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Pricing Central Tendency in Volatility

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PRICING CENTRAL TENDENCY IN VOLATILITY

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Abstract

It is widely accepted that there is a risk of fluctuating volatility. There is some evidence, analogously to long-term consumption risk literature or central tendency in interest rates, that there exists a slowly varying component in volatility. Volatility literature concentrates on investigation of two-factor volatility process, with one factor being very persistent. I propose a different parametrization of volatility process that includes this persistent component as a stochastic central tendency. The reparametrization is observationally equivalent but has compelling economic interpretation. I estimate the historical and risk-neutral parameters of the model jointly using GMM with the data on realized volatility and VIX volatility index and treating central tendency as completely unobservable. The main empirical result of the paper is that on average the volatility premium is mainly due to the premium on highly persistent shocks of the central tendency.

Keywords: stochastic volatility; central tendency; volatility risk premium; GMM

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1 Introduction

It is widely accepted that there is a risk of fluctuating volatility. There is some evidence, analogously to long-term consumption risk literature (Bansal & Yaron, 2004) or central tendency in interest rates (Balduzzi et al., 1998)¹, that there exists a slowly varying component in volatility. Bollerslev & Mikkelsen (1996) and Andersen & Bollerslev (1997) find strong evidence of high volatility persistence. Volatility literature concentrates on investigation of two-factor volatility process, with one factor being very persistent. I propose a different parametrization of volatility process that includes this persistent component as a stochastic drift or central tendency of market volatility. The reparametrization is likely observationally equivalent but has compelling economic interpretation. With this model I am able to price two volatility components separately.² I estimate the historical and risk-neutral parameters of the model jointly using GMM with the data on realized volatility and VIX volatility index and treating central tendency as completely unobservable.

The main result of the paper is that on average the volatility premium is at large extent comes from the premium on highly persistent shocks of the central tendency. Additional short lived but very volatile shocks bear a small but statistically significant premium. The volatility premium in most part compensates for the shocks in stochastic volatility drift rather than shocks of fast mean reversion to this central tendency. Hence, the role of shock persistence is crucial in determining the compensation for volatility risks.

The model I propose is very similar in structure to Duffie et al. (2000) and Bollerslev et al. (2010)³ who are generally concerned with memory patterns in stock market volatility and volatility premium. They employ continuous-time general equilibrium approach together with Epstein & Zin (1989) time non-separable preferences. These preferences are a crucial feature of the model that allows to separate volatility and volatility of volatility risk premia. In my model the second priced factor is central tendency, or stochastic drift of market volatility. Although the stochastic discount factor in my model does not come from the particular assumption on investor preference structure, it implies a similar compensation structure for different sources of volatility risk.

Analogously to continuous time interest rate model of Cox et al. (1985) it became wide spread in financial literature to model stochastic volatility as a mean-reverting process around constant mean level. The seminal work in this direction is done by Heston (1993) who proposed stochastic volatility continuous-time option pricing model. It is also well known that market

¹See also Andersen & Lund (1997); Reschreiter (2010, 2011).

²Adrian & Rosenberg (2008) find some evidence that volatility components risk, both short and long-run, are priced by analyzing a cross-section of portfolio returns.

³For analogous discrete time approach see Tauchen (2005) and Bollerslev et al. (2009).

volatility is highly persistent and has a thick tailed stationary distribution. Moreover, it is widely accepted that one-factor stochastic volatility models do not fit well and can not capture high persistence and thick tails at the same time. The idea of multi-factor volatility model dates back to [Engle & Lee \(1996\)](#) who consider several specifications of continuous stochastic volatility model. One of the specifications includes two additive volatility factors one of them being very persistent. The models are discretized using Euler approximation to match GARCH form and estimated with QML.

I propose a continuous-time model of market volatility where drift is not constant but rather stochastic and is driven by a separate mean-reverting stochastic process with its own random innovation. [Andersen & Lund \(1997\)](#) develop a continuous-time model of interest rates that has a stochastic drift instead of constant mean level inside of the interest rate dynamics. In this paper I do the same for stochastic volatility of the return. The appealing interpretation of such modeling approach, in contrast to additive component representation, is the interpretation of central tendency as a stochastic mean of volatility which determines the average level of volatility for a prolonged period of time.

[Engle & Lee \(1999\)](#) propose GARCH-like specification of stochastic volatility with unconditional mean replaced with slowly varying second GARCH component. In this specification the difference between two components is interpreted as transitory volatility component. One of the key limitations of this model is that only one innovation term drives both volatility components which does not play well with an idea of several sources of volatility risk. In a similar modeling approach [Christoffersen et al. \(2008\)](#) stress the result that a two-component model fits better than a one-component model with jumps.

The disadvantage of GARCH models is that they are not closed under temporal aggregation ([Drost & Nijman, 1993](#)) and parameter estimates are critically dependent on sampling interval. In this paper I derive exact discretization of the model with stochastic drift. Discretized joint model of volatility and central tendency is a vector autoregression of the order one with moving average heteroscedastic error structure of order one. The error structure is also kept in explicit form of stochastic integrals.

[Gallant et al. \(1999\)](#) estimate two-factor additive stochastic volatility model using Efficient Method of Moments and find that this model may successfully account for long memory effect. [Chernov et al. \(2003\)](#) evaluate empirically several continuous-time model specifications of stochastic volatility. In particular, some specifications include two additive volatility factors. One factor is responsible for tail thickness of returns, the other reflects volatility persistence. [Corsi \(2009\)](#) propose an additive cascade model of several volatility components that have different effect depending on the time horizon. He shows that despite the absence of genuine long memory the model is very successful in reproducing empirical characteristics of the returns.

Duffie et al. (2000) in conclusion to the paper propose a two-factor model of stochastic volatility model where one factor plays a role of stochastic trend rather than just an additional additive factor. They argue that given sufficiently small speed of mean reversion this factor may capture long memory in volatility which is argued to be evident in the data. Bates (2000) employ the continuous-time model of S&P500 index options with two-factor additive stochastic volatility. Volatility risk premium is assumed to be exogenous and proportional to current volatility. This assumption is consistent with simple log utility. Model parameters are estimated implicitly through minimization of option pricing errors.

When estimating stochastic volatility models the question of measurement is critical. Clearly, point-in-time volatility in continuous-time model is unobservable. Instead one has to use some approximations or implied measures. Andersen & Bollerslev (1998) give theoretical justification for approximation of integrated volatility using high-frequency return data. Bollerslev & Zhou (2002)⁴ propose an elegant approach to estimate parameters of structural continuous-time model of returns with stochastic volatility. The main idea is to express moment conditions in terms of integrated volatility rather than point-in-time values. Historical integrated volatility is measured as realized volatility or standard deviation of high frequency returns over daily period. Jiang & Oomen (2007) approach the problem of latent variable estimation from a different perspective but also on the basis of high-frequency data and GMM.

Joint estimation of the model parameters requires not only historical observation of market volatility but also a risk-neutral expectation of volatility. Risk-neutral measure relative to objective measure provides a link to investor preference parameters. Britten-Jones & Neuberger (2000) provide the theoretical justification for the model-free measure of integrated volatility which only requires current option prices. Carr & Wu (2009) generalize this approach and use it to analyze historical dynamics of variance risk premia of multiple indexes and individual stocks. The general idea of model-free measurement is to use a large set of option prices to construct a volatility measure. This measure is represented by a VIX volatility index.⁵

Given a particular stochastic discount factor (SDF) I link parameters of risk-neutral dynamics of volatility to its historical evolution. Theoretical model implies that risk-neutral volatility measure depends not only on historical structural parameters but also on risk prices. This connection logically requires joint estimation using both volatility measures. Garcia et al. (2011) estimate parameters of a continuous-time stochastic volatility model both for objective and risk-neutral distributions jointly. Risk-neutral measure of volatility is based on option price series expansion. In this paper I estimate joint model using the VIX index which is a broader and likely less noisy measure of volatility. Chernov & Ghysels (2000) use efficient method of

⁴See also Renault (2009)

⁵See also Jiang & Tian (2007) for detailed justification.

moments to estimate jointly historical and risk-neutral distribution parameters and filter out spot volatility. [Bollerslev et al. \(2011\)](#) also estimate joint volatility model but it lacks above mentioned multi-factor volatility specification.

Another contribution relative to methodology of [Bollerslev & Zhou \(2002\)](#) is that I keep the explicit definitions of model innovations in terms of stochastic integrals. This allows me to account for all possible interaction between the variables and innovations in analytical form. In particular, [Bollerslev & Zhou \(2002\)](#) rely on unbiased estimator of squared volatility (see [Renault, 2009](#)).

Inclusion of the stochastic drift in volatility model somewhat complicates econometric approach. First of all, integrated trend which shows up in discretized model is unobservable and there is no convenient proxy for it. Hence, I integrate it out which results in higher order ARMA structure for integrated volatility. But at the same time it preserves identification of structural model parameters and allows for the use of standard GMM procedure ([Hansen, 1982](#)).

The rest of the paper is organized as follows. [Section 2](#) states the continuous-time stochastic volatility model of the market return both for historical and risk-neutral distributions. [Section 3](#) shows how to discretize continuous-time model and represent it in terms of integrated variables. [Section 4](#) presents the decomposition of volatility premia and shows theoretical contribution of central tendency premia. [Section 5](#) outlines estimation strategy. [Section 6](#) describes empirical results. [Section 7](#) concludes.

2 The Model

In this section I present the continuous-time model of the stochastic volatility with the drift that is also stochastic. This drift represents the persistent central tendency of volatility or its slowly varying average level. The continuous-time diffusion is basically the extension of the square-root process used by [Heston \(1993\)](#) for option pricing. I show that with such a modification my model has a potential of matching the high persistence of volatility observed in the data as evident from the theoretical autocorrelation function of the spot volatility.

I assume a matching square-root form of stochastic discount factor (SDF) that assigns prices for shocks both in central tendency and volatility itself. Given the SDF it is easy to bridge historical distribution of returns, volatility, and central tendency with its equivalent risk-neutral distribution used for standard no-arbitrage pricing. Under this equivalent distribution the model form remains intact but most of the parameters are altered. In particular, assuming negative prices for volatility and central tendency risk, both processes become more persistent and have a higher unconditional mean level.

Consider the probability space (Ω, \mathcal{F}, P) which is a fundamental space of the stochastic

market price S_t . Assume that the log of stock price $p_t = \log S_t$ evolves according to the following stochastic differential equation:

$$dp_t = (\mu_r + \mu_\pi) dt + \sigma_t dW_t^r,$$

where constant parameters μ_r, μ_π and stochastic variables σ_t^2, y_t to be explained later. Here σ_t^2 plays a role of instantaneous variance of the market return. Define the integrated variance of the return over the h time interval,

$$\int_t^{t+h} d[p, p]_u = \int_t^{t+h} \sigma_u^2 du \equiv \mathcal{V}_{t,h}.$$

Also assume that this instantaneous volatility mean reverts to a stochastic central tendency which in turn mean reverts to a constant long-term mean of volatility. This volatility structure is similar in spirit to what is suggested by [Duffie et al. \(2000\)](#) in the end of the paper. These assumptions may be written in diffusion form as follows:

$$\begin{aligned} d\sigma_t^2 &= \kappa_\sigma (y_t - \sigma_t^2) dt + \eta_\sigma \sigma_t dW_t^\sigma, \\ dy_t &= \kappa_y (\mu - y_t) dt + \eta_y \sqrt{y_t} dW_t^y, \end{aligned} \tag{2.1}$$

where W_t^r, W_t^σ , and W_t^y are three standardized independent Brownian motion processes under the historical probability measure P . Under the suitable regularity conditions (see [Karatzas & Shreve, 1997](#)) the above multivariate diffusion has a unique strong solution on \mathbb{R}^+ . The parameter vector θ is assumed to lie within some compact set $\Theta \subset \mathbb{R}^d$. Provided that $2\mu\kappa_y \geq \eta_y^2$ the process y_t has a stationary Gamma distribution.

The reason I can call y_t a central tendency is the following. As I show in [Section B.2](#) the autocorrelation of the spot volatility is given by

$$\text{Corr}(\sigma_{t+h}^2, \sigma_t^2) = e^{-\kappa_\sigma h} + (e^{-\kappa_y h} - e^{-\kappa_\sigma h}) \frac{\kappa_\sigma}{\kappa_\sigma - \kappa_y} \frac{\eta_y^2}{\kappa_y} \left(\frac{\eta_y^2}{\kappa_y} + \frac{\kappa_\sigma + \kappa_y}{\kappa_\sigma} \frac{\eta_\sigma^2}{\kappa_\sigma} \right)^{-1}.$$

This formula shows that if mean reversion speed of y_t is much smaller than the mean reversion speed of spot volatility itself, then the autocorrelation function in the long horizons will be mainly due to component $e^{-\kappa_y h}$ which decays very slowly with small κ_y .

Now let the log stochastic discount factor (SDF) process $m_t = \log M_t$ be represented by the following SDE:

$$dm_t = -\mu_r dt - \frac{\mu_\pi}{\sigma_t} dW_t^r + \lambda_\sigma \sigma_t dW_t^\sigma + \lambda_y \sqrt{y_t} dW_t^y. \tag{2.2}$$

Here the vector $\left[\frac{\mu_\pi}{\sigma_t}, -\lambda_\sigma \sigma_t, -\lambda_y \sqrt{y_t} \right]$ is interpreted as a vector of risk prices arising from different

sources of uncertainty. The first element of the vector is the price of equity risk, the second is the price of volatility risk, and the third is the price of risk related to persistent stochastic mean of volatility.

Applying Girsanov's theorem to the model at hand I can define the new set of Brownian motion processes that are equivalent to the original

$$\begin{aligned} d\tilde{W}_t^r &= dW_t^r + \frac{\mu_\pi}{\sigma_t} dt, \\ d\tilde{W}_t^\sigma &= dW_t^\sigma - \lambda_\sigma \sigma_t dt, \\ d\tilde{W}_t^y &= dW_t^y - \lambda_y \sqrt{y_t} dt. \end{aligned}$$

This adjustment in Brownian innovations provides a new set of standard uncorrelated Brownian motions under risk-neutral probability measure Q on (Ω, \mathcal{F}) .

Under this new probability measure the model may be written as

$$\begin{aligned} dp_t &= \mu_r dt + \sigma_t d\tilde{W}_t^r, \\ d\sigma_t^2 &= \tilde{\kappa}_\sigma (\tilde{y}_t - \sigma_t^2) dt + \eta_\sigma \sigma_t d\tilde{W}_t^\sigma, \\ d\tilde{y}_t &= \tilde{\kappa}_y (\tilde{\mu} - \tilde{y}_t) dt + \tilde{\eta}_y \sqrt{\tilde{y}_t} d\tilde{W}_t^y, \end{aligned} \tag{2.3}$$

where the rescaled central tendency is

$$\tilde{y}_t = \frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} y_t,$$

and the modified parameters are

$$\tilde{\kappa}_\sigma = \kappa_\sigma - \lambda_\sigma \eta_\sigma, \quad \tilde{\kappa}_y = \kappa_y - \lambda_y \eta_y, \quad \tilde{\mu} = \mu \frac{\kappa_y \kappa_\sigma}{\tilde{\kappa}_y \tilde{\kappa}_\sigma}, \quad \tilde{\eta}_y = \eta_y \sqrt{\frac{\kappa_\sigma}{\tilde{\kappa}_\sigma}}.$$

Note that in general according to Girsanov's theorem the shift in drift does not alter the instantaneous diffusion parameters (in this case η_σ and η_y). It may seem that this rule is broken as the instantaneous diffusion parameter for the central tendency is a multiple of η_y . This modification is only due to rescaling of y_t itself. This rescaling preserves the interpretation of modified \tilde{y}_t as a central tendency under the risk-neutral measure.

3 Exact discretization

Clearly, the continuous-time model is a convenient theoretical construct. But in all of the empirical work we only deal with discretely observed data. In this section I show how continuous-

time stochastic volatility model with stochastic mean may be exactly discretized. Even after discretization the financial literature does not give a fool-proof recipe to measure spot volatility. It became a de facto standard that we can reliably measure only integrated volatility using high frequency data. Hence, I proceed in this section by deriving the dynamic discrete model where state variables are integrated volatility and central tendency. Besides, for further considerations of premia it is crucial to look at figures accumulated over some meaningful amount of time rather than instantaneous values.

It is well known that the exact discretization of continuous-time square-root process is a heteroscedastic first order autoregression. Since I have two interacting spot variables, the discretized system is likely to be of vector autoregressive form of order one. To make a transfer to the integrated state variables I also integrate the innovations which leads to the same order one vector autoregression but with more complicated moving average innovations of order one.

One more complication which will become evident later is the necessity to build the discrete model for variable integrated over a larger period of time than the lag in the autoregression. In this section I show that the model is not straight vector autoregression anymore but for estimation method of my choice it does not present a significant problem.

As Section [B.1](#) shows in more detail the spot volatility model is discretized as

$$\begin{aligned}\sigma_{t+h}^2 &= A_h^\sigma \sigma_t^2 + B_h^\sigma y_t + C_h^\sigma + \epsilon_{t,h}^\sigma, \\ y_{t+h} &= A_h^y y_t + C_h^y + \epsilon_{t,h}^y,\end{aligned}\tag{3.1}$$

where I define coefficients as

$$A_h^\sigma = \exp(-\kappa_\sigma h), \quad B_h^\sigma = \frac{\kappa_\sigma}{\kappa_\sigma - \kappa_y} (A_h^y - A_h^\sigma), \quad C_h^\sigma = \mu (1 - A_h^\sigma - B_h^\sigma),$$

and

$$A_h^y = \exp(-\kappa_y h), \quad C_h^y = \mu (1 - A_h^y).$$

Note that A_h^y and A_h^σ are multiplicative functions of time interval, that is $A_u^y A_v^y = A_{u+v}^y$. Subscripted notation for the error terms means that they are amalgamations of continuous Brownian innovations starting from the moment zero to h . In particular, the error structure of the discretized model is given as

$$\begin{aligned}\epsilon_{t,h}^\sigma &= \eta_\sigma \int_t^{t+h} \sigma_u A_{t+h-u}^\sigma dW_u^\sigma + \eta_y \int_t^{t+h} \sqrt{y_u} B_{t+h-u}^\sigma dW_u^y, \\ \epsilon_{t,h}^y &= \eta_y \int_t^{t+h} \sqrt{y_u} A_{t+h-u}^y dW_u^y.\end{aligned}$$

Note that the volatility innovation accumulates Brownian terms from both central tendency diffusion, and volatility diffusion itself. Also observe that the system in (3.1) is actually a bivariate vector autoregression of order one with state variables (σ_t^2, y_t) and heteroscedastic errors adapted to $\mathcal{F}_{t+h} = \sigma(W_u^\sigma, W_u^y | u \leq t+h)$.

Clearly, the same discretization technique may be applied to the risk-neutral model in (2.3).

So I have

$$\begin{aligned}\sigma_{t+h}^2 &= \tilde{A}_h^\sigma \sigma_t^2 + \tilde{B}_h^\sigma \tilde{y}_t + \tilde{C}_h^\sigma + \tilde{\epsilon}_{t,h}^\sigma, \\ \tilde{y}_{t+h} &= \tilde{A}_h^y \tilde{y}_t + \tilde{C}_h^y + \tilde{\epsilon}_{t,h}^y,\end{aligned}\tag{3.2}$$

with

$$\tilde{A}_h^\sigma = \exp(-\tilde{\kappa}_\sigma h), \quad \tilde{B}_h^\sigma = \frac{\tilde{\kappa}_\sigma}{\tilde{\kappa}_\sigma - \tilde{\kappa}_y} (\tilde{A}_h^y - \tilde{A}_h^\sigma), \quad \tilde{C}_h^\sigma = \tilde{\mu} (1 - \tilde{A}_h^\sigma - \tilde{B}_h^\sigma),$$

and

$$\tilde{A}_h^y = \exp(-\tilde{\kappa}_y h), \quad \tilde{C}_h^y = \tilde{\mu} (1 - \tilde{A}_h^y).$$

The error structure in the risk-neutral model is

$$\begin{aligned}\tilde{\epsilon}_{t,h}^\sigma &= \eta_\sigma \int_t^{t+h} \sigma_u \tilde{A}_{t+h-u}^\sigma d\tilde{W}_u^\sigma + \tilde{\eta}_y \int_t^{t+h} \sqrt{\tilde{y}_u} \tilde{B}_{t+h-u}^\sigma d\tilde{W}_u^y, \\ \tilde{\epsilon}_{t,h}^y &= \tilde{\eta}_y \int_t^{t+h} \sqrt{\tilde{y}_u} \tilde{A}_{t+h-u}^y d\tilde{W}_u^y.\end{aligned}$$

At this stage we already have exactly discretized model of spot volatility and central tendency, but my final target is the model for integrated volatility and central tendency. It is also argued later that instead of working with instantaneous variables it is more feasible to work with integrated variables. In order to make a transfer from instantaneous (σ_t^2, y_t) to integrated analog $(\mathcal{V}_{t,h}, \mathcal{Y}_{t,h})$ I integrate equations in (3.1) over the reference point in time as a dummy. Integrated volatility and integrated central tendency are defined as

$$\mathcal{V}_{t,h} \equiv \frac{1}{h} \int_t^{t+h} \sigma_u^2 du, \quad \mathcal{Y}_{t,h} \equiv \frac{1}{h} \int_t^{t+h} y_u du.$$

Here the first subscripted value denotes the beginning of the time interval, and the second denotes the length of this interval. In this particular case the integration interval starts at t and ends at $t+h$.

Integrate the linear system in (3.1) over t as a dummy of integration in the interval $[0, h]$:

$$\begin{aligned}\mathcal{V}_{t+h,h} &= A_h^\sigma \mathcal{V}_{t,h} + B_h^\sigma \mathcal{Y}_{t,h} + C_h^\sigma + \frac{1}{h} \int_0^h \epsilon_{t+s,h}^\sigma ds, \\ \mathcal{Y}_{t+h,h} &= A_h^y \mathcal{Y}_{t,h} + C_h^y + \frac{1}{h} \int_0^h \epsilon_{t+s,h}^y ds.\end{aligned}\tag{3.3}$$

This system of equations is again a bivariate vector autoregression of order one with respect to state vector $(\mathcal{V}_{t,h}, \mathcal{Y}_{t,h})$. But the error structure is a bit more complicated than it was for instantaneous state vector. Here the errors aggregate Brownian shocks not only over the period $[t+h, t+2h]$ but also from the previous period $[t, t+h]$. This fact makes the error structure a moving average of order one. These errors are measurable with respect to \mathcal{F}_{t+2h} but completely unpredictable with respect to \mathcal{F}_t . Hence the system in (3.3) is of almost VARMA(1,1) form. The error is a composition of shocks over the period $[t, t+h]$ and over $[t+h, t+2h]$. But error terms over both of these periods do not have the same distribution. That is why the system does not comply with the strict definition of VARMA(1,1).

Now before writing the integrated version for the risk-neutral model I have to clarify a more general approach to be used. For reasons to be seen in the estimation methodology section it is necessary to integrate the instantaneous discrete system in (3.2) over a larger time interval than h . So, denote another positive time variable $H \geq h$. the integration gives

$$\begin{aligned}\mathcal{V}_{t+h,H} &= \tilde{A}_h^\sigma \mathcal{V}_{t,H} + \tilde{B}_h^\sigma \tilde{\mathcal{Y}}_{t,H} + \tilde{C}_h^\sigma + \frac{1}{H} \int_0^H \tilde{\varepsilon}_{t+s,h}^\sigma ds, \\ \tilde{\mathcal{Y}}_{t+h,H} &= \tilde{A}_h^y \tilde{\mathcal{Y}}_{t,H} + \tilde{C}_h^y + \frac{1}{H} \int_0^H \tilde{\varepsilon}_{t+s,h}^y ds.\end{aligned}$$

Note the obvious notation $\tilde{\mathcal{Y}}_{t,H} = \frac{\kappa_\sigma}{\bar{\kappa}_\sigma} \mathcal{Y}_{t,H}$. This system looks very similar to historical version in (3.3) but with the following important distinctions. First, integration time intervals on the left $[t+h, t+h+H]$ and on the right $[t, t+H]$ clearly overlap. In other words, $\mathcal{V}_{t+h,H}$ and $\mathcal{V}_{t,H}$ have some common integrated volatility dynamics in the interval $[t+h, t+H]$.

4 Volatility premia

In this section I derive theoretical implications of the model for the premia related to different sources of risk. In general, a risk premium is defined as a difference between the objective and risk-neutral forecasts of the integrated risk factor. There is a large literature dealing with the premium associated with stochastic volatility. In this paper I hypothesized another source of risk, the shocks in slowly varying average level of volatility. Naturally, I define the premium for this risk as an excess forecast of the integrated central tendency under objective and risk-neutral probability measures. The most interesting question now is how this central tendency premium relates to volatility premium itself.

Thanks to [Andersen & Bollerslev \(1998\)](#) I have a reliable measurement instrument for integrated volatility. At this point it is beyond the scope of this paper to propose an analogous measure for the integrated central tendency. Hence, I will treat this factor as completely unob-

servable across the whole paper. Even though it is beyond my reach to quantify the dynamics of the volatility central tendency or its premium, I can still say a lot on the basis of the theoretical model. The information I have access to are unconditional moments of volatility and central tendency premia such as mean, standard deviation, cross-correlations, and autocorrelations. In this section I outline the methodology to derive these moments analytically. The method is based on the representation of time series as infinite integrals with respect to Brownian increments only. This approach together with stochastic calculus makes it very straightforward to derive the moments of interest.

Define two premia corresponding to both stochastic volatility and central tendency:

$$\begin{aligned} VP_{t,H} &= E_t^Q [\mathcal{V}_{t,H}] - E_t^P [\mathcal{V}_{t,H}], \\ CP_{t,H} &= E_t^Q [\tilde{\mathcal{Y}}_{t,H}] - E_t^P [\mathcal{Y}_{t,H}]. \end{aligned}$$

This means that a volatility risk premium is any excess expected integrated volatility under the risk-neutral measure over the expectation under the historical probability measure. In fact, the premium is always considered to be the negative of this quantity. It is also widely accepted that most of the times the premium associated with stochastic volatility is a negative value. Hence, just for exposition purposes I will consider the negative of this value.

The second premium above corresponds to the stochastic central tendency of volatility. Note that the genuine integrated central tendency under the risk-neutral measure is $\tilde{\mathcal{Y}}_{t,H} = \frac{\kappa_\sigma}{\bar{\kappa}_\sigma} \mathcal{Y}_{t,H}$. This rescaling is done to justify the use of term “central tendency” in application to the process under the risk-neutral measure.

With this definition it becomes clear how to quantify the importance of the premium associated with shocks in stochastic volatility drift. Define the difference between two premia as

$$TP_{t,H} = VP_{t,H} - CP_{t,H}.$$

This value may be interpreted as a transient premium.

The problem now is to characterize these three premia. In the estimation section I will argue that the only two values we observe or at least can find convincing proxies for are realized volatility and VIX volatility index. The first value proxies integrated volatility $\mathcal{V}_{t,h}$ over period h , and the second proxies risk-neutral expectation of integrated volatility over some larger period H . In this paper I eliminate any possibility of an error by leaving these values intact and not doing any forecasting of historical volatility. Forecasting approach was taken by [Eraker \(2009\)](#) with a very simple lagged realized volatility, [Bollerslev et al. \(2010\)](#) with HAR-RV, or [Todorov \(2010\)](#) with VAR-based forecast. This limitation immediately moves all premia into the rank of unobservables. On the other hand, my model allows for analytical expressions for

various analytical moments of premia. Below I briefly outline the approach I take to derive these moments. The proof with all necessary details is given in Section B.4.

First of all I represent both spot volatility and central tendency as an infinite stochastic integrals with respect to Brownian motions only. Under the historical measure the model becomes

$$y_t = \mu + \eta_y \int_{-\infty}^t \sqrt{y_v} A_{t-v}^y dW_v^y,$$

$$\sigma_t^2 = \mu + \eta_y \int_{-\infty}^t \sqrt{y_v} B_{t-v}^\sigma dW_v^y + \eta_\sigma \int_{-\infty}^t \sigma_v A_{t-v}^\sigma dW_v^\sigma.$$

Note that spot volatility accumulates shocks of both Brownian motions. Integrating this expression over the period $[t, t + H]$ I obtain another representation of integrated volatility and central tendency as infinite integrals:

$$H\mathcal{Y}_{t,H} = \mu H + \eta_y \int_t^{t+H} \sqrt{y_v} \left(\int_v^{t+H} A_{u-v}^y du \right) dW_v^y + \eta_y \int_{-\infty}^t \sqrt{y_v} \left(\int_t^{t+H} A_{u-v}^y du \right) dW_v^y,$$

$$H\mathcal{V}_{t,H} = \mu H + \eta_y \int_{-\infty}^t \sqrt{y_v} \left(\int_t^{t+H} B_{u-v}^\sigma du \right) dW_v^y + \eta_y \int_t^{t+H} \sqrt{y_v} \left(\int_v^{t+H} B_{u-v}^\sigma du \right) dW_v^y$$

$$+ \eta_\sigma \int_{-\infty}^t \sigma_v \left(\int_t^{t+H} A_{u-v}^\sigma du \right) dW_v^\sigma + \eta_\sigma \int_t^{t+H} \sigma_v \left(\int_v^{t+H} A_{u-v}^\sigma du \right) dW_v^\sigma.$$

Note that each expression above naturally breaks down into two parts. One accumulates random all shocks up to time t and the other shocks from t to $t + H$ only. Hence, taking expectations with respect to historical measure and information up to time t leaves only the first part of stochastic integrals intact. The second part is completely unpredictable with respect to \mathcal{F}_t .

The representation under the risk-neutral measure is very similar except to slight change of notation. Similarly, after taking expectation with respect to the measure Q I obtain the infinite integrals up to time t with respect to Brownian motions \tilde{W}^σ and \tilde{W}^y . In order to have a meaningful expression for the risk premia I have to measure these Brownian motions under that same measure I used for W^σ and W^y . This means that I have replace Brownian shocks under the risk-neutral measure by their equivalents under the physical measure. At this point it becomes meaningful to take the difference between risk-neutral and historical expectations of volatility and central tendency.

In the end I can show that the three above defined premia may be represented as

$$\begin{aligned}
CP_{t,H} &= E^P [CP_{t,H}] + \int_{-\infty}^t \sqrt{y_v} \varpi_{v,t}^C dW_v^y, \\
VP_{t,H} &= E^P [VP_{t,H}] + \int_{-\infty}^t \sigma_v \omega_{v,t} dW_v^\sigma + \int_{-\infty}^t \sqrt{y_v} \varpi_{v,t}^V dW_v^y, \\
TP_{t,H} &= E^P [TP_{t,H}] + \int_{-\infty}^t \sigma_v \omega_{v,t} dW_v^\sigma + \int_{-\infty}^t \sqrt{y_v} \varpi_{v,t}^T dW_v^y.
\end{aligned}$$

The first term in each expression above is an unconditional mean of the corresponding premium. The rest are stochastic integrals with respect to Brownian motion increments. Functions $\varpi_{v,t}$ and $\omega_{v,t}$ are completely deterministic and only depend on structural parameters of the model.

Unconditional means are

$$\begin{aligned}
E^P [VP_{t,H}] &= (\tilde{\mu} - \mu) - \frac{\mu}{H} \int_{-\infty}^t \int_t^{t+H} \left(\lambda_y \eta_y \frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} \tilde{B}_{u-v}^\sigma + \lambda_\sigma \eta_\sigma \tilde{A}_{u-v}^\sigma \right) dudv, \\
E^P [CP_{t,H}] &= (\tilde{\mu} - \mu) - \frac{\mu}{H} \int_{-\infty}^t \int_t^{t+H} \lambda_y \eta_y \frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} \tilde{A}_{s-v}^y dsdv, \\
E^P [TP_{t,H}] &= -\frac{\mu}{H} \int_{-\infty}^t \int_t^{t+H} \left(\lambda_y \eta_y \frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} \left(\tilde{B}_{s-v}^\sigma - \tilde{A}_{s-v}^y \right) + \lambda_\sigma \eta_\sigma \tilde{A}_{s-v}^\sigma \right) dsdv.
\end{aligned}$$

Note that both volatility and central tendency premia have one component in common, $(\tilde{\mu} - \mu) H$. If I normalize all premia by the length of the time interval, the common component is simply the difference between long-term mean of volatility, both spot and integrated, under two equivalent measures. This means that on average the volatility premium is not simply the difference between unconditional means of realized volatility and risk-neutral volatility measure.

Finally, using the representation above it is quite simple to compute unconditional moments of the three premia. For example, the variance of the premia are the following deterministic integrals:

$$\begin{aligned}
V^P [CP_{t,H}] &= \mu \int_{-\infty}^t \left(\varpi_{v,t}^C \right)^2 dv, \\
V^P [VP_{t,H}] &= \mu \int_{-\infty}^t \left[(\omega_{v,t})^2 + \left(\varpi_{v,t}^V \right)^2 \right] dv, \\
V^P [TP_{t,H}] &= \mu \int_{-\infty}^t \left[(\omega_{v,t})^2 + \left(\varpi_{v,t}^T \right)^2 \right] dv.
\end{aligned}$$

Correlations between premia and autocorrelations are derived analogously.

Since the above moments are hard to analyze analytically I will proceed to analyze them empirically by substituting estimates of structural parameters and using their variance-covariance

matrix to compute standard errors by delta method.

5 Estimation

In this section I describe how to jointly estimate parameters of continuous-time stochastic volatility model under the historic distribution in (2.1) and under the risk-neutral distribution in (2.3) with limited information.

The first subsection deals with a substantial hurdle for a financial econometrics, namely the measurement of unobservable factors such as volatility and introduced here central tendency. The first problem is how to measure volatility itself under two different probability measures. Under the historical measure Andersen & Bollerslev (1998) suggested⁶ to use intra-day high frequency data on the returns. They show that with an interval going to zero the almost sure limit of sum of squared returns is an integrated volatility. The daily realized volatility measure is readily reported and accessible from multiple sources. Under the risk-neutral measure the volatility is one of the main factors determining option prices. Using this fact Britten-Jones & Neuberger (2000) theoretically justify the use of a large set of option prices to construct a risk-neutral measure of volatility manifested in VIX volatility index.

A slight technical problem with these two data series is that they do not match with respect to the integration horizon of volatility. Realized volatility is a proxy for daily integrated volatility, and VIX is based on options with maturity of one month (22 business days). Moreover, realized volatility is a genuine volatility measure while VIX is a risk-neutral expectation of future integrated volatility. It is easy to synchronize these data by aggregating realized volatility over a month period but it is far more speculative to form its forecast. Different approaches lead to slightly different results (see Eraker, 2009; Todorov, 2010). For this paper I will stay out of this debate and will only use what I reliably have in the data and adjust econometric model appropriately. This adjustment amount to augmenting the model innovation with the forecast error of conditional expectation of volatility and central tendency. This approach reformulates the model in terms of conditional forecasts as state variables.

In Section 3 I have already derived the vector autoregression model for both historical and risk-neutral distributions where two factors are integrated volatility and central tendency. As I already stated I do not propose to measure the central tendency but rather treat it as unobservable and work around this complication. The trick to get rid of the integrated central tendency in the model is to marginalize it. By doing so with VAR(1,1) type of model I marginalize it into ARMA(2,2) model. This trick works for the realized volatility which is measured over one day and is lagged one day in the model. For the risk-neutral model the exact order of the moving

⁶See also Meddahi (2002)

average component is a bit hard to pin but, again, it does not play a significant role as long as I know the analytical structure of the innovations.

In the second part of this section I present the conditional moment I use to set up Generalized Method of Moments estimator (Hansen, 1982). It turns out that the GMM estimation of ARMA-type models is very unstable if used to estimate all structural parameters including the unconditional mean. To circumvent this problem I use the two step approach. First, I estimate the unconditional means of both historical and risk-neutral measures of volatility. This approach is known as variance targeting in GARCH literature and justified by Horvath et al. (2006) and Francq et al. (2009). On the second step I treat the mean as given and estimate the rest of the parameters by GMM. For that I compute analytically conditional mean and conditional second moment of integrated volatility explicitly taking into account all possible correlations between model innovations. The first moment only identifies speed of mean reversion parameter. The second moment is necessary to identify instantaneous diffusion parameters. In total this approach gives me four moments, two for each probability measure. As instruments I use lagged realized volatility, VIX, daily market return, and squared return.

5.1 Measurement

Clearly, the first problem an econometrician faces with these types of models is that they are formulated for the variables that are observed at best over discrete time intervals. To overcome this first obstacle I have derived the exact discretization of the same model. Since I am interested in estimating parameters under both historical and risk-neutral distributions, I have two discretized models, (3.1) and (3.2).

Now the next problem is that volatility σ_t^2 and its stochastic mean y_t are not observable variables even in discrete time intervals. Since point-in-time volatility is unobserved Andersen & Bollerslev (1998) suggested to estimate integrated volatility using high frequency observed return data. In particular, the following convergence result provides the basis for such estimation:

$$RV_{t,h} \equiv \sum_{j=1}^n r_{t+\frac{j-1}{n}h, t+\frac{j}{n}h}^2 \xrightarrow{a.s.} \int_t^{t+h} \sigma_u^2 du \equiv \mathcal{V}_{t,h}.$$

Since we can more or less reliably observe only integrated volatility, but not its point-in-time value, I have to resort to the methodology of Bollerslev & Zhou (2002). They transform all known results into relations between integrated volatility and apply GMM for estimation. Once again, I will adopt the measurement of integrated volatility under the physical measure P by the realized volatility.

Given discrete time observations on the theoretically reliable proxy of integrated volatility

it becomes natural to transform the model into (3.3) which is of VARMA(1,1) form with heteroscedastic errors and state vector $[\mathcal{V}_{t,h}, \mathcal{Y}_{t,h}]$. Another difficulty is that in the VARMA(1,1) model (3.3) the integrated central tendency $\mathcal{Y}_{t,h}$ is not observable. Even worse, there is no good proxy for this variable that I am aware of. Besides, the purpose of this paper is not to suggest a measure for this unobservable component but to circumvent this problem all together and estimate the model with what we reliably have in the data. To circumvent the absence of a good proxy for a latent variable I marginalize the observable and compute the moments in terms of what is known in the data. For more details see the following Section 5.2.

At this point I have formulated the model under the objective measure P for the integrated volatility which may be accurately proxied by the realized volatility. Realized volatility is normally constructed from the intra-day data to obtain a measure on a daily frequency, that is $h = 1$.

In order to estimate the risk-neutral model parameters I need the data on integrated volatility under the measure Q . Britten-Jones & Neuberger (2000) provide the result that connects option data with risk-neutral volatility forecast. Specifically,

$$VIX_{t,H} = 2 \int_0^\infty \frac{C(t+H, K) - \max(S_t - K, 0)}{K^2} dK = E_t^Q[\mathcal{V}_{t,H}],$$

where $C(t+H, K)$ is the price of call option maturing at time $t+H$ with strike price K , and $\max(S_t - K, 0)$ is the intrinsic value of this option at time t . If H is set at one month or 22 days period, then this expression is well proxied by the VIX index of volatility.

Note that I deliberately used time length H rather than h to stress the point that VIX is a forecast of integrated volatility over a period of 22 days rather than one days in case of realized volatility. So, even though VIX as a proxy for $E_t^Q[\mathcal{V}_{t,H}]$ is observed on a daily basis, it gives a prediction of volatility over 22 days in the future. In order to account for that in the theoretical model I integrate discrete point-in-time risk-neutral model in (3.2) over a period of time H rather than h as in the case of historical model. Even though this discrete-time model is not of simple familiar structure it is still manageable in terms of computing analytical moments and applying GMM procedure.

5.2 Moment conditions

In this section I compute analytically the first two moment conditions both for objective and risk-neutral measures of integrated volatility.

The first step is to eliminate unobservables from econometric model or, in other words, marginalize the observed variable represented by integrated volatility. Using the lag operator L

and taking the conditional expectation this system may be represented as

$$\begin{aligned}(1 - A_h^\sigma L) E_t^P [\mathcal{V}_{t+h,h}] &= B_h^\sigma E_t^P [\mathcal{Y}_{t,h}] + C_h^\sigma, \\ (1 - A_h^y L) E_t^P [\mathcal{Y}_{t+h,h}] &= C_h^y.\end{aligned}$$

Multiply the first equation by $(1 - A_h^y L)$, shift the time by h , and make a substitution from the second equation to obtain

$$(1 - A_h^y L) (1 - A_h^\sigma L) E_t^P [\mathcal{V}_{t+2h,h}] = B_h^\sigma C_h^y + (1 - A_h^y) C_h^\sigma,$$

or, in more common form,

$$E_t^P [\mathcal{V}_{t+2h,h} - (A_h^\sigma + A_h^y) \mathcal{V}_{t+h,h} + A_h^\sigma A_h^y \mathcal{V}_{t,h} - B_h^\sigma C_h^y - (1 - A_h^y) C_h^\sigma] = 0.$$

Analogously to the historical model I can marginalize integrated volatility and formulate the first moment completely analogously with a simple change in notation. One more problem I see here is that integrated volatility $\mathcal{V}_{t,H}$ under the risk-neutral distribution is not directly observable. What we have in the data is only a proxy for its risk-neutral forecast $E_t^Q [\mathcal{V}_{t,H}]$ given by VIX volatility index. So, introduce a new observable variable $\mathcal{V}_{t,H}^Q$ which is a good proxy for the risk-neutral conditional expectation:

$$\mathcal{V}_{t,H}^Q \approx E_t^Q [\mathcal{V}_{t,H}].$$

In this paper I do not argue for the accuracy of this measure but take it as given. Applying the law of iterated expectations this replacement of the latent variable does not actually change the moment:

$$E_t^Q [\mathcal{V}_{t+2h,H}^Q - (\tilde{A}_h^\sigma + \tilde{A}_h^y) \mathcal{V}_{t+h,H}^Q + \tilde{A}_h^\sigma \tilde{A}_h^y \mathcal{V}_{t,H}^Q - \tilde{B}_h^\sigma \tilde{C}_h^y - (1 - \tilde{A}_h^y) \tilde{C}_h^\sigma] = 0.$$

The first moments identify only mean and speed parameters of diffusion. In order to identify instantaneous variance and price of shocks one has to compute additional moments. As I show in Section B.3, the second conditional moments of integrated volatility and spot variables may

be written as

$$\begin{aligned}
E_t^P \left[(h\mathcal{V}_{t,h})^2 \right] &= A_1\sigma_t^2 + A_2y_t + A_3 + \left(a_h^\sigma\sigma_t^2 + b_h^\sigma y_t + c_h^\sigma \right)^2, \\
E_t^P \left[\sigma_{t+h}^4 \right] &= a_1\sigma_t^2 + a_2y_t + a_3 + \left(A_h^\sigma\sigma_t^2 + B_h^\sigma y_t + C_h^\sigma \right)^2, \\
E_t^P \left[\sigma_{t+h}^2 y_{t+h} \right] &= c_2y_t + c_3 + \left(A_h^\sigma\sigma_t^2 + B_h^\sigma y_t + C_h^\sigma \right) \left(A_h^y y_t + C_h^y \right), \\
E_t^P \left[y_{t+h}^2 \right] &= b_2y_t + b_3 + \left(A_h^y y_t + C_h^y \right)^2, \\
E_t^P \left[\sigma_{t+h}^2 \right] &= A_h^\sigma\sigma_t^2 + B_h^\sigma y_t + C_h^\sigma, \\
E_t^P \left[y_{t+h} \right] &= A_h^y y_t + C_h^y.
\end{aligned}$$

The only observable here is integrated volatility $\mathcal{V}_{t,h}^2$. The rest are latent variables which can be eliminated by taking appropriate lags and making substitutions. For example, the spot volatility and central tendency equations may be written as

$$\begin{aligned}
E_t^P \left[(1 - A_h^y L) y_{t+h} \right] &= C_h^y, \\
E_t \left[(1 - A_h^\sigma L) \sigma_{t+h}^2 \right] &= B_h^\sigma y_t + C_h^\sigma.
\end{aligned}$$

Multiply the last equation by $(1 - A_h^y L)$, shift the time by h using the law of iterated expectations, and finally substitute the first equation in to get

$$E_t^P \left[(1 - A_h^y L) (1 - A_h^\sigma L) \sigma_{t+2h}^2 \right] = B_h^\sigma C_h^y + (1 - A_h^y) C_h^\sigma.$$

The expression for the second moment of integrated volatility includes spot variables σ_t^4 , $\sigma_t^2 y_t$, y_t^2 , σ_t^2 , and y_t . Using the above approach each one of these variables is eliminated with the end result of

$$E_t^P \left[\left(1 - (A_h^\sigma)^2 L \right) \left(1 - A_h^\sigma A_h^y L \right) \left(1 - (A_h^y)^2 L \right) \left(1 - A_h^y L \right) \left(1 - A_h^\sigma L \right) \mathcal{V}_{t+5h,h}^2 \right] = M,$$

where constant M is defined in Section B.3 and its definition involves instantaneous variance parameters.

Parameters in the first and second moments of integrated volatility under two probability measures are implicitly given by the vectors $(\mu, \kappa_\sigma, \kappa_y, \eta_\sigma, \eta_y)$ and $(\mu, \tilde{\kappa}_\sigma, \tilde{\kappa}_y, \eta_\sigma, \eta_y)$, respectively. The connection between these parameters is

$$\tilde{\kappa}_\sigma = \kappa_\sigma - \lambda_\sigma \eta_\sigma, \quad \tilde{\kappa}_y = \kappa_y - \lambda_y \eta_y, \quad \tilde{\mu} = \mu \frac{\kappa_y \kappa_\sigma}{\tilde{\kappa}_y \tilde{\kappa}_\sigma}, \quad \tilde{\eta}_y = \eta_y \sqrt{\frac{\kappa_\sigma}{\tilde{\kappa}_\sigma}}.$$

If I estimate parameters of the two models jointly, then the parameter vector becomes

$$\theta = (\mu, \kappa_\sigma, \kappa_y, \eta_\sigma, \eta_y, \lambda_\sigma, \lambda_y),$$

which differs from historical set of parameters only by risk prices of volatility and central tendency innovations, λ_σ and λ_y .

Both first and second moments are expressed given fine information set \mathcal{F}_t which contains all past observations of point-in-time volatility σ_t^2 and y_t . Clearly, this information is not available to econometrician. Hence, an additional technical step of reduction in information set is necessary (Meddahi & Renault, 2004). Coarser information set includes only observations on past integrated volatilities.

Finally, the applicability of GMM is argued in Section B.5. There I claim that the moment restrictions, model innovations, integrated variables, their interactions are reducible to a simple stochastic integral. Consequently, for this variable I show the existence of finite fourth moment and central limit theorem.

6 Results

In this section I will describe the data I use for my empirical exercise. Then I will outline the results of estimation of continuous-time model parameters. The rest of the section is dedicated to analyzing the main result of the paper. This result states that for forecasting horizon of several days and more the volatility premium is mostly composed of the premium on central tendency. Part of it is composed of the premium on shocks that are relatively more volatile but short-lived. In other words, the additional source of risk, the one that drives volatility around its persistent stochastic drift, does bear some small but statistically significant premium. It is also interesting to note that the instantaneous variance of a central tendency is only a small fraction of instantaneous variance of additional volatility shocks. Hence, small shocks that preserve its effect long into the future are priced and make the largest portion of compensation.

In this study I use the following data. Daily volatility index, VIX, is constructed by CBOE⁷ for the period starting from 1990. This index proxies the integrated volatility forecast over the future 22 business days. Daily S&P500 index prices, SPX, and realized volatility measure, RV, are reported by Oxford-Man Institute⁸ for the period starting from 1996. The data for market index and log daily return are shown in Figure 1 on page 29. Both volatility indexes are shown in Figure 2 on page 30. I report descriptive statistics of the data in Table 1 on page 30. In addition, estimates of autocorrelation function are shown in Figure 3 on page 31.

⁷CBOE, VIX Historical Price Data, <http://www.cboe.com/micro/vix/historical.aspx>

⁸Oxford-Man Institute’s “realized library”, <http://realized.oxford-man.ox.ac.uk/home>

The results of several models estimation are given in Table 2 on page 31. The first four columns in this table correspond to a benchmark estimation of the univariate square-root diffusion to the daily realized volatility and VIX separately. The next four columns correspond to estimation of bivariate diffusion model with central tendency to RV and VIX separately. The last column is main estimation result of the paper as it fits the model with central tendency jointly to RV and VIX volatility measures. I used heteroscedasticity and autocorrelation robust estimator of moments covariance matrix (Newey & West, 1987) with Bartlett kernel and 5 lags for univariate models, and 50 lags for bivariate models. The instruments for the estimation in benchmark models are lagged variables themselves. For the main model estimation I used lags of realized volatility, VIX volatility index, and squared daily log-return.

First of all from Table 2 on page 31 it is clear that none of the models is not rejected by the J-test with p-value varying from 37% to 95% for benchmark cases, and 51% for the main model. In all benchmark cases the parameters are highly significant with an exception of diffusion parameter of central tendency.

The very first column suggests that the mean reversion of realized volatility is 0.0793. This number corresponds to half-life of a shock to a little under 4 days. This does not seem to lie close to roughly 30 days evident from estimated autocorrelation function in Figure 3 on page 31. The estimated mean reversion speed of VIX is 0.0152 which corresponds to half-life of 20 days which and also does not match well above 60 days evident from the figure. This observation clearly makes univariate models inadequate in view of the real data.

One can see that the modeling of the central tendency is relevant since both κ_y and η_y are significantly different from zero. Bivariate models bring an additional degree of freedom by introducing an additional shock which should cover the fat tails feature of the data. This is the most clearly seen in case of realized volatility. There the speed of mean reversion of the central tendency is estimated as 0.0218 which corresponds to almost 14 days of half-life. Even though it still does not reach the level seen in the data, is a certain improvement in the fit of the model. The central tendency mean reversion estimate for the VIX series went slightly up in comparison to the univariate model which is not a good sign.

Besides mean reversion, one can note some interesting patterns in diffusion parameter estimates. In univariate case the diffusion parameter of realized volatility is estimated as 0.1424, while in bivariate case two diffusion estimates are 0.1923 and 0.0258. This means that the shocks changing the persistent drift are much less volatile than shocks changing the movement of volatility around its long-term drift. The univariate estimate is somewhere in the middle. The same picture is observed in the case of VIX. Besides, diffusion parameter estimates are somewhat smaller for VIX than they are for RV. This together with higher estimated persistence of VIX is a reflection of smaller excess kurtosis but almost the same standard deviation of

VIX. Smaller diffusion parameter drives both second and fourth moment down, while smaller mean reversion speed increases standard deviation but does not seem to comparable increase the kurtosis.

The estimation of the joint model for RV and VIX does not improve standard errors of parameter estimates but rather somewhat decreases them. In particular, mean reversion parameter of volatility itself is not as precise as others. Central tendency estimates are still very significant. The main benefit of joint model estimation though are the risk prices. Here the prices of both shocks are significant at 5% level. Point estimates are very close to each other, 0.2714 and 0.2032, respectively. This means that central tendency shocks are slightly more expensive.

The model estimates imply that the risk-neutral speed of mean reversion of central tendency is $.0215 - .2714 \times .0279 = .0139$, and $.9040 - .2032 \times .1742 = .8686$ which are not far from estimates of VIX model. The speed of mean reversion of 0.9040 corresponds to 0.35 days of half-life. Hence, the shock on top of the central tendency is very very short lived but several time more volatile. Both of these factors contribute to a higher unconditional mean of volatility in the risk-neutral world, 21% against 13% daily. This observation is due to the identity $\tilde{\mu}\tilde{\kappa}_y\tilde{\kappa}_\sigma = \mu\kappa_y\kappa_\sigma$ which says that all else equal the higher unconditional mean of risk-neutral volatility requires higher persistence of either central tendency or volatility or both.

Given the estimates of model parameters I plug them in to analytical expressions for unconditional moments of volatility premium, central tendency premium, and their difference. By plugging in the estimates of these parameters I obtain point estimates of unconditional moments of unobservable risk premia. Standard errors for these estimates are computed using the delta method. Having the estimates $\hat{\theta}$ of structural parameters and their covariance matrix $\hat{\Omega}$ it is easy to compute estimates of $\gamma = f(\theta)$ and their standard errors. Point estimates are $\hat{\gamma} = f(\hat{\theta})$ and the covariance matrix is computed through delta method and given by

$$V(\hat{\gamma}) = \left[\frac{\partial f}{\partial \theta}(\hat{\theta}) \right] \hat{\Omega} \left[\frac{\partial f}{\partial \theta'}(\hat{\theta}) \right].$$

Implications for unconditional moments of volatility are given in Figure 4 on page 32 through Figure 6 on page 33. Risk premia mean and standard deviation are given in Figure 7 on page 33 and Figure 8 on page 34. These graphs, except for autocorrelations, plot unconditional moments over forecasting horizon H which I vary from 1 to 22 business days. The last number is the same as used in computing VIX volatility index. Everything below that is a shorter interval for volatility integration and is used to reveal the relative importance of central tendency over different forecasting horizons.

Figure 4 on page 32 shows that standard deviation of volatility, central tendency, and their

difference. There is no need for unconditional mean picture since it is known to be a constant given in Table 2 on page 31. The plot shows standard deviation normalized by a period length. There we see that the volatility and central tendency standard deviations are very close to each other across all horizons and vary slightly below 5% daily. It is interesting to note that with increase in aggregation interval the standard deviation decreases only marginally. This is due to high persistence of both series. The standard deviation of the difference is above 2% for 1 day of aggregation and below 0.5% for the upper end. This says that the transitory shocks in volatility are much more evident on the short forecasting horizons although still by order smaller than the effects of persistent component.

Figure 6 on page 33 shows the estimates of correlations between volatility and its components over different aggregation intervals. The most evident result is that the volatility itself is highly correlated with its central tendency. On the longer horizons the point estimate is very close to one but the estimate is much less precise. This explains close similarity in the second moment of volatility and central tendency. The correlation between volatility and its transitory innovations is much more pronounced at shorter aggregation intervals, above 50%, where transitory shocks are more noticeable as judged by its daily standard deviation. As expected due to assumption of uncorrelated Brownian innovations, the correlation between central tendency and transitory shocks is virtually zero.

Figure 6 on page 33 plots autocorrelations of three series for lags up to 60 days and one day of aggregation. The most interesting observation is that the integrated volatility autocorrelation somewhat understates the empirical estimates of autocorrelation of realized volatility given in Figure 3 on page 31. This observation is a positive reality check of the estimation results. Naturally, the autocorrelation of central tendency is even higher. And there is not much persistence in the difference between the two series.

Figure 7 on page 33 shows the most striking result. On this plot I present the comparison between unconditional means of volatility premium, central tendency premium, and their difference. The point estimates unambiguously suggest that the implied central tendency premium is actually larger on average than the volatility premium itself over all considered forecasting horizons. Volatility premium point estimate goes from 0.5% per day to roughly 1.5% per day for 22 days of integration. Central tendency point estimates are everywhere above. Confidence intervals (95%) for these two premia are approximately plus minus 0.5% daily.

The actual test for significant difference between the two premia is given by analyzing confidence intervals of the mean transitory premium. Point estimate goes from -0.3% for one day of forecasting to -0.1% for month long forecasting. In spite of relatively wide confidence intervals for the volatility and central tendency premia, the confidence interval for their difference is roughly plus minus 0.1-0.5% daily. This interval includes zero only at small aggregation intervals.

This suggests that the difference between the two premia is significant for forecasting horizons of several days and above. Hence, I can say that the unconditional means of the two premia are distinguishable for more than few days in the future. This also implies that the transitory shocks contribute a statistically significant though small weight into risk compensation. The basis for this result is evident from the previously considered graphs and estimation results. There it was clear that volatility and central tendency are very similar to each other. But the shocks that come on top of the central tendency are very volatile even though very short lived. This seems to be enough to make a contribution to the volatility premium unconditional level.

The next graph I want to analyze is given in Figure 8 on page 34. This plot represents daily unconditional standard deviation of the two premia and their difference over different forecasting horizons. Here we see monotonically increasing standard deviation of volatility and central tendency premia. Point estimates go smoothly from 0.1% for volatility and 0.2% for central tendency to 0.3-0.4% for 22 days forecast. This observation is natural simply because forecast is expected to deteriorate in efficiency with an increase in forecasting horizon. Moreover, doing rough calculation suggests that 95% of the times the volatility premium for the standard 22 days of volatility integration should stay inside of the interval of $1.5 \pm 0.4\%$ daily. This does not rule out the possibility of the volatility premium being negative. This goes in line with other studies that employ different forecasting techniques to characterize dynamic behavior of volatility premium and find it negative from time to time. The standard deviation of the difference between two premia is around 0.1% for one day to almost zero for 22 days. This means that the difference between the two premia is almost constant, especially on the longer horizons.

To conclude this section I want to stress the main result which says that the volatility premium on intervals longer than a few days is mainly due to compensation of highly persistent shocks that drive stochastic drift of volatility.

7 Conclusion

In this paper I proposed the continuous-time stochastic volatility model with varying central tendency. As a main result of the paper I argue that the major part of volatility risk premium is due to compensation for highly persistent shocks in volatility, those that drive central tendency. Additional short lived but very volatile shocks that drive volatility around its central tendency are associated with a small but significant negative premium.

My approach has several very conservative limitations. First, I treat central tendency as completely unobservable and for the purposes of estimation integrate it out. It would be a very promising avenue of future research to devise a reliable measure of integrated central tendency

analogous to integrated volatility. The second limitation is that I do not take a risk of speculation and do not propose a specific methodology to form a conditional forecast of historical volatility. This forecast could allow me to identify explicitly volatility premium. On the other hand, if I leave central tendency as unobservable the value of computing volatility premium by itself is doubtful. Hence, in order to gain insight into joint dynamics of volatility and central tendency premia it would take a substantial theoretical work. In spite of these limitations my approach does not preclude analytical computation of unconditional moments of different premia and their difference.

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A Tables and Figures

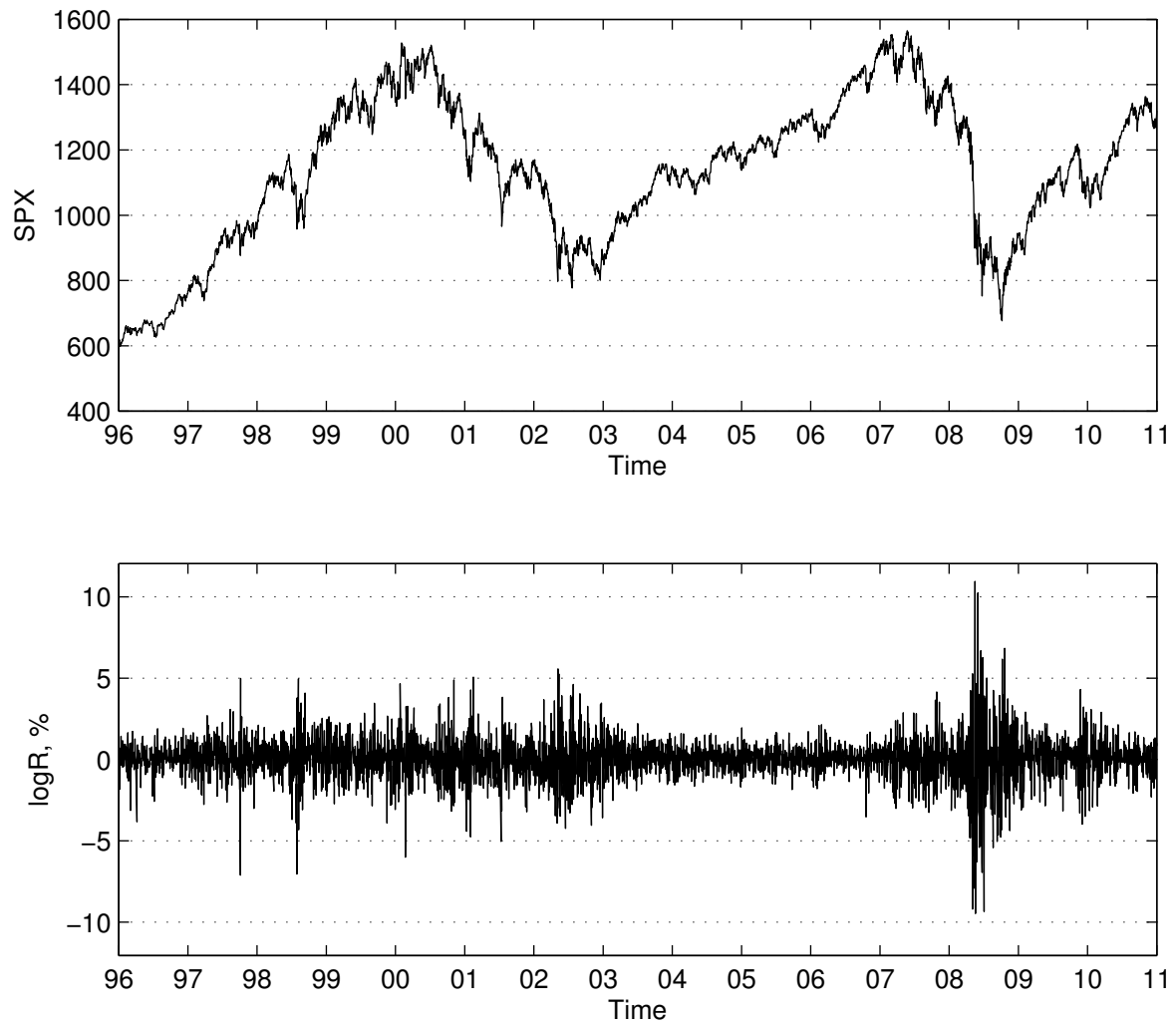


Figure 1: Daily S&P500 index (SPX) and market log return (logR).

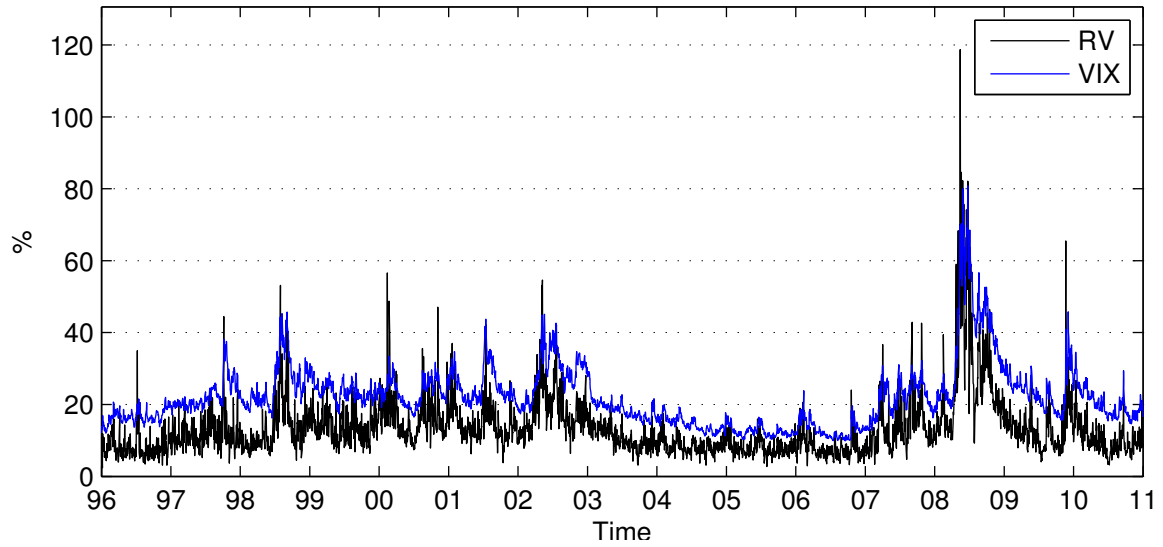


Figure 2: Daily option-based volatility index (VIX), and realized volatility (RV).

	Min	Max	Mean	Std	Skewness	Exc. Kurtosis
$100 \cdot \log(R)$	-9.47	10.96	0.019	1.31	-0.20	7.42
VIX	9.89	80.86	22.03	8.62	1.98	6.87
RV	2.38	118.75	13.34	8.36	3.34	19.87

Table 1: Descriptive statistics for market log returns ($\log R$), option-based volatility index (VIX), realized volatility (RV), and their difference, volatility risk premium (VRP).

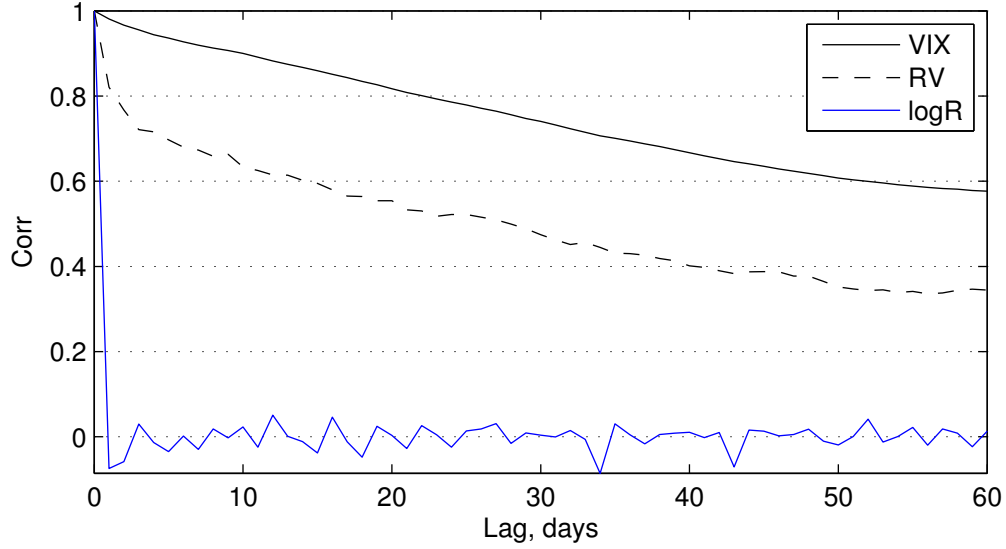


Figure 3: Autocorrelation function for option-based volatility index (VIX), realized volatility (RV), and log market return (logR).

θ	RV		VIX		RV		VIX		RV, VIX	
μ	0.1323	(0.0059)	0.2209	(0.0142)	0.1268	(0.0119)	0.2111	(0.0124)	0.1259	(0.0082)
κ_σ	0.0793	(0.0164)	0.0152	(0.0036)	0.9452	(0.0515)	0.9234	(0.0158)	0.9040	(0.0229)
η_σ	0.1424	(0.0141)	0.0552	(0.0029)	0.1923	(0.0037)	0.0509	(0.0049)	0.1742	(0.1216)
λ_σ									0.2032	(0.0686)
κ_y					0.0218	(0.0084)	0.0132	(0.0032)	0.0215	(0.0056)
η_y					0.0258	(0.0281)	0.0055	(0.0486)	0.0279	(0.0094)
λ_y									0.2714	(0.1212)
J	0.51		0.01		3.09		3.21		6.49	
p	0.47		0.95		0.39		0.36		0.26	
df	1		1		3		3		5	

Table 2: Estimation results of the model with stochastic trend. The parameters have the following interpretation: μ is the unconditional mean of historical volatility; κ_σ and κ_y are mean reversion speed parameters for volatility and central tendency, respectively, under historical measure; η_σ and η_y are instantaneous diffusion parameters; λ_σ and λ_y are risk prices. Risk-neutral speeds of mean reversion are $\tilde{\kappa}_\sigma = \kappa_\sigma - \lambda_\sigma \eta_\sigma$ and $\tilde{\kappa}_y = \kappa_y - \lambda_y \eta_y$. Risk-neutral volatility mean is given by $\tilde{\mu} = \frac{\kappa_\sigma \kappa_y}{\tilde{\kappa}_\sigma \tilde{\kappa}_y} \mu$.

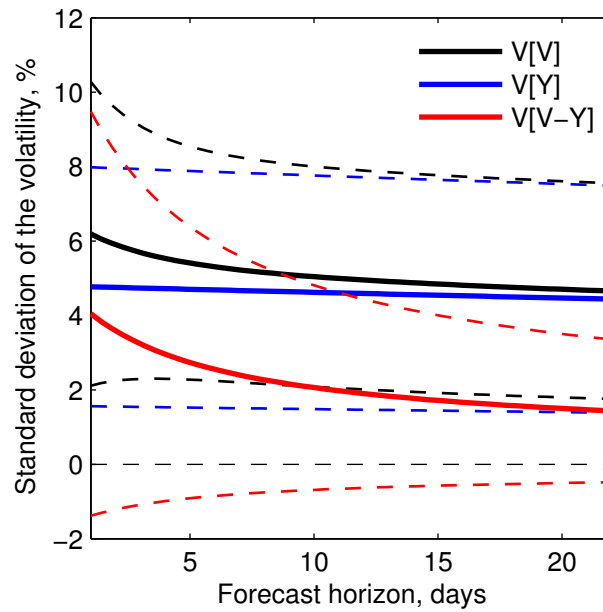


Figure 4: Implied standard deviations of daily volatility, $V[V]$, central tendency, $V[Y]$, and their difference, $V[V-Y]$. Implied 95% confidence intervals are given by dashed lines.

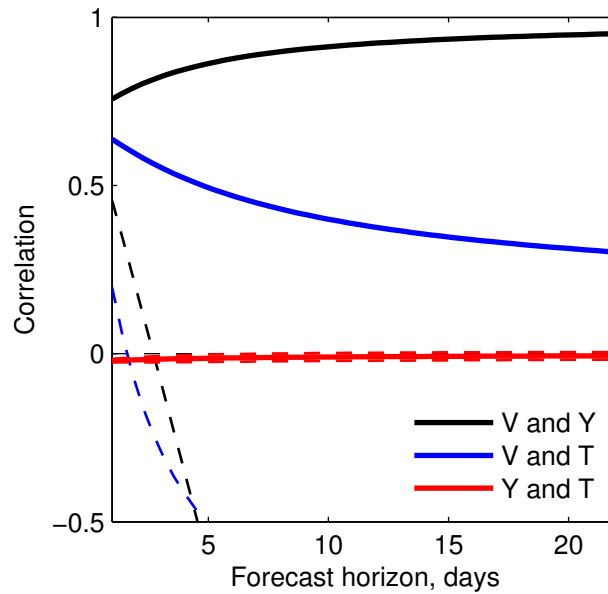


Figure 5: Implied correlations between volatility (V), central tendency (Y), and their difference (T). Implied 95% confidence intervals are given by dashed lines.

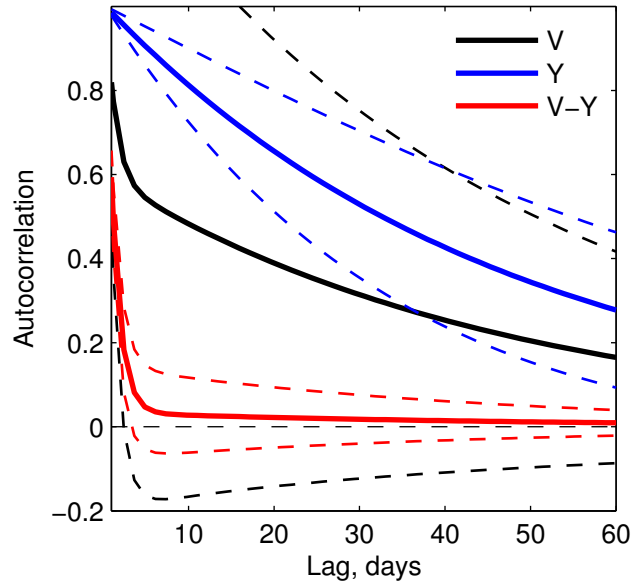


Figure 6: Implied autocorrelations of volatility (V), central tendency (Y), and their difference (T). Implied 95% confidence intervals are given by dashed lines.

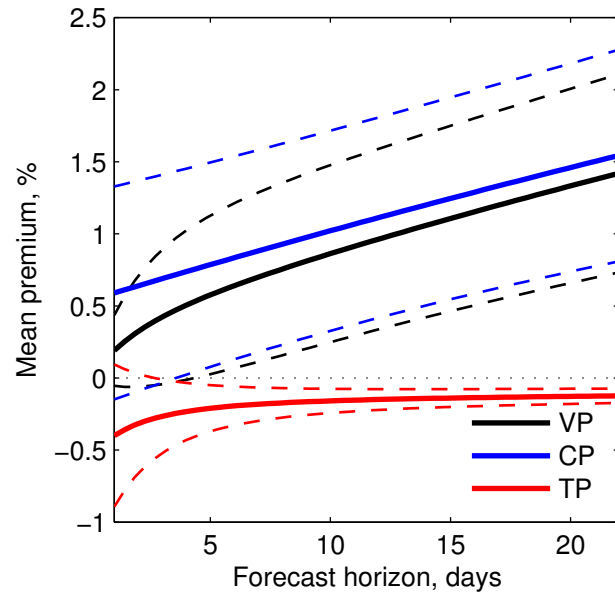


Figure 7: Implied means of daily volatility premium (VP), central tendency premium (CP), and their difference transitory premium (TP). Implied 95% confidence intervals are given by dashed lines.

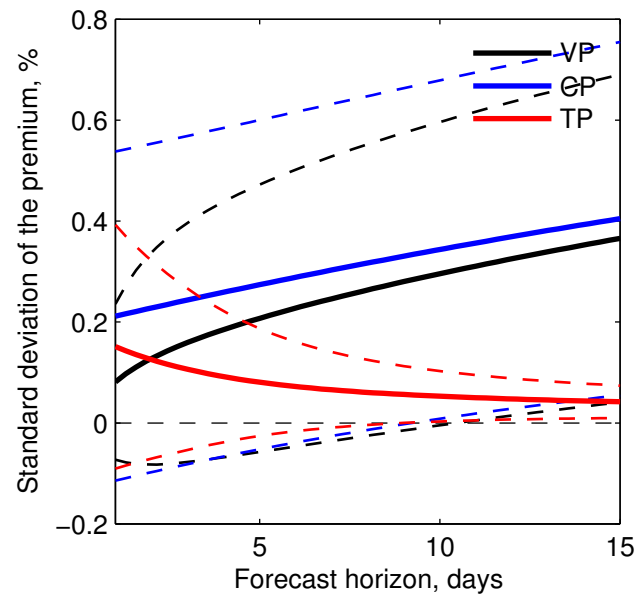


Figure 8: Implied standard deviations of daily volatility premium (VP), central tendency premium (CP), and their difference transitory premium (TP). Implied 95% confidence intervals are given by dashed lines.

B Proofs

B.1 Discretization of objective model

Given the model in (2.1) simple integration of y_t gives

$$\begin{aligned}
y_{t+h} &= y_t e^{-\kappa_y h} + \mu \kappa_y \int_t^{t+h} e^{-\kappa_y(t+h-v)} dv + \eta_y \int_t^{t+h} \sqrt{y_v} e^{-\kappa_y(t+h-v)} dW_v^y \\
&= y_t e^{-\kappa_y h} + \mu (1 - e^{-\kappa_y h}) + \eta_y \int_t^{t+h} \sqrt{y_v} e^{-\kappa_y(t+h-v)} dW_v^y \\
&= y_t A_h^y + \mu (1 - A_h^y) + \eta_y \int_t^{t+h} \sqrt{y_v} A_{t+h-v}^y dW_v^y.
\end{aligned}$$

For more compact notation I replaced $e^{-\kappa_y h}$ with A_h^y .

Now write the volatility process

$$\begin{aligned}
\sigma_{t+h}^2 &= \sigma_t^2 e^{-\kappa_\sigma h} + \kappa_\sigma \int_t^{t+h} y_u e^{-\kappa_\sigma(t+h-u)} du + \eta_\sigma \int_t^{t+h} \sigma_u e^{-\kappa_\sigma(t+h-u)} dW_u^\sigma \\
&= \sigma_t^2 A_h^\sigma + \kappa_\sigma \int_t^{t+h} y_u A_{t+h-u}^\sigma du + \eta_\sigma \int_t^{t+h} \sigma_u A_{t+h-u}^\sigma dW_u^\sigma \\
&= \sigma_t^2 A_h^\sigma + \kappa_\sigma \int_t^{t+h} \left[y_t A_{u-t}^y + \mu (1 - A_{u-t}^y) + \eta_y \int_t^u \sqrt{y_v} A_{u-v}^y dW_v^y \right] A_{t+h-u}^\sigma du \\
&\quad + \eta_\sigma \int_t^{t+h} \sigma_u A_{t+h-u}^\sigma dW_u^\sigma \\
&= \sigma_t^2 A_h^\sigma + \mu \kappa_\sigma \int_t^{t+h} A_{t+h-u}^\sigma du + \kappa_\sigma (y_t - \mu) \int_t^{t+h} A_{u-t}^y A_{t+h-u}^\sigma du \\
&\quad + \kappa_\sigma \eta_y \int_t^{t+h} \left(\int_t^u \sqrt{y_v} A_{u-v}^y A_{t+h-u}^\sigma dW_v^y \right) du + \eta_\sigma \int_t^{t+h} \sigma_u A_{t+h-u}^\sigma dW_u^\sigma \\
&= \sigma_t^2 A_h^\sigma + \mu \kappa_\sigma \int_t^{t+h} A_{t+h-u}^\sigma du + \kappa_\sigma (y_t - \mu) \int_t^{t+h} A_{u-t}^y A_{t+h-u}^\sigma du \\
&\quad + \kappa_\sigma \eta_y \int_t^{t+h} \sqrt{y_v} \left(\int_v^{t+h} A_{u-v}^y A_{t+h-u}^\sigma du \right) dW_v^y + \eta_\sigma \int_t^{t+h} \sigma_u A_{t+h-u}^\sigma dW_u^\sigma \\
&= \sigma_t^2 A_h^\sigma + \mu (1 - A_h^\sigma) + (y_t - \mu) \frac{\kappa_\sigma}{\kappa_\sigma - \kappa_y} (A_h^y - A_h^\sigma) \\
&\quad + \eta_y \frac{\kappa_\sigma}{\kappa_\sigma - \kappa_y} \int_t^{t+h} \sqrt{y_v} (A_{t+h-v}^y - A_{t+h-v}^\sigma) dW_v^y + \eta_\sigma \int_t^{t+h} \sigma_u A_{t+h-u}^\sigma dW_u^\sigma,
\end{aligned}$$

where it can be easily seen that

$$\begin{aligned}\int_t^{t+h} A_{t+h-u}^\sigma du &= \frac{1}{\kappa_\sigma} (1 - A_h^\sigma), \\ \int_t^{t+h} A_{u-t}^y A_{t+h-u}^\sigma du &= \frac{1}{\kappa_\sigma - \kappa_y} (A_h^y - A_h^\sigma), \\ \int_v^{t+h} A_{u-v}^y A_{t+h-u}^\sigma du &= \frac{1}{\kappa_\sigma - \kappa_y} (A_{t+h-v}^y - A_{t+h-v}^\sigma).\end{aligned}$$

Also denote

$$B_h^\sigma = \frac{\kappa_\sigma}{\kappa_\sigma - \kappa_y} (A_h^y - A_h^\sigma).$$

So the original process can be written as

$$\begin{aligned}\sigma_{t+h}^2 &= \mu (1 - A_h^\sigma - B_h^\sigma) + \sigma_t^2 A_h^\sigma + y_t B_h^\sigma \\ &\quad + \eta_y \int_t^{t+h} \sqrt{y_v} B_{t+h-u}^\sigma dW_v^y + \eta_\sigma \int_t^{t+h} \sigma_v A_{t+h-v}^\sigma dW_v^\sigma,\end{aligned}$$

as well as stochastic trend:

$$y_{t+h} = \mu (1 - A_h^y) + y_t A_h^y + \eta_y \int_t^{t+h} \sqrt{y_v} A_{t+h-v}^y dW_v^y.$$

So, the discretized model is represented by the following two equations:

$$\begin{aligned}\sigma_{t+h}^2 &= A_h^\sigma \sigma_t^2 + B_h^\sigma y_t + C_h^\sigma + \epsilon_{t,h}^\sigma, \\ y_{t+h} &= A_h^y y_t + C_h^y + \epsilon_{t,h}^y.\end{aligned}\tag{B.1}$$

The coefficients for volatility are

$$A_h^\sigma = \exp(-\kappa_\sigma h), \quad B_h^\sigma = \frac{\kappa_\sigma}{\kappa_\sigma - \kappa_y} (A_h^y - A_h^\sigma), \quad C_h^\sigma = \mu (1 - A_h^\sigma - B_h^\sigma),$$

and

$$A_h^y = \exp(-\kappa_y h), \quad C_h^y = \mu (1 - A_h^y).$$

Note that $A_{h_1}^y$ and $A_{h_2}^\sigma$ are multiplicative functions of time interval, that is $A_{h_1}^y A_{h_2}^y = A_{h_1+h_2}^y$.

The error structure is represented by

$$\begin{aligned}\epsilon_{t,h}^\sigma &= \eta_\sigma \int_t^{t+h} \sigma_u A_{t+h-u}^\sigma dW_u^\sigma + \eta_y \int_t^{t+h} \sqrt{y_u} B_{t+h-u}^\sigma dW_u^y, \\ \epsilon_{t,h}^y &= \eta_y \int_t^{t+h} \sqrt{y_u} A_{t+h-u}^y dW_u^y.\end{aligned}$$

Clearly, $E_t^P [\epsilon_{t,t+h}^\sigma] = 0$, and $E_t^P [\epsilon_{t,t+h}^y] = 0$.

Note that the same processes may be represented as infinite stochastic integrals with respect to Brownian motion only:

$$\begin{aligned} y_t &= \mu + \eta_y \int_{-\infty}^t \sqrt{y_v} A_{t-v}^y dW_v^y, \\ \sigma_t^2 &= \mu + \eta_y \int_{-\infty}^t \sqrt{y_v} B_{t-v}^\sigma dW_v^y + \eta_\sigma \int_{-\infty}^t \sigma_v A_{t-v}^\sigma dW_v^\sigma. \end{aligned} \tag{B.2}$$

B.2 Autocorrelation

Recall from (B.2) the representation of spot volatility and central tendency as infinite stochastic integrals with respect to Brownian motions only. Compute the autocovariance of spot volatility over the period h :

$$\begin{aligned} Cov(\sigma_{t+h}^2, \sigma_t^2) &= \eta_y E \left[\int_{-\infty}^{t+h} \sqrt{y_v} B_{t+h-v}^\sigma dW_v^y \int_{-\infty}^t \sqrt{y_v} B_{t-v}^\sigma dW_v^y \right] \\ &\quad + \eta_\sigma E \left[\int_{-\infty}^{t+h} \sigma_v A_{t+h-v}^\sigma dW_v^\sigma \int_{-\infty}^t \sigma_v A_{t-v}^\sigma dW_v^\sigma \right] \\ &= \eta_y^2 E \left[\int_{-\infty}^t y_v B_{t+h-v}^\sigma B_{t-v}^\sigma dv \right] + \eta_\sigma^2 E \left[\int_{-\infty}^t \sigma_v^2 A_{t+h-v}^\sigma A_{t-v}^\sigma dv \right] \\ &= \mu \eta_y^2 \int_{-\infty}^t B_{t+h-v}^\sigma B_{t-v}^\sigma dv + \mu \eta_\sigma^2 \int_{-\infty}^t A_{t+h-v}^\sigma A_{t-v}^\sigma dv. \end{aligned}$$

Here

$$\int_{-\infty}^t A_{t+h-v}^\sigma A_{t-v}^\sigma dv = \int_{-\infty}^t A_{2t+h-2v}^\sigma dv = \frac{1}{2\kappa_\sigma} A_h^\sigma.$$

One more integral is

$$\begin{aligned} \int_{-\infty}^t B_{t+h-v}^\sigma B_{t-v}^\sigma dv &= \left(\frac{\kappa_\sigma}{\kappa_\sigma - \kappa_y} \right)^2 \int_{-\infty}^t \left(A_{t+h-v}^y - A_{t+h-v}^\sigma \right) \left(A_{t-v}^y - A_{t-v}^\sigma \right) dv \\ &= \left(\frac{\kappa_\sigma}{\kappa_\sigma - \kappa_y} \right)^2 \left(\frac{1}{2\kappa_y} A_h^y - \frac{1}{\kappa_\sigma + \kappa_y} A_h^y - \frac{1}{\kappa_\sigma + \kappa_y} A_h^\sigma + \frac{1}{2\kappa_\sigma} A_h^\sigma \right) \\ &= \left(\frac{\kappa_\sigma}{\kappa_\sigma - \kappa_y} \right)^2 \left(\left(\frac{1}{2\kappa_y} - \frac{1}{\kappa_\sigma + \kappa_y} \right) A_h^y + \left(\frac{1}{2\kappa_\sigma} - \frac{1}{\kappa_\sigma + \kappa_y} \right) A_h^\sigma \right) \\ &= \left(\frac{\kappa_\sigma}{\kappa_\sigma - \kappa_y} \right)^2 \left(\frac{\kappa_\sigma - \kappa_y}{2\kappa_y (\kappa_\sigma + \kappa_y)} A_h^y - \frac{\kappa_\sigma - \kappa_y}{2\kappa_\sigma (\kappa_\sigma + \kappa_y)} A_h^\sigma \right) \\ &= \frac{\kappa_\sigma^2}{\kappa_\sigma^2 - \kappa_y^2} \left(\frac{1}{2\kappa_y} A_h^y - \frac{1}{2\kappa_\sigma} A_h^\sigma \right). \end{aligned}$$

Plugging in this leads to the following covariance:

$$\begin{aligned} Cov(\sigma_{t+h}^2, \sigma_t^2) &= \frac{\kappa_\sigma^2}{\kappa_\sigma^2 - \kappa_y^2} \left(\frac{\mu\eta_y^2}{2\kappa_y} A_h^y - \frac{\mu\eta_y^2}{2\kappa_\sigma} A_h^\sigma \right) + \frac{\mu\eta_\sigma^2}{2\kappa_\sigma} A_h^\sigma \\ &= \frac{\kappa_\sigma^2}{\kappa_\sigma^2 - \kappa_y^2} \frac{\mu\eta_y^2}{2\kappa_y} (A_h^y - A_h^\sigma) + \left(\frac{\mu\eta_\sigma^2}{2\kappa_\sigma} + \frac{\kappa_\sigma}{\kappa_\sigma + \kappa_y} \frac{\mu\eta_y^2}{2\kappa_y} \right) A_h^\sigma. \end{aligned}$$

Taking $h = 0$ I obtain the unconditional variance of spot volatility:

$$\begin{aligned} V(\sigma_t^2) &= \frac{\kappa_\sigma^2}{\kappa_\sigma^2 - \kappa_y^2} \left(\frac{\mu\eta_y^2}{2\kappa_y} - \frac{\mu\eta_y^2}{2\kappa_\sigma} \right) + \frac{\mu\eta_\sigma^2}{2\kappa_\sigma} \\ &= \frac{\kappa_\sigma}{\kappa_\sigma + \kappa_y} \frac{\mu\eta_y^2}{2\kappa_y} + \frac{\mu\eta_\sigma^2}{2\kappa_\sigma}. \end{aligned}$$

Hence, the autocorrelation is

$$\begin{aligned} Corr(\sigma_{t+h}^2, \sigma_t^2) &= A_h^\sigma + (A_h^y - A_h^\sigma) \frac{\kappa_\sigma^2}{\kappa_\sigma^2 - \kappa_y^2} \frac{\mu\eta_y^2}{2\kappa_y} \left(\frac{\kappa_\sigma}{\kappa_\sigma + \kappa_y} \frac{\mu\eta_y^2}{2\kappa_y} + \frac{\mu\eta_\sigma^2}{2\kappa_\sigma} \right)^{-1} \\ &= A_h^\sigma + (A_h^y - A_h^\sigma) \frac{\kappa_\sigma}{\kappa_\sigma - \kappa_y} \frac{\eta_y^2}{\kappa_y} \left(\frac{\eta_y^2}{\kappa_y} + \frac{\kappa_\sigma + \kappa_y}{\kappa_\sigma} \frac{\eta_\sigma^2}{\kappa_\sigma} \right)^{-1}. \end{aligned}$$

B.3 Second moment

In (B.1) replace h by another time indicator s and integrate from 0 to h which leads to the following expression for integrated volatility in terms of spot variables

$$h\mathcal{V}_{t,h} = a_h^\sigma \sigma_t^2 + b_h^\sigma y_t + c_h^\sigma + \int_0^h \epsilon_{t,s}^\sigma ds,$$

where I denote

$$a_h^\sigma = \int_0^h A_s^\sigma ds,$$

and analogously other coefficients. In particular, the error terms may be represented as

$$\begin{aligned} \int_0^h \epsilon_{t,s}^y ds &= \int_0^h \left(\eta_y \int_t^{t+s} \sqrt{y_u} A_{t+s-u}^y dW_u^y \right) ds \\ &= \eta_y \int_t^{t+h} \sqrt{y_u} \left(\int_{u-t}^h A_{t+s-u}^y ds \right) dW_u^y \\ &= \eta_y \int_t^{t+h} \sqrt{y_u} \left(\int_0^{t+h-u} A_s^y ds \right) dW_u^y \\ &= \eta_y \int_t^{t+h} \sqrt{y_u} a_{t+h-u}^y dW_u^y, \end{aligned}$$

ans

$$\begin{aligned}
\int_0^h \epsilon_{t,s}^\sigma ds &= \int_0^h \left(\eta_\sigma \int_t^{t+s} \sigma_u A_{t+s-u}^\sigma dW_u^\sigma + \eta_y \int_t^{t+s} \sqrt{y_u} B_{t+s-u}^\sigma dW_u^y \right) ds \\
&= \eta_\sigma \int_t^{t+h} \left(\int_{u-t}^h \sigma_u A_{t+s-u}^\sigma ds \right) dW_u^\sigma + \eta_y \int_t^{t+h} \left(\int_{u-t}^h \sqrt{y_u} B_{t+s-u}^\sigma ds \right) dW_u^y \\
&= \eta_\sigma \int_t^{t+h} \sigma_u a_{t+h-u}^\sigma dW_u^\sigma + \eta_y \int_t^{t+h} \sqrt{y_u} b_{t+h-u}^\sigma dW_u^y.
\end{aligned}$$

The conditional variance of integrated volatility is

$$\begin{aligned}
V_t^P [h\mathcal{V}_{t,h}] &= V_t^P \left[\int_0^h \epsilon_{t,s}^\sigma ds \right] \\
&= \eta_\sigma^2 \int_t^{t+h} E_t^P [\sigma_u^2] (a_{t+h-u}^\sigma)^2 du + \eta_y^2 \int_t^{t+h} E_t^P [y_u] (b_{t+h-u}^\sigma)^2 du \\
&= \eta_\sigma^2 \int_t^{t+h} (A_{u-t}^\sigma \sigma_t^2 + B_{u-t}^\sigma y_t + C_{u-t}^\sigma) (a_{t+h-u}^\sigma)^2 du \\
&\quad + \eta_y^2 \int_t^{t+h} (A_{u-t}^y y_t + C_{u-t}^y) (b_{t+h-u}^\sigma)^2 du \\
&= \sigma_t^2 \eta_\sigma^2 \int_0^h A_u^\sigma (a_{h-u}^\sigma)^2 du \\
&\quad + y_t \int_0^h \left(\eta_\sigma^2 B_u^\sigma (a_{h-u}^\sigma)^2 + \eta_y^2 A_u^y (b_{h-u}^\sigma)^2 \right) du \\
&\quad + \int_0^h \left(\eta_\sigma^2 C_u^\sigma (a_{h-u}^\sigma)^2 + \eta_y^2 C_u^y (b_{h-u}^\sigma)^2 \right) du.
\end{aligned}$$

To make the notation shorter,

$$V_t^P [h\mathcal{V}_{t,h}] = A_1 \sigma_t^2 + A_2 y_t + A_3.$$

Also, the conditional variance of future spot volatility is

$$\begin{aligned}
V_t^P [\sigma_{t+h}^2] &= V_t^P [\epsilon_{t,h}^\sigma] \\
&= \eta_\sigma^2 \int_t^{t+h} E_t^P [\sigma_u] (A_{t+h-u}^\sigma)^2 du + \eta_y^2 \int_t^{t+h} E_t^P [y_u] (B_{t+h-u}^\sigma)^2 du \\
&= \eta_\sigma^2 \int_t^{t+h} (A_{u-t}^\sigma \sigma_t^2 + B_{u-t}^\sigma y_t + C_{u-t}^\sigma) (A_{t+h-u}^\sigma)^2 du \\
&\quad + \eta_y^2 \int_t^{t+h} (A_{u-t}^y y_t + C_{u-t}^y) (B_{t+h-u}^\sigma)^2 du \\
&= \sigma_t^2 \eta_\sigma^2 \int_0^h A_u^\sigma (A_{h-u}^\sigma)^2 du \\
&\quad + y_t \int_0^h \left(\eta_\sigma^2 B_u^\sigma (A_{h-u}^\sigma)^2 + \eta_y^2 A_u^y (B_{h-u}^\sigma)^2 \right) du \\
&\quad + \int_0^h \left(\eta_\sigma^2 C_u^\sigma (A_{h-u}^\sigma)^2 + \eta_y^2 C_u^y (B_{h-u}^\sigma)^2 \right) du.
\end{aligned}$$

The conditional variance of future spot central tendency is

$$\begin{aligned}
V_t^P [y_{t+h}] &= V_t^P [\epsilon_{t,h}^y ds] \\
&= \eta_y^2 \int_t^{t+h} E_t^P [y_u] (A_{t+h-u}^y)^2 du \\
&= \eta_y^2 \int_t^{t+h} (A_{u-t}^y y_t + C_{u-t}^y) (A_{t+h-u}^y)^2 du \\
&= y_t \eta_y^2 \int_0^h A_u^y (A_{h-u}^y)^2 du + \eta_y^2 \int_0^h C_u^y (A_{h-u}^y)^2 du.
\end{aligned}$$

The conditional covariance of future spot volatility and central tendency is

$$\begin{aligned}
Cov_t^P [\sigma_{t+h}^2 y_{t+h}] &= E_t^P [\epsilon_{t,h}^\sigma \epsilon_{t,h}^y] \\
&= \eta_y^2 \int_t^{t+h} E_t^P [y_u] A_{t+h-u}^y B_{t+h-u}^\sigma du \\
&= \eta_y^2 \int_t^{t+h} (A_{u-t}^y y_t + C_{u-t}^y) A_{t+h-u}^y B_{t+h-u}^\sigma du \\
&= y_t \eta_y^2 \int_0^h A_u^y A_{h-u}^y B_{h-u}^\sigma du + \eta_y^2 \int_0^h C_u^y A_{h-u}^y B_{h-u}^\sigma du.
\end{aligned}$$

These expressions can be compactly written as

$$\begin{aligned} V_t^P [\sigma_{t+h}^2] &= a_1 \sigma_t^2 + a_2 y_t + a_3, \\ V_t^P [y_{t+h}] &= b_2 y_t + b_3, \\ Cov_t^P [\sigma_{t+h}^2 y_{t+h}] &= c_2 y_t + c_3, \end{aligned}$$

with obvious notation for parameters.

Now, the second conditional moments may be written as

$$\begin{aligned} E_t^P [(h\mathcal{V}_{t,h})^2] &= A_1 \sigma_t^2 + A_2 y_t + A_3 + (a_h^\sigma \sigma_t^2 + b_h^\sigma y_t + c_h^\sigma)^2, \\ E_t^P [\sigma_{t+h}^4] &= a_1 \sigma_t^2 + a_2 y_t + a_3 + (A_h^\sigma \sigma_t^2 + B_h^\sigma y_t + C_h^\sigma)^2, \\ E_t^P [\sigma_{t+h}^2 y_{t+h}] &= c_2 y_t + c_3 + (A_h^\sigma \sigma_t^2 + B_h^\sigma y_t + C_h^\sigma) (A_h^y y_t + C_h^y), \\ E_t^P [y_{t+h}^2] &= b_2 y_t + b_3 + (A_h^y y_t + C_h^y)^2, \\ E_t^P [\sigma_{t+h}^2] &= A_h^\sigma \sigma_t^2 + B_h^\sigma y_t + C_h^\sigma, \\ E_t^P [y_{t+h}] &= A_h^y y_t + C_h^y. \end{aligned}$$

The only observable here is integrated volatility $\mathcal{V}_{t,h}^2$. The rest are latent variables which can be eliminated by taking appropriate lags and making substitutions. For example, the spot volatility and central tendency equations may be written as

$$\begin{aligned} E_t^P [(1 - A_h^y L) y_{t+h}] &= C_h^y, \\ E_t [(1 - A_h^\sigma L) \sigma_{t+h}^2] &= B_h^\sigma y_t + C_h^\sigma. \end{aligned}$$

Multiply the last equation by $(1 - A_h^y L)$, shift the time by h using the law of iterated expectations, and finally substitute the first equation in to get

$$E_t^P [(1 - A_h^y L) (1 - A_h^\sigma L) \sigma_{t+2h}^2] = B_h^\sigma C_h^y + (1 - A_h^y) C_h^\sigma.$$

The expression for the second moment of integrated volatility includes spot variables σ_t^4 , $\sigma_t^2 y_t$, y_t^2 , σ_t^2 , and y_t . using the above approach each one of these variables is eliminated with the end result of

$$E_t^P [(1 - (A_h^\sigma)^2 L) (1 - A_h^\sigma A_h^y L) (1 - (A_h^y)^2 L) (1 - A_h^y L) (1 - A_h^\sigma L) \mathcal{V}_{t+5h,h}^2] = M.$$

The constant M may be obtained from computing the unconditional expectation of integrated

volatility

$$(1 - (A_h^\sigma)^2) (1 - A_h^\sigma A_h^y) (1 - (A_h^y)^2) (1 - A_h^y) (1 - A_h^\sigma) E^P [(h\mathcal{V}_{t,h})^2] = M,$$

and other spot variables:

$$\begin{aligned} E^P [(h\mathcal{V}_{t,h})^2] &= (a_h^\sigma)^2 E^P [\sigma_t^4] + 2a_h^\sigma b_h^\sigma E^P [\sigma_t^2 y_t] + (b_h^\sigma)^2 E^P [y_t^2] \\ &\quad + (A_1 + 2a_h^\sigma c_h^\sigma) \mu + (A_2 + 2b_h^\sigma c_h^\sigma) \mu + (A_3 + (c_h^\sigma)^2), \\ (1 - (A_h^\sigma)^2) E^P [\sigma_{t+h}^4] &= 2A_h^\sigma B_h^\sigma E^P [\sigma_t^2 y_t] + (B_h^\sigma)^2 E^P [y_t^2] \\ &\quad + (a_1 + 2A_h^\sigma C_h^\sigma) \mu + (a_2 + 2B_h^\sigma C_h^\sigma) \mu + (a_3 + (C_h^\sigma)^2), \\ (1 - A_h^\sigma A_h^y) E [\sigma_{t+h}^2 y_{t+h}] &= A_h^y B_h^\sigma E^P [y_t^2] + A_h^\sigma C_h^y \mu + (c_2 + B_h^\sigma C_h^y + A_h^y C_h^\sigma) \mu + (c_3 + C_h^\sigma C_h^y), \\ (1 - (A_h^y)^2) E [y_{t+h}^2] &= (b_2 + 2A_h^y C_h^y) \mu + (b_3 + (C_h^y)^2). \end{aligned}$$

B.4 Unconditional moments of the premia

Recall from (B.2) the representation of spot volatility and central tendency as infinite stochastic integrals with respect to Brownian motions only. Integrating it over time interval H I get

$$\begin{aligned} H\mathcal{Y}_{t,H} &= \int_t^{t+H} y_u du \\ &= \mu H + \eta_y \int_t^{t+H} \int_{-\infty}^u \sqrt{y_v} A_{u-v}^y dW_v^y du \\ &= \mu H + \eta_y \int_t^{t+H} \int_t^u \sqrt{y_v} A_{u-v}^y dW_v^y du + \eta_y \int_t^{t+H} \int_{-\infty}^t \sqrt{y_v} A_{u-v}^y dW_v^y du \\ &= \mu H + \eta_y \int_t^{t+H} \sqrt{y_v} \left(\int_v^{t+H} A_{u-v}^y du \right) dW_v^y + \eta_y \int_{-\infty}^t \sqrt{y_v} \left(\int_t^{t+H} A_{u-v}^y du \right) dW_v^y. \end{aligned}$$

And for integrated stochastic volatility:

$$\begin{aligned}
H\mathcal{V}_{t,H} &= \int_t^{t+H} \sigma_u^2 du \\
&= \mu H + \eta_y \int_t^{t+H} \int_{-\infty}^u \sqrt{y_v} B_{u-v}^\sigma dW_v^y du + \eta_\sigma \int_t^{t+H} \int_{-\infty}^u \sigma_v A_{u-v}^\sigma dW_v^\sigma du \\
&= \mu H + \eta_y \int_t^{t+H} \int_{-\infty}^t \sqrt{y_v} B_{u-v}^\sigma dW_v^y du + \eta_y \int_t^{t+H} \int_t^u \sqrt{y_v} B_{u-v}^\sigma dW_v^y du \\
&\quad + \eta_\sigma \int_t^{t+H} \int_{-\infty}^t \sigma_v A_{u-v}^\sigma dW_v^\sigma du + \eta_\sigma \int_t^{t+H} \int_t^u \sigma_v A_{u-v}^\sigma dW_v^\sigma du \\
&= \mu H + \eta_y \int_{-\infty}^t \sqrt{y_v} \left(\int_t^{t+H} B_{u-v}^\sigma du \right) dW_v^y + \eta_y \int_t^{t+H} \sqrt{y_v} \left(\int_v^{t+H} B_{u-v}^\sigma du \right) dW_v^y \\
&\quad + \eta_\sigma \int_{-\infty}^t \sigma_v \left(\int_t^{t+H} A_{u-v}^\sigma du \right) dW_v^\sigma + \eta_\sigma \int_t^{t+H} \sigma_v \left(\int_v^{t+H} A_{u-v}^\sigma du \right) dW_v^\sigma.
\end{aligned}$$

Define

$$\begin{aligned}
\tau_{t,t+H}^C &= \frac{1}{H} \eta_y \int_t^{t+H} A_{u-v}^y du, \\
\tau_{t,t+H}^V &= \frac{1}{H} \eta_y \int_t^{t+H} B_{u-v}^\sigma du, \\
s_{t,t+H} &= \frac{1}{H} \eta_\sigma \int_t^{t+H} A_{u-v}^\sigma du.
\end{aligned}$$

With this notation,

$$\begin{aligned}
\mathcal{Y}_{t,H} &= \mu + \int_t^{t+H} \sqrt{y_v} \tau_{v,t+H}^C dW_v^y + \int_{-\infty}^t \sqrt{y_v} \tau_{t,t+H}^C dW_v^y, \\
\mathcal{V}_{t,H} &= \mu + \int_{-\infty}^t \sqrt{y_v} \tau_{t,t+H}^V dW_v^y + \int_t^{t+H} \sqrt{y_v} \tau_{v,t+H}^V dW_v^y \\
&\quad + \int_{-\infty}^t \sigma_v s_{t,t+H} dW_v^\sigma + \int_t^{t+H} \sigma_v s_{v,t+H} dW_v^\sigma, \\
\mathcal{V}_{t,H} - \mathcal{Y}_{t,H} &= \int_{-\infty}^t \sqrt{y_v} (\tau_{t,t+H}^V - \tau_{t,t+H}^C) dW_v^y + \int_t^{t+H} \sqrt{y_v} (\tau_{v,t+H}^V - \tau_{v,t+H}^C) dW_v^y \\
&\quad + \int_{-\infty}^t \sigma_v s_{t,t+H} dW_v^\sigma + \int_t^{t+H} \sigma_v s_{v,t+H} dW_v^\sigma.
\end{aligned}$$

Under the risk-neutral measure the integrated central tendency is

$$H\tilde{\mathcal{Y}}_{t,H} = \tilde{\mu} H + \eta_y \frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} \int_t^{t+H} \sqrt{y_v} \left(\int_v^{t+H} \tilde{A}_{u-v}^y du \right) d\tilde{W}_v^y + \eta_y \frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} \int_{-\infty}^t \sqrt{y_v} \left(\int_t^{t+H} \tilde{A}_{u-v}^y du \right) d\tilde{W}_v^y,$$

and integrated volatility is

$$\begin{aligned} H\mathcal{V}_{t,H} = & \tilde{\mu}H + \eta_y \frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} \int_{-\infty}^t \sqrt{y_v} \left(\int_t^{t+H} \tilde{B}_{u-v}^\sigma du \right) d\tilde{W}_v^y + \eta_y \frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} \int_t^{t+H} \sqrt{y_v} \left(\int_v^{t+H} \tilde{B}_{u-v}^\sigma du \right) d\tilde{W}_v^y \\ & + \eta_\sigma \int_{-\infty}^t \sigma_v \left(\int_t^{t+H} \tilde{A}_{u-v}^\sigma du \right) d\tilde{W}_v^\sigma + \eta_\sigma \int_t^{t+H} \sigma_v \left(\int_v^{t+H} \tilde{A}_{u-v}^\sigma du \right) d\tilde{W}_v^\sigma. \end{aligned}$$

Now, define two kinds of premia, volatility and central tendency:

$$\begin{aligned} VP_{t,H} &= E_t^Q [\mathcal{V}_{t,H}] - E_t^P [\mathcal{V}_{t,H}], \\ CP_{t,H} &= E_t^Q [\tilde{\mathcal{Y}}_{t,H}] - E_t^P [\mathcal{Y}_{t,H}]. \end{aligned}$$

In order to compute these premia, find the respective conditional expectations. First, the historical expectation of central tendency:

$$E_t^P [H\mathcal{Y}_{t,H}] = \mu H + \eta_y \int_{-\infty}^t \sqrt{y_v} \left(\int_t^{t+H} A_{u-v}^y du \right) dW_v^y,$$

and the risk-neutral expectation:

$$\begin{aligned} E_t^Q [H\tilde{\mathcal{Y}}_{t,H}] &= \tilde{\mu}H + \eta_y \frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} \int_{-\infty}^t \sqrt{y_v} \left(\int_t^{t+H} \tilde{A}_{u-v}^y du \right) d\tilde{W}_v^y \\ &= \tilde{\mu}H + \eta_y \frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} \int_{-\infty}^t \sqrt{y_v} \left(\int_t^{t+H} \tilde{A}_{u-v}^y du \right) (dW_v^y - \lambda_y \sqrt{y_v} dv) \\ &= \tilde{\mu}H + \eta_y \frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} \int_{-\infty}^t \sqrt{y_v} \left(\int_t^{t+H} \tilde{A}_{u-v}^y du \right) dW_v^y - \lambda_y \eta_y \frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} \int_{-\infty}^t y_v \left(\int_t^{t+H} \tilde{A}_{u-v}^y du \right) dv. \end{aligned}$$

Now find the conditional expectations of integrated volatility under the P measure:

$$E_t^P [H\mathcal{V}_{t,H}] = \mu H + \eta_y \int_{-\infty}^t \sqrt{y_v} \left(\int_t^{t+H} B_{s-v}^\sigma ds \right) dW_v^y + \eta_\sigma \int_{-\infty}^t \sigma_v \left(\int_t^{t+H} A_{s-v}^\sigma ds \right) dW_v^\sigma,$$

and under the Q measure:

$$\begin{aligned} E_t^Q [H\mathcal{V}_{t,H}] &= \tilde{\mu}H + \eta_y \frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} \int_{-\infty}^t \sqrt{y_v} \left(\int_t^{t+H} \tilde{B}_{s-v}^\sigma ds \right) d\tilde{W}_v^y + \eta_\sigma \int_{-\infty}^t \sigma_v \left(\int_t^{t+H} \tilde{A}_{s-v}^\sigma ds \right) d\tilde{W}_v^\sigma \\ &= \tilde{\mu}H + \eta_y \frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} \int_{-\infty}^t \sqrt{y_v} \left(\int_t^{t+H} \tilde{B}_{s-v}^\sigma ds \right) (dW_v^y - \lambda_y \sqrt{y_v} dv) \\ &\quad + \eta_\sigma \int_{-\infty}^t \sigma_v \left(\int_t^{t+H} \tilde{A}_{s-v}^\sigma ds \right) (dW_v^\sigma - \lambda_\sigma \sigma_v dv). \end{aligned}$$

By taking the difference between two expectations of the central tendency I find the premium related to central tendency shocks:

$$\begin{aligned}
CP_{t,H} &= E_t^Q [\tilde{\mathcal{Y}}_{t,H}] - E_t^P [\mathcal{Y}_{t,H}] \\
&= (\tilde{\mu} - \mu) + \frac{1}{H} \eta_y \int_{-\infty}^t \sqrt{y_v} \left(\int_t^{t+H} \left(\frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} \tilde{A}_{u-v}^y - A_{u-v}^y \right) du \right) dW_v^y \\
&\quad - \frac{1}{H} \lambda_y \eta_y \frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} \int_{-\infty}^t y_v \left(\int_t^{t+H} \tilde{A}_{u-v}^y du \right) dv.
\end{aligned}$$

Write one of the terms separately and represent it as a sum of purely deterministic and purely stochastic integrals:

$$\begin{aligned}
\int_{-\infty}^t y_v \left(\int_t^{t+H} \tilde{A}_{s-v}^y ds \right) dv &= \int_{-\infty}^t \left(\mu + \eta_y \int_{-\infty}^v \sqrt{y_u} A_{v-u}^y dW_u^y \right) \left(\int_t^{t+H} \tilde{A}_{s-v}^y ds \right) dv \\
&= \mu \int_{-\infty}^t \left(\int_t^{t+H} \tilde{A}_{s-v}^y ds \right) dv \\
&\quad + \eta_y \int_{-\infty}^t \left(\int_{-\infty}^v \sqrt{y_u} A_{v-u}^y \left(\int_t^{t+H} \tilde{A}_{s-v}^y ds \right) dW_u^y \right) dv \\
&= \mu \int_{-\infty}^t \left(\int_t^{t+H} \tilde{A}_{s-v}^y ds \right) dv \\
&\quad + \eta_y \int_{-\infty}^t \sqrt{y_u} \left(\int_u^t A_{v-u}^y \left(\int_t^{t+H} \tilde{A}_{s-v}^y ds \right) dv \right) dW_u^y.
\end{aligned}$$

Making the substitution leads to the following expression for the central tendency premia:

$$\begin{aligned}
CP_{t,H} &= (\tilde{\mu} - \mu) - \frac{1}{H} \mu \lambda_y \eta_y \frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} \int_{-\infty}^t \int_t^{t+H} \tilde{A}_{s-v}^y ds dv \\
&\quad + \frac{1}{H} \eta_y \int_{-\infty}^t \sqrt{y_v} \left(\int_t^{t+H} \left(\frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} \tilde{A}_{u-v}^y - A_{u-v}^y \right) du \right) dW_v^y \\
&\quad - \frac{1}{H} \lambda_y \eta_y^2 \frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} \int_{-\infty}^t \sqrt{y_v} \left(\int_v^t A_{u-v}^y \left(\int_t^{t+H} \tilde{A}_{s-u}^y ds \right) du \right) dW_v^y.
\end{aligned}$$

Now compute the volatility premium:

$$\begin{aligned}
VP_{t,H} &= E_t^Q [\mathcal{V}_{t,H}] - E_t^P [\mathcal{V}_{t,H}] \\
&= (\tilde{\mu} - \mu) - \frac{1}{H} \lambda_y \eta_y \frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} \int_{-\infty}^t y_v \left(\int_t^{t+H} \tilde{B}_{s-v}^\sigma ds \right) dv - \frac{1}{H} \lambda_\sigma \eta_\sigma \int_{-\infty}^t \sigma_v^2 \left(\int_t^{t+H} \tilde{A}_{s-v}^\sigma ds \right) dv \\
&\quad + \frac{1}{H} \eta_y \int_{-\infty}^t \sqrt{y_v} \left(\int_t^{t+H} \left(\frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} \tilde{B}_{s-v}^\sigma - B_{s-v}^\sigma \right) ds \right) dW_v^y \\
&\quad + \frac{1}{H} \eta_\sigma \int_{-\infty}^t \sigma_v \left(\int_t^{t+H} \left(\tilde{A}_{s-v}^\sigma - A_{s-v}^\sigma \right) ds \right) dW_v^\sigma.
\end{aligned}$$

Represent next to last term as a sum of purely deterministic and purely stochastic integrals:

$$\begin{aligned}
\int_{-\infty}^t y_v \left(\int_t^{t+H} \tilde{B}_{s-v}^\sigma ds \right) dv &= \mu \int_{-\infty}^t \left(\int_t^{t+H} \tilde{B}_{s-v}^\sigma ds \right) dv \\
&\quad + \eta_y \int_{-\infty}^t \sqrt{y_u} \left(\int_u^t A_{v-u}^y \left(\int_t^{t+H} \tilde{B}_{s-v}^\sigma ds \right) dv \right) dW_u^y.
\end{aligned}$$

Do the same for the last term in volatility premium:

$$\begin{aligned}
\int_{-\infty}^t \sigma_v^2 \left(\int_t^{t+H} \tilde{A}_{s-v}^\sigma ds \right) dv &= \int_{-\infty}^t \left(\mu + \eta_y \int_{-\infty}^v \sqrt{y_u} B_{v-u}^\sigma dW_u^y \right. \\
&\quad \left. + \eta_\sigma \int_{-\infty}^v \sigma_u A_{v-u}^\sigma dW_u^\sigma \right) \left(\int_t^{t+H} \tilde{A}_{s-v}^\sigma ds \right) dv \\
&= \mu \int_{-\infty}^t \int_t^{t+H} \tilde{A}_{s-v}^\sigma ds dv \\
&\quad + \eta_y \int_{-\infty}^t \int_{-\infty}^v \sqrt{y_u} B_{v-u}^\sigma \left(\int_t^{t+H} \tilde{A}_{s-v}^\sigma ds \right) dW_u^y dv \\
&\quad + \eta_\sigma \int_{-\infty}^t \int_{-\infty}^v \sigma_u A_{v-u}^\sigma \left(\int_t^{t+H} \tilde{A}_{s-v}^\sigma ds \right) dW_u^\sigma dv \\
&= \mu \int_{-\infty}^t \int_t^{t+H} \tilde{A}_{s-v}^\sigma ds dv \\
&\quad + \eta_y \int_{-\infty}^t \sqrt{y_u} \left(\int_u^t B_{v-u}^\sigma \left(\int_t^{t+H} \tilde{A}_{s-v}^\sigma ds \right) dv \right) dW_u^y \\
&\quad + \eta_\sigma \int_{-\infty}^t \sigma_u \left(\int_u^t A_{v-u}^\sigma \left(\int_t^{t+H} \tilde{A}_{s-v}^\sigma ds \right) dv \right) dW_u^\sigma.
\end{aligned}$$

Making the substitution leads to the following representation of volatility premium:

$$\begin{aligned}
VP_{t,H} &= (\tilde{\mu} - \mu) - \frac{1}{H}\mu \int_{-\infty}^t \int_t^{t+H} \left(\lambda_y \eta_y \frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} \tilde{B}_{s-v}^\sigma + \lambda_\sigma \eta_\sigma \tilde{A}_{s-v}^\sigma \right) ds dv \\
&+ \frac{1}{H} \eta_\sigma \int_{-\infty}^t \sigma_v \left(\int_t^{t+H} \left(\tilde{A}_{s-v}^\sigma - A_{s-v}^\sigma \right) ds \right) dW_v^\sigma \\
&- \frac{1}{H} \lambda_\sigma \eta_\sigma^2 \int_{-\infty}^t \sigma_v \left(\int_v^t A_{u-v}^\sigma \left(\int_t^{t+H} \tilde{A}_{s-u}^\sigma ds \right) du \right) dW_v^\sigma \\
&+ \frac{1}{H} \eta_y \int_{-\infty}^t \sqrt{y_v} \left(\int_t^{t+H} \left(\frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} \tilde{B}_{s-v}^\sigma - B_{s-v}^\sigma \right) ds \right) dW_v^y \\
&- \frac{1}{H} \lambda_y \eta_y^2 \frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} \int_{-\infty}^t \sqrt{y_v} \left(\int_v^t A_{u-v}^y \left(\int_t^{t+H} \tilde{B}_{s-u}^\sigma ds \right) du \right) dW_v^y \\
&- \frac{1}{H} \lambda_\sigma \eta_\sigma \eta_y \int_{-\infty}^t \sqrt{y_v} \left(\int_v^t B_{u-v}^\sigma \left(\int_t^{t+H} \tilde{A}_{s-u}^\sigma ds \right) du \right) dW_v^y.
\end{aligned}$$

Taking into account that the unconditional means of three premia are

$$\begin{aligned}
E^P [VP_{t,H}] &= (\tilde{\mu} - \mu) - \frac{1}{H}\mu \int_{-\infty}^t \int_t^{t+H} \left(\lambda_y \eta_y \frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} \tilde{B}_{u-v}^\sigma + \lambda_\sigma \eta_\sigma \tilde{A}_{u-v}^\sigma \right) dudv, \\
E^P [CP_{t,H}] &= (\tilde{\mu} - \mu) - \frac{1}{H}\mu \int_{-\infty}^t \int_t^{t+H} \lambda_y \eta_y \frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} \tilde{A}_{s-v}^y ds dv, \\
E^P [TP_{t,H}] &= -\mu \int_{-\infty}^t \int_t^{t+H} \left(\lambda_y \eta_y \frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} \left(\tilde{B}_{s-v}^\sigma - \tilde{A}_{s-v}^y \right) + \lambda_\sigma \eta_\sigma \tilde{A}_{s-v}^\sigma \right) ds dv,
\end{aligned}$$

all in all the premia may be represented as

$$\begin{aligned}
CP_{t,H} &= E^P [CP_{t,H}] + \int_{-\infty}^t \sqrt{y_v} \varpi_{v,t}^C dW_v^y, \\
VP_{t,H} &= E^P [VP_{t,H}] + \int_{-\infty}^t \sigma_v \omega_{v,t} dW_v^\sigma + \int_{-\infty}^t \sqrt{y_v} \varpi_{v,t}^V dW_v^y, \\
TP_{t,H} &= E^P [TP_{t,H}] + \int_{-\infty}^t \sigma_v \omega_{v,t} dW_v^\sigma + \int_{-\infty}^t \sqrt{y_v} \varpi_{v,t}^T dW_v^y,
\end{aligned}$$

where

$$\begin{aligned}
H\omega_{v,t} &= \eta_\sigma \int_t^{t+H} \left(\tilde{A}_{s-v}^\sigma - A_{s-v}^\sigma \right) ds - \lambda_\sigma \eta_\sigma^2 \int_v^t A_{u-v}^\sigma \left(\int_t^{t+H} \tilde{A}_{s-u}^\sigma ds \right) du, \\
H\varpi_{v,t}^C &= \eta_y \int_t^{t+H} \left(\frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} \tilde{A}_{u-v}^y - A_{u-v}^y \right) du - \lambda_y \eta_y^2 \frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} \int_v^t A_{u-v}^y \left(\int_t^{t+H} \tilde{A}_{s-u}^y ds \right) du, \\
H\varpi_{v,t}^V &= \eta_y \int_t^{t+H} \left(\frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} \tilde{B}_{s-v}^\sigma - B_{s-v}^\sigma \right) ds \\
&\quad - \int_v^t \int_t^{t+H} \left(\lambda_y \eta_y^2 \frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} A_{u-v}^y \tilde{B}_{s-u}^\sigma + \lambda_\sigma \eta_\sigma \eta_y B_{u-v}^\sigma \tilde{A}_{s-u}^\sigma \right) ds du, \\
H\varpi_{v,t}^T &= \eta_y \int_t^{t+H} \left(\frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} \left(\tilde{B}_{s-v}^\sigma - \tilde{A}_{s-v}^y \right) - \left(B_{s-v}^\sigma - A_{s-v}^y \right) \right) ds \\
&\quad - \int_v^t \int_t^{t+H} \left(\lambda_y \eta_y^2 \frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} A_{u-v}^y \left(\tilde{B}_{s-u}^\sigma - \tilde{A}_{s-u}^y \right) + \lambda_\sigma \eta_\sigma \eta_y B_{u-v}^\sigma \tilde{A}_{s-u}^\sigma \right) ds du.
\end{aligned}$$

With the introduced notation it is actually quite easy to compute second moments. In particular, the variances are

$$\begin{aligned}
V^P [CP_{t,H}] &= \mu \int_{-\infty}^t \left(\varpi_{v,t}^C \right)^2 dv, \\
V^P [VP_{t,H}] &= \mu \int_{-\infty}^t \left[(\omega_{v,t})^2 + \left(\varpi_{v,t}^V \right)^2 \right] dv, \\
V^P [TP_{t,H}] &= \mu \int_{-\infty}^t \left[(\omega_{v,t})^2 + \left(\varpi_{v,t}^T \right)^2 \right] dv.
\end{aligned}$$

The covariances are

$$\begin{aligned}
Cov^P [CP_{t,H}, VP_{t,H}] &= \mu \int_{-\infty}^t \varpi_{v,t}^C \varpi_{v,t}^V dv, \\
Cov^P [CP_{t,H}, TP_{t,H}] &= \mu \int_{-\infty}^t \varpi_{v,t}^C \varpi_{v,t}^T dv, \\
Cov^P [VP_{t,H}, TP_{t,H}] &= \mu \int_{-\infty}^t \left[(\omega_{v,t})^2 + \varpi_{v,t}^V \varpi_{v,t}^T \right] dv.
\end{aligned}$$

The autocovariances are

$$\begin{aligned}
Cov^P [CP_{t,H}, CP_{t-h,H}] &= \mu \int_{-\infty}^{t-h} \overline{\omega}_{v,t}^C \overline{\omega}_{v,t-h}^C dv, \\
Cov^P [VP_{t,H}, VP_{t-h,H}] &= \mu \int_{-\infty}^{t-h} [\omega_{v,t} \omega_{v,t-h} + \overline{\omega}_{v,t}^V \overline{\omega}_{v,t-h}^V] dv, \\
Cov^P [TP_{t,H}, TP_{t-h,H}] &= \mu \int_{-\infty}^{t-h} [\omega_{v,t} \omega_{v,t-h} + \overline{\omega}_{v,t}^T \overline{\omega}_{v,t-h}^T] dv.
\end{aligned}$$

In addition I can compute unconditional moments of the integrated volatility and central tendency. The means are clearly both equal to μH . The variances are

$$\begin{aligned}
V^P [\mathcal{Y}_{t,H}] &= \mu \int_t^{t+H} (\tau_{v,t+H}^C)^2 dv + \mu \int_{-\infty}^t (\tau_{t,t+H}^C)^2 dv, \\
V^P [\mathcal{V}_{t,H}] &= \mu \int_t^{t+H} [(\tau_{v,t+H}^V)^2 + (\varsigma_{v,t+H})^2] dv + \mu \int_{-\infty}^t [(\tau_{t,t+H}^V)^2 + (\varsigma_{t,t+H})^2] dv, \\
V^P [\mathcal{V}_{t,H} - \mathcal{Y}_{t,H}] &= \mu \int_t^{t+H} [(\tau_{v,t+H}^V - \tau_{v,t+H}^C)^2 + (\varsigma_{v,t+H})^2] dv \\
&\quad + \mu \int_{-\infty}^t [(\tau_{t,t+H}^V - \tau_{t,t+H}^C)^2 + (\varsigma_{t,t+H})^2] dv.
\end{aligned}$$

The covariances are

$$\begin{aligned}
Cov^P [\mathcal{V}_{t,H}, \mathcal{Y}_{t,H}] &= \mu \int_{-\infty}^t \tau_{t,t+H}^V \tau_{t,t+H}^C dv + \mu \int_t^{t+H} \tau_{v,t+H}^V \tau_{v,t+H}^C dv, \\
Cov^P [\mathcal{V}_{t,H}, \mathcal{V}_{t,H} - \mathcal{Y}_{t,H}] &= V^P [\mathcal{V}_{t,H}] - Cov^P [\mathcal{V}_{t,H}, \mathcal{Y}_{t,H}], \\
Cov^P [\mathcal{Y}_{t,H}, \mathcal{V}_{t,H} - \mathcal{Y}_{t,H}] &= Cov^P [\mathcal{Y}_{t,H}, \mathcal{V}_{t,H}] - V^P [\mathcal{Y}_{t,H}].
\end{aligned}$$

For $h < H$ the autocovariances are

$$\begin{aligned}
Cov^P [\mathcal{Y}_{t,H}, \mathcal{Y}_{t-h,H}] &= \mu \int_t^{t-h+H} \tau_{v,t+H}^C \tau_{v,t-h+H}^C dv \\
&\quad + \mu \int_{t-h}^t \tau_{t,t+H}^C \tau_{v,t-h+H}^C dv + \mu \int_{-\infty}^{t-h} \tau_{t,t+H}^C \tau_{t-h,t-h+H}^C dv, \\
Cov^P [\mathcal{V}_{t,H}, \mathcal{V}_{t-h,H}] &= \mu \int_t^{t-h+H} \left(\tau_{v,t+H}^V \tau_{v,t-h+H}^V + \varsigma_{v,t+H} \varsigma_{v,t-h+H} \right) dv \\
&\quad + \mu \int_{t-h}^t \left(\tau_{t,t+H}^V \tau_{v,t-h+H}^V + \varsigma_{t,t+H} \varsigma_{v,t-h+H} \right) dv \\
&\quad + \mu \int_{-\infty}^{t-h} \left(\tau_{t,t+H}^V \tau_{t-h,t-h+H}^V + \varsigma_{t,t+H} \varsigma_{t-h,t-h+H} \right) dv, \\
Cov^P [\mathcal{V}_{t,H} - \mathcal{Y}_{t,H}, \mathcal{V}_{t-h,H} - \mathcal{Y}_{t-h,H}] &= \\
&= \mu \int_t^{t-h+H} \left(\tau_{v,t+H}^V - \tau_{v,t+H}^C \right) \left(\tau_{v,t-h+H}^V - \tau_{v,t-h+H}^C \right) dv \\
&\quad + \mu \int_t^{t-h+H} \varsigma_{v,t+H} \varsigma_{v,t-h+H} dv \\
&\quad + \mu \int_{t-h}^t \left(\tau_{t,t+H}^V - \tau_{t,t+H}^C \right) \left(\tau_{v,t-h+H}^V - \tau_{v,t-h+H}^C \right) dv \\
&\quad + \mu \int_{t-h}^t \varsigma_{t,t+H} \varsigma_{v,t-h+H} dv \\
&\quad + \mu \int_{-\infty}^{t-h} \left(\tau_{t,t+H}^V - \tau_{t,t+H}^C \right) \left(\tau_{t-h,t-h+H}^V - \tau_{t-h,t-h+H}^C \right) dv \\
&\quad + \mu \int_{-\infty}^{t-h} \varsigma_{t,t+H} \varsigma_{t-h,t-h+H} dv.
\end{aligned}$$

For $h \geq H$ the autocovariances are

$$\begin{aligned}
Cov^P [\mathcal{Y}_{t,H}, \mathcal{Y}_{t-h,H}] &= \mu \int_{t-h}^{t-h+H} \tau_{t,t+H}^C \tau_{v,t-h+H}^C dv + \mu \int_{-\infty}^{t-h} \tau_{t,t+H}^C \tau_{t-h,t-h+H}^C dv, \\
Cov^P [\mathcal{V}_{t,H}, \mathcal{V}_{t-h,H}] &= \mu \int_{t-h}^{t-h+H} \left(\tau_{t,t+H}^V \tau_{v,t-h+H}^V + \varsigma_{t,t+H} \varsigma_{v,t-h+H} \right) dv \\
&\quad + \mu \int_{-\infty}^{t-h} \left(\tau_{t,t+H}^V \tau_{t-h,t-h+H}^V + \varsigma_{t,t+H} \varsigma_{t-h,t-h+H} \right) dv, \\
Cov^P [\mathcal{V}_{t,H} - \mathcal{Y}_{t,H}, \mathcal{V}_{t-h,H} - \mathcal{Y}_{t-h,H}] &= \\
&= \mu \int_{t-h}^{t-h+H} \left(\tau_{t,t+H}^V - \tau_{t,t+H}^C \right) \left(\tau_{v,t-h+H}^V - \tau_{v,t-h+H}^C \right) dv \\
&\quad + \mu \int_{t-h}^{t-h+H} \varsigma_{t,t+H} \varsigma_{v,t-h+H} dv \\
&\quad + \mu \int_{-\infty}^{t-h} \left(\tau_{t,t+H}^V - \tau_{t,t+H}^C \right) \left(\tau_{t-h,t-h+H}^V - \tau_{t-h,t-h+H}^C \right) dv \\
&\quad + \mu \int_{-\infty}^{t-h} \varsigma_{t,t+H} \varsigma_{t-h,t-h+H} dv.
\end{aligned}$$

The covariances of the three premia with the central tendency are

$$\begin{aligned}
Cov^P [CP_{t,H}, \mathcal{Y}_{t,H}] &= \mu \int_{-\infty}^t \varpi_{v,t}^C \tau_{t,t+H}^C dv, \\
Cov^P [VP_{t,H}, \mathcal{Y}_{t,H}] &= \mu \int_{-\infty}^t \varpi_{v,t}^V \tau_{t,t+H}^C dv, \\
Cov^P [TP_{t,H}, \mathcal{Y}_{t,H}] &= \mu \int_{-\infty}^t \varpi_{v,t}^T \tau_{t,t+H}^C dv.
\end{aligned}$$

The covariances of the three premia with the volatility are

$$\begin{aligned}
Cov^P [CP_{t,H}, \mathcal{V}_{t,H}] &= \mu \int_{-\infty}^t \varpi_{v,t}^C \tau_{t,t+H}^V dv, \\
Cov^P [VP_{t,H}, \mathcal{V}_{t,H}] &= \mu \int_{-\infty}^t \left(\varpi_{v,t}^V \tau_{t,t+H}^V + \omega_{v,t} \varsigma_{t,t+H} \right) dv, \\
Cov^P [TP_{t,H}, \mathcal{V}_{t,H}] &= \mu \int_{-\infty}^t \left(\varpi_{v,t}^T \tau_{t,t+H}^V + \omega_{v,t} \varsigma_{t,t+H} \right) dv.
\end{aligned}$$

Finally, the covariances of the three premia with the difference between the volatility and central

tendency are

$$\begin{aligned}
Cov^P [CP_{t,H}, \mathcal{V}_{t,H} - \mathcal{Y}_{t,H}] &= \mu \int_{-\infty}^t \varpi_{v,t}^C (\tau_{t,t+H}^V - \tau_{t,t+H}^C) dv, \\
Cov^P [VP_{t,H}, \mathcal{V}_{t,H} - \mathcal{Y}_{t,H}] &= \mu \int_{-\infty}^t (\varpi_{v,t}^V (\tau_{t,t+H}^V - \tau_{t,t+H}^C) + \omega_{v,t} \varsigma_{t,t+H}) dv, \\
Cov^P [TP_{t,H}, \mathcal{V}_{t,H} - \mathcal{Y}_{t,H}] &= \mu \int_{-\infty}^t (\varpi_{v,t}^T (\tau_{t,t+H}^V - \tau_{t,t+H}^C) + \omega_{v,t} \varsigma_{t,t+H}) dv.
\end{aligned}$$

B.5 CLT for model innovations

Suppose that σ_t^2 adapted to $\mathcal{F}_t = \sigma \{ \sigma_\tau^2, \tau \leq t \}$ is a solution of the following square-root SDE:

$$d\sigma_t^2 = (\mu - \sigma_t^2) dt + \sigma_t dW_t,$$

where W_t is a standard Brownian motion on (Ω, \mathcal{F}, P) probability space. The solution may be written in recursive form as

$$\sigma_{t+h}^2 = \mu (1 - e^{-h}) + e^{-h} \sigma_t^2 + \int_t^{t+h} \sigma_v e^{v-t-h} dW_v, \quad (\text{B.3})$$

or in infinite stochastic integral representation

$$\sigma_t^2 = \mu + \int_{-\infty}^t \sigma_v e^{v-t} dW_v. \quad (\text{B.4})$$

Also define a variable t

$$X_t = \int_{t-1}^t \sigma_u e^{u-t} dW_u > 0,$$

I argue that the above representation is general enough for the model in my paper. All innovations, integrated variables, and their interactions may be reduced to the above form with a appropriate change of parameters and consequitive recursive substitutions.

It is true that X_t is a martingale difference sequence since trivially $E_{t-1} [X_t] = 0$. Also, X_t is L^1 -mixingale since unconditional expectation is equal to zero and I can take $c_t = 0$ and $\xi_m = 0$ so that

$$E |E_{t-m} [X_t]| \leq c_t \xi_m$$

for all t and $m \geq 0$.

Lemma 1. X_t is uniformly integrable.

Proof. Define

$$Y_t = \int_{-\infty}^t \sigma_u e^{u-t} dW_u.$$

Use Ito's Lemma (Karatzas & Shreve, 1997) for Y_t^2 :

$$Y_t^2 = 2 \int_{-\infty}^t \left(\int_{-\infty}^s \sigma_u e^{u-s} dW_u \right) \sigma_s e^{s-t} dW_s + \int_{-\infty}^t \sigma_s^2 e^{2s-2t} ds.$$

Take the expectation:

$$E[Y_t^2] = \frac{1}{2}\mu < \infty$$

for all t . Hence, $E[X_t^2] < E[Y_t^2] < \infty$. □

Lemma 2. Define $\bar{X}_T = T^{-1} \sum_{t=1}^T X_t$. Then, $\bar{X}_T \xrightarrow{P} 0$.

Proof. By Theorem 1 (Andrews, 1988, p.460) with the choice of $c_t = 0$ and taking into account uniform integrability of X_t I immediately have $T^{-1} \sum_{t=1}^T X_t \xrightarrow{P} 0$. □

Lemma 3. $E[X_t^2] = \mu(1 - e^{-2}) < \infty$.

Proof. Write

$$\begin{aligned} E[X_t^2] &= \mu - 2E \left[\int_{-\infty}^{t-1} \sigma_u e^{u-t} dW_u \int_{-\infty}^{t-1} \sigma_u e^{u-t} dW_u \right] \\ &= \mu - 2E \left[\int_{-\infty}^{t-1} \sigma_u^2 e^{2u-2t} du \right] \\ &= \mu - 2\mu \int_{-\infty}^{t-1} e^{2u-2t} du \\ &= \mu(1 - e^{-2}). \end{aligned}$$

□

Remark. It follows trivially from the previous Lemma that $T^{-1} \sum_{t=1}^T E[X_t^2] = \mu(1 - e^{-2}) > 0$.

Lemma 4. The fourth moment of X_t is finite, $E[X_t^4] < E[Y_t^4] = 3\mu < \infty$.

Proof. Using the Ito Lemma for Y_t^4 where $Y_t = \int_{-\infty}^t \sigma_u e^{u-t} dW_u$ I obtain

$$Y_t^4 = 4 \int_{-\infty}^t \left(\int_{-\infty}^s \sigma_u e^{u-s} dW_u \right)^3 \sigma_s e^{s-t} dW_s + 6 \int_{-\infty}^t \left(\int_{-\infty}^s \sigma_u e^{u-s} dW_u \right)^2 \sigma_s^2 e^{2s-2t} ds.$$

The unconditional expectation of this variable is

$$\begin{aligned}
E[Y_t^4] &= 6E \left[\int_{-\infty}^t \left(\int_{-\infty}^s \sigma_u e^{u-s} dW_u \right)^2 \sigma_s^2 e^{2s-2t} ds \right] \\
&= 6 \int_{-\infty}^t E \left[\left(\int_{-\infty}^s \sigma_u \sigma_s e^{u-s} dW_u \right)^2 \right] e^{2s-2t} ds \\
&= 6 \int_{-\infty}^t \left(\int_{-\infty}^s E[\sigma_u^2 \sigma_s^2] e^{2u-2s} du \right) e^{2s-2t} ds.
\end{aligned}$$

Write the expectation of the product of σ_u^2 and σ_s^2 separately using the representation in (B.3):

$$\begin{aligned}
E[\sigma_u^2 \sigma_s^2] &= E[\sigma_u^2 E_u[\sigma_s^2]] \\
&= E \left[\sigma_u^2 \left(\mu(1 - e^{s-u}) + e^{s-u} \sigma_u^2 + \int_u^s \sigma_v e^{s-v} dW_v \right) \right] \\
&= E \left[\mu^2(1 - e^{s-u}) + e^{s-u} \sigma_u^4 + \sigma_u^2 \int_u^s \sigma_v e^{s-v} dW_v \right] \\
&= \mu^2(1 - e^{s-u}) + e^{s-u} E[\sigma_u^4] + E \left[\sigma_u^2 E_u \left[\int_u^s \sigma_v e^{s-v} dW_v \right] \right].
\end{aligned}$$

The last expectation above is zero. The second moment of σ_u^2 is

$$\begin{aligned}
E[\sigma_u^4] &= E \left[\left(\int_{-\infty}^t \sigma_v e^{v-t} dW_v \right)^2 \right] \\
&= E \left[\int_{-\infty}^t \sigma_v^2 e^{2v-2t} dv \right] \\
&= \mu \int_{-\infty}^t e^{2v-2t} dv = \frac{1}{2} \mu.
\end{aligned}$$

Hence,

$$\begin{aligned}
E[\sigma_u^2 \sigma_s^2] &= \mu^2(1 - e^{s-u}) + \frac{1}{2} \mu e^{s-u} \\
&= \mu^2 + \left(\frac{1}{2} - \mu \right) \mu e^{s-u}.
\end{aligned}$$

Plug this back to the expression of the fourth moment of Y_t :

$$\begin{aligned}
E[Y_t^4] &= 6 \int_{-\infty}^t \left(\int_{-\infty}^s \left(\mu^2 + \left(\frac{1}{2} - \mu \right) \mu e^{s-u} \right) e^{2u-2s} du \right) e^{2s-2t} ds \\
&= 6 \int_{-\infty}^t \left(\int_{-\infty}^s \left(\mu^2 e^{2u-2s} + \left(\frac{1}{2} - \mu \right) \mu e^{u-s} \right) du \right) e^{2s-2t} ds \\
&= 6 \int_{-\infty}^t \left(\mu^2 + \left(\frac{1}{2} - \mu \right) \mu \right) e^{2s-2t} ds \\
&= 3\mu.
\end{aligned}$$

Hence, $E[Y_t^4] = 3\mu < \infty$. □

Remark. It follows from Lemma 4 that all moments below the fourth exist, hence part of the Lemma 3 is redundant.

Lemma 5. *It is true that $T^{-1} \sum_{t=1}^T X_t^2 \xrightarrow{p} \mu(1 - e^{-2})$.*

Proof. Write X_t as a following difference:

$$\begin{aligned}
X_t &= \int_{t-1}^t \sigma_u e^{u-t} dW_u \\
&= \int_{-\infty}^t \sigma_u e^{u-t} dW_u - e^{-1} \int_{-\infty}^{t-1} \sigma_u e^{u-t+1} dW_u \\
&= Z_t - e^{-1} Z_{t-1}.
\end{aligned}$$

The martingale Z_t may be represented recursively as

$$\begin{aligned}
Z_t &= \int_{-\infty}^t \sigma_u e^{u-t} dW_u \\
&= e^{-1} Z_{t-1} + \int_{t-1}^t \sigma_u e^{u-t} dW_u \\
&= e^{-1} Z_{t-1} + \eta_t.
\end{aligned}$$

Using Ito's Lemma for Z_t I can represent it as an SDE:

$$dZ_t = -Z_t dt + \sigma_t dW_t.$$

Using Ito's Lemma once again for Z_t^2 I can show that

$$\begin{aligned}
dZ_t^2 &= 2Z_t dZ_t + 2d[Z, Z]_t \\
&= 2(\sigma_t^2 - Z_t^2) dt + 2Z_t \sigma_t dW_t.
\end{aligned}$$

Plug in the infinite representation of σ_u^2 given in (B.4):

$$\begin{aligned} dZ_t^2 &= 2 \left(\mu + \int_{-\infty}^t \sigma_v e^{v-t} dW_v - Z_t^2 \right) dt + 2Z_t \sigma_t dW_t \\ &= 2 \left(\mu - Z_t^2 \right) dt + 2 \left(\int_{-\infty}^t \sigma_v e^{v-t} dW_v \right) dt + 2Z_t \sigma_t dW_t. \end{aligned}$$

Integrate this SDE from $t-1$ to t

$$\begin{aligned} Z_t^2 &= e^{-2} Z_{t-1}^2 + \mu \left(1 - e^{-2} \right) \\ &\quad + 2 \int_{t-1}^t e^{-2(t-s)} \left(\int_{-\infty}^s \left(\sigma_v \int_v^s e^{v-u} du \right) dW_v \right) ds + 2 \int_{t-1}^t e^{-2(t-s)} \left(\int_{-\infty}^s Z_u \sigma_u dW_u \right) ds \\ &= e^{-2} Z_{t-1}^2 + \mu \left(1 - e^{-2} \right) + \varepsilon_t \end{aligned}$$

Take the square of X_t and substitute recursive expressions for Z_t and Z_{t-1} :

$$\begin{aligned} X_t^2 &= Z_t^2 - 2e^{-1} Z_t Z_{t-1} + e^{-2} Z_{t-1}^2 \\ &= 2e^{-2} Z_{t-1}^2 + \mu \left(1 - e^{-2} \right) + \varepsilon_t - 2e^{-2} Z_{t-1}^2 - 2e^{-1} \eta_t Z_{t-1} \\ &= \mu \left(1 - e^{-2} \right) + \varepsilon_t - 2e^{-1} \eta_t Z_{t-1}. \end{aligned}$$

The first term above is the second moment of X_t . The variables ε_t and η_t can be shown to be mds and L^1 -mixingale analogously to X_t itself. The term $\eta_t Z_{t-1}$ is also a martingale and L^1 -mixingale simply by the fact that Z_{t-1} is measurable with respect to \mathcal{F}_{t-1} . This implies that both $T^{-1} \sum_{t=1}^T \varepsilon_t$ and $T^{-1} \sum_{t=1}^T \eta_t Z_{t-1}$ converge to zero in probability. This concludes the proof. \square

Proposition. *It is true that $\sqrt{T} \bar{X}_T \xrightarrow{d} N(0, \mu(1 - e^{-2}))$.*

Proof. There are four conditions to be checked. The first, that the second moment is finite for all t is proven in Lemma 3. The second condition that the average of second moments converges to some positive constant holds trivially since the second moment is the same constant for all t . The condition that $E|X_t|^r < \infty$ for some $r > 2$ and all t is shown in Lemma 4 for $r = 4$. The last condition, that the average of X_t squared converges in probability to the average of second moments, is proven in Lemma 5. Hence, all conditions of Corollary 5.25 (White, 1984, p.130) are checked. \square