*

The error variance $\bar{\Omega}$ corresponds to a steady state that balances an increase in error variance due to innovations $dB_G(t)$ in the true growth rate with a reduction in error variance due to (1) mean-reversion of the true growth rate at rate α_G and (2) arrival of new information with total precision $\bar{\tau}$.

Note that $\bar{\Omega}$ is not a "free parameter," but is instead determined as an endogenous function of the other parameters. Equation (A-9) implies that $\bar{\Omega}$ turns out to be the solution to the quadratic equation $\bar{\Omega}^{-1} = 2\, \alpha_G + \bar{\Omega}\, \sigma_G^2/\sigma_D^2 + \sum_{n=1}^{N} \bar{\tau}_n$. In equations (20) and (23), we scaled the units with which precision is measured by the endogenous parameter Ω because this leads to simpler filtering expressions which more clearly bring out intuition about signal processing.

From equation (A-10), the estimate $G(t)$ can be conveniently written as the weighted sum of $N+1$ sufficient statistics $H_n(t)$ corresponding to $N+1$ information flows dI_n. Define the sufficient statistics $H_n(t)$ by

$$\text{(A-13)} \qquad H_n(t) := \int_{u=-\infty}^{t} \mathrm{e}^{-(\alpha_G + \bar{\tau})\,(t-u)}\; dI_n(u), \quad n = 0, 1, \ldots, N,$$

which implies

$$\text{(A-14)} \qquad dH_n(t) = -(\alpha_G + \bar{\tau})\, H_n(t)\, dt + dI_n(t), \quad n = 0, 1, \ldots, N.$$

Then $G(t)$ becomes a linear combination of sufficient statistics $H_n(t)$ with weights proportional to the square roots of the precisions $\bar{\tau}_n^{1/2}$:

$$\text{(A-15)} \qquad G(t) = \sigma_G\, \bar{\Omega}^{1/2} \sum_{n=0}^{N} \bar{\tau}_n^{1/2}\, H_n(t).$$

The importance of each bit of information dI_n about the growth rate $G(t)$ decays exponentially at a rate $\alpha_G + \bar{\tau}$, which is the same for all of the signals. The half-life of a signal $\ln 2/(\alpha_G + \bar{\tau})$ decreases as the "aggregate precision" $\bar{\tau}$ increases. Even though the true unobserved growth rate may have a long half-life (small α_G), information about this growth rate may decay rapidly if aggregate precision $\bar{\tau}$ is large.

Note that equations (20), (23), and (A-10) imply that the estimate $G(t)$ mean-reverts to zero at a rate α_G while moving toward the true value G^* at rate $\bar{\tau}$:

$$\text{(A-16)} \qquad dG(t) = -\alpha_G \, G(t) \, dt + \bar{\tau} \, (G^* - G) \, dt + \sigma_G \, \bar{\Omega}^{1/2} \sum_{n=0}^{N} \bar{\tau}_n^{1/2} \, dB_n(t).$$

A.3. Proof of Theorem 3

Let $\mathrm{E}_t^n\{\ldots\}$ denote the conditional expectations operator $\mathrm{E}\{\ldots|\mathcal{F}_n(t)\}$ based on trader n's beliefs. Let $J^n(\mathcal{F}_n(t); X_n, C_n; X_m, m \neq n)$ denote the expected utility trader n receives as a function of his own consumption and trading strategies (C_n, X_n) and the $N-1$ other traders' trading strategies (X_m), conditional on his information set $\mathcal{F}_n(t)$. In this particular model, exponential utility functions with fixed interest rates make it unnecessary for $J^n(\ldots)$ to depend on other traders' consumption strategies. Define an *equilibrium* as a set of trading strategies $X_1^*, \ldots, X_N^*$ and consumption strategies $C_1^*, \ldots, C_N^*$ such that, for $n = 1, \ldots, N$, trader n's optimal consumption and trading strategies $X_n = X_n^*$ and $C_n = C_n^*$ solve the maximization problem

$$\text{(A-17)} \quad J^n(\mathcal{F}_n(t); X_n^*, C_n^*; X_m^*, m \neq n) = \max_{\{C_n, X_n\}} \quad \mathrm{E}_t^n \left\{ \int_{s=t}^{\infty} \mathrm{e}^{-\rho(s-t)} \; U(c_n(s)) \, ds \right\},$$

subject to inventories following

$$\text{(A-18)} \qquad dS_n(t) = x_n(t) \, dt$$

and money holdings following

$$\text{(A-19)} \qquad dM_n(t) = (r \, M_n(t) + S_n(t) \, D(t) - c_n(t) - P(t) \, x_n(t)) \; dt.$$

Conditional on any information set $\mathcal{F}_n(t)$, trader n's money and asset holdings must, with probability one, satisfy the budget constraint

$$\text{(A-20)} \qquad \liminf_{T \to \infty} \mathrm{E}_t^n \left\{ \mathrm{e}^{-r(T-t)} \; M_n(T) + \int_{u=T}^{\infty} \mathrm{e}^{-r(u-t)} \; D(u) \, S_n(T) \, du \right\} \geq 0.$$

In equations (A-18) and (A-19), the price $P(t)$, quantity $x_n(t)$, and consumption $c_n(t)$ are the abbreviations

(A-21)

$$P(t) := P[X_1, \ldots, X_N](t), \quad x_n(t) := \frac{dS_n(t)}{dt} = X_n(t, P(t); \mathcal{F}_n(t)), \quad c_n(t) := C_n(t; \mathcal{F}_n(t)).$$

This implies that trader n takes as given the strategies of the other traders when he chooses his optimal strategy. It also implies that when he chooses his optimal strategy, he takes into account how his strategy choice affects the price at which he trades and his trading opportunities in the future.

The budget constraint (A-20) says that if the trader calculates the fundamental value of his wealth using information at time t, then he does not engage in Ponzi finance. Also note that the optimal strategy will satisfy the transversality condition $\mathrm{E}_t^n\{\mathrm{e}^{-\rho(T-t)}\ J^n(\mathcal{F}_n(T), X_n^*, C_n^*; \ldots)\} \to 0$ as $T \to \infty$.

Instead of calculating the solution $J^n(\ldots)$ directly, we use the no regret approach, which assumes that trader n observes his residual supply schedule at each point in time, then picks an optimal point on the residual supply schedule. We then show that the solution to this less constrained problem also implements the optimal solution to the more constrained problem which defines $J^n(\ldots)$.

For the less constrained problem, we conjecture a steady-state value function $V(M_n, S_n, D, H_0, H_n, H_{-n})$, where M_n denotes trader n's cash holdings (measured in dollars) and S_n denotes trader n's holdings of the traded asset (measured in shares).

In a competitive model, a trader's value function depends on his wealth but does not depend on the decomposition of his wealth into his various security holdings. With imperfect competition, the decomposition of a trader's wealth into various security holdings does affect his value function because the trader cannot costlessly convert one security holding into cash or another security holding by trading at market prices. "Wealth" does not appear in the value function because wealth is not well-defined. Trader n is always influencing the mark-to-market value of his risky inventory by choosing his rate of trading. It is therefore necessary to keep track of the two components of wealth—cash M_n and inventories S_n—separately.

Also, we expect the asset price to be a linear combination of two components: (1) a dividend level component linear in dividend flow $D(t)$ and (2) a dividend-growth component linear in the variables $H_0(t)$, $H_n(t)$, and $H_{-n}(t)$. Given the symmetric linear conjectured form of the residual supply function, observing the average of other traders' signals $H_{-n}(t)$ is informationally equivalent to observing the intercept of the residual supply schedule (when $S_n(t) = S_n'(t) = 0$). Therefore we include $H_{-n}(t)$ as a state variable in the value function and omit the price $P(t)$.

In the derivations below, mathematical notation is simplified if the three state variables $H_0(t)$, $H_n(t)$, and $H_{-n}(t)$ are replaced with two "composite" signals, denoted $\hat{H}_n(t)$ and $\hat{H}_{-n}(t)$. Repeating equations (28), define the weighting constant $\hat{A}$ by

$$\hat{A} := \frac{\tau_0^{1/2}}{\tau_H^{1/2} + (N-1)\tau_L^{1/2}}, \tag{A-22}$$

and define the two composite signals $\hat{H}_n(t)$ and $\hat{H}_{-n}(t)$ by

$$\hat{H}_n(t) := H_n(t) + \hat{A}\, H_0(t), \tag{A-23}$$

$$\text{(A-24)} \qquad \hat{H}_{-n}(t) := H_{-n}(t) + \hat{A}\, H_0(t).$$

Trader n's estimate of the dividend growth rate can now be expressed as a function of the two composite signals $\hat{H}_n(t)$ and $\hat{H}_{-n}(t)$ as

$$\text{(A-25)} \qquad G_n(t) = \sigma_G\, \Omega^{1/2}\, \left(\tau_H^{1/2}\, \hat{H}_n(t) + (N-1)\tau_L^{1/2}\, \hat{H}_{-n}(t) \right).$$

Define the $N+1$ processes dB_0^n, dB_n^n, and dB_m^n, $m = 1, \ldots, N$, $m \neq n$, by

$$\text{(A-26)} \qquad dB_0^n(t) = \tau_0^{1/2}\, \frac{G^*(t) - G_n(t)}{\sigma_G\, \Omega^{1/2}}\, dt + dB_D(t),$$

$$\text{(A-27)} \qquad dB_n^n(t) = \tau_H^{1/2}\, \frac{G^*(t) - G_n(t)}{\sigma_G\, \Omega^{1/2}}\, dt + dB_n(t),$$

and

$$\text{(A-28)} \qquad dB_m^n(t) = \tau_L^{1/2}\, \frac{G^*(t) - G_n(t)}{\sigma_G\, \Omega^{1/2}}\, dt + dB_m(t).$$

The superscript n indicates conditioning on beliefs of trader n. Since trader n's forecast of the error $G^*(t) - G_n(t)$ is zero given his information set, these $N+1$ processes are independently-distributed Brownian motions from the perspective of trader n. In terms of these Brownian motions, trader n believes that signals change as follows:

$$\text{(A-29)} \qquad dH_0(t) = -(\alpha_G + \tau)\, H_0(t)\, dt + \tau_0^{1/2}\, \frac{G_n(t)}{\sigma_G\, \Omega^{1/2}}\, dt + dB_0^n(t),$$

$$\text{(A-30)} \qquad dH_n(t) = -(\alpha_G + \tau)\, H_n(t)\, dt + \tau_H^{1/2}\, \frac{G_n(t)}{\sigma_G\, \Omega^{1/2}}\, dt + dB_n^n(t),$$

$$\text{(A-31)} \qquad dH_{-n}(t) = -(\alpha_G + \tau)\, H_{-n}(t)\, dt + \tau_L^{1/2}\, \frac{G_n(t)}{\sigma_G\, \Omega^{1/2}}\, dt + \frac{1}{N-1} \sum_{\substack{m=1 \\ m \neq n}}^{N} dB_m^n(t).$$

Note that each signal drifts toward zero at rate $\alpha_G + \tau$ and drifts toward the optimal forecast $G_n(t)$ at a rate proportional to the square root of the signal's precision $\tau_0^{1/2}$, $\tau_H^{1/2}$, or $\tau_L^{1/2}$, respectively.

In terms of the composite variables $\hat{H}_n$ and $\hat{H}_{-n}$, we conjecture (and verify below) a steady-state value function of the form $V(M_n, S_n, D, \hat{H}_n, \hat{H}_{-n})$. Letting $(c_n(t), x_n(t))$ denote consumption and investment choices, write

$$\text{(A-32)} \qquad V(M_n, S_n, D, \hat{H}_n, \hat{H}_{-n}) := \max_{\{c_n(t), x_n(t)\}} \quad \mathrm{E}_t^n \left\{ \int_{s=t}^{\infty} - \mathrm{e}^{-\rho(s-t) - A\, c_n(s)}\, ds \right\},$$

where $P(x_n(t))$ is a linear function of $x_n(t)$ given by equation (31), dividends follow equation (18), inventories follow $dS_n(t) = x_n(t)\,dt$, the change in cash holdings $dM_n(t)$ is a quadratic function of $x_n(t)$ following

$$dM_n(t) = \left(r\,M_n(t) + S_n(t)\,D(t) - c_n(t) - P(x_n(t))\,x_n(t)\right)\,dt, \tag{A-33}$$

and signals $\hat{H}_n$ and $\hat{H}_{-n}$ follow a bivariate vector auto-regression given by

$$\begin{aligned} d\hat{H}_n(t) = & - \quad (\alpha_G + \tau)\,\hat{H}_n(t)\,dt \\ & + \quad (\tau_H^{1/2} + \hat{A}\tau_0^{1/2})\,\left(\tau_H^{1/2}\,\hat{H}_n(t) + (N-1)\tau_L^{1/2}\,\hat{H}_{-n}(t)\right)\,dt \\ & + \quad \hat{A}\,dB_0^n(t) + dB_n^n(t), \end{aligned} \tag{A-34}$$

$$\begin{aligned} d\hat{H}_{-n}(t) = & - \quad (\alpha_G + \tau)\,\hat{H}_{-n}(t)\,dt \\ & + \quad (\tau_L^{1/2} + \hat{A}\tau_0^{1/2})\,\left(\tau_H^{1/2}\,\hat{H}_n(t) + (N-1)\tau_L^{1/2}\,\hat{H}_{-n}(t)\right)\,dt \\ & + \quad \hat{A}\,dB_0^n(t) + \frac{1}{N-1}\sum_{\substack{m=1\\m\neq n}}^{N} dB_m^n(t). \end{aligned} \tag{A-35}$$

The dynamics of $\hat{H}_n$ and $\hat{H}_{-n}$ in equations (A-34) and (A-35) can be derived from equations (A-29), (A-30), and (A-31).

Note that the coefficient $\tau_H^{1/2} + \hat{A}\tau_0^{1/2}$ in the second line of equation (A-34) is different from the coefficient $\tau_L^{1/2} + \hat{A}\tau_0^{1/2}$ in the second line of equation (A-35). This difference is the key driving force behind the price-dampening effect resulting from the Keynesian beauty contest. It captures the fact that—in addition to disagreeing about the value of the asset in the present—traders also disagree about the dynamics of their future valuations. As shown in equations (C-33) and (C-34) in Appendix section C.3, these two different coefficients are the same in an otherwise similar private-values model. As a result, prices are not dampened in the private-values model.

Using the definition of $G_n(t)$ in equation (27) and the definition of $\hat{A}$ in equation (28), it can be shown that trader n believes the stochastic process $\hat{H}_n - \hat{H}_{-n}$ satisfies

$$\begin{aligned} d\left(\hat{H}_n - \hat{H}_{-n}\right) \quad = \quad & -(\alpha_G + \tau)\,\left(\hat{H}_n - \hat{H}_{-n}\right)\,dt + \frac{\tau_H^{1/2} - \tau_L^{1/2}}{\sigma_G\,\Omega^{1/2}}\,G_n(t)\,dt \\ & + dB_n^n(t) - \frac{1}{N-1}\sum_{\substack{m=1\\m\neq n}}^{N} dB_m^n(t). \end{aligned} \tag{A-36}$$

In equation (A-36), the term with $G_n(t)\,dt$ implies that each trader believes that $\hat{H}_n - \hat{H}_{-n}$ does not follow an AR-1 process. Because traders have different expectations $G_n(t)$, they agree in the present about how they will disagree in the future.

If traders only disagreed about the value of $G_n(t)$ in the present but agreed about the evolution of $\hat{H}_n - \hat{H}_{-n}$ in the future, then the coefficient of the $G_n(t)\,dt$ term would be zero, $\hat{H}_n - \hat{H}_{-n}$ would follow an AR-1 (Ornstein-Ühlenbeck) process, and traders would not disagree about the dynamics of process $\hat{H}_n - \hat{H}_{-n}$. In the otherwise similar model with private values, the term involving $G_n(t)\,dt$ becomes zero.

We conjecture and verify that the value function $V(M_n, S_n, D, \hat{H}_n, \hat{H}_{-n})$ has the specific quadratic exponential form

$$\text{(A-37)} \quad V(M_n, S_n, D, \hat{H}_n, \hat{H}_{-n}) = -\exp\Big(\psi_0 + \psi_M M_n + \tfrac{1}{2}\psi_{SS} S_n^2 + \psi_{SD} S_n D + \psi_{Sn}\, S_n \hat{H}_n + \psi_{Sx}\, S_n \hat{H}_{-n} + \tfrac{1}{2}\psi_{nn}\, \hat{H}_n^2 + \tfrac{1}{2}\psi_{xx}\, \hat{H}_{-n}^2 + \psi_{nx}\, \hat{H}_n \hat{H}_{-n}\Big).$$

The nine constants ψ_0, ψ_M, ψ_{SS}, ψ_{SD}, ψ_{Sn}, ψ_{Sx}, ψ_{nn}, ψ_{xx}, and ψ_{nx} have values consistent with a steady-state equilibrium. The term ψ_M measures the utility value of cash. The terms ψ_{SS}, ψ_{SD}, ψ_{Sn}, and ψ_{Sx} measure the utility value of risky asset holdings. The terms ψ_{nn}, ψ_{xx}, and ψ_{nx} capture the value of future trading opportunities based on current public and private information. The value of trading on innovations to future information is built into the constant term ψ_0.

The Hamilton-Jacobi-Bellman (HJB) equation corresponding to the conjectured value function $V(M_n, S_n, D, \hat{H}_n, \hat{H}_{-n})$ in equation (A-32) is

$$\text{(A-38)}$$

$$\begin{aligned} 0 \;=\; & \max_{c_n, x_n} \left\{ U(c_n) - \rho V + \frac{\partial V}{\partial M_n}\left(r M_n + S_n D - c_n - P(x_n)\, x_n\right) + \frac{\partial V}{\partial S_n} x_n \right\} \\ & + \frac{\partial V}{\partial D}\left(-\alpha_D D + \sigma_G \Omega^{1/2} \tau_H^{1/2} \hat{H}_n + \sigma_G \Omega^{1/2} (N-1) \tau_L^{1/2} \hat{H}_{-n}\right) \\ & + \frac{\partial V}{\partial \hat{H}_n}\left(-(\alpha_G + \tau)\hat{H}_n(t) + (\tau_H^{1/2} + \hat{A}\tau_0^{1/2})(\tau_H^{1/2}\hat{H}_n + (N-1)\tau_L^{1/2}\hat{H}_{-n})\right) \\ & + \frac{\partial V}{\partial \hat{H}_{-n}}\left(-(\alpha_G + \tau)\hat{H}_{-n}(t) + (\tau_L^{1/2} + \hat{A}\tau_0^{1/2})(\tau_H^{1/2}\hat{H}_n + (N-1)\tau_L^{1/2}\hat{H}_{-n})\right) + \tfrac{1}{2}\frac{\partial^2 V}{\partial D^2}\sigma_D^2 \\ & + \tfrac{1}{2}\frac{\partial^2 V}{\partial \hat{H}_n^2}\left(1 + \hat{A}^2\right) + \tfrac{1}{2}\frac{\partial^2 V}{\partial \hat{H}_{-n}^2}\left(\frac{1}{N-1} + \hat{A}^2\right) + \left(\frac{\partial^2 V}{\partial D \partial \hat{H}_n} + \frac{\partial^2 V}{\partial D \partial \hat{H}_{-n}}\right)\hat{A}\sigma_D + \frac{\partial^2 V}{\partial \hat{H}_n \partial \hat{H}_{-n}}\hat{A}^2. \end{aligned}$$

For the specific quadratic specification of the value function in equation (A-37), the HJB equation becomes

(A-39)

$$\begin{aligned}
0 \;=\; \min_{c_n, x_n} \; & \Big\{ -\frac{\mathrm{e}^{-Ac_n}}{V} - \rho + \psi_M (rM_n + S_n\, D - c_n - P(x_n)\, x_n) \\
& \quad + (\psi_{SS} S_n + \psi_{SD} D + \psi_{Sn} \hat{H}_n + \psi_{Sx} \hat{H}_{-n}) x_n \Big\} \\
& + \psi_{SD} S_n (-\alpha_D D + \sigma_G \Omega^{1/2} \tau_H^{1/2} \hat{H}_n + \sigma_G \Omega^{1/2} (N-1) \tau_L^{1/2} \hat{H}_{-n}) \\
& + \Big(\psi_{Sn} S_n + \psi_{nn} \hat{H}_n + \psi_{nx} \hat{H}_{-n} \Big) \\
& \qquad \Big(-(\alpha_G + \tau) \hat{H}_n(t) + (\tau_H^{1/2} + \hat{A} \tau_0^{1/2})(\tau_H^{1/2} \hat{H}_n + (N-1) \tau_L^{1/2} \hat{H}_{-n}) \Big) \\
& + \Big(\psi_{Sx} S_n + \psi_{xx} \hat{H}_{-n} + \psi_{nx} \hat{H}_n \Big) \\
& \qquad \Big(-(\alpha_G + \tau) \hat{H}_{-n}(t) + (\tau_L^{1/2} + \hat{A} \tau_0^{1/2})(\tau_H^{1/2} \hat{H}_n + (N-1) \tau_L^{1/2} \hat{H}_{-n}) \Big) \\
& + \tfrac{1}{2} \psi_{SD}^2 S_n^2 \sigma_D^2 + \tfrac{1}{2} \Big((\psi_{Sn} S_n + \psi_{nn} \hat{H}_n + \psi_{nx} \hat{H}_{-n})^2 + \psi_{nn} \Big) \Big(1 + \hat{A}^2 \Big) \\
& + \tfrac{1}{2} \Big((\psi_{Sx} S_n + \psi_{xx} \hat{H}_{-n} + \psi_{nx} \hat{H}_n)^2 + \psi_{xx} \Big) \left(\frac{1}{N-1} + \hat{A}^2 \right) \\
& + \Big((\psi_{Sn} + \psi_{Sx}) S_n + (\psi_{nn} + \psi_{nx}) \hat{H}_n + (\psi_{xx} + \psi_{nx}) \hat{H}_{-n} \Big) \; \psi_{SD} S_n \hat{A} \sigma_D \\
& + \Big((\psi_{Sn} S_n + \psi_{nn} \hat{H}_n + \psi_{nx} \hat{H}_{-n}) \, (\psi_{Sx} S_n + \psi_{xx} \hat{H}_{-n} + \psi_{nx} \hat{H}_n) + \psi_{nx} \Big) \; \hat{A}^2 .
\end{aligned}$$

The solution for optimal consumption is

(A-40)
$$c_n^*(t) = -\frac{1}{A} \, \log \Big(\frac{\psi_M \, V(t)}{A} \Big).$$

In the HJB equation (A-39), the price $P(x_n)$ is linear in x_n based on equation (31). Plugging $P(x_n)$ from equation (31) into the HJB equation (A-39) yields a quadratic function of x_n which captures the effect of trader n's trading rate x_n on prices. The optimal trading strategy is a linear function of the state variables given by

(A-41)
$$\begin{aligned}
x_n^*(t) = \tfrac{(N-1)\gamma_P}{2\psi_M} \; & \Big[\Big(\psi_{SD} - \tfrac{\psi_M \gamma_D}{\gamma_P} \Big) \; D(t) + \Big(\psi_{SS} - \tfrac{\psi_M \gamma_S}{(N-1)\gamma_P} \Big) \; S_n(t) \\
& + \psi_{Sn} \, \hat{H}_n(t) + \Big(\psi_{Sx} - \tfrac{\psi_M \gamma_H}{\gamma_P} \Big) \; \hat{H}_{-n}(t) \Big] .
\end{aligned}$$

Because the exponent of the conjectured value function is a quadratic function of the state variables, the best linear strategy will dominate any non-linear strategy or a mixed strategy.

The derivation of this optimal trading strategy assumes that trader n observes the values of $D(t)$, $S_n(t)$, $\hat{H}_n(t)$, and $\hat{H}_{-n}(t)$. Although trader n does not actually observe $\hat{H}_{-n}(t)$, he can implement the optimal quantity x_n^* by submitting an appropriate linear demand schedule. We can think of this demand schedule as a linear function of $P(t)$ whose intercept is a linear function of $D(t)$, $S_n(t)$, and $\hat{H}_n(t)$. Trader n can infer from the market-clearing condition (30) that $\hat{H}_{-n}$ is given by

$$\text{(A-42)}\quad \hat{H}_{-n}(t) = \frac{\gamma_P}{\gamma_H}\left(P(t) - D(t)\,\frac{\gamma_D}{\gamma_P}\right) - \frac{1}{(N-1)\gamma_H}\,x_n^*(t) - \frac{\gamma_S}{(N-1)\gamma_H}\,S_n(t).$$

Plugging equation (A-42) into equation (A-41) and solving for $x_n^*(t)$ implements the optimal trading strategy $x_n^*(t)$ as a linear demand schedule which depends on the price $P(t)$ and state variables $\hat{H}_n$, $S_n(t)$, and $D(t)$, which the trader directly observes. This schedule is given by

$$\begin{aligned}\text{(A-43)}\quad x_n^*(t) \;=\; & \frac{(N-1)\gamma_P}{\psi_M}\left(1 + \frac{\psi_{Sx}}{\psi_M}\frac{\gamma_P}{\gamma_H}\right)^{-1} \\ & \left[\left(\psi_{SD} - \psi_{Sx}\frac{\gamma_D}{\gamma_H}\right) D(t) + \left(\psi_{SS} - \psi_{Sx}\frac{\gamma_S}{(N-1)\gamma_H}\right) S_n(t) \right. \\ & \left. \quad + \psi_{Sn}\,\hat{H}_n(t) + \left(\psi_{Sx}\frac{\gamma_P}{\gamma_H} - \psi_M\right) P(t)\right].\end{aligned}$$

Symmetry requires that this demand schedule be the same as the demand schedule conjectured for the $N-1$ other traders. Equating the coefficients of $D(t)$, $\hat{H}_n(t)$, $S_n(t)$, and $P(t)$ in equation (A-43) to the conjectured coefficients γ_D, γ_H, $-\gamma_S$, and $-\gamma_P$ results in the following four restrictions that the values of the γ-parameters and ψ-parameters must satisfy in a symmetric equilibrium with linear trading strategies:

$$\text{(A-44)}\qquad \frac{(N-1)\gamma_P}{\psi_M}\left(1 + \frac{\psi_{Sx}}{\psi_M}\frac{\gamma_P}{\gamma_H}\right)^{-1}\left(\psi_{SD} - \psi_{Sx}\frac{\gamma_D}{\gamma_H}\right) = \gamma_D,$$

$$\text{(A-45)}\qquad \frac{(N-1)\gamma_P}{\psi_M}\left(1 + \frac{\psi_{Sx}}{\psi_M}\frac{\gamma_P}{\gamma_H}\right)^{-1}\psi_{Sn} = \gamma_H,$$

$$\text{(A-46)}\qquad \frac{(N-1)\gamma_P}{\psi_M}\left(1 + \frac{\psi_{Sx}}{\psi_M}\frac{\gamma_P}{\gamma_H}\right)^{-1}\left(\psi_{SS} - \psi_{Sx}\frac{\gamma_S}{(N-1)\gamma_H}\right) = -\gamma_S,$$

$$\text{(A-47)}\qquad \frac{(N-1)\gamma_P}{\psi_M}\left(1 + \frac{\psi_{Sx}}{\psi_M}\frac{\gamma_P}{\gamma_H}\right)^{-1}\left(\psi_{Sx}\frac{\gamma_P}{\gamma_H} - \psi_M\right) = -\gamma_P.$$

Note that it is not possible to solve this system for the four γ-parameters γ_H, γ_S, γ_D, and γ_P because this system of four equations can be written so that the four γ-parameters enter only as the three ratios γ_H/γ_P, γ_S/γ_P, and γ_D/γ_P. Therefore, we solve the system instead for the four unknowns ψ_{Sx}, γ_H, γ_S, and γ_D. The solution is

(A-48)
$$\psi_{Sx} = \frac{N-2}{2}\psi_{Sn}, \quad \gamma_H = \frac{N\gamma_P}{2\psi_M}\psi_{Sn}, \quad \gamma_S = -\frac{(N-1)\gamma_P}{\psi_M}\psi_{SS}, \quad \gamma_D = \frac{\gamma_P}{\psi_M}\psi_{SD}.$$

Define the constants C_L and C_G by

$$C_L := -\frac{\psi_{Sn}}{2\psi_{SS}}, \qquad C_G := \frac{\psi_{Sn}}{2\psi_M}\,\frac{N(r+\alpha_D)(r+\alpha_G)}{\sigma_G \Omega^{1/2}\left(\tau_H^{1/2} + (N-1)\tau_L^{1/2}\right)}. \tag{A-49}$$

When γ_D in equation (A-48) is plugged into equation (A-41), the coefficient on $D(t)$ zeros out; this implies that traders will not trade on public information. It is intuitively obvious that traders cannot trade on the basis of the public information $D(t)$ because all traders would want to trade in the same direction and this would be inconsistent with market clearing. Substituting equation (A-48) into equation (A-41) yields the solution for optimal strategy.

$$x_n^*(t) = \gamma_S \Big(C_L\left(H_n(t) - H_{-n}(t)\right) - S_n(t)\Big). \tag{A-50}$$

Define the average of traders' expected growth rates $\bar{G}(t)$ by

$$\bar{G}(t) := \frac{1}{N}\sum_{n=1}^{N} G_n(t), \tag{A-51}$$

Then, the equilibrium price is

$$P^*(t) = \frac{D(t)}{r+\alpha_D} + \frac{C_G\,\bar{G}(t)}{(r+\alpha_D)(r+\alpha_G)}. \tag{A-52}$$

One might expect that the solution of the maximization problem would yield solutions for the nine ψ-parameters as functions of the four γ-parameters. One might also expect that imposing symmetry by equating the four optimal γ-parameters (implied by trader n's optimal trading strategy) to the four conjectured γ-parameters would yield solutions for the four γ-parameters as functions of the nine ψ-parameters. In principle, one could then expect a solution to the thirteen equations in thirteen unknowns to describe a steady-state equilibrium, if one exists.

Although this is the intuition for the solution methodology, the solution does not work in this straightforward manner. As mentioned above, the four equations for the γ-parameters do not determine γ_P as a function of the nine ψ-parameters. Instead, the solution to the four γ-equations (A-48) implies a restriction on the ψ-parameters (the first of equations (A-48)), which must hold in a steady-state equilibrium. This restriction insures that the incentives to demand and supply liquidity are balanced, but it does not define a level of liquidity γ_P.

Plugging (A-40) and (A-41) back into the Bellman equation and setting the constant term and the coefficients of M_n, $S_n\,D$, S_n^2, $S_n\,\hat{H}_n$, $S_n\,\hat{H}_{-n}$, $\hat{H}_n^2$, $\hat{H}_{-n}^2$, and $\hat{H}_n\,\hat{H}_{-n}$ to be zero, we obtain nine equations. Using the first equation (A-48) to substitute ψ_{Sn} for ψ_{Sx}, there are in total nine equations in nine unknowns γ_P, $\psi_0, \psi_M, \psi_{SD}, \psi_{SS}, \psi_{Sn}, \psi_{nn}, \psi_{xx}$, and ψ_{nx}.

By setting the constant term, coefficient of M, and coefficient of SD to be zero, we obtain

$$\psi_M = -rA, \tag{A-53}$$

$$\psi_{SD} = -\frac{rA}{r+\alpha_D}, \tag{A-54}$$

(A-55)
$$\psi_0 = 1 - \log(r) + \frac{1}{r}\left(-\rho + \tfrac{1}{2}(1+\hat{A}^2)\psi_{nn} + \tfrac{1}{2}\left(\frac{1}{N-1} + \hat{A}^2\right)\psi_{xx} + \hat{A}^2\psi_{nx}\right).$$

In addition, by setting the coefficients of $S_n^2, S_n\,\hat{H}_n, S_n\,\hat{H}_{-n}, \hat{H}_n^2, \hat{H}_{-n}^2$ and $\hat{H}_n\,\hat{H}_{-n}$ to be zero, we obtain six polynomial equations in the six unknowns $\gamma_P, \psi_{SS}, \psi_{Sn}, \psi_{nn}, \psi_{xx}$, and ψ_{nx}. Defining the constants a_1, a_2, a_3, and a_4 by

$$\begin{aligned}
a_1 &:= -\alpha_G - \tau + \tau_H^{1/2}(\tau_H^{1/2} + \hat{A}\tau_0^{1/2}),\\
a_2 &:= -\alpha_G - \tau + (N-1)\tau_L^{1/2}(\tau_L^{1/2} + \hat{A}\tau_0^{1/2}),\\
a_3 &:= (\tau_H^{1/2} + \hat{A}\tau_0^{1/2})(N-1)\tau_L^{1/2},\\
a_4 &:= (\tau_L^{1/2} + \hat{A}\tau_0^{1/2})\tau_H^{1/2},
\end{aligned} \tag{A-56}$$

these six equations in six unknowns can be written

(A-57) $\underline{S_n^2}$:
$$\begin{aligned}
0 = {}& -\tfrac{1}{2}r\psi_{SS} - \frac{\gamma_P(N-1)}{rA}\psi_{SS}^2 + \frac{r^2A^2\sigma_D^2}{2(r+\alpha_D)^2} + \tfrac{1}{2}(1+\hat{A}^2)\psi_{Sn}^2\\
& + \tfrac{1}{2}\left(\frac{1}{N-1} + \hat{A}^2\right)\frac{(N-2)^2}{4}\psi_{Sn}^2 - \frac{rA}{r+\alpha_D}\hat{A}\sigma_D\frac{N}{2}\psi_{Sn} + \hat{A}^2\frac{N-2}{2}\psi_{Sn}^2,
\end{aligned}$$

(A-58) $\underline{S_n\hat{H}_n}$:
$$\begin{aligned}
0 = {}& -r\psi_{Sn} - \frac{\gamma_P(N-1)}{rA}\psi_{SS}\psi_{Sn} - \frac{rA}{r+\alpha_D}\sigma_G\Omega^{1/2}\tau_H^{1/2} + a_1\psi_{Sn}\\
& + \frac{N-2}{2}a_4\psi_{Sn} + (1+\hat{A}^2)\psi_{nn}\psi_{Sn} + \frac{N-2}{2}\left(\frac{1}{N-1} + \hat{A}^2\right)\psi_{nx}\psi_{Sn}\\
& - \frac{rA}{r+\alpha_D}\hat{A}\sigma_D\left(\psi_{nn} + \psi_{nx}\right) + \hat{A}^2\psi_{nx}\psi_{Sn} + \frac{N-2}{2}\hat{A}^2\psi_{nn}\psi_{Sn},
\end{aligned}$$

(A-59) $\underline{S_n \hat{H}_{-n}}$:

$$0 = -r\frac{N-2}{2}\psi_{Sn} + \frac{\gamma_P(N-1)}{rA}\psi_{SS}\psi_{Sn} - \frac{rA}{r+\alpha_D}\sigma_G\Omega^{1/2}(N-1)\tau_L^{1/2} + \Big(a_3 + \frac{N-2}{2}a_2\Big)\psi_{Sn} + (1+\hat{A}^2)\psi_{Sn}\psi_{nx} + \frac{N-2}{2}\left(\frac{1}{N-1} + \hat{A}^2\right)\psi_{xx}\psi_{Sn} - \frac{rA}{r+\alpha_D}\hat{A}\sigma_D\left(\psi_{xx} + \psi_{nx}\right) + \hat{A}^2\psi_{xx}\psi_{Sn} + \frac{N-2}{2}\hat{A}^2\psi_{nx}\psi_{Sn},$$

(A-60) $\underline{\hat{H}_n^2}$:

$$0 = -\frac{r}{2}\psi_{nn} - \frac{\gamma_P(N-1)}{4rA}\psi_{Sn}^2 + a_1\psi_{nn} + a_4\psi_{nx} + \tfrac{1}{2}(1+\hat{A}^2)\psi_{nn}^2 + \tfrac{1}{2}\left(\frac{1}{N-1} + \hat{A}^2\right)\psi_{nx}^2 + \hat{A}^2\psi_{nn}\psi_{nx},$$

(A-61) $\underline{\hat{H}_{-n}^2}$:

$$0 = -\frac{r}{2}\psi_{xx} - \frac{\gamma_P(N-1)}{4rA}\psi_{Sn}^2 + a_2\psi_{xx} + a_3\psi_{nx} + \frac{1+\hat{A}^2}{2}\psi_{nx}^2 + \tfrac{1}{2}\left(\frac{1}{N-1} + \hat{A}^2\right)\psi_{xx}^2 + \hat{A}^2\psi_{xx}\psi_{nx},$$

(A-62) $\underline{\hat{H}_n\hat{H}_{-n}}$:

$$0 = -r\psi_{nx} + \frac{\gamma_P(N-1)}{2rA}\psi_{Sn}^2 + a_3\psi_{nn} + a_4\psi_{xx} + (a_1+a_2)\psi_{nx} + (1+\hat{A}^2)\psi_{nn}\psi_{nx} + \left(\frac{1}{N-1} + \hat{A}^2\right)\psi_{xx}\psi_{nx} + \hat{A}^2\left(\psi_{nn}\psi_{xx} + \psi_{nx}^2\right).$$

We have not discovered a simple closed-form solution for equations (A-57)–(A-62); instead, we attempt to solve these equations numerically.

Equations (A-57)–(A-62) are necessary but not sufficient conditions for steady-state equilibrium with symmetric, linear flow-strategies. For a solution to the six polynomial equations to define a stationary equilibrium, it is sufficient for the solution to satisfy (1) a second-order condition implying $\gamma_P > 0$, (2) a stationarity condition implying $\gamma_S > 0$, (3) a transversality condition requiring $r > 0$, and (4) a budget constraint ruling out Ponzi schemes (implied by $r > 0$ and stationarity of inventories).

(1) The second order condition requires $\gamma_P > 0$. For the minimum in the optimization problem (A-39) to exist, the second order condition requires the 2×2 ma-

trix

$$\text{(A-63)} \qquad \begin{pmatrix} -\frac{A^2}{V} & 0 \\ 0 & \frac{2\,r\,A}{(N-1)\gamma_P} \end{pmatrix}$$

to be positive definite. Since the value function V is negative, this condition holds if and only if $\gamma_P > 0$. This is equivalent to requiring downward-sloping flow-demand schedules; it is also equivalent to requiring temporary price impact to be positive.

(2) If $\gamma_P > 0$ but $\gamma_S < 0$, then permanent price impact slopes the wrong way. Each trader's inventories grow exponentially over time, violating the requirement that inventories have a stationary distribution.

(3) The transversality condition for the value function $V(\ldots)$ is

$$\text{(A-64)} \qquad \lim_{T\to+\infty} \mathrm{E}_t^n\{\mathrm{e}^{-\rho(T-t)}\; V(M_n(T), S_n(T), D(T), \hat{H}_n(T), \hat{H}_{-n}(T))\} = 0.$$

From the HJB equation and equations (A-57)–(A-62), we have

$$\text{(A-65)} \qquad \mathrm{E}_t^n\{dV(M_n(t), S_n(t), D(t), \hat{H}_n(t), \hat{H}_{-n}(t))\} = \\ -(r-\rho)\, V(M_n(t), S_n(t), D(t), \hat{H}_n(t), \hat{H}_{-n}(t))\, dt.$$

This yields

$$\text{(A-66)} \qquad \mathrm{E}_t^n\{\mathrm{e}^{-\rho(T-t)}\, V(M_n(T), S_n(T), D(T), \hat{H}_n(T), \hat{H}_{-n}(T))\} = \\ \mathrm{e}^{-r(T-t)}\; V(M_n(t), S_n(t), D(t), \hat{H}_n(t), \hat{H}_{-n}(t)),$$

which implies that the transversality condition (A-64) is satisfied if $r > 0$.

(4) The budget constraint constraint ruling out Ponzi schemes (A-20) is automatically satisfied if $r > 0$ and the state variables are stationary.

Since the model assumes $r > 0$, the inequalities $\gamma_P > 0$ and $\gamma_S > 0$ are necessary and sufficient conditions for a solution to the six equations to characterize the desired equilibrium.

Under the assumptions $\gamma_P > 0$ and $\gamma_S > 0$, analytical results imply $\gamma_D > 0$, $\psi_M < 0$, and $\psi_{SD} < 0$, consistent with the intuition that traders prefer more to less; we also obtain $\psi_{SS} > 0$, consistent with the intuition that traders are averse to inventory risk. Our numerical results indicate that all endogenous parameters have the intuitively correct signs. For example, numerical results indicate that $\gamma_H > 0$, $\psi_{Sn} < 0$, $\psi_{Sx} < 0$, $\psi_{nn} < 0$, and $\psi_{xx} < 0$, consistent with the intuition that traders buy when they have bullish information, value greater expected dividends, and make greater profits (whether long or short) from more extreme signals. The sign of ψ_{nx} is intuitively and numerically ambiguous.

A.4. Proof of Theorem 4

Let a vector $(\gamma_P^*,\ \psi_{SS}^*,\ \psi_{Sn}^*,\ \psi_{nn}^*,\ \psi_{nx}^*,\ \psi_{xx}^*)$ be a solution to the system (A-57)–(A-62) for exogenous parameters A, σ_D, σ_G, r, α_G, α_D, τ_0, τ_L, and τ_H. If risk aversion is rescaled by factor F from A to A/F and other exogenous parameters are kept unchanged, then it is straightforward to show that a vector $(\gamma_P^* F,\ \ \psi_{SS}^*/F^2,\ \psi_{Sn}^*/F,\ \psi_{nn}^*,\ \psi_{nx}^*,\ \psi_{xx}^*)$ is the solution to the system (A-57)–(A-62). From equations (A-48), (A-49), and (32), it then follows that C_L changes to $C_L\, F$, λ changes to λ/F, κ changes to κ/F, but γ_S and C_G remain the same.

A.5. Proof of Corollary 1

With $\gamma_P = 0$, it is clear that $\psi_{nn} = \psi_{nx} = \psi_{xx} = 0$ solves the last three equations (A-60)–(A-62) of the six equations (A-57)–(A-62), consistent with the intuition that information has no value if there is no market liquidity. With $\gamma_P = \psi_{nn} = \psi_{xx} = \psi_{nx} = 0$, then the first three equations (A-57)–(A-59) become

$$\text{(A-67)}\qquad 0 = -\tfrac{1}{2} r\psi_{SS} + \frac{r^2 A^2 \sigma_D^2}{2(r+\alpha_D)^2} + \tfrac{1}{2}\left(\frac{1}{N-1} + \hat{A}^2\right)\frac{(N-2)^2}{4}\psi_{Sn}^2 + \tfrac{1}{2}(1+\hat{A}^2)\psi_{Sn}^2 - \frac{rA}{r+\alpha_D}\hat{A}\sigma_D \frac{N}{2}\psi_{Sn} + \hat{A}^2\frac{N-2}{2}\psi_{Sn}^2,$$

$$\text{(A-68)}\qquad 0 = -r\psi_{Sn} - \frac{rA}{r+\alpha_D}\sigma_G \Omega^{1/2}\tau_H^{1/2} + a_1\psi_{Sn} + \frac{N-2}{2}a_4\psi_{Sn},$$

$$\text{(A-69)}\qquad 0 = -r\frac{N-2}{2}\psi_{Sn} - \frac{rA}{r+\alpha_D}\sigma_G\Omega^{1/2}(N-1)\tau_L^{1/2} + \left(a_3 + \frac{N-2}{2}a_2\right)\psi_{Sn}.$$

Equations (A-68) and (A-69) are both linear equations in ψ_{Sn}. They have the same solution if and only if the existence condition is satisfied as an equality, $\Delta_H = 0$, in which case the unique solution for ψ_{Sn} is

$$\text{(A-70)}\qquad \psi_{Sn} = -\frac{rA\sigma_G\Omega^{1/2}\tau_H^{1/2}}{(r+\alpha_D)(r+\alpha_G)}.$$

Substituting (A-70) into (A-67) yields

$$\text{(A-71)}\qquad \psi_{SS} = \frac{rA^2}{(r+\alpha_D)^2}\left(\left(\sigma_D + \frac{\sigma_G\Omega^{1/2}\tau_0^{1/2}}{r+\alpha_G}\right)^2 + \frac{(\tau-\tau_0)\sigma_G^2\Omega}{(r+\alpha_G)^2}\right).$$

This implies $C_G = 1$:

$$\text{(A-72)}\qquad C_G = -\frac{\psi_{Sn}}{2rA}\,\frac{N(r+\alpha_D)(r+\alpha_G)}{\sigma_G\Omega^{1/2}\left(\tau_H^{1/2} + (N-1)\tau_L^{1/2}\right)} = \frac{N\tau_H^{1/2}}{2\left(\tau_H^{1/2} + (N-1)\tau_L^{1/2}\right)} = 1.$$

A.6. Limiting Case with $N \to \infty$, $\tau_L = 0$, and $\hat{A} \to 0$

Set $\tau_L = 0$, and then evaluate the solution in the limit as $N \to \infty$ and $\hat{A} \to 0$. We conjecture and verify that $\gamma_P = N\,\bar{\gamma}_P$, $\psi_{Sn} = N^{-1}\,\bar{\psi}_{Sn}$, $\psi_{SS} = N^{-1}\,\bar{\psi}_{SS}$, $\psi_{nn} = \bar{\psi}_{nn}$, $\psi_{nx} = \bar{\psi}_{nx}$, and $\psi_{xx} = \bar{\psi}_{xx}$, where $\bar{\gamma}_P$, $\bar{\psi}_{Sn}$, $\bar{\psi}_{SS}$, $\bar{\psi}_{nn}$, $\bar{\psi}_{nx}$, and $\bar{\psi}_{xx}$ are constants that do not depend on N.

Solving the system of equations (A-57)–(A-62) yields

$$\text{(A-73)} \qquad \bar{\psi}_{Sn} = -\frac{2Ar\Omega^{1/2}\sigma_G\tau_H^{1/2}}{(r+\alpha_D)(r+\alpha_G+\tau)}, \qquad \bar{\psi}_{SS} = \frac{A^2r^2\sigma_D^2}{(r+\alpha_D)^2(r+\alpha_G+\tau)},$$

$$\text{(A-74)} \qquad \bar{\gamma}_P = \frac{(r+\alpha_D)^2(r+\alpha_G+\tau)^2}{2Ar\sigma_D^2},$$

(A-75)

$$\bar{\psi}_{nn} = \tfrac{1}{2}\left(r + 2(\alpha_G+\tau-\tau_H) - \left((r+2(\alpha_G+\tau-\tau_H))^2 + \frac{4\Omega\sigma_G^2\tau_H}{\sigma_D^2}\right)^{1/2}\right),$$

$$\text{(A-76)} \qquad \bar{\psi}_{nx} = \frac{\Omega\sigma_G^2\tau_H/\sigma_D^2}{r+2(\alpha_G+\tau)-\tau_H-\bar{\psi}_{nn}},$$

$$\text{(A-77)} \qquad \bar{\psi}_{xx} = \frac{1}{r+2\alpha_G+2\tau}\left(\bar{\psi}_{nx}^2 - \frac{\Omega\sigma_G^2\tau_H}{\sigma_D^2}\right).$$

This implies that

$$\text{(A-78)} \qquad C_G \to \frac{r+\alpha_G}{r+\alpha_G+\tau} < 1, \qquad \lambda \to 0, \qquad \kappa \to 0,$$

$$\text{(A-79)} \qquad C_L = \frac{\Omega^{1/2}\sigma_G\tau_H^{1/2}(r+\alpha_D)}{Ar\sigma_D^2},$$

$$\text{(A-80)} \qquad \gamma_S = \frac{(N-1)\bar{\gamma}_P}{rA}\bar{\psi}_{SS} = \frac{(N-1)(r+\alpha_G+\tau)}{2} \to \infty.$$

A.7. Dampening Effect: The Present Value of Expected Cumulative Dividends and Cash Flow

From (A-25), (A-29), (A-30), and (A-31), we can derive the stochastic process for $G_n(t)$ and $G_{-n}(t) := \frac{1}{N-1}\sum_{m=1,\ldots,N;\; m\neq n} G_m(t)$ as follows:

(A-81)

$$dG_n(t) = -\alpha_G\,G_n(t)dt + \sigma_G\,\Omega^{1/2}\left(\tau_0^{1/2}dB_0^n(t) + \tau_H^{1/2}dB_n^n(t) + \tau_L^{1/2}\sum_{\substack{m=1\\m\neq n}}^{N} dB_m^n(t)\right),$$

(A-82)

$$dG_{-n}(t) = -(\alpha_G + \tau)\, G_{-n}(t)dt + \left(\tau_0 + \tau_L^{1/2}\left(2\tau_H^{1/2} + (N-2)\tau_L^{1/2}\right)\right) G_n(t)dt$$

$$+\sigma_G\, \Omega^{1/2}\left(\tau_0^{1/2} dB_0^n(t) + \tau_L^{1/2} dB_n^n(t) + \frac{\tau_H^{1/2} + (N-2)\tau_L^{1/2}}{N-1}\sum_{\substack{m=1\\ m\neq n}}^{N} dB_m^n(t)\right).$$

From (A-82), when $G_m(t) = G_n(t)$, trader n believes that other traders' estimates of expected growth rates $G_m(t)$ will mean-revert to zero at a rate $\alpha_G + (\tau_H^{1/2} - \tau_L^{1/2})^2 > \alpha_G$. From (A-81), trader n believes that his own estimate of expected growth rate $G_n(t)$ will mean-revert to zero at a rate α_G.

From (A-81), (A-82), and (18), the expected dynamics of $G_n(t)$, $G_{-n}(t)$, and $D(t)$ are given by

$$\mathrm{E}_0^n[G_n(t)] = \mathrm{e}^{-\alpha_G t}\, G_n(0), \tag{A-83}$$

(A-84)

$$\mathrm{E}_0^n[G_{-n}(t)] = \frac{\tau_0 + \tau_L^{1/2}\left(2\tau_H^{1/2} + (N-2)\tau_L^{1/2}\right)}{\tau}\left(\mathrm{e}^{-\alpha_G t} - \mathrm{e}^{-(\alpha_G+\tau)t}\right) G_n(0) + \mathrm{e}^{-(\alpha_G+\tau)t}\, G_{-n}(0),$$

$$\mathrm{E}_0^n[D(t)] = \frac{1}{\alpha_D - \alpha_G}\left(\mathrm{e}^{-\alpha_G t} - \mathrm{e}^{-\alpha_D t}\right) G_n(0) + \mathrm{e}^{-\alpha_D t}\, D(0). \tag{A-85}$$

The present value of expected cumulative dividends and cash flow from liquidating one share of the stock at date t using trader n's valuation is

$$PV_n(0,t) := \mathrm{E}_0^n\Big[\int_0^t \mathrm{e}^{-ru}\, D(u)du + \mathrm{e}^{-rt}\Big(\frac{D(t)}{r+\alpha_D} + \frac{G_n(t)}{(r+\alpha_D)(r+\alpha_G)}\Big)\Big]. \tag{A-86}$$

Substituting (A-83) and (A-85) into (A-86), it can be shown that (A-86) is equal to

$$F_n(0) = \frac{D(0)}{r+\alpha_D} + \frac{G_n(0)}{(r+\alpha_D)(r+\alpha_G)}. \tag{A-87}$$

The present value of expected cumulative dividends and cash flow from liquidating one share of the stock at date t using others' valuations $\sum_{m\neq n} F_m(t)/(N-1)$ is

$$PV_{-n}(0,t) := \mathrm{E}_0^n\Big[\int_0^t \mathrm{e}^{-ru}\, D(u)du + \mathrm{e}^{-rt}\Big(\frac{D(t)}{r+\alpha_D} + \frac{G_{-n}(t)}{(r+\alpha_D)(r+\alpha_G)}\Big)\Big]. \tag{A-88}$$

Assuming $G_m(0) = G_n(0) = \bar{G}(0)$ and substituting (A-83)–(A-85) into (A-88), it can be shown that equation (A-88) is equal to

$$PV_{-n}(0,t) = F_n(0) + \frac{(\tau_H^{1/2} - \tau_L^{1/2})^2}{\tau(r+\alpha_G)(r+\alpha_D)}\left(\mathrm{e}^{-(r+\alpha_G+\tau)t} - \mathrm{e}^{-(r+\alpha_G)t}\right) G_n(0). \tag{A-89}$$

Similarly, the present value of expected cumulative dividends and cash flow from liquidating one share of the stock at date t at the equilibrium price $P(t)$ is

$$\text{(A-90)}\quad PV_p(0,t) := \mathrm{E}_0^n\Big[\int_0^t \mathrm{e}^{-ru}\, D(u)du + \mathrm{e}^{-rt}\Big[\frac{D(t)}{r+\alpha_D} + \frac{C_G\,\bar{G}(t)}{(r+\alpha_D)(r+\alpha_G)}\Big)\Big].$$

Substituting (A-83)–(A-85) into (A-90), it can be shown that (A-90) is equivalent to
(A-91)

$$PV_p(0,t) = F_n(0) + \frac{C_G\Big(N - (\tau_H^{1/2} - \tau_L^{1/2})^2/\tau\,(N-1)\Big) - N}{N\,(r+\alpha_G)\,(r+\alpha_D)}\,\mathrm{e}^{-(r+\alpha_G)t}\;G_n(0)$$

$$+\frac{C_G\,(\tau_H^{1/2} - \tau_L^{1/2})^2/\tau\,(N-1)}{N\,(r+\alpha_G)\,(r+\alpha_D)}\;\mathrm{e}^{-(r+\alpha_G+\tau)t}\;G_n(0).$$

From (A-91), it follows that
(A-92)

$$\frac{dPV_p(0,t)}{dt} = \frac{G_n(0)\,\mathrm{e}^{-(r+\alpha_G)t}}{N\,(r+\alpha_G)\,(r+\alpha_D)}\bigg(\Big(N - C_G\Big(N - (\tau_H^{1/2} - \tau_L^{1/2})^2/\tau(N-1)\Big)\Big)(r+\alpha_G)$$

$$C_G\,(\tau_H^{1/2} - \tau_L^{1/2})^2/\tau(N-1)(r+\alpha_G+\tau)\,\mathrm{e}^{-\tau t}\bigg).$$

Clearly, (A-92) implies $\frac{dPV_p(0,t)}{dt} \to 0$ when $t \to \infty$. Define

$$\text{(A-93)}\qquad \hat{t} := -\ln\left(\left(1 + \frac{(1-C_G)N\tau}{C_G(\tau_H^{1/2} - \tau_L^{1/2})^2(N-1)}\right)\,\frac{r+\alpha_G}{r+\alpha_G+\tau}\right)/\tau.$$

Equation (A-92) implies $\frac{dPV_p(0,t)}{dt} > 0$ if and only if $t > \hat{t}$. It can be shown that $\hat{t} > 0$ if and only if $C_G > \hat{C}_G := \Big(1 + (1-1/N)(\tau_H^{1/2} - \tau_L^{1/2})^2/(r+\alpha_G)\Big)^{-1}$. This yields the following results:

- If $C_G \le \hat{C}_G$, then $\frac{dPV_p(0,t)}{dt} > 0$ for all $t > 0$.
- If $C_G > \hat{C}_G$, then $\frac{dPV_p(0,t)}{dt} = 0$ for $t = \hat{t}$, $\frac{dPV_p(0,t)}{dt} > 0$ for $t > \hat{t}$, and $\frac{dPV_p(0,t)}{dt} < 0$ for $t < \hat{t}$.

From (A-88), if follows that

$$\text{(A-94)}\quad \frac{dPV_{-n}(0,t)}{dt} = \frac{(\tau_H^{1/2} - \tau_L^{1/2})^2\,G_n(0)\;\mathrm{e}^{-(r+\alpha_G)t}}{\tau(r+\alpha_G)(r+\alpha_D)}\Big((r+\alpha_G) - (r+\alpha_G+\tau)e^{-\tau t}\Big).$$

(A-94) implies that $\frac{dPV_{-n}(0,t)}{dt} < 0$ iff $t < \ln\left(1 + \frac{\tau}{r+\alpha_G}\right)/\tau$.

A.8. Proof of Theorem 6

The proof of Part 1 is trivial and thus omitted here. If traders are correct on average, it follows that

(A-95)

$$d\hat{H}_n(t) - d\hat{H}_{-n}(t) = -(\alpha_G + \tau)(\hat{H}_n(t) - \hat{H}_{-n}(t))dt + dB_n(t) - \tfrac{1}{N-1} \sum_{m=1, m \neq n} dB_m(t).$$

Equations (38) and (39) imply the simple bivariate process of target inventories and actual inventories as in Part 2. Simple calculations yield

$$S_n^{TI}(t) = C_L \int_{-\infty}^{t} \mathrm{e}^{-(\alpha_G+\tau)(t-k)} \left(dB_n(k) - \frac{1}{N-1} \sum_{m=1, m \neq n} dB_m(k) \right), \tag{A-96}$$

(A-97)

$$S_n(t) = C_L \gamma_S \int_{-\infty}^{t} \frac{\mathrm{e}^{-(\alpha_G+\tau)(t-k)} - \mathrm{e}^{-\gamma_S(t-k)}}{\gamma_S - \alpha_G - \tau} \left(dB_n(k) - \frac{1}{N-1} \sum_{m=1, m \neq n} dB_m(k) \right).$$

From (A-96) and (A-97), simple calculations yield

$$\mathrm{Corr}\{S_n(t), S_n(t+\Delta t)\} = \frac{(\alpha_G + \tau)\ \mathrm{e}^{-\gamma_S\, \Delta t} - \gamma_S\ \mathrm{e}^{-(\alpha_G+\tau)\,\Delta t}}{\alpha_G + \tau - \gamma_S}, \tag{A-98}$$

$$\mathrm{Corr}\{S_n(t), S_n^{TI}(t)\} = \left(\frac{\gamma_S}{\gamma_S + \alpha_G + \tau} \right)^{1/2}. \tag{A-99}$$

B. A Competitive Model of Trading

In this section, we consider a model that is similar to the smooth trading model with the only difference that traders are perfectly competitive. The competitive equilibrium is different from the equilibrium with imperfect competition. Traders adjust their inventories immediately; they do not smooth out their trading over time. The one-period model is discussed next, followed by the continuous-time model of perfect competition.

B.1. One-Period Model

The setting is almost identical to the setting of our model of imperfect competition. For clarity, we repeat analogous assumptions here.

A risky asset with random liquidation value $v \sim N(0, 1/\tau_v)$ is traded for a safe numeraire asset. Each of N traders $n = 1, \ldots, N$ is endowed with S_n shares of a zero-net-supply risky asset, implying $\sum_{n=1}^{N} S_n = 0$. Traders observe signals about the normalized liquidation value $\tau_v^{1/2} v$. All traders observe a public signal $i_0 := \tau_0^{1/2} \, (\tau_v^{1/2} v) + e_0$ with $e_0 \sim N(0, 1)$. Each trader n observes a private signal $i_n := \tau_n^{1/2} \, (\tau_v^{1/2} v) + e_n$ with $e_n \sim N(0, 1)$. The asset payoff v, the public signal error e_0, and N private signal errors $e_1, \ldots, e_N$ are independently distributed.

Traders agree about the precision of the public signal τ_0 and agree to disagree about the precisions of private signals τ_n. Each trader is "relatively overconfident," believing his own signal has a high precision $\tau_n = \tau_H$ and other traders' signals have low precision $\tau_m = \tau_L$ for $m \neq n$, with $\tau_H > \tau_L \geq 0$.

Each trader submits a demand schedule $X_n(p) := X_n(i_0, i_n, S_n, p)$ to a single-price auction. An auctioneer calculates the market-clearing price $p := p[X_1, \ldots, X_N]$.

Trader n's terminal wealth is

$$W_n := v \, (S_n + X_n(p)) - p \, X_n(p). \tag{B-1}$$

The difference with equation (1) is that each trader n assumes that the price p does not depend on the quantities he trades. Each trader maximizes the same expected exponential utility function of wealth $\mathrm{E}^n\{-\,\mathrm{e}^{-A\, W_n}\}$ using his own beliefs about τ_H and τ_L to calculate the expectation.

Trader n maximizes his expected utility, or equivalently he maximizes $E^n\{W_n\} - \frac{1}{2} A \, Var^n\{W_n\}$. He chooses the quantity to trade x_n that solves the maximization problem

(B-2)

$$\max_{x_n} \left(\frac{\tau_v^{1/2}}{\tau} \left(\tau_0^{1/2} \, i_0 + \tau_H^{1/2} \, i_n + (N-1) \tau_L^{1/2} \, i_{-n} \right) (S_n + x_n) - p \, x_n - \frac{A}{2\tau} (S_n + x_n)^2 \right).$$

The first-order condition with respect to x_n yields

$$x_n^* = \frac{1}{A} \left(\tau_v^{1/2} \left(\tau_0^{1/2} \, i_0 + \tau_H^{1/2} \, i_n + (N-1) \tau_L^{1/2} \, i_{-n} \right) - p \, \tau \right) - S_n. \tag{B-3}$$

Suppose that traders submit symmetric linear demand schedules of the form

(B-4) $$x_n(i_0, i_n, S_n, p) = \alpha\, i_0 + \beta\, i_n - \gamma\, p - \delta\, S_n, \quad n = 1, \ldots, N.$$

The market-clearing condition $\sum_{n=1}^{N} x_n = 0$ implies

(B-5) $$p = \frac{\alpha}{\gamma}\, i_0 + \frac{\beta}{\gamma}\, \frac{1}{N} \sum_{n=1}^{N} i_n,$$

Substituting (B-5) into (B-4) yields

(B-6) $$x_n = \beta \left(i_n - \frac{1}{N} \sum_{m=1}^{N} i_m \right) - \delta\, S_n.$$

Thus, each trader trades on the difference between his signal i_n and the average of all N signals and also trades out of his current inventory S_n.

Substituting (B-5) into (B-3), the equilibrium strategy x_n^* can be expressed as a linear function of i_0, i_n, $\sum_{n=1}^{N} i_n$, and S_n. Equating coefficients to the corresponding coefficients in equation (B-6) yields

(B-7) $$\delta = 1, \qquad \alpha = \frac{\tau_v^{1/2} \tau_0^{1/2} \left(\tau_H^{1/2} - \tau_L^{1/2} \right)}{A \left(\tau_H^{1/2} + (N-1) \tau_L^{1/2} \right)},$$

$$\beta = \frac{1}{A} \tau_v^{1/2} \left(\tau_H^{1/2} - \tau_L^{1/2} \right), \qquad \gamma = \frac{\tau \left(\tau_H^{1/2} - \tau_L^{1/2} \right)}{A \left(\tau_H^{1/2} + (N-1) \tau_L^{1/2} \right)}.$$

Substituting α, β and γ into equations (B-5) and (B-6) yields

(B-8) $$x_n^* = \tfrac{1}{A} \left(1 - \tfrac{1}{N} \right) \tau_v^{1/2} (\tau_H^{1/2} - \tau_L^{1/2})(i_n - i_{-n}) - S_n.$$

Define the target inventory as

(B-9) $$S_n^{TI} = \tfrac{1}{A} \left(1 - \tfrac{1}{N} \right) \tau_v^{1/2} \, (\tau_H^{1/2} - \tau_L^{1/2}) \, (i_n - i_{-n}).$$

Equation (B-8) is similar to equation (11), except for the endogenous constant $\delta = 1$, *i.e*, each trader trades toward to his "target inventory" S_n^{TI} immediately in the competitive model. Note that target inventories are identical to target inventories (13) in the model with imperfect competition.

As in equation (12) for the case of imperfect competition, the equilibrium price p^* is equal to the average of traders' valuations:

(B-10) $$p^* = \tfrac{1}{N} \sum_{n=1}^{N} \mathrm{E}^n\{v\} = \frac{\tau_v^{1/2}}{\tau} \left(\tau_0^{1/2}\, i_0 + \tfrac{\tau_H^{1/2} + (N-1) \tau_L^{1/2}}{N} \sum_{n=1}^{N} i_n \right).$$

To summarize, in the model of perfect competition, both the target inventories and equilibrium price are the same as in our smooth trading model with imperfect competition. The key difference is that in the model of perfect competition, traders trade toward their target inventories immediately instead of gradually.

B.2. A Continuous-time Model of Perfect Competition

For the competitive equilibrium, we use the same notations and information structure as in our smooth trading model of imperfect competition. The only difference from our smooth trading model is that traders do not take into account price impact when solving for their optimal demand. For all dates $t > -\infty$, the optimal strategies S_n^* and C_n^* solve trader n's maximization problem

$$\max_{\{C_n, S_n\}} \quad E_t^n \left\{ \int_{s=t}^{\infty} e^{-\rho(s-t)}\, U(c_n(s))\, ds \right\}, \tag{B-11}$$

where the wealth $W_n(t)$ follows the process

$$dW_n(t) = r\, W_n(t)\, dt + S_n(t)\ \left(dP(t) + D(t)\, dt - r\, P(t)\, dt\right) - c_n(t)\, dt. \tag{B-12}$$

These two equations are similar to equations (A-17) and (22), but there are several differences. First, traders take prices in equation (B-12) as given. Second, in the model with perfect competition traders can costlessly transfer funds from their money account to stock account. It is therefore sufficient to keep track only of aggregate wealth dynamics, rather than separately keep track of a money account and a stock account.

Traders use the history of the dividend process, the history of their own private signals, and the average of all signals, as inferred from prices, to obtain their estimates of the growth rate. The inference problem is identical to the one in the smooth trading model.

To solve the equilibrium, we conjecture that price is a linear function of $D(t)$ and $\bar{G}(t)$, specifically,

$$P(t) = \frac{D(t)}{r + \alpha_D} + C_G\, \frac{\bar{G}(t)}{(r + \alpha_D)(r + \alpha_G)}. \tag{B-13}$$

It can be shown that

$$\begin{aligned}
dP(t) = \quad & - \quad \frac{1}{r + \alpha_D} \left(\alpha_D D(t) - \sigma_G \Omega^{1/2} \left(\tau_H^{1/2} \hat{H}_n(t) + (N-1) \tau_L^{1/2} \hat{H}_{-n}(t) \right) \right) dt \\
& + \quad \frac{C_G \sigma_G \Omega^{1/2} (\tau_H^{1/2} + (N-1)\tau_L^{1/2})}{N (r + \alpha_D)(r + \alpha_G)} ((a_1 + (N-1)a_4) \hat{H}_n(t) + (a_3 + (N-1)a_2) \hat{H}_{-n}(t)) dt \\
& + \quad \frac{1}{r + \alpha_D} \left((G^*(t) - G_n(t)) dt + \sigma_D dB_D \right) \\
& + \quad \frac{C_G \sigma_G \Omega^{1/2} (\tau_H^{1/2} + (N-1)\tau_L^{1/2})}{N (r + \alpha_D)(r + \alpha_G)} \left(N \hat{A} dB_0^n(t) + dB_n^n(t) + \sum_{m=1; m \neq n} dB_m^n(t) \right).
\end{aligned}$$

We conjecture and verify that the value function $V(W_n, \hat{H}_n, \hat{H}_{-n})$ has the specific quadratic exponential form

$$\text{(B-14)}\quad V(W_n, \hat{H}_n, \hat{H}_{-n}) = -\exp\Big(\psi_0+\psi_W W_n+\tfrac{1}{2}\psi_{nn}\hat{H}_n^2+\tfrac{1}{2}\psi_{xx}\hat{H}_{-n}^2+\psi_{nx}\hat{H}_n\hat{H}_{-n}\Big).$$

As in our smooth trading model, the five constants ψ_0, ψ_W, ψ_{nn}, ψ_{xx}, and ψ_{nx} have values consistent with a steady-state equilibrium. The terms ψ_{nn}, ψ_{xx}, and ψ_{nx} capture the value of future trading opportunities based on current public and private information. The value of trading on innovations to future information is built into the constant term ψ_0. Equation (B-14) is similar to equation (A-37), except that it has a simpler form because the five terms M_n, S_n^2, $S_n D$, $S_n\hat{H}_n$, and $S_n\hat{H}_{-n}$ are effectively replaced by one term, W_n.

The Hamilton-Jacobi-Bellman (HJB) equation corresponding to the conjectured value function $V(W_n, \hat{H}_n, \hat{H}_{-n})$ in equation (B-14) is

$$\begin{aligned}
0 \;=\; \min_{c_n, s_n}\; & -\frac{e^{-Ac_n}}{V} - \rho + \psi_W \Bigg(rW_n + S_n D(t) - c_n - rP(t)S_n(t) - \frac{\alpha_D}{r+\alpha_D}D(t)S_n \\
& + \frac{\sigma_G \Omega^{1/2}}{r+\alpha_D}\Big(\tau_H^{1/2}\hat{H}_n(t) + (N-1)\tau_L^{1/2}\hat{H}_{-n}(t)\Big) S_n \\
& + \frac{C_G\sigma_G\Omega^{1/2}(\tau_H^{1/2}+(N-1)\tau_L^{1/2})}{N(r+\alpha_D)(r+\alpha_G)}\Big((a_1+(N-1)a_4)\hat{H}_n(t) + (a_3+(N-1)a_2)\hat{H}_{-n}(t)\Big) S_n \Bigg) \\
& + \Big(\psi_{nn}\hat{H}_n(t) + \psi_{nx}\hat{H}_{-n}(t)\Big)\Big(-(\alpha_G+\tau)\hat{H}_n + (\tau_H^{1/2}+\hat{A}\tau_0^{1/2})(\tau_H^{1/2}\hat{H}_n + (N-1)\tau_L^{1/2}\hat{H}_{-n})\Big) \\
& + \Big(\psi_{xx}\hat{H}_{-n}(t) + \psi_{nx}\hat{H}_n(t)\Big)\Big(-(\alpha_G+\tau)\hat{H}_{-n} + (\tau_L^{1/2}+\hat{A}\tau_0^{1/2})(\tau_H^{1/2}\hat{H}_n + (N-1)\tau_L^{1/2}\hat{H}_{-n})\Big) \\
& + \tfrac{1}{2}\psi_W^2 S_n^2 \left(\frac{C_G^2\sigma_G^2\Omega(N\hat{A}^2+1)(\tau_H^{1/2}+(N-1)\tau_L^{1/2})^2}{N(r+\alpha_D)^2(r+\alpha_G)^2} + \frac{\sigma_D^2}{(r+\alpha_D)^2} + \frac{2C_G\sigma_G\sigma_D\Omega^{1/2}\tau_0^{1/2}}{(r+\alpha_D)^2(r+\alpha_G)}\right) \\
& + \tfrac{1}{2}\Big((\psi_{nn}\hat{H}_n(t) + \psi_{nx}\hat{H}_{-n}(t))^2 + \psi_{nn}\Big)\Big(1+\hat{A}^2\Big) \\
& + \tfrac{1}{2}\Big((\psi_{xx}\hat{H}_{-n}(t) + \psi_{nx}\hat{H}_n(t))^2 + \psi_{xx}\Big)\left(\frac{1}{N-1}+\hat{A}^2\right) \\
& + \psi_W S_n \Big((\psi_{nn}+\psi_{nx})\hat{H}_n(t) + (\psi_{xx}+\psi_{nx})\hat{H}_{-n}(t)\Big) \\
& \qquad \left(\frac{C_G\sigma_G\Omega^{1/2}}{N(r+\alpha_D)(r+\alpha_G)}(\tau_H^{1/2}+(N-1)\tau_L^{1/2})(N\hat{A}^2+1) + \frac{\sigma_D\hat{A}}{r+\alpha_D}\right) \\
& + \Big((\psi_{nn}\hat{H}_n(t) + \psi_{nx}\hat{H}_{-n}(t))\,(\psi_{xx}\hat{H}_{-n}(t) + \psi_{nx}\hat{H}_n(t)) + \psi_{nx}\Big)\;\hat{A}^2,
\end{aligned}$$

where constants a_1, a_2, a_3, and a_4 are defined as

$$\begin{aligned}
a_1 &:= -\alpha_G - \tau + \tau_H^{1/2}(\tau_H^{1/2} + \hat{A}\tau_0^{1/2}), \\
a_2 &:= -\alpha_G - \tau + (N-1)\tau_L^{1/2}(\tau_L^{1/2} + \hat{A}\tau_0^{1/2}), \\
a_3 &:= (\tau_H^{1/2} + \hat{A}\tau_0^{1/2})(N-1)\tau_L^{1/2}, \\
a_4 &:= (\tau_L^{1/2} + \hat{A}\tau_0^{1/2})\tau_H^{1/2}.
\end{aligned} \tag{B-15}$$

As in the smooth trading model, the solution for optimal consumption is

$$c_n^*(t) = -\frac{1}{A}\,\log\Big(\frac{\psi_W\,V(t)}{A}\Big). \tag{B-16}$$

Plugging optimal consumption and $P(t)$ from equation (B-13) into the HJB equation yields a quadratic function of S_n. It can be shown that the optimal trading strategy is a linear function of the state variables $\hat{H}_n(t)$ and $\hat{H}_{-n}(t)$,

$$\begin{aligned}
S_n^*(t) \;=\; C\; \Bigg(& C_G\sigma_G\Omega^{1/2}\Big(\tau_H^{1/2} + (N-1)\tau_L^{1/2}\Big)\Big((r - a_1 - (N-1)a_4)\hat{H}_n(t) \\
& \qquad + ((N-1)(r-a_2) - a_3)\hat{H}_{-n}(t)\Big) \\
& - \sigma_G\Omega^{1/2}(r+\alpha_G)N\Big(\tau_H^{1/2}\hat{H}_n(t) + (N-1)\tau_L^{1/2}\hat{H}_{-n}(t)\Big) \\
& - \Big((\psi_{nn}+\psi_{nx})\hat{H}_n(t) + (\psi_{xx}+\psi_{nx})\hat{H}_{-n}(t)\Big) \\
& \Big(C_G\sigma_G\Omega^{1/2}(\tau_H^{1/2} + (N-1)\tau_L^{1/2})(N\hat{A}^2+1) + \sigma_D\hat{A}N(r+\alpha_G)\Big)\Bigg),
\end{aligned} \tag{B-17}$$

where

$$C := \frac{(r+\alpha_D)(r+\alpha_G)/\psi_W}{C_G^2\sigma_G^2\Omega(\tau_H^{1/2} + (N-1)\tau_L^{1/2})^2(N\hat{A}^2+1) + N\sigma_D^2(r+\alpha_G)^2 + 2N(r+\alpha_G)\sigma_D C_G\sigma_G\Omega^{1/2}\tau_0^{1/2}}.$$

Market clearing, $\sum_{n=1}^N S_n^*(t) = 0$, implies

$$C_G = \frac{N(r+\alpha_G)\Big(\sigma_G\Omega^{1/2} + \sigma_D\hat{A}(\psi_{nn}+\psi_{xx}+2\psi_{nx})/(\tau_H^{1/2} + (N-1)\tau_L^{1/2})\Big)}{\sigma_G\Omega^{1/2}\Big(N(r+\alpha_G) + (N-1)(\tau_H^{1/2} - \tau_L^{1/2})^2 - (1+N\hat{A}^2)(\psi_{nn}+\psi_{xx}+2\psi_{nx})\Big)}. \tag{B-18}$$

Combining equations (B-17) and (B-18) yields

$$S_n^*(t) = C_L\,(\hat{H}_n - \hat{H}_{-n}), \tag{B-19}$$

where the constant C_L is defined as

(B-20)

$$C_L := C\,\Big(\sigma_G\Omega^{1/2}\,\Big(C_G(\tau_H^{1/2} + (N-1)\tau_L^{1/2})(r - a_1 - (N-1)a_4) - N\tau_H^{1/2}(r+\alpha_G)\Big)$$

$$-\left(\psi_{nn}+\psi_{nx}\right)\left(C_G\sigma_G\Omega^{1/2}(\tau_H^{1/2}+(N-1)\tau_L^{1/2})(1+N\hat{A}^2)+\sigma_D\hat{A}N(r+\alpha_G)\right)\Big).$$

Plugging (B-16) and (B-19) back into the Bellman equation and setting the constant term and the coefficients of W_n, $\hat{H}_n^2$, $\hat{H}_{-n}^2$, and $\hat{H}_n\,\hat{H}_{-n}$ to be zero yields five equations, which can be solved for the five unknown parameters ψ_0, ψ_W, ψ_{nn}, ψ_{nx}, and ψ_{xx}.

Equating the constant term and the coefficient of W_n to zero yields

(B-21)
$$\psi_W=-rA,$$

(B-22)
$$\psi_0=1-\log(r)+\frac{1}{r}\left(-\rho+\tfrac{1}{2}(1+\hat{A}^2)\psi_{nn}+\tfrac{1}{2}\left(\frac{1}{N-1}+\hat{A}^2\right)\psi_{xx}+\hat{A}^2\psi_{nx}\right).$$

Equating the coefficients of $\hat{H}_n^2$, $\hat{H}_{-n}^2$, and $\hat{H}_n\,\hat{H}_{-n}$ to zero results in three polynomial equations in the three unknowns ψ_{nn}, ψ_{xx}, and ψ_{nx}. Defining c_1, c_2, c_3, and c_4 by

$$c_1:=\frac{C_G^2\sigma_G^2\Omega(N\hat{A}^2+1)(\tau_H^{1/2}+(N-1)\tau_L^{1/2})^2}{N(r+\alpha_D)^2(r+\alpha_G)^2}+\frac{\sigma_D^2}{(r+\alpha_D)^2}+\frac{2C_G\sigma_G\sigma_D\Omega^{1/2}\tau_0^{1/2}}{(r+\alpha_D)^2(r+\alpha_G)},$$

$$c_2:=\frac{C_G\sigma_G\Omega^{1/2}}{N(r+\alpha_D)(r+\alpha_G)}(\tau_H^{1/2}+(N-1)\tau_L^{1/2})(N\hat{A}^2+1)+\frac{\sigma_D\hat{A}}{r+\alpha_D},$$

$$c_3:=\frac{rA\sigma_G\Omega^{1/2}C_L}{r+\alpha_D}\left(\frac{C_G(\tau_H^{1/2}+(N-1)\tau_L^{1/2})(r-a_1-(N-1)a_4)}{N(r+\alpha_G)}-\tau_H^{1/2}\right),$$

$$c_4:=\frac{rA\sigma_G\Omega^{1/2}C_L}{r+\alpha_D}\left(\frac{C_G(\tau_H^{1/2}+(N-1)\tau_L^{1/2})(r-a_2-\frac{a_3}{N-1})}{N(r+\alpha_G)}-\tau_L^{1/2}\right),$$

these three equations in three unknowns can be written as follows:

(B-23) $\underline{\hat{H}_n^2}$:

$$0=-\frac{r}{2}\psi_{nn}+a_1\psi_{nn}+a_4\psi_{nx}-rAC_Lc_2(\psi_{nn}+\psi_{nx})+\tfrac{1}{2}(1+\hat{A}^2)\psi_{nn}^2$$
$$+\tfrac{1}{2}\left(\frac{1}{N-1}+\hat{A}^2\right)\psi_{nx}^2+\hat{A}^2\psi_{nn}\psi_{nx}+c_3+\tfrac{1}{2}r^2A^2c_1C_L^2,$$

(B-24) $\underline{\hat{H}_{-n}^2}$:

$$0=-\frac{r}{2}\psi_{xx}+a_2\psi_{xx}+a_3\psi_{nx}+rAC_Lc_2(\psi_{xx}+\psi_{nx})+\frac{1+\hat{A}^2}{2}\psi_{nx}^2$$
$$+\tfrac{1}{2}\left(\frac{1}{N-1}+\hat{A}^2\right)\psi_{xx}^2+\hat{A}^2\psi_{xx}\psi_{nx}-(N-1)c_4+\tfrac{1}{2}r^2A^2c_1C_L^2,$$

(B-25) $\underline{\hat{H}_n \hat{H}_{-n}}$:

$$0 = -r\psi_{nx} + (a_1 + a_2)\psi_{nx} + a_3\psi_{nn} + a_4\psi_{xx} + rAC_L c_2(\psi_{nn} - \psi_{xx})$$
$$+(1 + \hat{A}^2)\psi_{nn}\psi_{nx} + \left(\frac{1}{N-1} + \hat{A}^2\right)\psi_{xx}\psi_{nx} + \hat{A}^2\left(\psi_{nn}\psi_{xx} + \psi_{nx}^2\right)$$
$$+(N-1)c_4 - c_3 - r^2A^2c_1C_L^2.$$

To summarize, optimal consumption is defined in (B-16), the optimal trading strategy is defined in (B-19), and the endogenous coefficient C_L is defined in (B-20). The equilibrium price is defined in (B-13), and the endogenous coefficient C_G is defined in (B-18). The parameters ψ_W and ψ_0 are presented in (B-21) and (B-22). The parameters ψ_{nn}, ψ_{nx}, and ψ_{xx} are obtained from numerical solution of the system of the three equations (B-23)–(B-25). These results are stated in theorem 5.

Information has no value if there is no trading, so that $\psi_{nn} = \psi_{nx} = \psi_{xx} = 0$ solves the three equations (B-23)–(B-25) when there is no liquidity. This implies

(B-26)

$$c_3 + \tfrac{1}{2}r^2A^2c_1C_L^2 = 0, \quad -(N-1)c_4 + \tfrac{1}{2}r^2A^2c_1C_L^2 = 0, \quad (N-1)c_4 - c_3 - r^2A^2c_1C_L^2 = 0.$$

These equations imply that liquidity vanishes when $\tau_H = \tau_L$ and $C_G = 1$. This is different from our smooth trading model of disagreement with imperfect competition, in which market liquidity vanishes when $\tau_H^{1/2}/\tau_L^{1/2} = 2 + 2/(N-2)$ and $C_G = 1$.

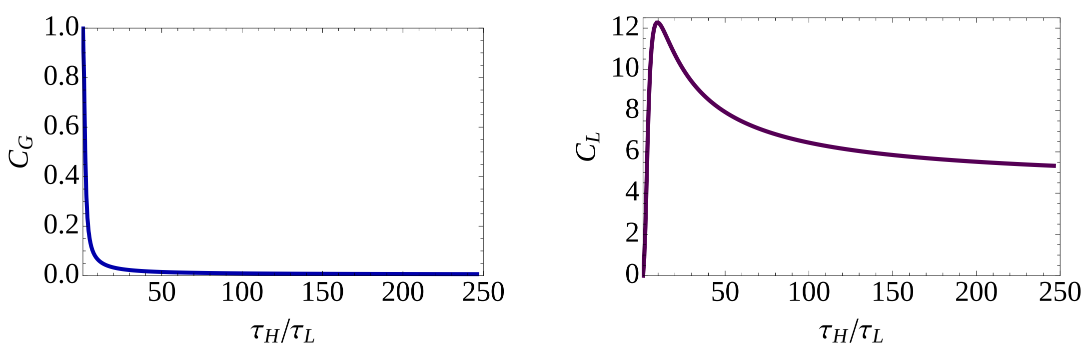


FIGURE 9. COEFFICIENTS C_G AND C_L AGAINST τ_H/τ_L WHILE FIXING $\tau = 7.4$.

Figure 9 shows the effect of changes in the degree of overconfidence τ_H/τ_L on the endogenous parameters C_G and C_L. To compare the results with our smooth trading model, we use the same exogenous parameter values as in figure 2 and panel (a) of figure 7. The horizontal axis shows the ratio τ_H/τ_L. As this ratio increases, τ_H is increasing and τ_L is decreasing so that the total precision τ is fixed (and other exogenous parameters are also fixed). Higher values of the ratio τ_H/τ_L correspond to higher degrees of overconfidence. As disagreement τ_H/τ_L increases, the left panel shows that the parameter C_G declines monotonically, while the right panel shows that the parameter C_L, which governs the size of target inventories, first increases and then decreases.

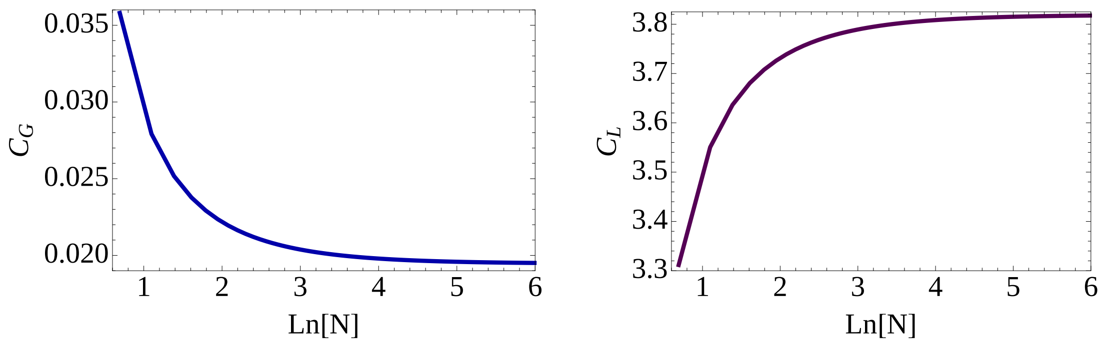


FIGURE 10. COEFFICIENTS C_G, C_L AGAINST N WHILE FIXING $\tau = 1.4$ AND $\tau_L = 0$.

For finite N, figure 10 shows the effect of changes in the number of traders N on C_G and C_L, using the same exogenous parameter values as in figure 3 and panel (b) of figure 7. As N increases, the left panel shows that C_G decreases monotonically toward a constant asymptote, and the right panel shows that C_L increases monotonically toward a constant asymptote. When N is large, our numerical results show that our smooth trading model of imperfect competition converges to the equilibrium of the competitive model.

As in the smooth trading model, we find a closed-form solution when we set $\tau_L = 0$, and then evaluate the solution in the limit as $N \to \infty$ and $\hat{A} \to 0$. We conjecture and verify that $\psi_{nn} = \bar{\psi}_{nn}$, $\psi_{nx} = \bar{\psi}_{nx}$, and $\psi_{xx} = \bar{\psi}_{xx}$, where $\bar{\psi}_{nn}$, $\bar{\psi}_{nx}$, and $\bar{\psi}_{xx}$ are constants that do not depend on N.

Solving the system of equations (B-23)–(B-25) yields

(B-27)
$$\bar{\psi}_{nn} = \tfrac{1}{2}\left(r + 2(\alpha_G + \tau - \tau_H) - \left((r + 2(\alpha_G + \tau - \tau_H))^2 + \frac{4\Omega\sigma_G^2\tau_H}{\sigma_D^2}\right)^{1/2}\right),$$

(B-28)
$$\bar{\psi}_{nx} = \frac{\Omega\sigma_G^2\tau_H/\sigma_D^2}{r + 2(\alpha_G + \tau) - \tau_H - \bar{\psi}_{nn}},$$

(B-29)
$$\bar{\psi}_{xx} = \frac{1}{r + 2\alpha_G + 2\tau}\left(\bar{\psi}_{nx}^2 - \frac{\Omega\sigma_G^2\tau_H}{\sigma_D^2}\right).$$

Equations (B-18) and (B-20) imply that

(B-30)
$$C_G \to \frac{r + \alpha_G}{r + \alpha_G + \tau} < 1, \qquad C_L = \frac{\Omega^{1/2}\sigma_G\tau_H^{1/2}(r + \alpha_D)}{Ar\sigma_D^2}.$$

These results are exactly the same as the limiting case when $N \to \infty$ and $\hat{A} \to 0$ in the smooth trading model. This confirms that our smooth trading model of imperfect competition converges to the competitive model when $N \to \infty$.

C. A Continuous-Time Model of Smooth Trading with Private Values

In this section, we consider an alternative smooth trading model in which private values with a common prior replace disagreement (with different priors) as the modeling device which makes trade possible in equilibrium. We show that optimal trading strategies balance the trade-off between the temporary price impact costs of a trader's own trades and the decay of his private information resulting from the permanent price impact of other traders trading on similar information. When investors put enough weight on their private values, an equilibrium exists, prices immediately reveal a weighted average of all traders' signals and private values, and traders continue to trade gradually toward their target inventories.

The model has the following key features: (1) There is only one type of trader, a strategic informed trader; there are no noise traders or market makers. (2) Each trader has private information about the same underlying fundamental value; the "noise" in their signals is uncorrelated. (3) All signals have the same precision; the structure of the model is common knowledge; traders share a common prior and apply Bayes law correctly and consistently. (4) Each trader gains private value from investing in the asset; the private value is uncorrelated with the fundamental value. (5) Traders trade strategically, correctly taking into account how the permanent and temporary price impact of their trades affects the trading of other traders. (6) Random variables are jointly normally distributed and traders have additive exponential utility functions. (7) Traders are "symmetric" in the sense that they have the same utility functions and symmetrically different beliefs about the information structure in the economy. (8) All model state variables are stationary.

We describe an "almost closed form" steady-state equilibrium with "smooth trading" characterized precisely by endogenous parameters solving a set of five polynomial equations in five unknowns. We show that an equilibrium exists when traders put large enough weight on their private values. Although it is necessary to solve numerically for an endogenous factor by which noisy private values lower the precision of signals inferred from prices, other endogenous parameters are obtained as closed-form functions of this endogenous factor.

The equilibrium in the model with private values is similar to the model with overconfidence. There is, however, one important difference: In the model with private values, there is no price dampening associated with the "Keynesian beauty contest."

In the model with private values—unlike the model based on disagreement—even though traders have different valuations of the asset at present, they do not disagree about the dynamics of how those valuations will change in the future; this makes prices equal to a noisy weighted average of traders' buy-and-hold valuations, with the weights summing exactly to one, not to a dampened value less than one.

In the model with disagreement, traders not only trade because they disagree with the average of other traders' valuations in the present, but they also trade based on disagreement concerning their predictions about how the average of other traders' valuations will change in the future. This makes prices equal to a weighted average of traders' buy-and-hold valuations, with the weights summing to a constant less than one.

C.1. Model Set-Up

There are N risk averse oligopolistic traders who trade a risky zero-net-supply asset against a risk-free asset, which earns constant risk-free rate $r > 0$.

The risky asset is traded at price $P(t)$ and pays out dividends at continuous rate $D(t)$. Dividends follow a stochastic process with mean-reverting stochastic growth rate $G^*(t)$, constant instantaneous volatility $\sigma_D > 0$, and constant rate of mean reversion $\alpha_D > 0$,

$$dD(t) := -\alpha_D \, D(t) \, dt + G^*(t) \, dt + \sigma_D \, dB_D(t). \tag{C-1}$$

The growth rate $G^*(t)$ follows an AR-1 process with mean reversion α_G and volatility σ_G:

$$dG^*(t) := -\alpha_G \, G^*(t) \, dt + \sigma_G \, dB_G(t). \tag{C-2}$$

The dividend is publicly observable, but the growth rate $G^*(t)$ is not observed by any trader. This structure of payoffs is similar to equations (18) and (19) in the model of disagreement.

The information structure is slightly different from the model with disagreement. Each trader n observes a continuous stream of private information $I_n(t)$ about a common value $G^*(t)$,

$$dI_n(t) := \tau_I^{1/2} \, \frac{G^*(t)}{\sigma_G \, \Omega^{1/2}} \, dt + dB_{In}(t), \quad n = 1, \ldots, N. \tag{C-3}$$

Since the drift $\tau_I^{1/2} \, G^*(t)/(\sigma_G \Omega^{1/2})$ is proportional to $G^*(t)$, each increment $dI_n(t)$ in the process $I_n(t)$ is a noisy observation of the unobserved growth rate $G^*(t)$. The denominator $\sigma_G \, \Omega^{1/2}$ scales $G^*(t)$ so that the conditional scaled error variance is one. This simplifies intuitive interpretation of the model. The parameter Ω measures the steady-state error variance in units of time, as discussed below. The "precision" parameter τ_I measures the informativeness of the signal $dI_n(t)$ as a signal-to-noise ratio describing how fast the information flow generates a signal of a given level of statistical significance. Since traders agree on how much information τ_I each signal contains, the traders share a common prior. In the similar equation (20) for the model with disagreement, each trader assigns a higher precision τ_H to his own signal and lower precision τ_L to signals of others; therefore, traders do not share a common prior.

Using the scaling parameter Ω, the information content of the publicly observable dividend $D(t)$ can be expressed in a form consistent with the notation for private signals $I_n(t)$ in equation (C-3). Define $dI_0(t) := \left[\alpha_D\, D(t)\, dt + dD(t)\right]/\sigma_D$ and $\tau_0 := \Omega\, \sigma_G^2/\sigma_D^2$ with $dB_0 := dB_D$. Then the public information $I_0(t)$ in the divided stream (C-1) can equivalently be written

$$\text{(C-4)} \qquad dI_0(t) := \tau_0^{1/2}\, \frac{G^*(t)}{\sigma_G\, \Omega^{1/2}}\, dt + dB_0(t).$$

Observing the process $I_0(t)$ is informationally equivalent to observing the dividend process $D(t)$. The quantity τ_0 measures the precision of the dividend process in units analogous to the units of precision for private signals. We assume that $dB_D(t), dB_G(t), dB_{I1}(t), \ldots, dB_{IN}(t), dB_{J1}(t), \ldots, dB_{JN}(t)$ are independently-distributed, standardized Brownian motions. This notation simplifies the filtering formulas we are about to derive.

Unlike in the model with disagreement, the risky asset generates privately-observed private benefits for traders owning it; this assumption helps to generate trade. Specifically, we assume that the risky asset generates a cash flow $D(t) + \pi_J\, H_n^J(t)$, where the first component is a publicly-observed, common-value cash dividend—as in the model with disagreement—and the additional second component is a privately-observed cash-equivalent of the private benefit trader n receives from holding the risky asset. We assume that the trader n's private benefit $H_n^J(t)$ follows an AR-1 process with the mean reversion rate δ_J,

$$\text{(C-5)} \qquad dH_n^J(t) = -\delta_J H_n^J(t) dt + dB_{Jn}(t), \quad n = 1, \ldots, N.$$

where π_J and δ_J are constants. In order to keep the number of state variables the same as the number of state variables in the model of disagreement, it is necessary to set the mean reversion rate δ_J to equal a specific value. As shown below, this specific value equates the mean-reversion rate of private values δ_J to the mean reversion rate of private signals. Since there are no *a priori* reasons to believe that private value and information flow share similar dynamics, this assumption is a key limitation of the smooth-trading model with private values.

Each trader's information set at time t, denoted $\mathcal{F}_n(t)$, consists of the histories of the publicly-observed dividend process $D(s)$, the trader's own private information $I_n(s)$, the trader's private observation of his own private value $H_n^J(t)$, and the market price $P(s)$, $s \in (-\infty, t]$. All traders process information rationally.

Let $S_n(t)$ denote the inventory of trader n at time t. Assume the risky asset is in zero net supply, implying $\sum_{n=1}^N S_n(t) = 0$. Each trader's trading strategy X_n is assumed to be a mapping from his information set $\mathcal{F}_n(t)$ at time t into a "flow-demand schedule" which defines the derivative of his inventory $x_n(t) := X_n(t, P(t); \mathcal{F}_n(t))$ ("trading intensity") as a function of the market-clearing price $P(t)$. An auctioneer continuously calculates the market-clearing price $P(t) := P[X_1, \ldots, X_N](t)$ such that the market-clearing condition $\sum_{n=1}^N x_n(t) = 0$ is satisfied. Let $\mathrm{E}_t^n\{\ldots\}$ denote the conditional expectations operator $\mathrm{E}\{\ldots|\mathcal{F}_n(t)\}$ based on trader n's beliefs.

Each trader has time-additively-separable exponential utility function $U(c_n(s)) := -\,\mathrm{e}^{-A\, c_n(s)}$ with constant-absolute-risk-aversion parameter A and the time preference parameter ρ. Trader n's consumption strategy C_n defines a consumption rate $c_n(t) := C_n(t; \mathcal{F}_n(t))$.

We define an *equilibrium* as a set of trading strategies $X_1^*, \ldots, X_N^*$ and consumption strategies $C_1^*, \ldots, C_N^*$ such that, for $n = 1, \ldots, N$, trader n's optimal consumption and trading strategies $X_n = X_n^*$ and $C_n = C_n^*$ solve his maximization problem taking as given the optimal strategies of the other traders. Trader n's maximization problem is

$$\text{(C-6)} \quad J^n(\mathcal{F}_n(t); X_n^*, C_n^*; X_m^*, m \neq n) = \max_{\{C_n, X_n\}} \quad \mathrm{E}_t^n \left\{ \int_{s=t}^{\infty} \mathrm{e}^{-\rho(s-t)} \; U(c_n(s)) \, ds \right\},$$

where inventories follow the process $dS_n(t) = x_n(t)\, dt$ and money holdings $M_n(t)$ follow the process

$$\text{(C-7)} \quad dM_n(t) = \big(r\, M_n(t) + S_n(t)\, \big(D(t) + \pi_J\, H_n^J(t)\big) - c_n(t) - P(t)\, x_n(t)\big)\; dt.$$

Equation (C-7) is similar to equation (22) for the model with disagreement, except for the term $\pi_J H_n^J(t)$, which measures the cash-equivalent of the private benefit of owning the asset as a "convenience yield."

Note that the price $P(t)$, quantity $x_n(t)$, and consumption $c_n(t)$ are the abbreviations

(C-8)

$$P(t) := P[X_1, \ldots, X_N](t), \quad x_n(t) := \frac{dS_n(t)}{dt} = X_n(t, P(t); \mathcal{F}_n(t)), \quad c_n(t) := C_n(t; \mathcal{F}_n(t)).$$

When solving the maximization problem, trader n takes as given the trading strategies X_m, $m \neq n$, for the other $N-1$ traders; in doing so, he exercises market power by taking into account how his own trading strategy affects equilibrium prices $P(t)$ and future trading opportunities. The optimal strategy must satisfy the transversality condition $\mathrm{E}_t^n\{\mathrm{e}^{-\rho(T-t)}\; J^n(\mathcal{F}_n(T), X_n^*, C_n^*; \ldots)\} \to 0$ as $T \to \infty$.

Innovations in private values show up as noise in prices, as a result of which traders infer from prices only a noisy version of the average of other traders' signals. We will show next that each trader can infer from the equilibrium prices only the average of a linear combination $\frac{1}{N-1}\sum_{m\neq n}(I_m(t) + k\, B_{Jm}(t))$ of other traders' private signals $I_m(t)$ and private values $B_{Jm}(t)$. The value of the weight k on private values is determined endogenously in equilibrium.

C.2. Bayesian Updating

Let $G_n(t) := \mathrm{E}_t^n\{G^*(t)\}$ denote trader n's estimate of the unobserved growth rate $G^*(t)$ conditional on his information set at time t. This information set consists of dividend information $I_0(s)$, the trader's private signal $I_n(s)$, the trader's private value $H_n^J(t)$, and the noisy average of other traders' signals inferred from prices $\frac{1}{N-1}\sum_{m\neq n}(I_m(s) + k\, B_{Jm}(s))$, $s \in (-\infty, t]$.

Define Ω as the error variance $\Omega := \mathrm{Var}^n\{(G^*(t) - G_n(t))/\sigma_G\}$. We assume a symmetric steady state in which Ω is a constant which does not depend on time t or trader n. There are simple and intuitive formulas for information processing:

LEMMA 2: *Let τ denote the sum of precisions*

$$\tau := \tau_0 + \tau_I + (N-1)\,\frac{1}{1+k^2}\,\tau_I. \tag{C-9}$$

Then Ω and $dG_n(t)$ satisfy

$$\Omega^{-1} := \left(\mathrm{Var}^n\left\{\frac{G^*(t) - G_n(t)}{\sigma_G}\right\}\right)^{-1} = 2\,\alpha_G + \tau, \tag{C-10}$$

(C-11)

$$dG_n(t) = -\left(\alpha_G + \tau\right) G_n(t)dt + \sigma_G \Omega^{1/2}\left(\tau_0^{1/2} dI_0(t) + \tau_I^{1/2} dI_n(t) + \frac{\tau_I^{1/2}}{1+k^2}\sum_{m \neq n}^{N}\left(dI_m(t) + k\,dB_{Jm}(t)\right)\right).$$

The proof is in Appendix section C.8. This lemma is similar to Lemma 1 in Appendix section A.2, except trader n attributes a precision τ_I to his own signal $dI_n(t)$ and a lower precision $\tau_I/(1+k^2)^2$ to other traders' signals $dI_m(t)+k\,dB_{Jm}(t)$, since those signals are contaminated by trading due to private values. The total precision of information τ is not quasi-exogenous, as in the model of disagreement, but rather depends on the endogenous factor k, whose value will be derived below.

Note that Ω is not a "free parameter;" instead, it is determined as an endogenous function of the other parameters. Equation (C-10) implies that Ω is the solution to the quadratic equation $\Omega^{-1} = 2\,\alpha_G + \Omega\,\sigma_G^2/\sigma_D^2 + \tau$. In equations (C-3) and (C-4), we scale the units with which precision is measured by the endogenous parameter Ω because this leads to simpler Kalman filtering expressions which more clearly bring out the intuition of signal processing.

Similar to equations (25) and (A-30), define statistics $H_n^I(t)$ corresponding to information flow dI_n as

$$H_n^I(t) := \int_{u=-\infty}^{t} \mathrm{e}^{-(\alpha_G+\tau)\,(t-u)}\; dI_n(u), \quad n = 0, 1, ...N, \tag{C-12}$$

which implies

$$dH_n^I(t) = -(\alpha_G + \tau)\; H_n^I(t)\; dt + dI_n(t), \quad n = 0, 1, ...N. \tag{C-13}$$

A trader also infers a noisy average of other traders' signals $H_m^I(t) + k\,H_m^J(t)$ from equilibrium prices. To prevent intractability resulting from an exploding number of state variables and to keep the number of state variables in both models the same, it is necessary to make the restrictive assumption that the private signals $H_n^I(t)$ and the private values $H_n^J(t)$ mean-revert to zero at the same rate; this requires the assumption $\delta_J := \alpha_G + \tau$.

Define signals $H_n(t)$ and $H_{-n}(t)$, adjusted to reflect private values, by

$$\text{(C-14)}\quad H_n(t) := H_n^I(t) + k\, H_n^J(t), \qquad H_{-n}(t) := \frac{1}{N-1} \sum_{m \neq n} \left(H_m^I(t) + k\, H_m^J(t) \right).$$

Equation (C-11) implies that the estimate $G_n(t)$ can be conveniently written as a linear combination of sufficient statistics $H_0^I(t)$, $H_n^I(t)$, and $H_{-n}(t)$:

$$\text{(C-15)}\quad G_n(t) = \sigma_G\, \Omega^{1/2} \left(\tau_0^{1/2}\, H_0^I(t) + \tau_I^{1/2}\, H_n^I(t) + (N-1)\, \frac{\tau_I^{1/2}}{1+k^2}\, H_{-n}(t) \right).$$

This equation is similar to equation (27) in the model with disagreement.

As we show below, trader n's optimal trading strategy depends on several variables. First, it depends on trader n's estimates of the unobserved growth rate $G^*(t)$. Second, it depends on the dynamic statistical relationship between this growth rate and the signals $H_0^I(t)$ and $H_n^I(t)$, which reflect his public and private information about fundamental value. Third, it depends on $H_n^J(t)$, which reflects his own private value. Finally, it depends on $H_{-n}(t)$, which reflects the noisy private information of other traders that trader n extracts from prices with contamination from "noise" associated with their private values. We next examine the dynamics of some of these variables.

Define the $N+1$ processes dB_0^n, dB_{In}^n, and dB_m^n, $m = 1, \ldots N$, $m \neq n$, by

$$\text{(C-16)}\qquad dB_0^n(t) = \tau_0^{1/2}\, \frac{G^*(t) - G_n(t)}{\sigma_G\, \Omega^{1/2}}\, dt + dB_D(t),$$

$$\text{(C-17)}\qquad dB_{In}^n(t) = \tau_I^{1/2}\, \frac{G^*(t) - G_n(t)}{\sigma_G\, \Omega^{1/2}}\, dt + dB_{In}(t),$$

and

$$\text{(C-18)}\qquad dB_m^n(t) = \tau_I^{1/2}\, \frac{G^*(t) - G_n(t)}{\sigma_G\, \Omega^{1/2}}\, dt + dB_{Im}(t) + k\, dB_{Jm}(t).$$

The superscript n indicates conditioning on the information set of trader n. Since trader n's forecast of the error $G^*(t) - G_n(t)$ is zero given his information set, these $N+1$ processes are independently-distributed Brownian motions from the perspective of trader n. In terms of these Brownian motions, trader n believes that signals change as follows:

$$\text{(C-19)}\qquad dH_0^I(t) = -(\alpha_G + \tau)\, H_0^I(t)\, dt + \tau_0^{1/2}\, \frac{G_n(t)}{\sigma_G\, \Omega^{1/2}}\, dt + dB_0^n(t),$$

$$\text{(C-20)}\qquad dH_n^I(t) = -(\alpha_G + \tau)\, H_n^I(t)\, dt + \tau_I^{1/2}\, \frac{G_n(t)}{\sigma_G\, \Omega^{1/2}}\, dt + dB_{In}^n(t),$$

(C-21) $$dH_{-n}(t) = -(\alpha_G+\tau)H_{-n}(t)dt+\tau_I^{1/2}\frac{G_n(t)}{\sigma_G\,\Omega^{1/2}}dt+\frac{1}{N-1}\sum_{m=1,..N;m\neq n} dB_m^n(t).$$

Note that each signal drifts toward zero at rate $\alpha_G+\tau$ and drifts toward the optimal forecast $G_n(t)$ at a rate proportional to the square root of the signal's precisions $\tau_0^{1/2}$ or $\tau_I^{1/2}$, respectively.

C.3. Utility Maximization with Market Power

We use the no regret approach to calculate the value function $J^n(\ldots)$. We assume that trader n observes his residual supply schedule $P(.) := P_n(.,t)$ at each point in time and picks an optimal point on the residual supply schedule. We then show that the solution to this less constrained problem implements the optimal solution to the more constrained problem which defines $J^n(\ldots)$.

For the less constrained problem, we conjecture a steady-state value function $V(M_n, S_n, D, H_0^I, H_n^I, H_n^J, H_{-n})$, where M_n denotes trader n's cash holdings (measured in dollars) and S_n denotes trader n's holdings of the traded asset (measured in shares).

We expect the asset price to be a linear combination of two components: (1) a dividend level component linear in dividends $D(t)$ and (2) a dividend-growth component linear in the variables $H_0^I(t)$, $H_n^I(t)$, $H_n^J(t)$, and $H_{-n}(t)$. The symmetric linear conjectured form of the residual supply function implies that observation of the average of other traders' signals $H_{-n}(t)$ is informationally equivalent to observation of the intercept of the trader's residual supply schedule. We therefore include $H_{-n}(t)$ as a state variable in the value function and omit the price $P(t)$.

In deriving the equilibrium, the problem is simplified if the three state variables $H_0^I(t)$, $H_n^I(t)$, and $H_{-n}(t)$ are replaced with two "composite" signals, which we denote $\hat{H}_n^I(t)$ and $\hat{H}_{-n}(t)$. Define the weighting constant $\hat{A}$ by

(C-22) $$\hat{A} := \frac{\tau_0^{1/2}}{\tau_I^{1/2}\,(1+(N-1)/(1+k^2))}.$$

Define the two composite signals $\hat{H}_n^I(t)$ and $\hat{H}_{-n}(t)$ by

(C-23) $$\hat{H}_n^I(t) := H_n^I(t) + \hat{A}\,H_0^I(t),$$

(C-24) $$\hat{H}_{-n}(t) := H_{-n}(t) + \hat{A}\,H_0^I(t).$$

These composite signals incorporate public information contained in the dividend stream. Define

(C-25) $$\hat{H}_n(t) := \hat{H}_n^I(t) + k\,H_n^J(t).$$

Trader n's estimate of dividend growth rate can be expressed as a function of the two composite signals $\hat{H}_n^I(t)$ and $\hat{H}_{-n}(t)$,

$$G_n(t) = \sigma_G \, \Omega^{1/2} \left(\tau_I^{1/2} \, \hat{H}_n^I(t) + (N-1) \, \frac{1}{1+k^2} \, \tau_I^{1/2} \, \hat{H}_{-n}(t) \right). \tag{C-26}$$

Note that this estimate does not depend on trader n's private value $H_n^J(t)$, since the term $H_n^J(t)$ captures the private benefit of owning the risky asset, not information about its common fundamental value.

We conjecture (and verify below) a steady-state value function of the form $V(M_n, S_n, D, \hat{H}_n, \hat{H}_{-n})$. Letting $(c_n(t), x_n(t))$ denote the optimal consumption and investment policy, we have

$$V(M_n, S_n, D, \hat{H}_n, \hat{H}_{-n}) := \max_{\{c_n(t), x_n(t)\}} \quad \mathrm{E}_t^n \left\{ \int_{s=t}^{\infty} - \, \mathrm{e}^{-\rho(s-t) - A \, c_n(s)} \; ds \right\}. \tag{C-27}$$

The six state variables satisfy six stochastic differential equations

$$dM_n(t) = \big(r \, M_n(t) + S_n(t) \, (D(t) + \pi_J \, H_n^J(t)) - c_n(t) - P(x_n(t)) \, x_n(t) \big) \; dt, \tag{C-28}$$

$$dS_n(t) = x_n(t) \, dt, \tag{C-29}$$

$$dD(t) = -\alpha_D \, D(t) \, dt + G_n(t) \, dt + \sigma_D \, dB_0^n(t), \tag{C-30}$$

$$dH_n^J(t) = -(\alpha_G + \tau) \, H_n^J(t) dt + dB_{Jn}(t), \tag{C-31}$$

$$\begin{aligned} d\hat{H}_n^I(t) = \; & - \; (\alpha_G + \tau) \, \hat{H}_n^I(t) \, dt \\ & + \; (\tau_I^{1/2} + \hat{A} \tau_0^{1/2}) \, \tau_I^{1/2} \, \left(\hat{H}_n^I(t) + \frac{N-1}{1+k^2} \, \hat{H}_{-n}(t) \right) \, dt \\ & + \; \hat{A} \, dB_0^n(t) + dB_{In}^n(t), \end{aligned} \tag{C-32}$$

$$\begin{aligned} d\hat{H}_{-n}(t) = \; & - \; (\alpha_G + \tau) \, \hat{H}_{-n}(t) \, dt \\ & + \; (\tau_I^{1/2} + \hat{A} \tau_0^{1/2}) \, \tau_I^{1/2} \, \left(\hat{H}_n^I(t) + \frac{N-1}{1+k^2} \, \hat{H}_{-n}(t) \right) \, dt \\ & + \; \hat{A} \, dB_0^n(t) + \frac{1}{N-1} \sum_{m=1,..N; m \neq n} dB_m^n(t). \end{aligned} \tag{C-33}$$

The dynamics of $\hat{H}_n^I(t)$ and $\hat{H}_{-n}(t)$ in equations (C-32) and (C-33) can be derived from equations (C-19), (C-20), and (C-21). It can be shown that the value function conveniently depends on state variables $\hat{H}_n^I(t)$ and $H_n^J(t)$ only through $\hat{H}_n(t)$, and

$$\text{(C-34)} \qquad \begin{aligned} d\hat{H}_n(t) = \; & - \quad (\alpha_G + \tau)\, \hat{H}_n(t)\, dt \\ & + \quad (\tau_I^{1/2} + \hat{A}\tau_0^{1/2})\, \tau_I^{1/2} \left(\hat{H}_n^I(t) + \frac{N-1}{1+k^2}\, \hat{H}_{-n}(t) \right) dt \\ & + \quad \hat{A}\, dB_0^n(t) + dB_{In}^n(t) + k\, dB_{Jn}(t). \end{aligned}$$

This system of equations is similar to the system of equations (A-32)–(A-35).

Equation (C-28), describing the dynamics of cash $M(t)$, differs from equation (A-33) by including an additional term related to private benefits $\pi_J\, H_n^J(t)$.

Furthermore, in the second lines of equations (C-33) and (C-34), the factors $\tau_I^{1/2} + \hat{A}\tau_0^{1/2}$ are the same in both equations. In the otherwise similar model based on disagreement, these two factors are different; the factor is equal to $\tau_H^{1/2} + \hat{A}\tau_0^{1/2}$ in equation (A-34) and $\tau_L^{1/2} + \hat{A}\tau_0^{1/2}$ in equation (A-35). The equality of these two factors in the model based on private values ultimately leads to an important difference between the disagreement model and the private-values model with common prior. The model with private values does not generate "price dampening," which is associated with the logic of a Keynesian beauty contest in the model based on disagreement.

More specifically, in the model with disagreement, each trader believes that his own signal drifts toward the fundamental value at a rate reflecting his own high precision τ_H, while the average of other traders' signals drifts toward the fundamental value at a rate reflecting a lower precision τ_L (equations (A-34) and (A-35)). In the model with private values, by contrast, each trader believes that both his own signal and the noisy signal of other traders, inferred from prices, drift toward the fundamental value at a rate reflecting the higher precision τ_I, not the lower precision affected by noise added by private values (equations (C-33) and (C-34)). Thus, this noise affects the precision of the signal inferred from prices as an estimate of fundamental value in the present, *i.e.*, $G_n(t)$ in equation (C-15), but it does not affect the drift of this estimate. In the model with disagreement, equation (A-36) shows that trader n believes that $H_n - H_{-n}$ decays at rate $\alpha_G + \tau$ but also drifts in a direction proportional to $G_n(t)$. In the model with private values, each trader believes that the quantity equivalent to $H_n - H_{-n}$ follows an AR-1 process and the drift term proportional to $G_n(t)$ becomes zero.

The value function $V(\,)$ satisfies the transversality condition

$$\text{(C-35)} \qquad \lim_{T \to +\infty} \mathrm{E}_t^n \{ \mathrm{e}^{-\rho(T-t)}\; V(M_n(T), S_n(T), D(T), \hat{H}_n(T), \hat{H}_{-n}(T)) \} = 0.$$

C.4. Linear Conjectured Strategies

Based on his information set, each trader submits a flow-demand schedule for the rate at which he will buy the asset at time t as a function of the market-clearing price. Trader n conjectures that the other $N-1$ traders, $m = 1, \ldots N$, $m \neq n$, submit symmetric linear demand schedules of the form

$$X_m(t) = \frac{dS_n(t)}{dt} = \gamma_D\, D(t) + \gamma_H\, \hat{H}_m(t) - \gamma_S\, S_m(t) - \gamma_P\, P(t), \tag{C-36}$$

where $\hat{H}_m(t) := \hat{H}^I_m + k\, H^J_m$ sums together both private information about the fundamental value and the privately-observed private value. The demand schedules are defined by the four constants γ_D, γ_H, γ_S, and γ_P.

Let $x_n(t) = X_n(t, P(t)) = dS_n(t)/dt$ denote the "flow-quantity" traded by trader n. From the market-clearing condition and the linear conjecture for demand schedules of other traders, it follows that

$$x_n(t) + \sum_{m=1,..N;m\neq n} \Big(\gamma_D\, D(t) + \gamma_H\, \hat{H}_m(t) - \gamma_S\, S_m(t) - \gamma_P\, P(t)\Big) = 0. \tag{C-37}$$

Using zero net supply $\sum_{m=1}^{N} S_m(t) = 0$, this can be solved for trader n's conjectured price impact function (written $P(.)$ instead of $P(., t)$)

$$P(x_n(t)) = \frac{\gamma_D}{\gamma_P}\, D(t) + \frac{\gamma_H}{\gamma_P}\, \hat{H}_{-n}(t) + \frac{\gamma_S}{\gamma_P} \frac{1}{N-1}\, S_n(t) + \frac{1}{(N-1)\gamma_P}\, x_n(t). \tag{C-38}$$

Plugging the price impact function (C-38) into the optimization problem (C-27), trader n solves for his optimal consumption and demand schedule.

C.5. Conjectured Value Function

We conjecture and verify that the value function $V(M_n, S_n, D, \hat{H}_n, \hat{H}_{-n})$ has the specific quadratic exponential form

$$\begin{aligned} V(M_n, S_n, D, \hat{H}_n, \hat{H}_{-n}) = -\exp\Big(&\psi_0 + \psi_M M_n + \tfrac{1}{2}\psi_{SS} S_n^2 + \psi_{SD} S_n D \\ &+ \psi_{Sn}\, S_n \hat{H}_n + \psi_{Sx}\, S_n \hat{H}_{-n} + \tfrac{1}{2}\psi_{nn}\, (\hat{H}_n - \hat{H}_{-n})^2\Big). \end{aligned} \tag{C-39}$$

The seven constants ψ_0, ψ_M, ψ_{SS}, ψ_{SD}, ψ_{Sn}, ψ_{Sx}, and ψ_{nn} have values consistent with a steady-state equilibrium.

The term ψ_M measures the utility value of cash. The terms ψ_{SS}, ψ_{SD}, ψ_{Sn}, and ψ_{Sx} measure the utility value of risky asset holdings. The term ψ_{nn} captures the value of future trading opportunities based on current public and private information, as well as private values. The value of trading on innovations to future information is built into the constant term ψ_0.

The value function (C-39) for the model with private values has a simpler form than the value function (A-37) for the model with disagreement. In the model with private values, the value of future profit opportunities can be conveniently written as $\frac{1}{2}\psi_{nn}\, (\hat{H}_n - \hat{H}_{-n})^2$. In the model of disagreement, the value of future trading opportunities takes the more complicated form of a linear combination of separate terms $\hat{H}_n^2$, $\hat{H}_{-n}^2$, and $\hat{H}_n \hat{H}_{-n}$, with three different coefficients $\frac{1}{2}\psi_{nn}$, $\frac{1}{2}\psi_{xx}$, and ψ_{nx}. The intuition is that the price dampening effect due to the Keynesian beauty contest makes calculations of future profit opportunities more complicated in the model with disagreement

C.6. Characterization of Steady-State Symmetric Equilibrium with Linear Trading Strategies and Quadratic Value Functions

To solve for a steady-state equilibrium, it is necessary to determine simultaneously values for the four γ-parameters defining the optimal demand schedule in equation (C-36), the seven ψ-parameters defining the value function in equation (C-39), and the parameter k quantifying the weight on private signals in equation (C-9).

The solution to these equations is discussed in Appendix section C.9. We obtain the following theorem:

THEOREM 7: ***Characterization of Equilibrium.*** *There exists a steady-state, Bayesian-perfect equilibrium with symmetric, linear flow-strategies with positive trading volume if and only if the five polynomial equations (C-64)–(C-68) have a solution satisfying $\gamma_P > 0$ and $\gamma_S > 0$. Such an equilibrium has the following properties:*

1) *There is an endogenously determined constant $C_L := -\frac{\psi_{Sn}}{2\psi_{SS}} > 0$, such that trader n's optimal flow-strategy $x_n^*(t)$ makes time-differentiable inventories $S_n(t)$ change at rate*

$$x_n^*(t) = \frac{dS_n(t)}{dt} = \gamma_S \left(C_L \left(\hat{H}_n(t) - \hat{H}_{-n}(t) \right) - S_n(t) \right). \tag{C-40}$$

2) *The equilibrium price is*

$$P^*(t) = \frac{D(t)}{r + \alpha_D} + \frac{\bar{G}(t) + \sigma_G \Omega^{1/2} k \tau_I^{1/2} \frac{1}{N} \sum_{n=1}^{N} H_n^J(t)}{(r + \alpha_D)(r + \alpha_G)}, \tag{C-41}$$

where $\bar{G}(t)$ denotes the average of traders' expected growth rates,

(C-42)

$$\bar{G}(t) + \sigma_G \Omega^{1/2} k \tau_I^{1/2} \frac{1}{N} \sum_{n=1}^{N} H_n^J(t) := \sigma_G \Omega^{1/2} \frac{1}{N} \sum_{n=1}^{N} \left(\tau_I^{1/2} \hat{H}_n(t) + (N-1) \frac{1}{1+k^2} \tau_I^{1/2} \hat{H}_{-n}(t) \right).$$

Note there is always a trivial no-trade equilibrium. If each trader submits a no-trade demand schedule $X_n(t, .) \equiv 0$, then such a no-trade demand schedule is optimal for all traders. This is not a symmetric linear equilibrium in which an auctioneer can establish a meaningful market price.

Equations (C-40) and (C-41) imply that the equilibrium with trade has a surprisingly simple structure in which quantities adjust to new information slowly, while prices adjust instantaneously. Equation (C-40) is similar to equation (33) in the model with disagreement. It implies that each trader has a target inventory proportional to the difference between his own private signal $\hat{H}_n(t)$ and the average of other traders' private signals $\hat{H}_{-n}(t)$ inferred from prices; note that these private signals are sums of fundamental-information components and private-values components. Each trader continuously moves his inventory toward his target inventory so that the difference decays at rate γ_S.

The equation (C-41) is similar to the equation (34) in the model with disagreement. It implies that the price is a linear function of the weighted average of all traders' expected growth rates, adjusted by adding terms representing their private values. The equilibrium price can be also written as the precision-weighted average of the N composite signals $\hat{H}_n(t)$,

(C-43)

$$P^*(t) = \frac{D(t)}{r+\alpha_D} + \frac{\sigma_G\,\Omega^{1/2}}{(r+\alpha_D)(r+\alpha_G)}\,\frac{\tau_I^{1/2}\,(1+(N-1)/(1+k^2))}{N}\sum_{n=1,..N}\hat{H}_n(t).$$

The price responds instantaneously to innovations in each trader's private information and private value reflected in variables $\hat{H}_n(t) := \hat{H}_n^I(t) + kH_n^J(t)$, so that the average of all signals is immediately revealed. This occurs despite the fact that, to reduce trading costs resulting from adverse selection, each trader intentionally slows down his trading to reduce other traders' estimates of the magnitude of his private signal. Note also that equation (C-41) does not have a price dampening multiplier $C_G < 1$, unlike the model with disagreement.

Another difference from the model with disagreement is that the total precision τ in the information flow depends on the factor k, which is endogenously derived in equation (C-70).

Mathematical intuition and numerical calculations (as discussed below) suggest that the existence condition for the continuous-time model is the following:

CONJECTURE 2: ***Existence Condition.*** *An equilibrium with trade exists if and only if*

(C-44)
$$k^2 > \frac{N}{N-2},$$

Equation (C-44) implies that the existence condition is $1 + k^2 > 2 + \frac{2}{N-2}$, which is equivalent to the existence condition $\tau_H^{1/2}/\tau_L^{1/2} > 2 + \frac{2}{N-2}$ in (35) in our smooth trading model with disagreement. It is worth emphasizing that the weight k on private benefits in signals inferred from prices is endogenously determined in the model with private values, whereas $\tau_H^{1/2}/\tau_L^{1/2}$ is the ratio of exogenously specified parameters in the model with disagreement. It can be shown that k is approximately proportional to the coefficient on private benefits π_J, when π_J is large, as illustrated numerically in figure 11. When the private benefit of holding the risky asset is larger, all traders trade on it more intensely, this reduces the precision of other traders' information inferred from prices, and the total information revealed in prices (C-9) becomes smaller (because k increases).

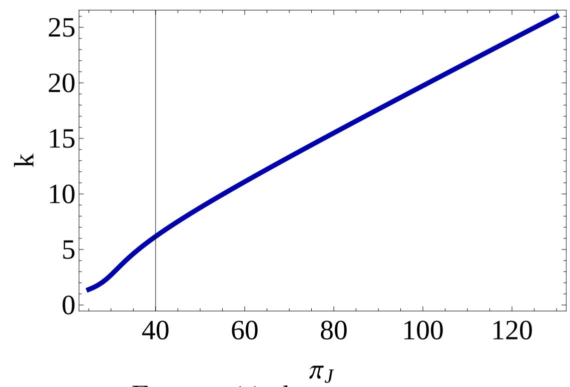


FIGURE 11. k AGAINST π_J.

The existence condition can be expressed in terms of exogenous parameters. Replacing k with the exogenous parameter π_J, it follows that an equilibrium with trade exists if and only if

$$\pi_J > \frac{N^{1/2}\,\sigma_G\,\Omega^{1/2}\,\tau_I^{1/2}}{(N-2)^{1/2}\,(r+\alpha_D)}\left(1+\frac{\tau}{r+\alpha_G}\right), \tag{C-45}$$

where

$$\Omega = \frac{\sigma_D^2}{2\sigma_G^2}\left(-\left(2\alpha_G+\frac{N}{2}\tau_I\right)+\left(\left(2\alpha_G+\frac{N}{2}\tau_I\right)^2+\frac{4\sigma_G^2}{\sigma_D^2}\right)^{1/2}\right). \tag{C-46}$$

Although we have not been able to prove analytically the conditions under which equilibrium exists, extensive numerical analysis supports the following intuitive argument. We expect equilibrium with trade to exist only if traders put enough weight on their private values. If π_J is very large (and thus k is very large), an equilibrium should exist. As π_J falls toward some critical value, the parameter γ_P—which measures the liquidity of the market—should fall to a value close to zero, the equilibrium should involve very little trade, and the value function should resemble a no-trade equilibrium. The value of k such that $\gamma_P = 0$ defines a "critical" value k^* such that equilibrium exists if and only if $k > k^*$.

This intuitive argument leads to a mathematically precise existence condition derived from the five equations in five unknowns (C-64)–(C-68) in Appendix section C.9. This equilibrium is derived by plugging $\gamma_P = 0$, representing the case with no market liquidity, into these equations. With $\gamma_P = 0$, it is clear that $\psi_{nn} = 0$ solves the last equation (C-68), consistent with the intuition that private information has no value if there is no market liquidity. It is also straightforward to show that a solution to the first four equations (C-64)–(C-67) requires the critical value k^* to satisfy $1+(k^*)^2 = 2+2/(N-2)$. We therefore conjecture that an equilibrium with trade, consistent with theorem 7, exists if and only if condition (C-44) holds.

Our extensive examination of numerical solutions to the five equations (C-64)–(C-68) supports this conjecture. We have found that precisely one solution with downward-sloping demand schedules ($\gamma_P > 0$) is discovered when the existence condition (C-44) is satisfied. Although k requires a numerical solution of (C-70), as shown in Appendix section C.9, we can solve for ψ_{sn}, ψ_{SS}, ψ_{nn}, and γ_P as closed-form functions of k. For the limiting case $\pi_J \to \infty$, we can also obtain a closed-form solution for all the endogenous parameters.

C.7. Conclusion

We describe a symmetric continuous-time model of trading among oligopolistic informed traders with asymmetric information and private values. This framework is tractable, and we obtain an "almost-closed-form" solution. We show that, with enough weight on the private value, an equilibrium exists in which prices immediately reveal the average of all traders' private signals (defined as the sum of fundamental signals and private-values multiplied by an endogenous factor k), but traders continue to trade gradually toward target inventories. In contrast to the model with overconfidence, prices do not reflect a "Keynesian beauty contest."

C.8. Proof of Lemma 2

Applying the Stratonovich-Kalman-Bucy filter to the filtering problem summarized by equation (C-2) for signals and by equations (C-3) and (C-4) for observations, we find that the filtering estimate is defined by the Itô differential equation

$$\text{(C-47)} \qquad \begin{aligned} dG(t) = -\alpha_G\, G(t)\, dt + \sigma_G \Omega^{1/2} \Big\{ & \tau_0^{1/2} \Big(dI_0(t) - G(t)\, \frac{\tau_0^{1/2}}{\sigma_G\, \Omega^{1/2}}\, dt \Big) \\ & + \tau_I^{1/2} \Big(dI_n(t) - G(t)\, \frac{\tau_I^{1/2}}{\sigma_G\, \Omega^{1/2}}\, dt \Big) \\ & + \frac{\tau_I^{1/2}}{1+k^2} \sum_{m \neq n} \Big(dI_m(t) - G(t)\, \frac{\tau_I^{1/2}}{\sigma_G\, \Omega^{1/2}}\, dt + k\, dB_{Jm} \Big) \Big\}. \end{aligned}$$

The mean-square filtering error of the estimate $G(t)$, denoted $\sigma_G^2\, \Omega(t)$, is defined by the Riccati differential equation

$$\text{(C-48)} \qquad \sigma_G^2\, \frac{d\Omega(t)}{dt} = -2\alpha_G\, \sigma_G^2\, \Omega(t) + \sigma_G^2 - \sigma_G^2\, \Omega(t) \Big(\tau_0 + \tau_I + \frac{\tau_I}{1+k^2} \Big).$$

Rearranging terms in the first equation yields equation (C-11). Using the steady-state assumption that $d\Omega/dt = 0$ and solving the second equation for the steady state value $\Omega = \Omega(t)$ yields equation (C-10).

C.9. Proof of Theorem 7

Suppressing a subscript n for notational simplicity, the HJB equation corresponding to the conjectured value function $V(M_n, S_n, D, \hat{H}_n, \hat{H}_{-n})$ in equation (C-27) is

(C-49)

$$\begin{aligned}
0 \;=\; & \max_{c_n, x_n} \left\{ U(c_n) - \rho V + \frac{\partial V}{\partial M_n}(r M_n + S_n(D + \pi_J H_n^J) - c_n - P(x_n)\, x_n) + \frac{\partial V}{\partial S_n} x_n \right\} \\
& + \frac{\partial V}{\partial D}\left(-\alpha_D D + \sigma_G \Omega^{1/2} \tau_I^{1/2} \left(\hat{H}_n^I + (N-1)/(1+k^2)\, \hat{H}_{-n}\right)\right) \\
& + \frac{\partial V}{\partial \hat{H}_n}\left(-(\alpha_G + \tau)\hat{H}_n(t) + (\tau_I^{1/2} + \hat{A}\tau_0^{1/2})\, \tau_I^{1/2}\, (\hat{H}_n^I + (N-1)/(1+k^2)\, \hat{H}_{-n})\right) \\
& + \frac{\partial V}{\partial \hat{H}_{-n}}\left(-(\alpha_G + \tau)\hat{H}_{-n}(t) + (\tau_I^{1/2} + \hat{A}\tau_0^{1/2})\, \tau_I^{1/2}\, (\hat{H}_n^I + (N-1)/(1+k^2)\, \hat{H}_{-n})\right) \\
& + \tfrac{1}{2}\frac{\partial^2 V}{\partial D^2}\sigma_D^2 + \tfrac{1}{2}\frac{\partial^2 V}{\partial \hat{H}_n^2}\left(1 + \hat{A}^2 + k^2\right) + \tfrac{1}{2}\frac{\partial^2 V}{\partial \hat{H}_{-n}^2}\left(\frac{1}{N-1}\,(1+k^2) + \hat{A}^2\right) \\
& + \left(\frac{\partial^2 V}{\partial D \partial \hat{H}_n} + \frac{\partial^2 V}{\partial D \partial \hat{H}_{-n}}\right)\hat{A}\,\sigma_D + \frac{\partial^2 V}{\partial \hat{H}_n \partial \hat{H}_{-n}}\hat{A}^2.
\end{aligned}$$

For the specific quadratic specification of the value function in equation (C-39), the HJB equation becomes

(C-50)

$$\begin{aligned}
0 \;=\; & \min_{c_n, x_n} \left\{ -\frac{\mathrm{e}^{-A c_n}}{V} - \rho + \psi_M(r M_n + S_n\,(D + \pi_J H_n^J) - c_n - P(x_n)\, x_n) \right. \\
& \qquad \left. + \,(\psi_{SS} S_n + \psi_{SD} D + \psi_{Sn}\hat{H}_n + \psi_{Sx}\hat{H}_{-n})\, x_n \right\} \\
& + \psi_{SD} S_n \left(-\alpha_D D + \sigma_G \Omega^{1/2} \tau_I^{1/2} \left(\hat{H}_n^I + (N-1)/(1+k^2)\, \hat{H}_{-n}\right)\right) \\
& + \left(\psi_{Sn} S_n + \psi_{nn}(\hat{H}_n - \hat{H}_{-n})\right) \\
& \qquad \left(-(\alpha_G + \tau)\hat{H}_n(t) + (\tau_I^{1/2} + \hat{A}\tau_0^{1/2})\, \tau_I^{1/2}\, (\hat{H}_n^I + (N-1)/(1+k^2)\, \hat{H}_{-n})\right) \\
& + \left(\psi_{Sx} S_n + \psi_{nn}(\hat{H}_{-n} - \hat{H}_n)\right) \\
& \qquad \left(-(\alpha_G + \tau)\hat{H}_{-n}(t) + (\tau_I^{1/2} + \hat{A}\tau_0^{1/2})\, \tau_I^{1/2}\, (\hat{H}_n^I + (N-1)/(1+k^2)\, \hat{H}_{-n})\right) \\
& + \tfrac{1}{2}\psi_{SD}^2 S_n^2 \sigma_D^2 + \tfrac{1}{2}\left((\psi_{Sn} S_n + \psi_{nn}(\hat{H}_n - \hat{H}_{-n}))^2 + \psi_{nn}\right)\left(1 + \hat{A}^2 + k^2\right) \\
& + \tfrac{1}{2}\left((\psi_{Sx} S_n + \psi_{nn}(\hat{H}_{-n} - \hat{H}_n))^2 + \psi_{nn}\right)\left(\frac{1}{N-1}\,(1+k^2) + \hat{A}^2\right) \\
& + (\psi_{Sn} + \psi_{Sx})\, S_n\, \psi_{SD}\, S_n\, \hat{A}\, \sigma_D \\
& + \left(\left(\psi_{Sn} S_n + \psi_{nn}(\hat{H}_n - \hat{H}_{-n})\right)\left(\psi_{Sx} S_n + \psi_{nn}(\hat{H}_{-n} - \hat{H}_n)\right) - \psi_{nn}\right)\hat{A}^2.
\end{aligned}$$

The solution for optimal consumption is

$$\text{(C-51)} \qquad c_n^*(t) = -\frac{1}{A}\,\log\Big(\frac{\psi_M\,V(t)}{A}\Big).$$

In the HJB equation (C-50), the price $P(x_n)$ is linear in x_n based on equation (C-38). Plugging $P(x_n)$ from equation (C-38) into the HJB equation (C-50) yields a quadratic function of x_n, which captures the effect of trader n's trading rate x_n on prices. Because the exponent of the conjectured value function is a quadratic function of the state variables, the optimal trading strategy is a linear function of the state variables given by

$$\text{(C-52)} \quad x_n^*(t) = \tfrac{(N-1)\gamma_P}{2\psi_M}\,\Big[\Big(\psi_{SD} - \tfrac{\psi_M\gamma_D}{\gamma_P}\Big)\,D(t) + \Big(\psi_{SS} - \tfrac{\psi_M\gamma_S}{(N-1)\gamma_P}\Big)\,S_n(t) + \psi_{Sn}\,\hat{H}_n(t) + \Big(\psi_{Sx} - \tfrac{\psi_M\gamma_H}{\gamma_P}\Big)\,\hat{H}_{-n}(t)\Big].$$

The derivation of this optimal trading strategy assumes that trader n observes the values of $D(t)$, $S_n(t)$, $\hat{H}_n(t)$, and $\hat{H}_{-n}(t)$. Although trader n does not actually observe $\hat{H}_{-n}(t)$, he can implement the optimal quantity $x_n^*(t)$ by submitting an appropriate linear demand schedule. We can think of this demand schedule as a linear function of $P(t)$ whose intercept is a linear function of $D(t)$, $S_n(t)$, and $\hat{H}_n(t)$. Trader n can infer from the market-clearing condition (C-37) that $\hat{H}_{-n}$ is given by

$$\text{(C-53)} \quad \hat{H}_{-n}(t) = \frac{\gamma_P}{\gamma_H}\,\Big(P(t) - D(t)\,\frac{\gamma_D}{\gamma_P}\Big) - \frac{1}{(N-1)\gamma_H}\,x_n^*(t) - \frac{\gamma_S}{(N-1)\gamma_H}\,S_n(t).$$

Plugging equation (C-53) into equation (C-52) and solving for $x_n^*(t)$ implements the optimal trading strategy $x_n^*(t)$ as a linear demand schedule which depends on the price $P(t)$ and state variables $\hat{H}_n$, $S_n(t)$, and $D(t)$, which the trader directly observes. This schedule is given by

$$\begin{aligned}\text{(C-54)}\, x_n^*(t) \quad = \quad & \frac{(N-1)\gamma_P}{\psi_M}\,\Big(1 + \frac{\psi_{Sx}}{\psi_M}\frac{\gamma_P}{\gamma_H}\Big)^{-1} \\ & \Big[\Big(\psi_{SD} - \psi_{Sx}\frac{\gamma_D}{\gamma_H}\Big)\,D(t) + \Big(\psi_{SS} - \psi_{Sx}\frac{\gamma_S}{(N-1)\gamma_H}\Big)\,S_n(t) \\ & + \psi_{Sn}\,\hat{H}_n(t) + \Big(\psi_{Sx}\frac{\gamma_P}{\gamma_H} - \psi_M\Big)\,P(t)\Big].\end{aligned}$$

Symmetry requires that this demand schedule be the same as the demand schedule conjectured for the $N-1$ other traders. Equating the coefficients of $D(t)$, $\hat{H}_n(t)$, $S_n(t)$, and $P(t)$ in equation (C-54) to the conjectured coefficients γ_D, γ_H, $-\gamma_S$, and $-\gamma_P$ results in the following four restrictions that the values of the γ-parameters and ψ-parameters must satisfy in a symmetric equilibrium with linear trading strategies:

$$\frac{(N-1)\gamma_P}{\psi_M}\left(1+\frac{\psi_{Sx}}{\psi_M}\frac{\gamma_P}{\gamma_H}\right)^{-1}\left(\psi_{SD}-\psi_{Sx}\frac{\gamma_D}{\gamma_H}\right)=\gamma_D, \tag{C-55}$$

$$\frac{(N-1)\gamma_P}{\psi_M}\left(1+\frac{\psi_{Sx}}{\psi_M}\frac{\gamma_P}{\gamma_H}\right)^{-1}\psi_{Sn}=\gamma_H, \tag{C-56}$$

$$\frac{(N-1)\gamma_P}{\psi_M}\left(1+\frac{\psi_{Sx}}{\psi_M}\frac{\gamma_P}{\gamma_H}\right)^{-1}\left(\psi_{SS}-\psi_{Sx}\frac{\gamma_S}{(N-1)\gamma_H}\right)=-\gamma_S, \tag{C-57}$$

$$\frac{(N-1)\gamma_P}{\psi_M}\left(1+\frac{\psi_{Sx}}{\psi_M}\frac{\gamma_P}{\gamma_H}\right)^{-1}\left(\psi_{Sx}\frac{\gamma_P}{\gamma_H}-\psi_M\right)=-\gamma_P. \tag{C-58}$$

Solving this system, we obtain four equations in terms of the four unknowns ψ_{Sx}, γ_H, γ_S, and γ_D. The solution is

(C-59)

$$\psi_{Sx}=\frac{N-2}{2}\psi_{Sn},\quad \gamma_H=\frac{N\gamma_P}{2\psi_M}\psi_{Sn},\quad \gamma_S=-\frac{(N-1)\gamma_P}{\psi_M}\psi_{SS},\quad \gamma_D=\frac{\gamma_P}{\psi_M}\psi_{SD}.$$

Plugging the last equation into equation (C-52) implies that traders will not trade on public information. It is intuitively obvious that traders cannot trade on the basis of the public information $D(t)$ because all traders would want to trade in the same direction. Substituting equation (C-59) into equation (C-52) yields the solution for optimal strategy.

$$x_n^*(t)=\gamma_S\left(C_L\left(\hat{H}_n(t)-\hat{H}_{-n}(t)\right)-S_n(t)\right). \tag{C-60}$$

The four equations for the γ-parameters do not determine γ_P as a function of the nine ψ-parameters. Instead, the solution to the four γ-equations implies a restriction on the ψ-parameters which must hold in a steady state-equilibrium.

Plug (C-51) and (C-52) back into the Bellman equation and set the constant term and the coefficients of M_n, $S_n\,D$, S_n^2, $S_n\,\hat{H}_n$, $S_n\,\hat{H}_{-n}$, and $(\hat{H}_n-\hat{H}_{-n})^2$ to be zero. In addition, set the coefficient of $S_n\,H_n^J$ equal to the coefficient of $S_n\,\hat{H}_n^I$ multiplying k so that the value function only depends on $\hat{H}_n^I$ and H_n^J through state variable $\hat{H}_n$. There are in total eight equations in eight unknowns γ_P, ψ_0, ψ_M, ψ_{SD}, ψ_{SS}, ψ_{Sn}, ψ_{nn}, and k.

Setting the constant term, coefficient of M, and coefficient of SD to be zero yields

$$\psi_M=-rA, \tag{C-61}$$

$$\psi_{SD} = -\frac{rA}{r+\alpha_D}, \tag{C-62}$$

$$\psi_0 = 1 - \log(r) + \frac{1}{r}\left(-\rho + \tfrac{1}{2}\,\frac{N}{N-1}\,(1+k^2)\,\psi_{nn}\right). \tag{C-63}$$

In addition, combining $S_n\,\hat{H}_n^I$ with $S_n\,H_n^J$ and setting the coefficients of S_n^2, $S_n\,\hat{H}_n$, $S_n\,\hat{H}_{-n}$, and $(\hat{H}_n - \hat{H}_{-n})^2$ to zero yields five polynomial equations in the five unknowns γ_P, ψ_{SS}, ψ_{Sn}, ψ_{nn}, and k. These five equations in five unknowns can be written

(C-64) $\underline{S_n^2}:$

$$\begin{aligned} 0 = &-\tfrac{1}{2}r\psi_{SS} - \frac{\gamma_P(N-1)}{rA}\psi_{SS}^2 + \frac{r^2A^2\sigma_D^2}{2(r+\alpha_D)^2} + \tfrac{1}{2}(1+\hat{A}^2+k^2)\psi_{Sn}^2 \\ &+\tfrac{1}{2}\left(\frac{1+k^2}{N-1} + \hat{A}^2\right)\frac{(N-2)^2}{4}\psi_{Sn}^2 - \frac{rA}{r+\alpha_D}\hat{A}\sigma_D\frac{N}{2}\psi_{Sn} + \hat{A}^2\frac{N-2}{2}\psi_{Sn}^2, \end{aligned}$$

(C-65) $\underline{S_nH_n^I, S_nH_n^J}:$

$$0 = -(r+\alpha_G+\tau)\psi_{Sn} - \frac{\gamma_P(N-1)}{rA}\psi_{SS}\psi_{Sn} - \frac{rA\pi_J}{k} + \frac{N(1+k^2)}{2(N-1)}\psi_{nn}\psi_{Sn},$$

(C-66) $\underline{S_n\hat{H}_n}:$

$$\begin{aligned} 0 = &-(r+\alpha_G+\tau)\psi_{Sn} - \frac{\gamma_P(N-1)}{rA}\psi_{SS}\psi_{Sn} - \frac{rA}{r+\alpha_D}\sigma_G\Omega^{1/2}\tau_I^{1/2} \\ &+\frac{N}{2}(\tau_I^{1/2} + \hat{A}\tau_0^{1/2})\tau_I^{1/2}\psi_{Sn} + (1+k^2)\frac{N}{2(N-1)}\psi_{nn}\psi_{Sn}, \end{aligned}$$

(C-67) $\underline{S_n\hat{H}_{-n}}:$

$$\begin{aligned} 0 = &-(r+\alpha_G+\tau)\frac{N-2}{2}\psi_{Sn} + \frac{\gamma_P(N-1)}{rA}\psi_{SS}\psi_{Sn} - \frac{N(1+k^2)}{2(N-1)}\psi_{Sn}\psi_{nn} \\ &-\frac{rA}{r+\alpha_D}\sigma_G\Omega^{1/2}(N-1)\frac{\tau_I^{1/2}}{(1+k^2)} + \frac{N}{2}(\tau_I^{1/2} + \hat{A}\tau_0^{1/2})\tau_I^{1/2}\frac{N-1}{1+k^2}\psi_{Sn}, \end{aligned}$$

(C-68) $\underline{(\hat{H}_n - \hat{H}_{-n})^2}:$

$$0 = -(\frac{r}{2} + \alpha_G + \tau)\psi_{nn} - \frac{\gamma_P(N-1)}{4rA}\psi_{Sn}^2 + \frac{1+k^2}{2}\frac{N}{N-1}\psi_{nn}^2.$$

We describe next how to solve the system (C-64) and (C-68). Equations (C-66) and (C-67) imply

$$\text{(C-69)} \qquad \psi_{Sn} = -\frac{2r\, A\, \sigma_G\, \Omega^{1/2}\, \tau_I^{1/2}\, (1 + (N-1)/(1+k^2))}{N\,(r+\alpha_D)\,(r+\alpha_G)}.$$

Equations (C-65) and (C-66) imply that the constant k is given by

$$\text{(C-70)} \qquad k = \frac{(r+\alpha_D)\,\pi_J}{\sigma_G\,\Omega^{1/2}\,\tau_I^{1/2}\,\left(1+\frac{\tau}{r+\alpha_G}\right)}.$$

From (C-65), solve for ψ_{SS} as a function of γ_P and ψ_{nn} to obtain

$$\text{(C-71)} \quad \psi_{SS} = \frac{rA}{\gamma_P\,(N-1)} \left(\frac{N(1+k^2)\,\psi_{nn}}{2(N-1)} - (r+\alpha_G+\tau)\left(1 - \frac{N(1+k^2)}{2(N+k^2)}\right)\right).$$

From (C-68), solve for γ_P as a function of ψ_{nn} to obtain
(C-72)

$$\gamma_P = \frac{N^2\,(r+\alpha_D)^2\,(r+\alpha_G)^2}{(N-1)\,rA\,\sigma_G^2\,\Omega\,\tau_I\,(1+\frac{N-1}{1+k^2})^2} \left(\frac{N}{N-1}\,\frac{1+k^2}{2}\,\psi_{nn}^2 - (\tfrac{1}{2}r+\alpha_G+\tau)\,\psi_{nn}\right).$$

Then substitute both γ_P and ψ_{nn} into (C-64) to obtain a quadratic equation for ψ_{nn}. This equation has two real roots. Take the negative root, which implies private signals have positive value, to obtain

$$\text{(C-73)} \qquad \psi_{nn} = \frac{-b-(b^2-4ac)^{1/2}}{2a},$$

where

$$a := \left(\tfrac{1}{2}\sigma_D^2 + \frac{\hat{A}\,\sigma_D\,\sigma_G\,\Omega^{1/2}\,\tau_I^{1/2}\,(1+\frac{N-1}{1+k^2})}{r+\alpha_G}\right) \frac{N^3(r+\alpha_G)^2(1+k^2)}{2(N-1)\,\sigma_G^2\,\Omega\,\tau_I\,(1+\frac{N-1}{1+k^2})^2} + \frac{N^2\,(1+k^2)\,(1+k^2+N\,\hat{A}^2)}{4\,(N-1)},$$

$$b := \left(\tfrac{1}{2}\sigma_D^2 + \frac{\hat{A}\,\sigma_D\,\sigma_G\,\Omega^{1/2}\,\tau_I^{1/2}\,(1+\frac{N-1}{1+k^2})}{r+\alpha_G}\right) \frac{N^2\,(r+\alpha_G)^2\,(\frac{1}{2}r+\alpha_G+\tau)}{\sigma_G^2\,\Omega\,\tau_I\,(1+\frac{N-1}{1+k^2})^2} - \frac{N^2\,(1+k^2+(N-1)\,\hat{A}^2)}{2\,(N-1)}\,(\tfrac{1}{2}r+\alpha_G+\tau) - \frac{rN\,(1+k^2)}{4\,(N-1)} + \frac{N\,(1+k^2)\,(r+\alpha_G+\tau)}{N-1}\,\frac{N+(2-N)\,k^2}{2\,(N+k^2)},$$

$$c := \tfrac{1}{2}r\,(r+\alpha_G+\tau)\,\frac{N+(2-N)\,k^2}{2\,(N+k^2)} - (r+\alpha_G+\tau)^2\,\frac{(N+(2-N)\,k^2)^2}{4\,(N+k^2)^2}.$$

Substituting ψ_{nn} into (C-71) and (C-72) yields solutions for γ_P and ψ_{SS}.

To summarize, even though k is determined numerically from equation (C-70), since the total precision τ itself in that equation depends on k, other unknowns can be written as explicit functions of k. When π_J and thus k are very large, k is approximately proportional to π_J, with

$$\text{(C-74)} \qquad k \approx \frac{r + \alpha_D}{\sigma_G \Omega^{1/2} \tau_I^{1/2} (1 + \frac{\tau_0 + \tau_I}{r + \alpha_G})} \, \pi_J;$$

this gives a closed-form solution when $\pi_J \to \infty$.

The transversality condition is equivalent to $r > 0$: The HJB equation and equations (C-64)–(C-68) imply

$$\text{(C-75)} \qquad \mathrm{E}_t^n\{dV(M_n(t), S_n(t), D(t), \hat{H}_n(t), \hat{H}_{-n}(t))\} = \\ -(r - \rho)\, V(M_n(t), S_n(t), D(t), \hat{H}_n(t), \hat{H}_{-n}(t))\, dt.$$

This yields

$$\text{(C-76)} \qquad \mathrm{E}_t^n\{\mathrm{e}^{-\rho(T-t)}\, V(M_n(T), S_n(T), D(T), \hat{H}_n(T), \hat{H}_{-n}(T))\} = \\ \mathrm{e}^{-r(T-t)}\; V(M_n(t), S_n(t), D(t), \hat{H}_n(t), \hat{H}_{-n}(t)),$$

which implies that the transversality condition (C-35) is indeed satisfied if $r > 0$.