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Scalarization Methods and Expected Multi-Utility Representations

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Abstract

I characterize the class of (possibly incomplete) preference relations over lotteries which can be represented by a compact set of (continuous) expected utility functions that preserve both indifferences and strict preferences. This finding contrasts with the representation theorem of Dubra, Maccheroni and Ok (2004) which typically delivers some functions which do not respect strict preferences. For a preference relation of the sort that I consider in this paper, my representation theorem reduces the problem of recovering the associated choice correspondence over convex sets of lotteries to a scalar-valued, parametric optimization exercise. By utilizing this scalarization method, I also provide characterizations of some solution concepts. Most notably, I show that in an otherwise standard game with incomplete preferences, the collection of *pure* strategy equilibria that one can find using this scalarization method corresponds to a refinement of the notion of Nash equilibrium that requires the (deterministic) action of each player be undominated by any *mixed* strategy that she can follow, given others' actions.

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1. Introduction

Starting with Aumann (1962), early research on representation of incomplete preference relations under risk explored sufficient conditions that allow one to extend a preference relation by a *single* expected utility function. Put precisely, given a (possibly incomplete) preference relation $\succsim$ over the set of lotteries on a prize space X, the purpose of a typical work in this early literature is to find a von Neumann-Morgenstern function u such that

$$p \succ q \text{ implies } \int_X u\,dp > \int_X u\,dq, \quad \text{and,} \quad p \sim q \text{ implies } \int_X u\,dp = \int_X u\,dq. \tag{1}$$

As also noted by Aumann (1962, p. 448), the main merit of this representation notion is that maximization of such an expected utility function over a set will always lead to a maximal lottery in that set.[1] Thereby, in every choice set, we can identify *a* lottery that the decision maker in question can possibly select from that set.

However, when studying economic phenomena related to indecisiveness, the researcher often needs to recover the choice correspondence induced by an incomplete preference relation in its entirety. Indeed, the best-known behavioral consequences of indecisiveness include (i) a certain degree of randomness in choices, which, as Mandler (2005) notes, may reflect itself with intransitivity of observed choice behavior; and (ii) the multiplicity of alternatives that might be chosen in a given situation, which is the focus of Rigotti and Shannon (2005) in their work on indeterminacy of equilibria in security markets. The study of how an agent may or should resolve her indecisiveness is a related area of research.[2] Moreover, it has been recently observed that a variety of interesting behavioral phenomena can be explained by two-stage choice procedures where in the first stage the agent identifies a *collection* of maximal alternatives in a given choice set (with respect to an endogenously determined incomplete preference relation), and then makes her final choice among these maximal alternatives according to a secondary criterion.[3]

[1] In fact, only the first part of property (1) is crucial for this conclusion.

[2] For example, Ok, Ortoleva and Riella (2011) propose a model in which the choice between two incomparable alternatives, say x and y, depends on other options in a certain way: the presence of a third alternative z that is asymmetrically dominated by x or y increases the decision maker's tendency to choose the dominating alternative. In turn, Danan (2010) studies the problem of "how to choose in the absence of preference" from a normative point of view.

[3] Various reference-dependent choice models, for instance, *necessitate* the use of incomplete preferences in such a procedural context (Masatlioglu and Ok, 2005; Apesteguia and Ballester, 2009). Another example is the procedural model of Manzini and Mariotti (2007) that accounts for intransitive choice behavior. A longer list of indecisiveness-related phenomena includes preference for flexibility and choice deferral (Danan and Ziegelmeyer, 2006; Kopylov, 2009), preference for commitment (Danan, Guerdjikova and Zimper, 2012), and several implications for political games (Roemer, 1999; Levy, 2004).

The problem of recovering the choice correspondence induced by an incomplete preference relation gave rise to the literature on multi-utility representations which provide a set of utility functions that fully characterize a given preference relation. In fact, it seems fair to argue that the virtue of such a representation theorem lies in its potential use as an analytical tool that can facilitate the exercise of identifying the choice correspondence associated with a preference relation which satisfies certain behavioral axioms. The performance of a representation theorem in this regard depends, in turn, on the properties of the set of utility functions that it delivers.

The main finding of the present paper is an expected multi-utility representation theorem that delivers a compact and convex set U of von Neumann-Morgenstern functions each satisfying the property (1) (see Theorem 3 below).[4] Given a preference relation that admits such a set U, by a well-known "theorem of alternative," one can show that an element of a *convex* set K of lotteries is maximal in K if, and only if, it maximizes over K the expectation of a function in U (see Proposition 1). Thus, for a preference relation of the sort that I consider, my representation theorem reduces the problem of recovering the associated choice correspondence over convex sets of lotteries to a *scalar-valued*, parametric optimization exercise. In turn, when applied to a choice problem with a non-convex set of lotteries, (in the absence of the completeness axiom) this scalarization method characterizes a mode of choice behavior that corresponds to a refinement of the traditional definition of "rationalizability" based on binary comparisons. (More on this below.)

The axioms in my representation theorem are quite weak. If the strict upper and lower contour sets associated with the preference relation are open, standard independence properties and a further continuity axiom on the symmetric part of the preference relation imply the representation.

By now, there is a sizable literature on multi-utility representations of preference relations. My main result is most closely related to the expected-multi utility representation of Dubra, Maccheroni and Ok (2004) (henceforth, DMO). While both models focus on the same structural framework, my representation theorem is logically distinct from theirs because, unlike DMO, I do not assume that the preference relation is closed. In fact, a preference relation that can be represented as in my theorem *cannot* be closed unless its strict and/or incomplete part are empty. Put differently, a set of von Neumann-Morgenstern functions that represents a preference relation $\succsim$ in the sense of DMO will, typically, be either non-compact or contain at least one function u such that

[4]Throughout the paper, I assume that the prize space X is a compact metric space. In turn, the set U delivered by my representation consists of continuous functions on X, while compactness of U refers to sup-norm.

$\int_X u\,dp = \int_X u\,dq$ for a pair of lotteries p, q with $p \succ q$ (see Observation 2 in Section 3). This, in turn, implies that under the axioms of DMO, the aforementioned scalarization method will often fail to recover the associated choice correspondence. Two dual difficulties jointly drive this conclusion. First, if a function u does not satisfy the first part of property (1), maximization of the expectation of u over a choice set may deliver non-maximal lotteries. Second, if the set of utility functions in question is non-compact, there may be maximal lotteries which do not maximize the expectation of any of these functions. (Related examples can be found in Section 3 and Appendix A.)

Given a preference relation of the type considered by DMO, their representation theorem transforms the problem of identifying the induced choice correspondence to a *vector-valued* optimization exercise that is equivalent to the problem of finding the "Pareto-frontier" of a utility possibility set. Moreover, this "utility possibility set" that one has to deal with often consists of *infinite dimensional* utility vectors even when there are only finitely many riskless prizes. While attacking such a problem directly would typically seem to be an extremely elusive exercise, in fact, even in social choice problems with a finite dimensional utility possibility set, the classical methods of identifying the set of Pareto optimal allocations also build upon scalarization techniques. For example, Mas-Colell, Whinston and Green (1995) suggest two methods within classical consumer theory. The first method is simply the scalarization method that I discussed above, applied to classical consumer theory: one maximizes a weighted average of consumers' utility functions over the set of available allocations (Mas-Colell et al., 1995, p. 560). The second method is to maximize the utility function of a particular consumer while keeping constant the values of the utility functions of all other consumers (Mas-Colell et al., 1995, p. 562).

It should be noted, however, that the problem of finding Pareto optimal allocations in classical consumer theory has many special features, which improve the performance of scalarization methods in that particular set-up. For example, if consumers' utility functions are strictly concave, the associated utility possibility set becomes strictly convex. This, in turn, implies that an allocation that maximizes a weighted average of consumers' utility functions is necessarily Pareto optimal. This conclusion holds even if the objective function assigns zero weight to some of the utility functions, and despite the fact that such an objective function would not be strictly increasing with respect to the Pareto order. On the other hand, in decision problems under risk, there seems to be no reason to restrict our attention to strictly convex "utility possibility sets," for an expected utility function is linear in lotteries. This is one of the difficulties that underlie the weaker performance of DMO approach with respect to the first method of scalarization. In turn, when applied to decision problems under risk, the second method is also of limited use, for controlling the values of some of the utility functions that represent the preference relation in question may very well lead to a constrained optimization problem that is not compatible with the method of Lagrange multipliers. (More on this and related difficulties in Section 3 and Appendix B.)

In view of these remarks, compared with DMO approach, my representation theorem seems to be more suitable for the standard tools of economists. While the present paper is mainly motivated by this tractability concern, it is also possible to draw a conceptual line between my representation and that of DMO. More specifically, my approach can be seen as a *multi-self* representation of a decision maker in the sense that there is a one to one correspondence between the utility functions that my representation delivers and different patterns of choice behavior that the decision maker might actually follow (at least, if the set of feasible lotteries is convex). By contrast, the behavior of an agent who can be described à la DMO is analogous to that of a *coalition of distinct individuals* who respect the Pareto rule. (Naturally, in both cases the corresponding multi-person interpretation refers to a set of agents who respect the completeness axiom.)

In this paper, I also utilize the first method of scalarization that I discussed above to provide characterizations of some solution concepts in individual choice theory, game theory and social choice theory. Most notably, in a normal-form game in which the players' preferences satisfy the hypotheses of my representation theorem, the collection of *pure* strategy equilibria that one can find using this scalarization method corresponds to a refinement of the notion of Nash equilibrium that requires the (deterministic) action of each player be undominated by any *mixed* strategy that she can follow, given others' actions. An analogous notion of "rationalizability" in individual choice problems has been suggested by Heller (2012) in a concurrent work, which builds upon the findings of the present paper.

1.1 Literature Review

Bewley's (1986) seminal work in the Anscombe-Aumann framework also focuses on preference relations with open strict-contour sets.[5] Though Bewley's original approach proved particularly useful in applications (see, e.g., Rigotti and Shannon, 2005), in the subsequent theoretical work scholars' attention shifted to closed preorders. Such works include Ghirardato, Maccheroni, Marinacci and Siniscalchi (2003) in a Savagean framework; Gilboa, Maccheroni, Marinacci and Schmeidler (2010), and Ok, Ortoleva, and Riella (forthcoming) in the Anscombe-Aumann framework; Evren and Ok (2011) in

[5]Bewley is concerned with "indecisiveness in beliefs," as opposed to "indecisiveness in tastes," which is the subject of the present paper.

the ordinal framework; and DMO type representations of Baucells and Shapley (2008), and Evren (2008).

To the best of my knowledge, for decision problems under risk, Manzini and Mariotti (2008) and Galaabaatar and Karni (forthcoming a) are the only other papers concerned with characterization of preference relations with open strict-contour sets. The representation theorem of Manzini and Mariotti is based on utility intervals, instead of a set of utility functions. Their approach requires an independence axiom (called Non-Comparability Sure Thing) on the incomplete part of the preference relation, which is not suitable for expected multi-utility representations. A particular implication of this axiom is that there cannot exist pairwise incomparable lotteries p, q, r such that $\frac{1}{2}p + \frac{1}{2}q \succ r$. While I do allow for such pattern of preference, the refined notion of "rationalizability" that I discuss in Section 7 rules out the *choice* of r among such three lotteries.[6] Furthermore, the representation of Manzini and Mariotti implies that for any p, q, r with $p \succ q$, the independence property $\alpha r + (1-\alpha)p \succ \alpha r + (1-\alpha)q$ will *typically fail* for large $\alpha \in (0, 1)$.

In turn, Galaabaatar and Karni (forthcoming a) is a concurrent paper that is more closely related to my approach. In fact, their representation of a strict preference relation coincides with mine, except that their theorem focuses on a finite dimensional mixture space.[7] The distinctive feature of Galaabaatar and Karni is that the asymmetric part of a (weak) preference relation in their sense does not coincide with the strict preference relation, which is the primitive object in their model. In the present paper, following the traditional approach, I do not make such a distinction. A more detailed discussion of Galaabaatar and Karni can be found in Section 5.1 and Appendix E.

2. Notation and Terminology

Given a compact metric space Y, I denote by $\mathbf{C}(Y)$ the space of continuous, real functions on Y endowed with the sup-norm $\|\cdot\|_\infty$. In turn, $\Delta(Y)$ stands for the set of all (Borel) probability measures on Y. I write $E(p, u)$ for the expectation of $u \in \mathbf{C}(Y)$ with respect to $p \in \Delta(Y)$; that is, $E(p, u) := \int_Y u\, dp$. As usual, $\Delta(Y)$ is equipped with the topology of weak convergence: a sequence (p_n) in $\Delta(Y)$ converges to $p \in \Delta(Y)$ iff $E(p_n, u) \to E(p, u)$ for every $u \in \mathbf{C}(Y)$.

[6] It may be useful to note that in the absence of the completeness axiom, there are alternative ways of relating the choice behavior of the decision maker to her "psychological" preference relation, each giving rise to a particular method to recover preferences from the the observed choice data (see, e.g., Eliaz and Ok, 2006; and Heller, 2012).

[7] The results of the two papers are independent, and the proof techniques are quite distinct. It is also worth noting a companion paper of Galaabaatar and Karni (forthcoming b), which provides more refined versions of their representation in the Anscombe-Aumann framework.

Following the standard conventions, by a **binary relation** $\mathcal{R}$ on a set $\mathfrak{A}$ I mean a subset of $\mathfrak{A}^2$, and write $a\mathcal{R}b$ instead of $(a,b) \in \mathcal{R}$. If $\mathfrak{A}$ is a topological space, when I say that $\mathcal{R}$ is **closed** or **open**, I will be referring to the product topology on $\mathfrak{A}^2$. A **preorder** refers to a reflexive and transitive binary relation, which is said to be a **partial order** if it is also antisymmetric. If $\mathcal{R}$ is a preorder on $\mathfrak{A}$, given any $K \subseteq \mathfrak{A}$, I say that a point $a \in K$ is **$\mathcal{R}$-maximal** in K if there does not exist $b \in K$ such that $b\mathcal{R}a$ and not $a\mathcal{R}b$.

Throughout the paper, X stands for a compact metric space of riskless prizes, and $\Delta(X)$ for the set of lotteries. In some parts of the paper, I take as primitive a preorder $\succsim$ on $\Delta(X)$, which represents the (weak) **preference relation** of a decision maker. When I follow this approach, I denote by $\succ$ and $\sim$ the **asymmetric** and **symmetric** parts of $\succsim$, respectively, defined as usual: $p \succ q$ iff $p \succsim q$ and not $q \succsim p$, while $p \sim q$ iff $p \succsim q$ and $q \succsim p$. The **incomplete part** of $\succsim$, denoted $\bowtie$, is defined by $p \bowtie q$ iff neither $p \succsim q$ nor $q \succsim p$. When $p \bowtie q$, I say that p and q are **$\succsim$-incomparable**, meaning that the decision maker is indecisive between p and q. The preference relation $\succsim$ is said to be **complete** if $\bowtie = \emptyset$, and **incomplete** otherwise. In turn, I say that $\succsim$ is **nontrivial** if $p \succ q$ for some p, q in $\Delta(X)$, and **trivial** otherwise. The **open-continuity property** refers to the requirement that the sets $\{p \in \Delta(X) : p \succ q\}$ and $\{p \in \Delta(X) : q \succ p\}$ be open in $\Delta(X)$ for each $q \in \Delta(X)$.

Given a preorder $\succsim$ on $\Delta(X)$ and a function $u \in \mathbf{C}(X)$, by a slight abuse of terminology, I will say that u is **strictly $\succsim$-increasing** if the associated expectation operator is strictly $\succsim$-increasing, meaning that $E(p,u) > E(q,u)$ whenever $p \succ q$. Similarly, when I say that u is **indifference preserving** I mean that $E(p,u) = E(q,u)$ whenever $p \sim q$. If u is both strictly $\succsim$-increasing and indifference preserving, I will refer to it as an **Aumann utility** (for $\succsim$).

Finally, for a set $K \subseteq \Delta(X)$, I denote by $\mathcal{M}(\succsim, K)$ the set of $\succsim$-maximal elements of K.

3. Scalarization Methods and Representation Notions

Using the terminology that I have just introduced, Aumann's (1962) representation notion consists of a single "Aumann utility" for a given preference relation $\succsim$ on $\Delta(X)$. As I noted earlier, the appeal of this representation notion mainly stems from the fact that a lottery which maximizes the expectation of a strictly $\succsim$-increasing function over a set of lotteries is guaranteed to be a $\succsim$-maximal element of that set.

On the other hand, the exercise of finding a single Aumann utility for a preference relation is of limited use, for such a function simply *extends* the relation in question to a complete preorder, but does not *characterize* it. In particular, this approach ceases to be useful when one wishes to understand among which sorts of alternatives the decision maker in question is indecisive, or to determine the associated choice correspondence in its entirety.

To overcome this difficulty, DMO identified necessary-sufficient conditions which allow one to find a set of functions $U \subseteq \mathbf{C}(X)$ such that, for every p, q in $\Delta(X)$,

$$p \succsim q \quad \text{if and only if} \quad E(p, u) \geq E(q, u) \text{ for every } u \in U. \tag{2}$$

Note that in this representation, the set U is allowed to contain functions that are not strictly $\succsim$-increasing. More precisely, for some $u \in U$ and $p, q \in \Delta(X)$, we may have $E(p, u) = E(q, u)$ while $p \succ q$.

When viewed as an analytical tool, the representation notion of DMO transforms the problem of preference maximization to a vector-valued optimization exercise. Specifically, given $\succsim$ and U as above, an element p of a set $K \subseteq \Delta(X)$ is $\succsim$-maximal in K if and only if the utility vector $(E(p, u))_{u \in U}$ is a $\geq$-maximal element of the set

$$\left\{(E(q, u))_{u \in U} : q \in K\right\},$$

where $\geq$ stands for the usual partial order on $\mathbb{R}^U$.

The set U in (2) may well be infinite, even when the prize space X is finite. In such cases, attacking this sort of a vector-valued optimization problem directly can be extremely tedious. To understand how elusive such an exercise can be, it would suffice here to note that such a problem is similar to identifying the (strong) Pareto-frontier of a utility possibility set in a social choice problem that involves infinitely many agents.

In fact, even in optimization problems with *finitely* many objective functions, often it proves very useful to transform the problem at hand to a suitable scalar-valued optimization exercise, instead of attacking it directly. As I mentioned earlier, a remarkable example of such techniques in classical consumer theory is to transform the problem of finding the set of Pareto optimal allocations into the problem of maximizing the utility function of a particular consumer while keeping constant the values of the utility functions of all other consumers (Mas-Colell, Whinston and Green, 1995, p. 562). Then, one uses the method of Lagrange multipliers to solve the transformed, scalar-valued problem. On the other hand, one of the special features of this classical problem is that if there are n consumers, there exists at least n variables, each corresponding to the consumption of a given consumer, and (typically) n active constraints, one for economic feasibility and $n-1$ for controlled utility functions. Moreover, the consumers are assumed to care only about their own consumption, while the feasibility constraint depends on all variables.

Thereby, one ensures that the derivatives of active constraints are linearly independent.

As in this classical problem, if the preference relation $\succsim$ admits a DMO type representation, we can think of transforming the problem of finding $\succsim$-maximal elements of a set K into a scalar-valued, constrained optimization exercise (see Lemma 2 in Appendix B). However, in this case, the method of Lagrange multipliers may cease to be useful. The trouble is that a convex combination of some functions in a representing set U would typically be collinear with the normal vector of K at a given $\succsim$-maximal lottery p (provided that the boundary of K is smooth at p). In turn, the utility functions that appear in this convex combination may well be a subset of the active constraints. This would directly violate the classical constraint qualification once we describe the boundary of K with suitable constraint functions. (More on this and related issues in Appendix B.)

Therefore, one would often find it much easier to maximize a single expected utility function over a predetermined set K, without further constraints. For example, in such a problem, the method of Lagrange multipliers would be readily applicable if X is finite, and K is a convex set that contains an interior point relative to $\Delta(X)$ and that can be expressed with finitely many, continuously differentiable inequalities.

In view of these remarks, a natural question that follows is when we can represent a preference relation $\succsim$ by a set $U \subseteq \mathbf{C}(X)$ such that (i) for each $u \in U$, maximization of $E(\cdot, u)$ over a set of lotteries K delivers a $\succsim$-maximal element of K; (ii) by varying u in U, we can recover all $\succsim$-maximal elements of K; so that

$$\mathcal{M}(\succsim, K) = \bigcup_{u \in U} \arg\max_{r \in K} E(r, u). \tag{3}$$

In what follows, with a slight abuse of terminology, I will refer to this method of recovering $\mathcal{M}(\succsim, K)$ as **unconstrained scalarization method**.

The main finding of the present paper is an expected multi-utility representation theorem that is consistent with the equality (3) whenever K is a convex subset of $\Delta(X)$ (Theorem 3 below). This result characterizes the class of preference relations $\succsim$ that can be represented by a compact and convex set U of Aumann utilities as follows: For every p, q in $\Delta(X)$,

$$\begin{aligned} p \succ q \quad &\text{if and only if} \quad E(p, u) > E(q, u) \text{ for every } u \in U, \\ p \sim q \quad &\text{if and only if} \quad E(p, u) = E(q, u) \text{ for every } u \in U. \end{aligned} \tag{4}$$

This representation notion has two key features that make it compatible with the unconstrained scalarization method: (a) each u in U is strictly $\succsim$-increasing; (b) the set

U is compact.

To see the importance of the point (a), let $u \in \mathbf{C}(X)$ and $p, q \in \Delta(X)$ be such that $E(p,u) = E(q,u)$ while $p \succ q$. Also assume that the usual independence property holds so that $\alpha p + (1-\alpha)r \succ \alpha q + (1-\alpha)r$ for every $\alpha \in (0,1)$ and $r \in \Delta(X)$. Then, the lottery p is the only $\succsim$-maximal element of the interval $K := \{\alpha p + (1-\alpha)q : \alpha \in [0,1]\}$, while $\arg\max_{r \in K} E(r,u) = K$. Thus, for any set U that contains u, the left side of (3) is a proper subset of its right side.

In turn, lack of compactness of U leads to the converse problem: the right side of (3) may not contain its left side. In Appendix A, to demonstrate this point, I will prove the following observation by means of an example.

Observation 1. *Let $\succsim$ be a preorder on $\Delta(X)$ which admits a convex but non-compact set $U \subseteq \mathbf{C}(X)$ that represents $\succsim$ as in (2) or (4). Then:*

(i) *$\mathcal{M}(\succsim, \Delta(X))$ may contain lotteries which do not maximize the expectation of any strictly $\succsim$-increasing function in $\mathbf{C}(X)$.*[8]

(ii) *In fact, with $X := [0,1]$, such lotteries in $\mathcal{M}(\succsim, \Delta(X))$ may even be a dense subset of $\Delta(X)$, while $\mathcal{M}(\succsim, \Delta(X))$ is a proper subset of $\Delta(X)$.*

While the set of utility functions in the representation theorem of DMO may be non-compact, as a by product of my main representation theorem, in Appendix D, I provide an axiomatic characterization of a DMO type representation with a compact set of utility functions. However, this refinement only makes more transparent the other difficulty of DMO approach: a compact set of utility functions that represents a preorder $\succsim$ in the sense of DMO *necessarily* contains some functions which are not strictly $\succsim$-increasing, unless the preorder is trivial or complete. This is the content of the next observation.

Observation 2. *If U and $\succsim$ satisfy (2) for every p, q in $\Delta(X)$, and if U is a compact subset of $\mathbf{C}(X)$ that consists of strictly $\succsim$-increasing functions, then the preorder $\succsim$ is either complete or trivial.*[9]

I prove this observation in Appendix C. The proof builds upon Schmeidler's (1971) theorem which shows that on a connected set, a nontrivial preorder that satisfies the open-continuity property cannot be closed unless it is actually complete. Indeed, if $\succsim$ admits a DMO type representation, it must be closed. Moreover, if the set U in (2)

[8]Aumann (1962, 1964) shows that this conclusion cannot hold for a polyhedral set K of lotteries. Consequently, the set X in Observation 1(i) must be infinite, but even when X is finite, the conclusion of the observation may indeed hold for a non-polyhedral set $K \subseteq \Delta(X)$. (See Example 3 in Appendix A.)

[9]In fact, the same conclusion obtains even if U is weakly-compact. (The proof is available upon request.)

consists of strictly $\succsim$-increasing functions, for each p, q in $\Delta(X)$ we must in fact have

$$p \succ q \quad \text{if and only if} \quad E(p,u) > E(q,u) \text{ for every } u \in U.$$

If the set U is also compact, this characterization of $\succ$ readily implies that $\succsim$ also satisfies the open-continuity property, making it subject to Schmeidler's theorem.

Finally, it should be noted that in the literature on multi-objective optimization, it is a well-known problem that Pareto type orders (as in the set-up of DMO) are, in general, incompatible with the unconstrained scalarization method. As a partial remedy, scholars have sought conditions under which one can recover a dense set of maximal elements by maximizing functions that are strictly increasing with respect to the preorder in question. A classical result of this sort is the density theorem of Arrow, Barankin and Blackwell (1953), which focuses on the usual order of an Euclidean space. More recently, Makarov and Rachovski (1996) have proved a more general density result for a partial order in a topological vector space.

When applied to DMO type preorders, this density-based approximation method brings about two difficulties. First, after finding a dense set of maximal lotteries, say $\mathcal{M}_0$, it is not clear at all how to recover the whole set of maximal lotteries. In particular, the set $\mathcal{M}(\succsim, K)$ may not be closed, even if K is compact and convex. (While Observation 1(ii) already demonstrates this point, one can also provide finite dimensional examples in the same direction, along the lines of Arrow et al. (1953, Section 3).) Consequently, applying the closure operator to $\mathcal{M}_0$ may deliver non-maximal lotteries. In fact, for a preference relation $\succsim$ as in Observation 1(ii), the closure of a dense subset of $\mathcal{M}(\succsim, \Delta(X))$ is simply the entire space $\Delta(X)$, although a plethora of lotteries may be non-maximal (see Lemma 1 in Appendix A). The second problem is that if we stop searching for further elements of $\mathcal{M}(\succsim, K)$ upon recovering a dense subset $\mathcal{M}_0$, we may as well be leaving unidentified a large set of maximal lotteries in K. In particular, the set $\mathcal{M}(\succsim, K)\backslash\mathcal{M}_0$ may also be a dense subset of K as in Observation 1(ii).

4. Representation of Strict Preference Relations

In this section, I focus on a transitive and irreflexive binary relation $\succ$ on $\Delta(X)$, which I think of as a **strict preference relation** of a decision maker.[10] The main finding of this section is a representation result (Theorem 1 below) that serves as the main building block of my representation of preorders in the following section.

[10]When a strict preference relation $\succ$ is taken as primitive, incompleteness of the weak preference relation of the decision maker can be deduced from the lack of negative-transitivity of $\succ$.

I say that $\succ$ is an **open-continuous strict preference relation** if it satisfies the following axioms.

Open-Continuity. The sets $\{p \in \Delta(X) : p \succ q\}$ and $\{p \in \Delta(X) : q \succ p\}$ are open in $\Delta(X)$, for each q in $\Delta(X)$.

Independence. For every p, q, r in $\Delta(X)$ and $\alpha \in (0, 1)$,

$$p \succ q \quad \text{if and only if} \quad \alpha p + (1-\alpha)r \succ \alpha q + (1-\alpha)r.$$

Strict Preorder. $\succ$ is irreflexive and transitive.

Nontriviality. $p^\bullet \succ q^\bullet$ for some $p^\bullet, q^\bullet$ in $\Delta(X)$.

The following preliminary observation proves useful in what follows.

Observation 3. *Let $\succ$ be an open-continuous strict preference relation. Then, $\succ$ is an open subset of $\Delta(X)^2$, and it is an asymmetric binary relation.*

Proof. By the independence axiom, $p \succ q$ implies $p \succ \frac{1}{2}p+\frac{1}{2}q \succ q$. In turn, by transitivity of $\succ$, applying the open-continuity axiom to the pairs $\left(p, \frac{1}{2}p + \frac{1}{2}q\right)$ and $\left(\frac{1}{2}p + \frac{1}{2}q, q\right)$ yields a neighborhood N_p of p and a neighborhood N_q of q such that $r \succ w$ for every $(r, w) \in N_p \times N_q$. This shows that $\succ$ is open. Moreover, $p \succ q$ and $q \succ p$ would imply $p \succ p$ by transitivity, which contradicts irreflexivity of $\succ$. Thus, $\succ$ is also asymmetric. □

The next theorem shows that an open-continuous strict preference relation $\succ$ can be represented by a compact set of $\succ$-increasing functions.[11]

Theorem 1. *Let X be a compact metric space. A binary relation $\succ$ on $\Delta(X)$ is an open-continuous strict preference relation if and only if there exists a nonempty compact set $U \subseteq \mathbf{C}(X)$ such that:*
(i) *For every p, q in $\Delta(X)$,*

$$p \succ q \quad \textit{if and only if} \quad E(p, u) > E(q, u) \textit{ for every } u \in U.$$

(ii) *$E(p^\bullet, u) > E(q^\bullet, u)$ for every $u \in U$ and some $p^\bullet, q^\bullet$ in $\Delta(X)$.*

I will prove this theorem in Appendix C. It suffices to note here that the main step in the proof (of the "only if" part) is to show that the cone $\{\gamma(p-q) : p \succ q \text{ and } \gamma > 0\}$ is a relatively open subset of its span in a suitable topology (which is known as the bounded

[11] An **$\succ$-increasing function** refers to an element u of $\mathbf{C}(X)$ such that $E(p, u) > E(q, u)$ whenever $p \succ q$.

weak*-topology).

Next, I provide a few definitions which will be useful in what follows.

Definition 1. If $\succ$ admits a set $U \subseteq \mathbf{C}(X)$ as in Theorem 1, I will say that U is a **utility set** (for $\succ$). Given a pair of lotteries $p^\bullet, q^\bullet$ with $p^\bullet \succ q^\bullet$, a $(p^\bullet, q^\bullet)$**-normalized utility set** refers to a utility set U such that $E(p^\bullet, u) = 1$ and $E(q^\bullet, u) = 0$ for every $u \in U$. If the choice of a particular pair $(p^\bullet, q^\bullet)$ is immaterial, I will simply talk about a "normalized utility set." In turn, given a nonempty, compact set $U \subseteq \mathbf{C}(X)$, I will denote by $\succ_U$ the binary relation on $\Delta(X)$ defined by U as in part (i) of Theorem 1.

In the proof of Theorem 1, I will show that, in fact, given any pair of lotteries $p^\bullet, q^\bullet$ with $p^\bullet \succ q^\bullet$, we can find a $(p^\bullet, q^\bullet)$-normalized utility set. It is also important to note that if U is a utility set for $\succ$, so is any closed subset V of $\mathbf{C}(X)$ such that $\overline{\text{co}}\,(V) = \overline{\text{co}}\,(U)$.[12] By the uniqueness result of DMO, it can be shown that the converse is also true if we focus on normalized utility sets:

Theorem 2. *Let $U \subseteq \mathbf{C}(X)$ be a $(p^\bullet, q^\bullet)$-normalized utility set for an open-continuous strict preference relation. Then $V \subseteq \mathbf{C}(X)$ is another such set if and only if V is closed and $\overline{\text{co}}\,(V) = \overline{\text{co}}\,(U)$.*

Theorem 2 shows that a $(p^\bullet, q^\bullet)$-normalized utility set is *unique* up to closed-convex hull. An immediate implication is that, depending on the choice of $(p^\bullet, q^\bullet)$, there exists a unique, *convex* $(p^\bullet, q^\bullet)$-normalized utility set. It is also clear that this is the *largest* $(p^\bullet, q^\bullet)$-normalized utility set. Moreover, by some well-known results in functional analysis, it can be shown that the closure of the set of extreme points of this largest set gives us the *smallest* $(p^\bullet, q^\bullet)$-normalized utility set. I conclude this section with these observations.

Observation 4. *Let $\succ$ be an open-continuous strict preference relation, and pick any two lotteries $p^\bullet, q^\bullet$ with $p^\bullet \succ q^\bullet$. Then, there exist largest and smallest $(p^\bullet, q^\bullet)$-normalized utility sets, U_+ and U_-, respectively. Here, $U_+ = \overline{\text{co}}\,(U_-)$ and U_- is the closure of the set of extreme points of U_+.*

5. Extension to Preorders

In this section, my main purpose is to give a suitable extension of Theorem 1 that allows us to distinguish between the notions of indifference and indecisiveness embodied in a preorder $\succsim$.

[12]Throughout the paper, co stands for the convex hull operator, while $\overline{\text{co}}$ denotes the closure of co.

Let $\succsim$ be a binary relation on $\Delta(X)$, which represents the weak preference relation of a decision maker. As usual, I will denote by $\sim$ and $\succ$ the symmetric and asymmetric parts of $\succsim$, respectively. Consider the following two axioms:

Indifference Independence (II). For every p, q, r in $\Delta(X)$ and $\alpha \in (0,1)$,

$$p \sim q \quad \text{if and only if} \quad \alpha p + (1-\alpha) r \sim \alpha q + (1-\alpha) r.$$

Symmetric Closedness (SC). For every p, q, in $\Delta(X)$, if p belongs to the closures of both $\{r \in \Delta(X) : r \succ q\}$ and $\{r \in \Delta(X) : q \succ r\}$, then $p \sim q$.

(II) and the independence axiom (on $\succ$) are jointly equivalent to the usual statement "$p \succsim q$ iff $\alpha p + (1-\alpha) r \succsim \alpha q + (1-\alpha) r$, for $\alpha \in (0,1)$." On the other hand, (SC) simply says that if p can be approximated by a sequence of lotteries strictly preferred to q and another sequence of lotteries strictly worse than q, then p must be indifferent to q. It is worth noting that if both of the sets $\{r \in \Delta(X) : r \succsim q\}$ and $\{r \in \Delta(X) : q \succsim r\}$ were closed for any lottery q, then (SC) would trivially hold.[13] In particular, every DMO type preorder satisfies (SC).

The following theorem is my main result, which completes the task of characterizing the class of preorders that can fully be described by a compact set of Aumann utilities.

Theorem 3. *Let X be a compact metric space. For a binary relation $\succsim$ on $\Delta(X)$ the following two statements are equivalent.*

(i) *$\succsim$ is a preorder that satisfies* (II) *and* (SC)*, and its asymmetric part $\succ$ is an open-continuous strict preference relation.*

(ii) *There exists a utility set $U \subseteq \mathbf{C}(X)$ for $\succ$ such that, for every p, q in $\Delta(X)$,*

$$p \sim q \quad \textit{if and only if} \quad E(p,u) = E(q,u) \textit{ for every } u \in U. \tag{5}$$

Moreover, upon normalization, the set U is unique up to closed convex hull.

The next observation provides a property that is equivalent to (SC) for preorders that satisfy all other hypotheses in part (i) of Theorem 3.

Observation 5. *Let $\succsim$ be a preorder on $\Delta(X)$ that satisfies* (II)*. Assume further that the asymmetric part of $\succsim$ is an open-continuous strict preference relation. Then, $\succsim$*

[13]The said closedness property, however, is too strong for my purposes due to Schmeidler's (1971) theorem.

satisfies (SC) *if and only if the following property holds:*

$$\left.\begin{array}{l}\{r \in \Delta(X) : r \succ p\} = \{r \in \Delta(X) : r \succ q\} \neq \emptyset \\ \textit{and} \\ \{r \in \Delta(X) : p \succ r\} = \{r \in \Delta(X) : q \succ r\} \neq \emptyset\end{array}\right\} \quad \textit{imply} \quad p \sim q. \qquad (*)$$

In view of this observation, in the statement of Theorem 3, instead of (SC) we can equally well utilize property $(*)$. In words, this property says that if the strict upper and lower contour sets of p coincide with those of q, respectively, and if these sets are nonempty, then p and q must be indifferent. It is a simple task to show directly that (SC) implies property $(*)$. In turn, the easiest way to establish the converse implication is to invoke Theorem 1 (see Appendix C). In the statement of Theorem 3, I have chosen to utilize (SC) as it facilitates a closer comparison of my approach with that of DMO.

As I discussed earlier, Theorem 3 is motivated mainly by tractability concerns. However, the conceptual content of this result, as a "multi-self representation," is also remarkable. In the present context, it seems reasonable to view a function $u \in \mathbf{C}(X)$ as a description of a possible self of the agent defined by $\succsim$ if, in principle, the agent defined by $\succsim$ might behave as if her choices are guided by maximization of $E(\cdot, u)$.[14] In formal terms, this is equivalent to requiring that maximization of $E(\cdot, u)$ over any set $K \subseteq \Delta(X)$ should return $\succsim$-maximal elements of K. In this precise sense, Theorem 3 is a multi-self representation thanks to the fact that it delivers strictly $\succsim$-increasing functions.

It is also worth noting that, given a set U as in Theorem 3, whenever $E(p, u) = E(q, u)$ for some $u \in U$ we cannot have $p \succ q$ (as each function in U is $\succ$-increasing). This is a logical requirement for the validity of the multi-self interpretation above. Indeed, whenever $E(p, u) = E(q, u)$ for some $u \in U$, it would follow that a "self" of the agent may choose q when p is available, while $p \succ q$ would imply that the agent "herself" would never behave in the same way. Put differently, in the present model, whenever $E(p, u) = E(q, u)$ for some $u \in U$, the agent defined by $\succsim$ may choose either alternative from the set $\{p, q\}$.[15]

Remark 1. In contrast to the observations above, given a nonempty set $U \subseteq \mathbf{C}(X)$ that represents a preorder $\succsim^*$ in the sense of DMO, the choice behavior induced by $\succsim^*$

[14]**The agent defined by** $\succsim$ refers to a decision maker who might select a lottery from a choice set $K \subseteq \Delta(X)$ if and only if that lottery is a $\succsim$-maximal element of K. (In Section 7, I will discuss an alternative choice behavior for those cases in which the choice set is non-convex, which corresponds to a stronger notion of maximality.)

[15]In particular, p and q are $\succsim$-incomparable whenever $E(p, u) = E(q, u)$ and $E(p, v) > E(q, v)$ for some u, v in U. (More on this in the following subsection.)

is analogous to that of a coalition of distinct individuals, as defined by U:

$$p \succ^* q \quad \text{iff} \quad E(p,u) \geq E(q,u) \text{ for all } u \in U \text{ with strict inequality for some } u \in U.$$

Hence, a typical function $u \in U$ may not bear sufficient information to determine a $\succsim^*$-consistent choice among two lotteries p and q. (That is, we may have $E(p,u) = E(q,u)$ even if $p \succ^* q$.)

5.1 Relations to Galaabaatar-Karni

In a concurrent paper, Galaabaatar and Karni (forthcoming a) prove a finite dimensional version of Theorem 1 for a strict preference relation that admits best and worst lotteries.[16] They also propose an alternative method of extending this representation to preorders, by describing the weak preference relation of the decision maker by means of her strict preference relation. As a corollary of my findings, in Appendix E, I will generalize the approach of Galaabaatar and Karni to include an infinite prize space X. This extension also allows for an arbitrary number of maximal lotteries in $\Delta(X)$.

The approach of Galaabaatar and Karni is motivated by the fact that the weak preference relation induced by their method of extension is closed, while the primitive strict preference relation is assumed to be open. On the other hand, in view of Schmeidler's (1971) theorem, these two continuity properties cannot hold simultaneously in the absence of the completeness axiom if, following the traditional approach, we do not distinguish between the asymmetric part of the agent's weak preference relation and her strict preference relation.[17] Indeed, these two preference relations differ in the approach of Galaabaatar and Karni, which gives rise to an important question: how would the corresponding agent behave when faced with a choice problem? More specifically, the question is whether the agent's choice correspondence would consist of maximal lotteries with respect to her strict or weak preference relation.

In my approach, these two methods of defining the agent's choice correspondence are equivalent, for, following the traditional practice, I do not make a distinction between the agent's strict preference relation and the asymmetric part of her weak preference relation. Within the approach of Galaabaatar and Karni, if one defines the agent's choices by means of her strict preference relation, in terms of the implied choice behavior their approach becomes equivalent to mine, for in both models the agent's strict

[16] From the analysis of Galaabaatar and Karni, it also follows that when X is finite, the open-continuity axiom is equivalent to an Archimedean property, assuming that the independence axiom holds.

[17] Recently, Dubra (2011) proved a variant of Schmeidler's theorem in a finite dimensional setting, which combines the independence axiom with algebraic continuity properties instead of topological ones. Galaabaatar and Karni focus on Dubra's algebraic approach. Karni (2011) provides a more detailed discussion of the continuity properties of the preference relations considered by Galaabaatar and Karni.

preference relation satisfies the conditions of Theorem 1. In particular, in this case, the unconstrained scalarization method can be utilized equally well within the approach of Galaabaatar and Karni (see Section 6 below) to recover the agent's choice correspondence. On the other hand, if one defines the choice behavior of a decision maker considered by Galaabaatar and Karni by means of her weak preference relation, then the two models imply distinct behavior. In fact, a weak preference relation in the sense of Galaabaatar and Karni admits a DMO type representation, which is not compatible with the unconstrained scalarization method as I discussed earlier.

From a normative point of view, a disadvantage of my approach is that the agent's strict preference relation does not respect the Pareto order induced by the associated utility set. That is, given $\succsim$ and U as in Theorem 3, even if $E(p,u) \geq E(q,u)$ for all $u \in U$, we may not have $p \succsim q$ unless all inequalities are strict. The approach of Galaabaatar and Karni brings about a similar difficulty if one defines the agent's choice behavior by means of her strict preference relation. Specifically, in this case, the agent's choice behavior would not respect the associated Pareto order, just as in my approach. Thus, it appears that, at some level, the said difficulty is a necessary price to pay for the unconstrained scalarization method. This seems to be a reasonable cost especially if one views a representation theorem as an analytical tool, rather than a normative statement.

6. More on Scalarization and Maximal Elements

In this section, I will show that the choice correspondence induced by an open-continuous strict preference relation can be recovered by the unconstrained scalarization method. I will then investigate continuity properties of such choice correspondences.

In the remainder of the paper, the relation between weak and strict preferences of the decision maker are irrelevant for my purposes so long as the choice behavior of the decision maker is guided by an open-continuous strict preference relation. Hence, I simply focus on a strict preference relation $\succ$ on $\Delta(X)$.

In what follows, $\mathcal{M}(\succ, K)$ denotes the set of **$\succ$-maximal** elements of a set $K \subseteq \Delta(X)$; that is $\mathcal{M}(\succ, K) := \{p \in K : \text{there does not exist } q \in K \text{ such that } q \succ p\}$. Throughout this section, I interpret $\mathcal{M}(\succ, K)$ as the set of lotteries that the decision maker may chose from K. Moreover, without further mention I assume that X is a compact metric space.

As I noted several times, it is plain that maximization of the expectation of an $\succ$-increasing function on a set $K \subseteq \Delta(X)$ would deliver a $\succ$-maximal element of K. A more interesting question is the converse: Given a utility set U for $\succ$, is it true that each element of $\mathcal{M}(\succ, K)$ maximizes the expectation of a function u in U? The next proposition shows that the answer is affirmative if K is convex and if one focuses on a convex utility set.

Proposition 1. *Suppose that $\succ$ is an open-continuous strict preference relation on $\Delta(X)$, and let $U \subseteq \mathbf{C}(X)$ be a utility set for $\succ$. Then, for any convex subset K of $\Delta(X)$,*

$$\mathcal{M}(\succ, K) = \bigcup_{v \in \overline{\text{co}}(U)} \arg\max_{q \in K} E(q, v).$$

In particular, if U is a convex utility set, $\mathcal{M}(\succ, K) = \bigcup_{u \in U} \arg\max_{q \in K} E(q, u)$.

On occasion, it may be of interest to focus on a smaller (i.e., non-convex) utility set U, and express every function in $\overline{\text{co}}(U)$ as a weighted average of functions in U.[18] This approach is feasible, thanks to compactness of a utility set:

Observation 6. *Let U be a compact subset of $\mathbf{C}(X)$. Then, an element v of $\mathbf{C}(X)$ belongs to $\overline{\text{co}}(U)$ if and only if there exists a $\varphi \in \Delta(U)$ such that $E(q, v) = \int_U E(q, u)\, d\varphi(u)$ for every $q \in \Delta(X)$.*

Proof. By a version of Choquet's theorem (see Phelps, 2001, Proposition 1.2, p. 4), an element v of $\mathbf{C}(X)$ belongs to the closed-convex hull of a compact set $U \subseteq \mathbf{C}(X)$ if and only if there is a Borel probability measure φ on U such that $\mathfrak{T}(v) = \int_U \mathfrak{T}(u)\, d\varphi(u)$ for every continuous, linear functional $\mathfrak{T}$ on $\mathbf{C}(X)$. Moreover, by the well-known representation theorem of Riesz, a continuous linear functional on $\mathbf{C}(X)$ is none but a function of the form $v \to \int_X v\, d\eta$ for a signed measure η on X. The desired conclusion follows from the fact such a signed measure η can be expressed as an algebraic combination of elements of $\Delta(X)$. $\square$

It should also be noted that, given a compact set $U \subseteq \mathbf{C}(X)$ and any $\varphi \in \Delta(U)$, the system of equalities $E(q, v) = \int_U E(q, u)\, d\varphi(u)$ $(q \in \Delta(X))$ has a unique solution $v_\varphi \in \mathbf{C}(X)$ which is defined by $v_\varphi(x) := \int_U u(x)\, d\varphi(u)$ for $x \in X$. In view of these observations, Proposition 1 is equivalent to the following statement:

Proposition 1'. *Suppose that $\succ$ is an open-continuous strict preference relation on $\Delta(X)$, and let $U \subseteq \mathbf{C}(X)$ be a utility set for $\succ$. Then, for any convex subset K of $\Delta(X)$, we have*

$$\mathcal{M}(\succ, K) = \bigcup_{\varphi \in \Delta(U)} \arg\max_{q \in K} \int_U E(q, u)\, d\varphi(u).$$

[18] As I noted in Introduction, similar approaches are frequently used in classical consumer theory (e.g., Mas-Colell et al., 1995, Proposition 16.E.2).

The proof of Proposition 1' follows the logic of a theorem of alternative due to Fan, Glicksberg and Hoffman (1957). In passing, I sketch the argument for the sake of completeness.

Proof of Proposition 1'. Since the other inclusion is trivial, it suffices to show that $\mathcal{M}(\succ, K) \subseteq \bigcup_{\varphi \in \Delta(U)} \arg\max_{q \in K} \int_U E(q,u)\, d\varphi(u)$. Let $q^* \in \mathcal{M}(\succ, K)$, and note that for each $q \in K$, the function $u \to E(q - q^*, u)$ is continuous on U. Since $q \to E(q - q^*, \cdot)$ is an affine operator, convexity of the set K implies that $\widetilde{K} := \{E(q - q^*, \cdot) : q \in K\} \subseteq \mathbf{C}(U)$ is also convex. Moreover, by $\succ$-maximality of q^* on K, we have $\widetilde{K} \cap \mathbf{C}(U)_{++} = \varnothing$ where $\mathbf{C}(U)_{++} := \{f \in \mathbf{C}(U) : f(u) > 0 \text{ for every } u \in U\}$. Since $\mathbf{C}(U)_{++}$ is an open convex cone,[19] by standard separation and duality arguments we conclude that there exists a $\varphi \in \Delta(U)$ such that $\int_U f(u)\, d\varphi(u) \leq 0$ for every $f \in \widetilde{K}$. $\square$

6.1. Continuity Properties of $\mathcal{M}(\succ, K)$

The next item in my agenda is to show that the choice correspondence induced by an open-continuous strict preference relation is upper hemicontinuous.

Given a sequence (K_n) of subsets of $\Delta(X)$, define

$$\liminf K_n := \{\lim p_n : (p_n) \text{ converges and } p_n \in K_n \text{ for every } n\},$$

and

$$\limsup K_n := \bigcup \liminf K_{n_m},$$

where the union is taken over the collection of all subsequences of (K_n) with a generic member (K_{n_m}). When $\liminf K_n = K = \limsup K_n$, the set K is said to be the **Kuratowski limit** of (K_n). Since $\Delta(X)$ is compact, on the collection of nonempty closed subsets of $\Delta(X)$ (denoted as $\mathcal{K}$), the notion of Kuratowski convergence coincides with convergence in the Hausdorff metric, d_H.

Upper hemicontinuity of a choice correspondence induced by a strict preference relation demands, in fact, nothing more than openness of that relation:

Observation 7. *Let $\succ$ be an open subset of $\Delta(X)^2$. Then:*
(i) *For any $K \subseteq \Delta(X)$, the set $\mathcal{M}(\succ, K)$ is relatively closed in K.*
(ii) *Given a sequence (K_n) of subsets of $\Delta(X)$, we have*

$$\liminf \mathcal{M}(\succ, K_n) \subseteq \mathcal{M}(\succ, \limsup K_n). \tag{6}$$

[19]Throughout the paper, by a **convex cone** I mean a convex subset of a vector space that is closed under positive scalar multiplication.

In particular, for any $K \subseteq \limsup K_n$*,*

$$K \cap \liminf \mathcal{M}(\succ, K_n) \subseteq \mathcal{M}(\succ, K).$$

That is, for any convergent sequence (p_n) *with* $p_n \in \mathcal{M}(\succ, K_n)$ *for every* n*, whenever* $\lim p_n$ *belongs to a set* $K \subseteq \limsup K_n$*, it also belongs to* $\mathcal{M}(\succ, K)$*.*
(iii) $K \rightrightarrows \mathcal{M}(\succ, K)$ *is an upper hemicontinuous correspondence from the metric space* $(\mathcal{K}, d_H)$ *into* $\Delta(X)$*.*

Here, the key observation is (6), which immediately implies the other conclusions in (ii). Moreover, (i) is a trivial consequence of (ii), and (iii) also follows immediately because (ii) implies that the graph of the correspondence $K \rightrightarrows \mathcal{M}(\succ, K)$ is a closed subset of $\mathcal{K} \times \Delta(X)$ (while the range $\Delta(X)$ is compact). On the other hand, (6) readily follows from definitions: If $q \succ \lim p_n$ for a lottery q and a convergent sequence $(p_n) \in \mathcal{M}(\succ, K_1) \times \mathcal{M}(\succ, K_2) \times \cdots$, then q cannot belong to $\limsup K_n$, for otherwise openness of $\succ$ would imply that $q_n \succ p_n$ for some large n and $q_n \in K_n$.

In contrast to Observation 7(i), as I noted in Section 3, for a DMO type preorder $\succsim^*$, the set $\mathcal{M}(\succsim^*, K)$ need not be closed even if $K \subseteq \Delta(X)$ is compact and convex. Moreover, typically, the correspondence $\mathcal{M}(\succsim^*, \cdot)$ is not upper hemicontinuous. For example, in Figure 1, the increasing sequence of closed convex sets (K_n) converges to K_∞. But with $U := \{u, v\}$, the lottery p is the unique maximal element of K_∞ with respect to the DMO type preorder $\succsim^*$ induced by U, although the lottery q belongs to $\mathcal{M}(\succsim^*, K_n)$ for every n.

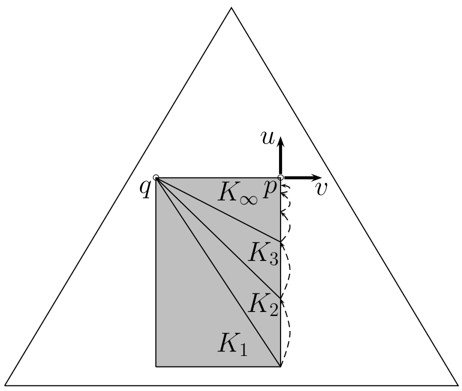

Figure 1
Lack of upper hemicontinuity in DMO

7. Characterizations of some Solution Concepts

7.1. Incomplete Preferences and Non-binary Choice Behavior

Let $\succ$ represent the strict preference relation of a decision maker who has to choose a lottery from a set $K \subseteq \Delta(X)$. Following the traditional approach, so far I have assumed that such an agent might choose any element of $\mathcal{M}(\succ, K)$. However, analogously to the use of a mixed strategy in a game-theoretic framework, in principle, our agent can condition her choice from the set K to the outcome of a random experiment such as flipping a coin or rolling a die. If we consider all such randomization devices that can return finitely many outcomes, we can say that, effectively, the choice set available to our agent is equal to $\text{co}(K)$.[20] More generally, if we also allow randomization over infinitely many alternatives in K, the "effective" choice set would become $\overline{\text{co}}(K)$.

In the absence of the completeness axiom, this observation has profound implications because $\succ$-maximality of a lottery in K does not guarantee its $\succ$-maximality in $\text{co}(K)$. Thus, the agent may have a reason to avoid choosing some elements of $\mathcal{M}(\succ, K)$. In decision theory literature, that indecisiveness may give rise to such non-binary choice behavior has been first noted by Nehring (1997).[21]

The next example, which follows the logic of Nehring's Example 1, illustrates the issue.

Example 1. Let $X := \{x, y, z\}$ and pick a number $\varepsilon \in (0, 1/2)$. Consider the open-continuous strict preference relation $\succ_U$ on $\Delta(X)$ induced by the set $U := \{u, v\}$ where u and v are the real functions on X defined as in the following table:

	u	v
x	1	0
y	ε	ε
z	0	1

Then, δ_y is $\succ_U$-maximal in $\{\delta_x, \delta_y, \delta_z\}$, but we have $\frac{1}{2}\delta_x + \frac{1}{2}\delta_z \succ_U \delta_y$.[22] □

One can think of various real-life choice situations in concert with this example. Suppose, for instance, that x, y and z are three different restaurants. While x and z are specialized in vegetarian and meat dishes, respectively, y offers both types of dishes, but at a lower quality. Our decision maker, Klaus, is an academic. He is supposed to

[20]As usual, I view the convex combination $\alpha_1 p_1 + \cdots + \alpha_n p_n$ as a compound lottery that yields the lottery p_i with probability α_i. By Choquet's theorem, we can similarly interpret the elements of $\overline{\text{co}}(K)$ provided that K is a compact subset of $\Delta(X)$.

[21]A more detailed discussion of the related literature can be found in Alcantud (2006).

[22]Throughout the paper, δ_x stands for the degenerate lottery supported at x.

make a reservation in a restaurant for himself and a guest, who has been invited for a seminar talk. The guest may prefer meat or vegetarian dishes, but Klaus does not know her tastes. Klaus' preferences over restaurants reflect his (incomplete) knowledge of the guest's tastes. Consequently, Klaus is indecisive between any pair of restaurants. What would be the potential choices of Klaus? Example 1 shows that selecting x or z randomly may make Klaus strictly better off than choosing the intermediate option y.

Motivated by similar observations, recently Heller (2012) proposed an alternative notion of rationalizable choice behavior. According to Heller's approach, it is "rational" to select a lottery p from a choice set K if and only if $p \in K \cap \mathcal{M}(\succ, \overline{\text{co}}(K))$. Heller's main finding is a characterization of choice correspondences that can be rationalized in this stronger sense. He also provides a representation theorem as a corollary of my findings. Heller's representation is an immediate consequence of the following observation:

Corollary 1. *Suppose that $\succ$ is an open-continuous strict preference relation on $\Delta(X)$ for a compact metric space X, and let $U \subseteq \mathbf{C}(X)$ be a convex utility set for $\succ$. Then, for every nonempty $K \subseteq \Delta(X)$,*

$$\bigcup_{u \in U} \arg\max_{q \in K} E(q, u) = K \cap \mathcal{M}(\succ, \text{co}(K)) = K \cap \mathcal{M}(\succ, \overline{\text{co}}(K)).$$

I omit the proof of Corollary 1, as it is an obvious consequence of Proposition 1.

It should be noted that Heller's notion of rationalizability may be equally interesting for DMO type preference relations. (In particular, one could easily modify Example 1 for such a preference relation.) However, the unconstrained scalarization method cannot be utilized to characterize the induced choice correspondence for such a preference relation.

In the next section, along the lines of Heller (2012), I will propose a refinement of the notion of pure-strategy Nash equilibrium for normal-form games with incomplete preference relations. I will then show that for open-continuous strict preference relations, this refined equilibrium notion corresponds to the set of pure-strategy equilibria that we can find using the unconstrained scalarization method.

7.2. On Nash Equilibria of Games with Incomplete Preferences

Consider a finite set of players $\mathcal{T} := \{1, ..., T\}$ with a generic element t. The set of pure strategies available to player t is a compact metric space X_t. Thus, the set $X := X_1 \times \cdots \times X_T$ of pure strategy profiles is also a compact, metrizable space. Each player t has a strict preference relation $\succ_t$ on the set $\Delta(X)$. A mixed strategy profile is a generic element $\mathbf{p} := (p_1, ..., p_T)$ of the set $\mathbf{\Delta} := \Delta(X_1) \times \cdots \times \Delta(X_T)$. I denote by $\mathfrak{B}(X_t)$ the collection of all Borel subsets of X_t. Each mixed strategy profile $\mathbf{p}$ induces a probability measure $\mathbf{p}^{\otimes}$ on X, which is the unique element of $\Delta(X)$ that satisfies $\mathbf{p}^{\otimes}(A_1 \times \cdots \times A_T) = \prod_{t=1}^{T} p_t(A_t)$ for every $(A_1, ..., A_T) \in \mathfrak{B}(X_1) \times \cdots \times \mathfrak{B}(X_T)$.

A **mixed-strategy (Nash) equilibrium** is a mixed strategy profile $\mathbf{p} := (p_1, ..., p_T)$ such that $\mathbf{p}^{\otimes} \in \mathcal{M}\left(\succ_t, \left\{(q_t, p_{-t})^{\otimes} : q_t \in \Delta(X_t)\right\}\right)$ for each t, where $p_{-t} := (p_i)_{i \in \mathcal{T} \setminus \{t\}}$. Similarly, a **pure-strategy equilibrium** is an element $x := (x_1, ..., x_T)$ of X such that $\delta_x \in \mathcal{M}\left(\succ_t, \left\{\delta_{(y_t, x_{-t})} : y_t \in X_t\right\}\right)$, where $x_{-t} := (x_i)_{i \in \mathcal{T} \setminus \{t\}}$.

As we have seen in Section 7.1, even with a single player, a pure-strategy equilibrium may not remain as an equilibrium upon the introduction of mixed strategies. This motivates the following definition.

Definition 2. An element x of X is a **randomization-proof (pure-strategy) equilibrium** if

$$\delta_x \in \mathcal{M}\left(\succ_t, \left\{\left(q_t, \delta_{x_{-t}}\right)^{\otimes} : q_t \in \Delta(X_t)\right\}\right) \text{ for each player } t.$$

Let $RE(\succ)$ stand for the set of all randomization-proof equilibria, and $ME(\succ)$ for the set of all mixed-strategy equilibria, where $\succ$ stands for the preference profile $\succ_1, ..., \succ_T$. By definitions, it is clear that

$$RE(\succ) = \{x \in X : \delta_x \in ME(\succ)\}. \tag{7}$$

The next item in my agenda is to provide some characterizations of $RE(\succ)$ and $ME(\succ)$ by utilizing the unconstrained scalarization method. More specifically, I will show that these sets can be expressed as a suitable union of Nash equilibria of games induced by the utility functions that characterize agents' preference relations.

In what follows, $ME(u_1, ..., u_T)$ stands for the set of all mixed-strategy equilibria of a modified version of the above game in which each player t's preference relation is complete and admits a (single) von Neumann-Morgenstern utility function u_t. Similarly, $PE(u_1, ..., u_T)$ denotes the set of pure-strategy equilibria of this modified game with complete preference relations.

Notice that the set $\{(q_t, p_{-t})^{\otimes} : q_t \in \Delta(X_t)\}$ is a convex subset of $\Delta(X)$ for each $\mathbf{p} \in \mathbf{\Delta}$. Thus, the following characterization of mixed-strategy equilibria is an obvious consequence of Proposition 1.

Corollary 2. *For each $t \in \mathcal{T}$, suppose that $\succ_t$ is an open-continuous strict preference relation on $\Delta(X)$, and let $U_t \subseteq \mathbf{C}(X)$ be a convex utility set for $\succ_t$. Then,*

$$ME(\succ) = \bigcup ME\left(u_1, ..., u_T\right),$$

where the union is taken over $(u_1, ..., u_T) \in U_1 \times \cdots \times U_T$.

My final result in this section is a game-theoretic version of Corollary 1, which shows that applying the unconstrained scalarization method to pure strategies filters the set of randomization-proof equilibria.

Corollary 3. *For each* $t \in T$, *let* $\succ_t$ *and* U_t *be as in Corollary 2. Then:*

$$RE(\succ) = \bigcup PE(u_1, ..., u_T), \tag{8}$$

where the union is taken over $(u_1, ..., u_T) \in U_1 \times \cdots \times U_T$.

Proof. By (7), if x belongs to $RE(\succ)$, then δ_x belongs to $ME(\succ)$. Thus, in this case, Corollary 2 implies that $\delta_x \in ME(u_1, ..., u_T)$ for some $(u_1, ..., u_T) \in U_1 \times \cdots \times U_T$. It immediately follows that $x \in PE(u_1, ..., u_T)$. Hence, the left side of (8) is contained in the right side.

For the converse inclusion, let $x \in PE(u_1, ..., u_T)$ for some $(u_1, ..., u_T) \in U_1 \times \cdots \times U_T$. Then, $\delta_x \in ME(u_1, ..., u_T)$ since the preferences are complete in the game defined by $u_1, ..., u_T$. Hence, Corollary 2 and equation (7) imply that $x \in RE(\succ)$. □

In passing, it may be useful to note that if each X_t is a convex subset of a vector space, and if the utility functions that represent the strict preference relation of any given player are concave in pure strategies available to that player, then the set of randomization-proof equilibria coincides with the set of pure-strategy equilibria.[23] However, if the utility functions are only quasi-concave, even in games with no strategic interactions (such as general equilibrium models in consumer theory), the notion of a randomization-proof equilibrium provides a genuine refinement of the notion of a pure-strategy equilibrium. (For brevity, I omit the proofs of these claims, which are available upon request.)

7.3. Weak Pareto Optimality and Social Planning with Incompletely Known Preferences

Let $\mathcal{T}$ be a society that consists of finitely many agents, and X a compact metric space of social alternatives. Assume that each agent t has a strict preference relation $\succ_t$

[23] Bade (2005, Theorem 3) proves a related result, which shows that if each player's preference relation admits a DMO type representation with finitely many utility functions that are *strictly* concave in pure strategies available to that player, then the unconstrained scalarization method delivers the set of *pure-strategy equilibria*. In turn, my related observation above shows that if players have open-continuous strict preference relations, only the concavity of the utility functions would suffice for the same conclusion.

on $\Delta(X)$. Consider the following notion of domination: for every p, r in $\Delta(X)$,

$$p \succ r \quad \text{if and only if} \quad p \succ_t r \text{ for every } t \in \mathcal{T}.$$

The notion of efficiency induced by this domination relation $\succ$ is often referred to as **weak Pareto optimality.** The next result provides a characterization of this efficiency notion for incomplete preference relations**.**

Corollary 4. *For each $t \in \mathcal{T}$, suppose that $\succ_t$ is an open-continuous strict preference relation on $\Delta(X)$, and let $U_t \subseteq \mathbf{C}(X)$ be a convex utility set for $\succ_t$. Then, for every convex $K \subseteq \Delta(X)$,*

$$\mathcal{M}(\succ, K) = \bigcup \arg\max_{q \in K} E\left(q, \sum_{t \in \mathcal{T}} \alpha_t u_t\right),$$

where the union is taken over $(\alpha_t, u_t)_{t \in \mathcal{T}} \in \mathbb{R}_+^{\mathcal{T}} \times \mathbf{C}(X)^{\mathcal{T}}$ such that $\sum_{t \in \mathcal{T}} \alpha_t = 1$ and $u_t \in U_t$ for every $t \in \mathcal{T}$.

Proof. It is clear that $\succ$ is equal to the open-continuous strict preference relation $\succ_U$ induced by the set $U := \bigcup_{t \in \mathcal{T}} U_t$. Moreover, $\text{co}(U)$ is a compact set that consists of all functions of the form $\sum_{t \in \mathcal{T}} \alpha_t u_t$ for some $(\alpha_t, u_t)_{t \in \mathcal{T}} \in \mathbb{R}_+^{\mathcal{T}} \times \mathbf{C}(X)^{\mathcal{T}}$ such that $\sum_{t \in \mathcal{T}} \alpha_t = 1$ and $u_t \in U_t$ for every $t \in \mathcal{T}$. Thus, the proof follows from Proposition 1.[24]
□

There exists an alternative interpretation of Corollary 4, which may be useful on occasion. Suppose that each agent's preference relation is complete, but the planner has an incomplete knowledge of agents' preferences. Then, we can think of $\succ_t$ as a binary relation that represents the knowledge of the social planner about the strict preference relation of agent t.

When viewed from this perspective, Corollary 4 resembles the efficiency theorems of McLennan (2002) and Carroll (2010). However, the present approach differs from theirs in several respects. First, I do not directly assume that planner's knowledge about a given agent can be summarized by a set of utility functions. Rather, I derive this conclusion from the properties of the binary relations that model planner's knowledge. Second, I allow X to be infinite and do not restrict my attention to the grand set $K = \Delta(X)$. On the other hand, these advantages of the present approach come at a cost: Weak Pareto optimality is a weaker notion of efficiency compared to that of McLennan and Carroll.

[24]To be more precise, note that the statement of Proposition 1 does not include the cases in which $\succ$ is trivial, but in such cases we can utilize an obvious generalization of Proposition 1 to obtain the desired conclusion.

(A lottery p dominates a lottery r in the sense of McLennan and Carroll if there exists an agent t such that $E(p,u) > E(r,u)$ for every $u \in U_t$, while $E(p,u) \geq E(r,u)$ for every $u \in U_i$ and $i \in \mathcal{T}\backslash\{t\}$. Here, U_t is an exogenously given set of utility functions that represents the planner's knowledge about agent t.)

Appendix

A. Negative Examples on Unconstrained Scalarization Method

I start with the case of a DMO type preorder induced by a non-compact set of utility functions. Set $X := [0,1]$, and

$$\hat{U} := \{u \in \mathbf{C}(X) : u(0) = 0,\ u(1) = 1 \text{ and } \|u\|_\infty \leq 2\}.$$

Let $\succsim^\wedge$ be the preorder on $\Delta(X)$ induced by $\hat{U}$ via the rule (2). The next lemma lists some interesting properties of $\succsim^\wedge$.

Lemma 1.(i) *For any $p \in \Delta(X)$ and $\alpha \in (1/2, 1]$, we have $\delta_1 \succ^\wedge \alpha\delta_0 + (1-\alpha)p$.*
(ii) *Any lottery r on X with $r(\{0\}) = 0$ is $\succsim^\wedge$-maximal on $\Delta(X)$.*
(iii) *In particular, if $r(\{0\}) = 0$ and $r(I) > 0$ for every nondegenerate interval I in X, then $r \in \mathcal{M}(\succsim^\wedge, \Delta(X))$. But whenever such an r belongs to $\arg\max_{q\in\Delta(X)} E(q,u)$ for some $u \in \mathbf{C}(X)$, then u is a constant function.*
(iv) *The set of elements of $\mathcal{M}(\succsim^\wedge, \Delta(X))$ which do not maximize the expectation of any strictly $\succsim^\wedge$- increasing function in $\mathbf{C}(X)$ is a dense subset of $\Delta(X)$.*

Part (i) of Lemma 1 shows that, geometrically, almost "one half" of the space $\Delta(X)$ consists of lotteries that are not $\succsim^\wedge$-maximal. Thus, $\Delta(X)\backslash\mathcal{M}(\succsim^\wedge, \Delta(X))$ is a substantially large set. Let $\mathcal{M}_0(\succsim^\wedge)$ be the set of lotteries in $\Delta(X)$ which maximize the expectation of a strictly $\succsim^\wedge$-increasing function in $\mathbf{C}(X)$. Part (iv) shows that the set $\mathcal{M}(\succsim^\wedge, \Delta(X))\backslash\mathcal{M}_0(\succsim^\wedge)$ is dense in $\Delta(X)$, in line with Observation 1(ii). Moreover, by part (iii), the set $\mathcal{M}(\succsim^\wedge, \Delta(X))\backslash\mathcal{M}_0(\succsim^\wedge)$ contains some lotteries which do not maximize the expectation of any non-constant, continuous function on X.

It is also worth noting that $\succsim^\wedge$ is a partial order on $\Delta(X)$. Thus, by a suitable application of the density theorem of Makarov and Rachovski (1996), it can be shown that $\mathcal{M}_0(\succsim^\wedge)$ is a dense subset of $\mathcal{M}(\succsim^\wedge, \Delta(X))$. However, if one applies the closure operator to the set $\mathcal{M}_0(\succsim^\wedge)$ in order to recover the (dense) set $\mathcal{M}(\succsim^\wedge, \Delta(X))\backslash\mathcal{M}_0(\succsim^\wedge)$, one would end up with the entire space $\Delta(X)$, which contains all the "bad" lotteries in $\Delta(X)\backslash\mathcal{M}(\succsim^\wedge, \Delta(X))$. These observations verify my concluding remarks in Section 3.

Part (ii) is the key claim in Lemma 1, which I prove in Appendix C. In turn, part (iii) follows from part (ii) immediately, while part (i) is a trivial consequence of definitions.

Finally, to see why part (iv) holds, note that each neighborhood of a given lottery on $[0,1]$ contains a lottery r such that (a) $r(\{0\}) = 0$, and (b) $r(I) > 0$ for every nondegenerate interval I in $[0,1]$ that contains 0 or 1. Moreover, if such a lottery r maximizes $E(\cdot, u)$ on $\Delta([0,1])$ for some $u \in \mathbf{C}([0,1])$, then $u(0) = u(1)$, although we have $\delta_1 \succ^{\wedge} \delta_0$. Thus, part (iv) also follows from part (ii).

As I noted earlier, compactness of the set of utility functions in my representation notion is also essential for applicability of the unconstrained scalarization method. The next lemma demonstrates this point.

Lemma 1'. *Let X and $\hat{U}$ be as above, and denote by $\succsim_o^{\wedge}$ the preorder on $\Delta(X)$ induced by the set $\hat{U}$ via the rule* (4). *Then, the conclusions of* Lemma1(i)-(iv) *also hold for $\succsim_o^{\wedge}$.*

Lemmas 1 and 1' jointly prove Observation 1. I conclude this appendix with two further examples which show that the DMO approach is not compatible with the unconstrained scalarization method even when the preorder in question can be represented by finitely many utility functions.

Example 2. Let us denote a generic element of $\mathbb{R}_+^3$ as $x := (x_1, x_2, x_3)$. Set

$$X := \left\{x \in \mathbb{R}_+^3 : x_1 + x_2 + x_3 \leq 4\right\} \quad \text{and} \quad U := \{u, v\},$$

where, for every $x \in X$,

$$u(x) := (x_1 + x_2)^{1/2} (x_3)^{1/2} \quad \text{and} \quad v(x) := 2 (x_1)^{1/2} (x_2)^{1/2}.$$

Let $\succsim$ stand for the preorder on $\Delta(X)$ induced by U via the rule (2). It is clear that

$$\arg\max_{x \in X} u(x) = \{x \in X : x_1 + x_2 = 2,\ x_3 = 2\}, \quad \text{and}$$
$$\arg\max_{q \in \Delta(X)} E(q, u) = \left\{q \in \Delta(X) : q\left(\arg\max_{x \in X} u(x)\right) = 1\right\}.$$

However, $x^* := (1, 1, 2)$ is the unique maximizer of v on $\arg\max_{x \in X} u(x)$, implying that the lottery δ_{x^*} is the only element of $\arg\max_{q \in \Delta(X)} E(q, u)$ that is $\succsim$-maximal on $\Delta(X)$. Hence, $\arg\max_{q \in \Delta(X)} E(q, u)$ is not contained in $\mathcal{M}(\succsim, \Delta(X))$.

Moreover, δ_{x^*} does not maximize the expectation of any strictly $\succsim$-increasing function $f \in \mathbf{C}(X)$. To see this, take any such f. Note that U is normalized in the sense that $u(x^*) = v(x^*) = 2$ and $u(\mathbf{0}) = v(\mathbf{0}) = 0$, where $\mathbf{0} := (0,0,0)$. Thus, by normalizing f accordingly, we can assume that $f \in \operatorname{co}(U)$. (For more on this argument, see the proof of Theorem 2 below). As neither u nor v are strictly $\succsim$-increasing, we can in fact write

$f = \alpha u + (1-\alpha)v$ for some $\alpha \in (0,1)$. It easily follows that $\frac{\partial f}{\partial x_1}(x^*) > \frac{\partial f}{\partial x_3}(x^*)$, but the normal vector of the set X at x^* equals $(1,1,1)$. Hence, $x^* \notin \arg\max_{x\in X} f(x)$. $\square$

Example 3. Consider a set X that consists of three alternatives, and let B denote a closed ball in the interior of $\Delta(X)$. Pick any non-constant $u \in \mathbb{R}^3$ as a utility vector. Then, there exists a unique a lottery p^* that maximizes $E(\cdot, u)$ on B. Now, pick any $p' \in \Delta(X)$, distinct from p^*, such that $E(p', u) = E(p^*, u)$. Put $v := p^* - p'$, $U := \{u, v\}$ and $K := \text{co}\,(\{p'\} \cup \{q \in B : E(q,v) \geq E(p^*, v)\})$. (As usual, I identify $\Delta(X)$ with the unit simplex in $\mathbb{R}^3$.) Then, both p^* and p' maximize $E(\cdot, u)$ on K, but only p^* is a maximal element of K with respect to the DMO type preorder $\succsim$ induced by U. Moreover, there does not exist a strictly $\succsim$-increasing $f \in \mathbb{R}^3$ such that $p^* \in \arg\max_{q\in K} E(q, f)$. As Figure 2 illustrates, this scenario simply replicates a well-known problem related to the identification of the Pareto frontier of a utility possibility set contained in a Euclidean space.

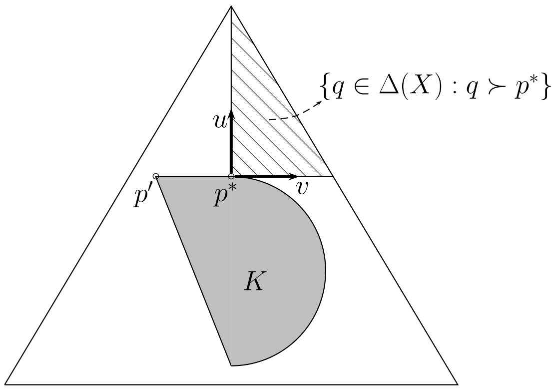

Figure 2

Failure of unconstrained scalarization in Exp. 3

B. A Constrained Scalarization Method for DMO Type Preorders

The next lemma provides a constrained scalarization method for DMO type preorders. A notable difference with classical consumer theory is that in the present set-up, we have to pick an objective function that is strictly increasing with respect to the preorder in question. The existence of such a function is assured by Proposition 3 of DMO, provided that the prize space is a compact metric space.

Lemma 2. *Let X be a compact metric space, and $\succsim$ a preorder on $\Delta(X)$. Suppose that $\succsim$ admits a set $U \subseteq \mathbf{C}(X)$ that represents $\succsim$ as in* (2)*. Pick an Aumann utility f_A for $\succsim$. Then, for any $K \subseteq \Delta(X)$, an element p of K belongs to $\mathcal{M}(\succsim, K)$ if, and only if,* *there exists a function* $c : U \to \mathbb{R}$ *such that*

$$p \in \arg\max \{E(q, f_A) : q \in K,\ E(q, u) \geq c(u)\ \forall u \in U\}.$$

Proof. Pick any $p \in K$. First, suppose that $p \in \mathcal{M}(\succsim, K)$. Put $c(u) := E(p, u)$ for every $u \in U$. Then, for any $q \in K$, whenever $E(q, u) \geq c(u)$ for every $u \in U$, we have $q \sim p$, and hence, $E(q, f_A) = E(p, f_A)$. Thus, p maximizes $E(\cdot, f_A)$ among such q's. Conversely, if p maximizes $E(\cdot, f_A)$ over a set of the form $\{q \in K : E(q, u) \geq c(u)\ \forall u \in U\}$ for some $c : U \to \mathbb{R}$, then p must also belong to $\mathcal{M}(\succsim, K)$ because $E(\cdot, f_A)$ is strictly $\succsim$-increasing while $E(\cdot, u)$ is weakly $\succsim$-increasing for every $u \in U$. $\square$

Although Lemma 2 provides a clear-cut characterization of maximal lotteries for DMO type preorders, the class of constrained optimization problems described in this lemma may not be so tractable, as we may not be able to utilize Kuhn-Tucker theorem. Notice that in the first part of the proof above, the specification of $c(\cdot)$ shrinks the constraint set to the equivalence class of the maximal lottery p. In fact, such tight selections of $c(\cdot)$ would *typically* lead to the failure of the classical constraint qualification. To understand the problem, suppose that the set U in Lemma 2 is finite, and let us write $U = \{u_i : i = 1, ..., m\}$. Assume also that the set K is convex. Then, as in the proof of Proposition 1', a $\succsim$-maximal element p of K should maximize over K a function of the form $E(\cdot, \alpha_1 u_{i_1} + \cdots + \alpha_k u_{i_k})$ for some $\alpha_1, ..., \alpha_k > 0$. Under usual regularity conditions, this implies that the vector $\alpha_1 u_{i_1} + \cdots + \alpha_k u_{i_k}$ is tangent to the set K at the point p. If the expected utility constraints induced by the functions $u_{i_1}, ..., u_{i_k}$ are active at p (that is, if $E(p, u_{i_j}) = c(u_{i_j})$ for $j = 1, ..., k$), it would follow that there exist $k+1$ active constraints (the last one describing the boundary of K at p) with linearly dependent derivatives. This, in turn, would violate the classical constraint qualification.

It is also worth noting that in Lemma 2, if the set U is finite and K is convex, at least one expected utility constraint must be active at the maximal lottery in question unless this lottery is already an element of $\arg\max_K E(\cdot, f_A)$. In other words, if the *un*constrained scalarization method is not readily applicable, the constrained scalarization method proposed in Lemma 2 will force one to deal with some active expected utility constraints.

In view of these remarks, it will come as no surprise to see that in Examples 2 and 3 from Appendix A, for relevant specifications of $c(\cdot)$ the maximal points in question would not satisfy the first order conditions in constrained optimization problems as in Lemma 2. Indeed, in Example 3, the maximal lottery p^* can maximize the expectation of an Aumann utility f_A over a set of the form $\{q \in K : E(q, u) \geq c(u),\ E(q, v) \geq c(v)\}$ only if $E(p^*, u) = c(u)$. The derivative of this active constraint is simply the vector u, which also coincides with the normal vector of the set K at the point p^*. If $E(p^*, v) > c(v)$, this implies that f_A (i.e., the derivative of the expected utility function that acts as the objective function) cannot be expressed as a linear combination of the derivatives of the two active constraints (contrary to the conclusion of Kuhn-Tucker theorem). Similarly, in Example 2, the point x^* can maximize an Aumann utility f_A over a set of the form $\{x \in X : u(x) \geq c(u),\ v(x) \geq c(v)\}$ only if $u(x^*) = c(u)$. Moreover, the derivative of $u(\cdot)$ at x^* equals $(1/2, 1/2, 1/2)$, while the normal vector of X at x^* equals $(1, 1, 1)$. But the derivative of f_A at the point x^* cannot be collinear with $(1, 1, 1)$ (as I noted in Example 2). Thus, if $v(x^*) > c(v)$, we again see that x^* would not satisfy the first order conditions in this constrained optimization problem.

C. Omitted Proofs

Proof of Lemma 1. As I noted in Appendix A, only part (ii) of Lemma 1 requires a proof. Let $r \in \Delta(X)$ be such that $r(\{0\}) = 0$. To prove that r is $\succsim^{\wedge}$-maximal, take any $q \in \Delta(X)$ with $q \neq r$. Then, there is a Borel set $X_0 \subseteq X$ such that $r(X_0) > q(X_0)$.

First assume $r(\{1\}) \leq q(\{1\})$. Then, as we also have $r(\{0\}) \leq q(\{0\})$, it follows that $r(X_0 \backslash \{0, 1\}) > q(X_0 \backslash \{0, 1\})$. Hence, by normality of countably additive measures on a metric space, there exists a closed set F contained in $X_0 \backslash \{0, 1\}$ such that $r(F) > q(F)$ (cf. Aliprantis and Border, 1999, Theorem 17.24).

For each $\varepsilon > 0$, set $\mathcal{B}_\varepsilon := \{x \in X : |x - y| < \varepsilon \text{ for some } y \in F \cup \{0\}\}$. Note that by Tietze extension theorem, there exists a function $u_\varepsilon \in \hat{U}$ such that, for any $x \in [0, 1]$,

$$u_\varepsilon(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \in \{1\} \cup (X \backslash \mathcal{B}_\varepsilon), \\ 2 & \text{if } x \in F. \end{cases}$$

It is plain that, for every $p \in \Delta(X)$,

$$\lim_{\varepsilon \to 0} \int_{\mathcal{B}_\varepsilon \backslash F} u_\varepsilon \, dp = \lim_{\varepsilon \to 0} \left(u_\varepsilon(0) p(\{0\}) + \int_{\mathcal{B}_\varepsilon \backslash (F \cup \{0\})} u_\varepsilon \, dp \right) = 0.$$

Hence,

$$\lim_{\varepsilon \to 0} E(q, u_\varepsilon) = \lim_{\varepsilon \to 0} \int_{F \cup (X \backslash \mathcal{B}_\varepsilon)} u_\varepsilon \, dq = 2q(F) + q(X \backslash (F \cup \{0\})) \leq q(F) + 1,$$

$$\lim_{\varepsilon \to 0} E(r, u_\varepsilon) = \lim_{\varepsilon \to 0} \int_{F \cup (X \backslash \mathcal{B}_\varepsilon)} u_\varepsilon \, dr = 2r(F) + r(X \backslash (F \cup \{0\})) = r(F) + 1.$$

It follows that $E(r, u_\varepsilon) > E(q, u_\varepsilon)$ for all sufficiently small ε.

Suppose now $r(\{1\}) > q(\{1\})$. For each $\varepsilon \in (0,1)$, pick any $v_\varepsilon \in \hat{U}$ such that $v_\varepsilon(x) = 0$ for $x \in [0, 1-\varepsilon]$. Then, as $v_\varepsilon(1) = 1$ for every $\varepsilon \in (0,1)$, we obviously have $\lim_{\varepsilon \to 0} E(r, v_\varepsilon) = r(\{1\})$ and $\lim_{\varepsilon \to 0} E(q, v_\varepsilon) = q(\{1\})$, implying that $E(r, v_\varepsilon) > E(q, v_\varepsilon)$ for all sufficiently small ε. This completes the proof of (ii). $\square$

The proof of Lemma 1' is identical with the proof of Lemma 1, and these two results jointly imply Observation 1. In what follows, X denotes an arbitrary, compact metric space. Next, I prove a basic fact:

Claim 1. *If $U \subseteq \mathbf{C}(X)$ is a compact set, $\{(p,q) : E(p,u) > E(q,u)\ \forall u \in U\}$ is an open subset of $\Delta(X)^2$.*

Proof. Let U be a compact subset of $\mathbf{C}(X)$, and take any $p, q \in \Delta(X)$ such that $E(p,u) - E(q,u) > 0$ for every $u \in U$. Since the function $u \to E(p,u) - E(q,u)$ is continuous on $\mathbf{C}(X)$, it attains its minimum on the compact set U. Thus, there exists a $\gamma > 0$ such that $E(p,u) - E(q,u) > \gamma$ for every $u \in U$. Moreover, since $u \to E(\cdot, u)$ is a continuous map from $\mathbf{C}(X)$ to $\mathbf{C}(\Delta(X))$, the set $\{E(\cdot,u) : u \in U\} \subseteq \mathbf{C}(\Delta(X))$ is also compact. Hence, by Arzelà-Ascoli theorem (cf. Dunford and Schwartz, 1958, Theorem IV.6.7), there exists a neighborhood N_p of p such that $E(p',u) - E(p,u) > -\gamma/2$ for every $p' \in N_p$ and $u \in U$. Similarly, there exists a neighborhood N_q of q such that $E(q',u) - E(q,u) < \gamma/2$ for every $q' \in N_q$ and $u \in U$. Then, $E(p',u) - E(q',u) > 0$ for every $(p',q') \in N_p \times N_q$, as we seek. $\square$

Proof of Observation 2. If U and $\succsim$ satisfy (2) for every p, q in $\Delta(X)$, and if U consists of strictly $\succsim$-increasing functions, then it readily follows that, for every p, q in $\Delta(X)$,

$$p \succ q \quad \text{if and only if} \quad E(p,u) - E(q,u) > 0 \text{ for every } u \in U.$$

From Claim 1 it thus follows that if U is compact, then $\succ$ must be open. Hence, the proof follows from Schmeidler's (1971) theorem. $\square$

I will present the proof of Theorem 1 at the end of this appendix. I proceed with:

Proof of Theorem 2. The "if" part of the theorem is a routine exercise. For the "only if" part, let U and V be $(p^\bullet, q^\bullet)$-normalized utility sets for an open-continuous strict preference relation $\succ$. I shall first show that, for every p, q in $\Delta(X)$,

$$E(p,u) \geq E(q,u) \quad \forall u \in U \quad \text{imply} \quad E(p,v) \geq E(q,v) \quad \forall v \in V. \tag{9}$$

By definition of U, for each $\alpha \in (0,1)$ the former set of inequalities imply $\alpha p^\bullet +$

$(1-\alpha)\,p \succ \alpha q^{\bullet}+(1-\alpha)\,q$. Then, $E\left(\alpha p^{\bullet}+(1-\alpha)\,p,v\right) > E\left(\alpha q^{\bullet}+(1-\alpha)\,q,v\right)$ for every $v\in V$, by definition of V. Passing to limit as $\alpha\to 0$ yields the desired conclusion: $E(p,v)\geq E(q,v)$ for every $v\in V$.

By the proof of the uniqueness result of DMO, (9) implies that V is contained in $\mathrm{cl}\left(\mathrm{cone}\,(U)+\{\beta\mathbf{1}_X:\beta\in\mathbb{R}\}\right)$, where $\mathrm{cone}\,(U)\subseteq\mathbf{C}(X)$ is the smallest convex cone that contains U while cl stands for the closure operator. Clearly, we can write $\mathrm{cone}\,(U)=\bigcup_{\gamma>0}\gamma\,\mathrm{co}\,(U)$. Hence, for each $v\in V$, there exist real sequences $(\beta_n),(\gamma_n)$ and a sequence (u_n) in $\mathrm{co}\,(U)$ such that $\lim_{n\to\infty}\left\|(\gamma_n u_n+\beta_n\mathbf{1}_X)-v\right\|_\infty=0$. Since $E(q^{\bullet},v)=0=E(q^{\bullet},u_n)$ for every n, it follows that $\lim\beta_n=\lim E(q^{\bullet},\gamma_n u_n+\beta_n\mathbf{1}_X)=0$. Thus, $\lim\left\|\gamma_n u_n-v\right\|_\infty=0$. Since $E(p^{\bullet},v)=1=E(p^{\bullet},u_n)$ for every n, it then follows that $\lim\gamma_n=\lim E(p^{\bullet},\gamma_n u_n)=1$. We therefore conclude that $\lim\left\|u_n-v\right\|_\infty=0$. Hence, $V\subseteq\overline{\mathrm{co}}\,(U)$, and we similarly have $U\subseteq\overline{\mathrm{co}}\,(V)$. $\square$

Proof of Observation 4. Let U be a $(p^{\bullet},q^{\bullet})$-normalized utility set for $\succ$, and denote by $\mathcal{E}$ the set of extreme points of $\overline{\mathrm{co}}(U)$. From Theorem 2 it immediately follows that $\overline{\mathrm{co}}(U)$ is the largest $(p^{\bullet},q^{\bullet})$-normalized utility set for $\succ$. In particular, $\overline{\mathrm{co}}(U)$ is also a compact set (cf. Dunford and Schwartz, 1958, Theorem V.2.6). Thus, by Krein-Milman theorem (cf. Dunford and Schwartz, 1958, Theorem V.8.4), we have $\overline{\mathrm{co}}(\mathrm{cl}(\mathcal{E}))=\overline{\mathrm{co}}(U)$. Moreover, by a theorem of Milman (known as the partial converse of Krein-Milman theorem) any closed set V such that $\overline{\mathrm{co}}(V)=\overline{\mathrm{co}}(U)$ contains $\mathrm{cl}(\mathcal{E})$ (cf. Dunford and Schwartz, 1958, Theorem V.8.5). Hence, Theorem 2 implies that $\mathrm{cl}(\mathcal{E})$ is the smallest $(p^{\bullet},q^{\bullet})$-normalized utility set for $\succ$. $\square$

Proof of Theorem 3. Since the other implication is fairly obvious, I will only prove that (i) implies (ii). Fix a preorder $\succsim$ on $\Delta(X)$ that satisfies (II) and (SC). Also assume that $\succ$ is an open-continuous strict preference relation, and let U be a utility set for $\succ$. To verify (5), pick any pair of lotteries p,q. As $\succ$ is nontrivial, there exists a pair $p^{\bullet},q^{\bullet}$ in $\Delta(X)$ with $p^{\bullet}\succ q^{\bullet}$.

Suppose first that $p\sim q$. Fix any $u\in U$. Then, for any $\alpha\in(0,1)$, the independence axiom implies $\alpha p^{\bullet}+(1-\alpha)p\succ\alpha q^{\bullet}+(1-\alpha)p$, while (II) implies $\alpha q^{\bullet}+(1-\alpha)p\sim\alpha q^{\bullet}+(1-\alpha)q$. Since $\succsim$ is transitive, it follows that $\alpha p^{\bullet}+(1-\alpha)p\succ\alpha q^{\bullet}+(1-\alpha)q$ for every $\alpha\in(0,1)$. As in the proof of (9) above, invoking the definition of U and passing to limit as $\alpha\to 0$ yield $E\,(p,u)\geq E\,(q,u)$. Similarly, we also have $E\,(p,u)\leq E\,(q,u)$. Hence, we conclude that $E\,(p,u)=E\,(q,u)$ for every $u\in U$.

Conversely, assume now that $E\,(p,u)=E\,(q,u)$ for every $u\in U$. Since $p^{\bullet}\succ q^{\bullet}$, the independence axiom implies $p^{\bullet}\succ\frac{1}{2}p^{\bullet}+\frac{1}{2}q^{\bullet}\succ q^{\bullet}$. Hence, by open-continuity of $\succ$, there exists an $\varepsilon\in(0,1)$ such that $p^{\bullet}\succ\varepsilon p+(1-\varepsilon)\left(\frac{1}{2}p^{\bullet}+\frac{1}{2}q^{\bullet}\right)\succ q^{\bullet}$. Put $p':=\varepsilon p+(1-\varepsilon)\left(\frac{1}{2}p^{\bullet}+\frac{1}{2}q^{\bullet}\right)$, $q':=\varepsilon q+(1-\varepsilon)\left(\frac{1}{2}p^{\bullet}+\frac{1}{2}q^{\bullet}\right)$, and note that $E\,(p',u)=E\,(q',u)$ for every $u \in U$. Thus, from the definition of U it follows that for any $r \in \Delta(X)$, we have $r \succ p'$ iff $r \succ q'$, and $p' \succ r$ iff $q' \succ r$. Moreover, by the independence axiom, $\alpha p' + (1-\alpha)p^\bullet \succ p' \succ \alpha p' + (1-\alpha)q^\bullet$ for every $\alpha \in (0,1)$. Passing to limit as $\alpha \to 1$ implies that p' belongs to both of the sets $\text{cl}\,\{r \in \Delta(X) : r \succ p'\}$ and $\text{cl}\,\{r \in \Delta(X) : p' \succ r\}$. But, as I have just noted, these sets coincide with $\text{cl}\,\{r \in \Delta(X) : r \succ q'\}$ and $\text{cl}\,\{r \in \Delta(X) : q' \succ r\}$, respectively. Hence, (SC) implies $p' \sim q'$. By (II), we obtain the desired conclusion: $p \sim q$. □

Proof of Observation 5. To see why (SC) implies property $(*)$, consider a pair of lotteries p, q that satisfy the hypotheses of property $(*)$. Then, $\{r \in \Delta(X) : r \succ p\}$ and $\{r \in \Delta(X) : p \succ r\}$ are nonempty, and hence, the independence axiom clearly implies that p belongs to the closures of both of these sets. But, by hypotheses of property $(*)$, this means that p belongs to the closures of $\{r \in \Delta(X) : r \succ q\}$ and $\{r \in \Delta(X) : q \succ r\}$. Thus, (SC) implies $p \sim q$, which verifies the property $(*)$. To prove the converse implication, suppose that property $(*)$ holds, and let U be a utility set for $\succ$. Note that if p belongs to the closures of $\{r \in \Delta(X) : r \succ q\}$ and $\{r \in \Delta(X) : q \succ r\}$, then $E(p,u) = E(q,u)$ for every $u \in U$. Thus, in this case, the hypotheses of property $(*)$ also hold, implying that $p \sim q$. □

I will now proceed to the proof of Theorem 1. (The proofs of the remaining non-trivial results can be found in text.)

C.1. Proof of Theorem 1

First, I need to introduce a bit of notation and terminology. $ca(X)$ stands for the space of signed measures on X equipped with the usual setwise algebraic operations. As is well-known, when endowed with the total-variation norm $\|\cdot\|$, the space $ca(X)$ is isometrically isomorphic to the norm-dual of $\mathbf{C}(X)$. In turn, the weak*-topology on $ca(X)$ is the coarsest topology that makes continuous every functional of the form $\eta \to \int_X u \, d\eta$ for some $u \in \mathbf{C}(X)$. Thus, the relative weak*-topology on $\Delta(X)$ coincides with the topology of weak-convergence. I will denote by τ the *bounded weak*-topology* on $ca(X)$. This is the finest topology that coincides with the weak*-topology on every set of the form $B_\lambda := \{\eta \in ca(X) : \|\eta\| \leq \lambda\}$ for $\lambda > 0$. Hence, a set $\mathfrak{D} \subseteq ca(X)$ is τ-open (resp. τ-closed) if and only if $\mathfrak{D} \cap B_\lambda$ is relatively weak*-open (resp. weak*-closed) in B_λ for every $\lambda > 0$.

Throughout the proof, I will write $\widetilde{u}(\eta)$ instead of $\int_X u \, d\eta$. In turn, for any nonempty set $N \subseteq \Delta(X)$ and $r \in \Delta(X)$, by $N \succ r$ I will mean that $w \succ r$ for every $w \in N$. The expression $r \succ N$ is understood analogously.

Note that the necessity of the open-continuity axiom for the representation is an immediate consequence of Claim 1, while the remainder of the "if" part of Theorem 1 is trivial.

To prove the "only if" part, let $\succ$ be an open-continuous strict preference relation on $\Delta(X)$. Put $\mathcal{C} := \{\gamma(p-q) : p \succ q,\ \gamma > 0\}$ and let $\mathcal{S}$ stand for the span of $\Delta(X)-\Delta(X)$. From Jordan decomposition theorem, it readily follows that $\mathcal{S} = \{\eta \in ca(X) : \eta(X) = 0\}$.

I omit the proof of the following claim, which is a routine exercise.

Claim 2. *$\mathcal{C}$ is a convex cone such that for every p, q in $\Delta(X)$, we have $p \succ q$ if and only if $p - q \in \mathcal{C}$.*

The next claim will be my main tool in what follows.

Claim 3. *For any $\lambda > 0$, the set $\mathcal{C} \cap B_\lambda$ is relatively weak*-open in $\mathcal{S} \cap B_\lambda$.*

Proof. Since $(ca(X), \|\cdot\|)$ is isometrically isomorphic to the topological dual of the separable Banach space $\mathbf{C}(X)$, the weak*-topology of B_λ is metrizable (cf. Dunford and Schwartz, 1958, Theorem V.5.1). Let σ stand for a compatible metric.

Suppose by contradiction that $\mathcal{C} \cap B_\lambda$ is not relatively weak*-open in $\mathcal{S} \cap B_\lambda$ for some $\lambda > 0$. Then there exists a point $\mu \in \mathcal{C} \cap B_\lambda$ such that, for every natural number n, we have $\sigma(\mu, \mu_n) < 1/n$ for some $\mu_n \in (\mathcal{S} \cap B_\lambda) \backslash \mathcal{C}$. Note that $\mu \neq 0$ since $\succ$ is irreflexive. Hence, by passing to a subsequence if necessary, we can assume that $\mu_n \neq 0$ for every n. By Jordan decomposition theorem, this implies that $\mu_n = \gamma_n(p_n - q_n)$ for some mutually singular p_n, q_n in $\Delta(X)$ and $\gamma_n > 0$. By mutual singularity, we have $\|p_n - q_n\| = 2$ for every n, and hence, $\gamma_n \leq \lambda/2$. Since $\Delta(X)$ is compact and (γ_n) is bounded, it follows that there is an increasing self-map $k \to n_k$ on $\mathbb{N}$ such that (γ_{n_k}), (p_{n_k}) and (q_{n_k}) are convergent subsequences. Let the corresponding limits be γ, p and q, respectively. Then, by construction, as $k \to \infty$ the sequence $\gamma_{n_k}(p_{n_k} - q_{n_k}) = \mu_{n_k}$ converges to both $\gamma(p-q)$ and μ in weak*-topology. It follows that $\gamma(p-q) = \mu$, while $\gamma > 0$ and $p - q = \mu/\gamma \in \mathcal{C}$. So, by Claim 2, we have $p \succ q$. Moreover, $\succ$ is an open subset of $\Delta(X)^2$ by Observation 3. From the definitions of p and q, it follows that $p_{n_k} \succ q_{n_k}$ for all large k, implying that $\mu_{n_k} \in \mathcal{C}$, a contradiction. $\square$

Claim 3 leads to the following conclusion in terms of the bounded weak*-topology.

Claim 4. *$\mathcal{C}$ is a relatively τ-open subset of $\mathcal{S}$.*

Proof. Fix any $\lambda > 0$, and note that $(\mathcal{S}\backslash\mathcal{C}) \cap B_\lambda = (\mathcal{S} \cap B_\lambda)\backslash(\mathcal{C} \cap B_\lambda)$. Therefore, Claim 3 implies that $(\mathcal{S}\backslash\mathcal{C}) \cap B_\lambda$ is a relatively weak*-closed subset of $\mathcal{S} \cap B_\lambda$. Since $\mathcal{S}$ and B_λ are both weak*-closed sets, so is $\mathcal{S} \cap B_\lambda$. It thus follows that $(\mathcal{S}\backslash\mathcal{C}) \cap B_\lambda$ is, in fact, a weak*-closed set. As λ is arbitrary, we conclude that $\mathcal{S}\backslash\mathcal{C}$ is a τ-closed set. This immediately implies the desired conclusion: $\mathcal{C}$ is a relatively τ-open subset of $\mathcal{S}$. $\square$

It is known that τ is a locally convex linear topology, and a linear functional on $ca(X)$ is τ-continuous if and only if it is weak*-continuous. These observations lead to Krein-Šmulian theorem: a convex subset of $ca(X)$ is τ-closed if and only if it is weak*-closed.[25] Thus, τ-closure of $\mathcal{C}$ coincides with its weak*-closure. In what follows, $\mathrm{cl}(\mathcal{C})$ denotes this set. It is also worth noting that since $\mathcal{S}$ is τ-closed, taking τ-closure of $\mathcal{C}$ relative to $\mathcal{S}$ also leads to the same set, $\mathrm{cl}(\mathcal{C})$. I will use these observations without further mention throughout the remainder of the proof. Moreover, I fix a pair of lotteries $p^\bullet, q^\bullet$ with $p^\bullet \succ q^\bullet$, and set $\eta^\bullet := p^\bullet - q^\bullet$.

Claim 5. *There exists a nonempty, compact set $U \subseteq \mathbf{C}(X)$ such that:*
(i) $\widetilde{u}(p^\bullet) = 1$ *and* $\widetilde{u}(q^\bullet) = 0$ *for every* $u \in U$;
(ii) $\mathrm{cl}(\mathcal{C}) = \{\eta \in \mathcal{S} : \widetilde{u}(\eta) \geq 0$ *for every* $u \in U\}$.

I will prove Claim 5 momentarily. Together with the next claim, this will complete the proof of Theorem 1.

Claim 6. *Given a set U as in* Claim 5, *for every p, q in $\Delta(X)$, we have $p \succ q$ if and only if $\widetilde{u}(p) > \widetilde{u}(q)$ for every $u \in U$.*

Proof. Consider a pair of lotteries p, q, and put $\mu := p - q$. Suppose first that $\widetilde{u}(\mu) > 0$ for every $u \in U$. Then, since U is compact, there exists a number $\beta > 0$ such that $\widetilde{u}(\mu) \geq \beta$ for every $u \in U$. Now pick any $\alpha \in (0, \beta)$. I shall show that μ belongs to the τ-interior of $\mathrm{cl}(\mathcal{C})$ (relative to $\mathcal{S}$). To this end, first note that, by Claim 4, the set $\mathcal{C} - \alpha\eta^\bullet$ is a τ-neighborhood of the origin. Thus, $\mu + (\mathcal{C} - \alpha\eta^\bullet)$ is a τ-neighborhood of μ. Moreover, any element η of this set is of the form $\eta = \mu + (\mu_1 - \alpha\eta^\bullet)$ for some $\mu_1 \in \mathcal{C}$. From the properties of U, it thus follows that $\widetilde{u}(\eta) \geq \beta - \alpha > 0$ for every $\eta \in \mu + (\mathcal{C} - \alpha\eta^\bullet)$ and $u \in U$. Then, applying part (ii) of Claim 5 yields $\mu + (\mathcal{C} - \alpha\eta^\bullet) \subseteq \mathrm{cl}(\mathcal{C})$. This implies that μ belongs to the τ-interior of $\mathrm{cl}(\mathcal{C})$, as we sought. But since $\mathcal{C}$ is a τ-open convex set, the τ-interior of $\mathrm{cl}(\mathcal{C})$ simply equals $\mathcal{C}$. Thus, $\mu \in \mathcal{C}$, that is, $p \succ q$.

Conversely, suppose now $p \succ q$ so that $\mu \in \mathcal{C}$. Take any $u \in U$. Since $\mathcal{C}$ is τ-open, it is also algebraically open. Thus, there exists an $\alpha > 0$ such that $\mu - \alpha\eta^\bullet \in \mathcal{C}$. By Claim 5(ii), we therefore have $\widetilde{u}(\mu - \alpha\eta^\bullet) \geq 0$, that is, $\widetilde{u}(\mu) \geq \alpha$. $\square$

Proof of Claim 5. Let us define $\mathfrak{G} := \{u \in \mathbf{C}(X) : \widetilde{u}(\eta) \geq 0$ for every $\eta \in \mathrm{cl}(\mathcal{C})\}$, $U := \{u \in \mathfrak{G} : \widetilde{u}(p^\bullet) = 1,\ \widetilde{u}(q^\bullet) = 0\}$ and $\mathcal{C}^+ := \{\eta \in \mathcal{S} : \widetilde{u}(\eta) \geq 0$ for every $u \in U\}$.

[25] These results apply on the topological dual of any Banach space. For a detailed discussion, see Dunford and Schwartz (1957, Section V.5), in particular Corollary V.5.5, Theorems V.5.6 and V.5.7.

Note that $\mathfrak{G}$ is closed, and as a closed subset of $\mathfrak{G}$, the set U is also closed. Hence, by Arzelà-Ascoli theorem, to verify compactness of U it suffices to show that this set is bounded and equicontinuous.

Since the weak*-topology is coarser than the norm-topology of $ca(X)$, and since $\Delta(X)$ is a norm-bounded set, applying the open-continuity axiom to the lotteries $p^\bullet, q^\bullet$ yields an $\alpha \in (0,1)$, close enough to 1, such that $p^\bullet \succ \alpha q^\bullet + (1-\alpha)\,\Delta(X)$ and $\alpha p^\bullet + (1-\alpha)\,\Delta(X) \succ q^\bullet$. In particular, we have $p^\bullet \succ \alpha q^\bullet + (1-\alpha)\,\delta_x$ and $\alpha p^\bullet + (1-\alpha)\,\delta_x \succ q^\bullet$ for each $x \in X$. Thus, by definition of U, we see that $\frac{1}{1-\alpha} \geq u(x) \geq \frac{-\alpha}{1-\alpha}$ for every $u \in U$ and $x \in X$. Hence, the set U is bounded.

Now, fix an $x \in X$ and $\varepsilon > 0$. Pick an $\alpha \in (0,1)$ such that $\frac{\alpha}{1-\alpha} < \varepsilon$. Since $\alpha p^\bullet + (1-\alpha)\,\delta_x \succ \alpha q^\bullet + (1-\alpha)\,\delta_x$, clearly, the open-continuity axiom implies that there is an open set $O \subseteq X$, which contains x, such that $\alpha p^\bullet + (1-\alpha)\,\delta_z \succ \alpha q^\bullet + (1-\alpha)\,\delta_x$ and $\alpha p^\bullet + (1-\alpha)\,\delta_x \succ \alpha q^\bullet + (1-\alpha)\,\delta_z$ for every $z \in O$. It readily follows that $|u(x) - u(z)| \leq \frac{\alpha}{1-\alpha} < \varepsilon$ for every $z \in O$ and $u \in U$. Hence, U is also equicontinuous, as we sought.

It remains to show that $\mathcal{C}^+ = \mathrm{cl}(\mathcal{C})$ and U is nonempty. That $\mathcal{C}^+ \supseteq \mathrm{cl}(\mathcal{C})$ follows from definitions immediately. To prove the converse inclusion, first note that since $\mathrm{cl}(\mathcal{C})$ is a weak*-closed convex cone, by standard separation and duality arguments, for each $\eta \in \mathcal{S} \setminus \mathrm{cl}(\mathcal{C})$ we can find a function $u \in \mathfrak{G}$ such that $\widetilde{u}(\eta) < 0$.

I shall now show that $q^\bullet - p^\bullet$ does not belong to $\mathrm{cl}(\mathcal{C})$. To this end, note that by Claim 4, the set $\mathcal{C} - \frac{\eta^\bullet}{2}$ is a τ-neighborhood of the origin (relative to $\mathcal{S}$). Thus, $-\eta^\bullet - \left(\mathcal{C} - \frac{\eta^\bullet}{2}\right)$ is a τ-neighborhood of $-\eta^\bullet$. This set does not intersect $\mathcal{C}$, for otherwise we would have $-\eta^\bullet - \left(\mu_1 - \frac{\eta^\bullet}{2}\right) = \mu_2$ for some μ_1, μ_2 in $\mathcal{C}$. In turn, this would imply $-\eta^\bullet = 2\,(\mu_1 + \mu_2) \in \mathcal{C}$, and hence, $q^\bullet \succ p^\bullet$, which contradicts asymmetry of $\succ$. Thereby, we have shown that there exists a τ-neighborhood of $-\eta^\bullet = q^\bullet - p^\bullet$ that does not intersect $\mathcal{C}$. This simply means that $q^\bullet - p^\bullet$ does not belong to $\mathrm{cl}(\mathcal{C})$, as we sought.

By combining the observations above, we see that $\widetilde{u_\bullet}(q^\bullet - p^\bullet) < 0$ for some $u_\bullet \in \mathfrak{G}$. To complete the proof that $\mathcal{C}^+ \subseteq \mathrm{cl}(\mathcal{C})$, let $\eta \in \mathcal{S} \setminus \mathrm{cl}(\mathcal{C})$ and pick a $u \in \mathfrak{G}$ such that $\widetilde{u}(\eta) < 0$. Fix a sufficiently small $\alpha > 0$ such that $\widetilde{u}(\eta) + \alpha \widetilde{u_\bullet}(\eta) < 0$. Notice that $u_1 := u + \alpha u_\bullet$ belongs to $\mathfrak{G}$. Moreover, $\widetilde{u_1}(q^\bullet - p^\bullet) < 0$, for $\widetilde{u}(q^\bullet - p^\bullet) \leq 0$ by definition of $\mathfrak{G}$. Now, set $v_1 := \frac{1}{\widetilde{u_1}(p^\bullet - q^\bullet)}\,(u_1 - \widetilde{u_1}(q^\bullet)\mathbf{1}_X)$. It readily follows that v_1 is an element of U such that $\widetilde{v_1}(\eta) = \frac{\widetilde{u_1}(\eta)}{\widetilde{u_1}(p^\bullet - q^\bullet)} < 0$. Hence, $\eta \notin \mathcal{C}^+$, which shows that $\mathcal{C}^+ \subseteq \mathrm{cl}(\mathcal{C})$.

Finally, note that since $q^\bullet - p^\bullet \in \mathcal{S} \setminus \mathrm{cl}(\mathcal{C})$, the argument above also shows that the set U is nonempty. $\square$

D. A DMO Type Representation with a Compact set of Utility Functions

Consider the following axiom imposed on the asymmetric part of a preorder $\succsim^*$ on $\Delta(X)$.

Directional Open-Continuity. There exist a pair of lotteries $p^\bullet, q^\bullet$ such that for each $r \in \Delta(X)$ and $\alpha \in (0,1]$, we have $N_1 \succ^* \alpha q^\bullet + (1-\alpha)r$ and $\alpha p^\bullet + (1-\alpha)r \succ^* N_2$ for a neighborhood N_1 of $\alpha p^\bullet + (1-\alpha)r$ and a neighborhood N_2 of $\alpha q^\bullet + (1-\alpha)r$.

This axiom means that the open-continuity property holds on every pair of compound lotteries ρ^1, ρ^2 whenever ρ^1 can be obtained from ρ^2 by shifting a positive weight from a lottery $q^\bullet$ to a "strongly better" lottery $p^\bullet$. Here, the term "strongly better" corresponds to an open-continuous strict preference relation that is a subset of $\succ^*$ (which can be defined in an obvious way, building upon the statement of the axiom).

Let us also recall the independence axiom utilized by DMO:

Independence*. $p \succsim^* q$ implies $\alpha p + (1-\alpha)r \succsim^* \alpha q + (1-\alpha)r$ for every $p, q, r \in \Delta(X)$ and $\alpha \in [0,1]$.

The following DMO type representation theorem is a side payoff of my main findings, which delivers a compact set of utility functions.

Theorem D. *Let X be a compact metric space. A binary relation $\succsim^*$ on $\Delta(X)$ is a closed preorder that satisfies Directional Open-Continuity and Independence* if, and only if, there exists a nonempty compact set $U \subseteq \mathbf{C}(X)$ such that:*
(i) *For every p, q in $\Delta(X)$, we have $p \succsim^* q$ if and only if $E(p,u) \geq E(q,u)$ for every $u \in U$.*
(ii) *$E(p^\bullet, u) > E(q^\bullet, u)$ for every $u \in U$ and some $p^\bullet, q^\bullet$ in $\Delta(X)$.*

The proof of Theorem D follows the proof of Claim 5 above. The only remarkable difference is that we should replace $\mathrm{cl}(\mathcal{C})$ with the set $\{\gamma(p-q) : p \succsim^* q,\ \gamma \geq 0\}$, which is shown to be weak*-closed by DMO. Moreover, upon normalization of the representing set of utility functions, in the present set-up we can also obtain a uniqueness theorem analogous to Theorem 2.

E. An Extension of Galaabaatar-Karni Representation

Let $\succ$ be an open-continuous strict preference relation on $\Delta(X)$. If we consider $\succ$ as a primitive object, how can we define the weak preference relation of the decision maker? Theorem 3 of the present paper and the representation theorem of Galaabaatar and Karni (forthcoming a) provide different solutions to this question.

If we follow Theorem 3, the weak preference relation should be defined as the union of $\succ$ with an indifference relation. In turn, in view of Observation 5, there are two equivalent methods of defining the indifference relation:

(i) $p \sim q$ if there exists an $\alpha \in (0,1]$ and $w \in \Delta(X)$ such that $\alpha p + (1-\alpha)w$ belongs to the closures of both $\{r \in \Delta(X) : r \succ \alpha q + (1-\alpha)w\}$ and $\{r \in \Delta(X) : \alpha q + (1-\alpha)w \succ r\}$.

(ii) $p \sim q$ if for each $\alpha \in [0,1]$ and $r, w \in \Delta(X)$, we have $r \succ \alpha p + (1-\alpha)w$ iff $r \succ \alpha q + (1-\alpha)w$, while $\alpha p + (1-\alpha)w \succ r$ iff $\alpha q + (1-\alpha)w \succ r$.

It should be noted that in these statements, "there exists" and "for each" clauses as well as the mixture operations are motivated by the fact that sets of the form $\{r \in \Delta(X) : r \succ q\}$ and $\{r \in \Delta(X) : q \succ r\}$ may be empty. Moreover, as can easily be seen, both of the definitions *(i)* and *(ii)* imply that $p \sim q$ iff $E(p,u) = E(q,u)$ for every $u \in U$, where U is a utility set that represents $\succ$.

As I discussed earlier, the most important difference of Galaabaatar-Karni approach is that the asymmetric part of a weak preference relation in their sense does not coincide with the primitive, strict preference relation, $\succ$. They also avoid the emptiness problem that I discussed above by assuming the existence of a pair of lotteries p^*, p_* such that $p^* \succ q \succ p_*$ for every $q \in \Delta(X)$ that is distinct from p^* and p_*. Upon relaxation of this condition, a weak preference relation in the sense of Galaabaatar and Karni can be defined as follows:

$$p \succsim_{GK} q \quad \text{if} \quad r \succ \alpha p + (1-\alpha)w \text{ implies } r \succ \alpha q + (1-\alpha)w \text{ for every } r, w \in \Delta(X) \text{ and } \alpha \in [0,1].$$

The next theorem is an extension of the representation theorem of Galaabaatar and Karni (for preference relations over risky prospects) that also allows for infinitely many prizes.

Theorem E. *Let X be a compact metric space. For a pair of binary relation $\succ$ and $\succsim^*$ on $\Delta(X)$ the following two statements are equivalent:*
(i) *$\succ$ is an open-continuous strict preference relation and $\succsim^* = \succsim_{GK}$.*
(ii) *$\succ$ admits a utility set $U \subseteq \mathbf{C}(X)$ that also represents $\succsim^*$ in the sense of DMO.*

Proof. In view of Theorem 1, it suffices to show that if $U \subseteq \mathbf{C}(X)$ is a utility set for an open-continuous strict preference relation $\succ$, then for every $p, q \in \Delta(X)$,

$$p \succsim_{GK} q \quad \text{iff} \quad E(p,u) \geq E(q,u) \text{ for every } u \in U.$$

Indeed, if $E(p,u) \geq E(q,u)$ for every $u \in U$, then $E(\alpha p + (1-\alpha)w, u) \geq E(\alpha q + (1-\alpha)w, u)$ for every $u \in U$, $\alpha \in [0,1]$ and $w \in \Delta(X)$. Hence, by definitions, it readily follows that $p \succsim_{GK} q$. Conversely, suppose $p \succsim_{GK} q$. Pick a pair of lotteries $p^\bullet, q^\bullet$ such that $p^\bullet \succ q^\bullet$. Then, for all $\alpha \in (0,1)$, we have $\alpha p + (1-\alpha)p^\bullet \succ \alpha p + (1-\alpha)q^\bullet$. By definition of $\succsim_{GK}$, this implies $\alpha p + (1-\alpha)p^\bullet \succ \alpha q + (1-\alpha)q^\bullet$ for every $\alpha \in (0,1)$.

Invoking the definition of U, we then see that $E(\alpha p+(1-\alpha)p^{\bullet}, u) > E(\alpha q+(1-\alpha)q^{\bullet}, u)$ for every $u \in U$ and $\alpha \in (0,1)$. Passing to limit as $\alpha \to 1$ yields the desired conclusion: $E(p,u) \geq E(q,u)$ for every $u \in U$. $\square$

Remark E1. The original definition of Galaabaatar and Karni reads as follows:

$$p \succsim_{GK} q \quad \text{if} \quad r \succ p \text{ implies } r \succ q \text{ for every } r \in \Delta(X). \tag{10}$$

With this definition, the conclusion of Theorem E may fail unless there exist p^*, p_* as I described above. For example, let u_1 and u_2 be distinct functions in a utility set U that represents $\succ$. Pick a lottery p in $\arg\max_{r\in\Delta(X)} E(r, u_1)$ and a (possibly distinct) lottery q in $\arg\max_{r\in\Delta(X)} E(r, u_2)$. Then, according to definition (10), we voidly have $p \sim_{GK} q$, but in general $E(p,u)$ may well be different than $E(q,u)$ for some $u \in U$.

Remark E2. In view of the proof of Theorem E, definition *(ii)* is equivalent to the following statement: $p \sim q$ if for each $\alpha \in [0,1]$ and $r, w \in \Delta(X)$, we have $r \succ \alpha p + (1-\alpha)w$ iff $r \succ \alpha q + (1-\alpha)w$.

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