Centre for
Economic
and Financial
Research
at
New Economic
School

November 2009

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Working Paper No 139

CEFIR / NES Working Paper series

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20 November 2009

Abstract

We study efficiency properties of the irrevocable exit English auction in a setting with interdependent values. Maskin (1992) shows that the pairwise single-crossing condition is sufficient for efficiency of the English auction with two bidders and suggests that it is also a necessary condition. This paper clarifies and extends Maskin's results to the case of N bidders. We introduce the *generalized single crossing condition*—a fairly intuitive extension of the pairwise single-crossing condition—and prove that it is *essentially* a *necessary* and *sufficient* condition for the existence of an efficient equilibrium of the N-bidder English auction.

Keywords: English auction, efficient auction, ex post equilibrium, single-crossing, interdependent values.

1 Introduction

How to sell a good to the buyer who values it the most is one of the main questions of the theory of auctions. When each buyer knows how much she values the good, any common auction format is efficient. The task becomes

*We are grateful to Murali Agastya, Paul Dower, Glenn Ellison, Vijay Krishna, Muhamet Yildiz, an associate editor and two anonymous referees for many comments and suggestions.

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harder as the informational environment gets more complex. When the valuations of the buyers are asymmetric and depend on the private information of the other buyers, the set of efficient mechanisms is quite limited. Among these is the open ascending price, the English auction. Milgrom & Weber (1982) introduce the irrevocable exit model of the English auction. This standard model is known to possess an efficient equilibrium when value functions satisfy certain conditions. The set of minimal (necessary and sufficient) conditions for efficiency of the English auction remains a long-standing problem. This paper provides a solution.

Milgrom & Weber (1982) show that with symmetric interdependent values the English auction has an efficient equilibrium and, if signals (the bidders' private information) are affiliated, it also generates higher expected revenue than other common auction forms. Maskin (1992) suggests that with asymmetric values the pairwise single-crossing condition is necessary for the efficiency of the English auction and shows that it is a sufficient condition when there are two bidders. However, Perry & Reny (2005) provide an example with three bidders, where the pairwise single-crossing is satisfied but no efficient equilibrium exists. Krishna (2003) presents a pair of sufficient conditions for efficiency of the N-bidder English auction—the average-crossing and cyclical-crossing conditions.[1]

We introduce the *generalized single-crossing (GSC)* condition, a property of the value functions that is a natural extension of the pairwise single-crossing (SC) condition. At any signal profile call the bidders with equal and maximal values the winners' circle. SC states that if the signal of any bidder from the winners' circle is slightly increased, she belongs to the new winners' circle. This implies that a bidder's own private information affects her own valuation more than the valuations of her competitors. Our new condition, GSC states that if the signals of any subset of the winners' circle are slightly increased, the resulting winners' circle contains at least one of the bidders whose signal was increased. GSC implies, and in the case of two bidders, reduces to SC.

There are two main results. If GSC is violated, except in three spe-

[1] The average-crossing condition requires that starting from a signal profile where the values of several bidders are equal and maximal, if the signal of one of them is increased, the corresponding increments to the values of the others are lower than the average increment. The cyclical-crossing requires that the increments to the values are ranked in the prespecified cyclical order—the effect on the own value is the largest and decreases for each subsequent bidder in the cycle.

cial cases, no efficient equilibrium of the N-bidder English auction exists. Conversely, if GSC is satisfied at every signal profile, the N-bidder English auction admits an efficient ex post equilibrium. In two of the exceptions, illustrated by examples 2 and 3 below, an efficient equilibrium exists despite a violation of SC in the interior of the signals' domain. This clarifies the conventional wisdom that SC is necessary for efficiency of any mechanism.[2]

The English auction is not the only efficient mechanism in the interdependent values setting, and GSC is not the weakest condition for efficiency of arbitrary mechanisms. For instance, a generalized Vickrey auction is efficient if SC holds. In a Vickrey auction the bidders report their signals to the auctioneer, who assigns the object to the bidder with the highest value based on the reports and prescribes the payment. To implement the mechanism the auctioneer has to know all that the bidders commonly know, and the bidders have to trust the auctioneer to run the mechanism correctly. In contrast, to run the English auction the auctioneer only has to observe which bidders are active at the current price. A transparent set of rules and the strategic simplicity make the English auction attractive to the bidders as well. In the English auction, even if the values are interdependent, the strategy in the efficient equilibrium is nothing but "drop out when the price reaches what you believe your value is."[3]

Updating the beliefs about the signals of the other bidders in an efficient equilibrium of the English auction requires solving a vector-system $p \cdot \mathbf{1} = \mathbf{V}(\mathbf{s})$ (equating the price and the valuations of the bidders; see Section 3 for details). An efficient equilibrium exists if the solution $\mathbf{s}(p)$ is non-decreasing, and GSC is the exact condition guaranteeing this. As such, it can be exploited directly or with appropriate modifications in the analysis of any economic system described by a vector-system $\mathbf{p} = \mathbf{f}(\mathbf{x})$ that requires the monotonicity of the solution $\mathbf{x}(\mathbf{p})$. In the international trade context, for example, such systems may be prices or quantities of final goods as functions of factor prices or factor demands.

Independently from us Dubra, Echenique & Manelli (2009) also look for

[2] See, for instance, Maskin (1992), pp. 127-128; Dasgupta & Maskin (2000), pp. 348-349.

[3] Kirchkamp & Moldovanu (2004) experimentally compare an English auction with a sealed-bid second-price auction in the setting with interdependent and asymmetric values. Even though not all the subjects follow their efficient equilibrium strategies, overall the bidding in the English auction is close enough to the equilibrium so that the English auction is significantly more efficient.

the conditions under which the English auction is efficient. Their main research questions are different form ours. In particular, they study the connection between the theories of auctions and trade mentioned above and attempt to relax the assumption of differentiality of the value functions, typical in the literature on efficient auctions. In the final version, Dubra et al. (2009) present a condition, an *own effect property (OEP)* and use it to generalize the Stolper-Samuelson theorem. They also show that it is a sufficient condition for efficiency of the English auction. In essence, OEP requires GSC for all possible increases of the signals (not only infinitesimal).[4] Dubra et al. (2009) also show that if OEP does not hold, then *there is* a distribution of the bidders' signals such that an efficient equilibrium in the English auction does not exist. Our result is much stronger, we show that if GSC does not hold, then for *any* distribution of the signals no efficient equilibrium exists.

Recall that in the standard model of the English auction exits are irrevocable. Izmalkov (2003) proposes an alternative model of the English auction. In his model the bidders are allowed to re-enter even if they have previously dropped out. The English auction with reentry is efficient under conditions that are weaker than GSC. (GSC would imply that no reentry happens in an efficient equilibrium.) At the same time the possibility of reentry substantially enriches the strategy space and provides opportunities to exchange messages, which may allow bidders to coordinate on a collusive outcome. In contrast, the irrevocable-exit English auction is robust to collusion within the auction since the only way a bidder can send a message is by exiting.

Recently several mechanisms were designed to allocate multiple units in the interdependent values setting efficiently. All of these mechanisms are remarkable constructions, however, in their single unit version they are significantly more complex than the English auction. The "contingent bid" mechanism of Dasgupta & Maskin (2000) requires each buyer to submit a price she is willing to pay given the realized values of the others, a $(N-1)$-variable function. This auction is efficient if SC holds.[5] Perry & Reny (2002)

[4]Dubra et al. (2009) do not require differentiability of the value functions. Yet, their other assumptions—the continuity and increasing in ties condition imply partial differentiability along the equilibrium path of the efficient equilibrium, which is enough for our proof of sufficiency.

[5]Jehiel & Moldovanu (2001) study the setting with allocative externalities. When private information is one-dimensional Jehiel & Moldovanu (2001) offer a sufficient for efficiency congruence condition. This condition reduces to the pairwise single-crossing without the allocative externalities. When private information is multi-dimensional Jehiel

and Perry & Reny (2005) design two elegant mechanisms which incorporate a concept of "directed bids." Every buyer bids against every other buyer, thus managing $N-1$ bids simultaneously. These auctions require the strong form of SC for efficiency.[6]

The rest of the paper is organized as follows. Section 2 describes the environment and introduces GSC. Its sufficiency is proven in Section 3. The examples clarifying the role of GSC and the proof of its necessity are in Section 4.

2 Preliminaries

2.1 The model

There is a single indivisible good to be auctioned among a set $\mathcal{N} = \{1, 2, ..., N\}$ of bidders. Prior to the auction each bidder j privately observes a real valued signal $s_j \in [0, 1]$. Signals are distributed according to a joint density function $f(\mathbf{s})$, where $\mathbf{s} = (s_1, s_2, ..., s_N)$ denotes the profile of the signals of all the bidders.[7] It is assumed that f has full support and is strictly positive on its interior.

If the realized signals are $\mathbf{s}$, the value of the object to bidder j is $V_j(\mathbf{s})$. Note that a bidder's value may depend on the information obtained by the other bidders. The sale of an oil track is a typical example of such an environment—a firm's estimate of the worth of the track may depend on the results of the "off-site" drilling conducted by a rival that owns an adjacent track, see Porter (1995).

The value functions $\mathbf{V} = (V_1, V_2, ..., V_N)$ are assumed to have the following properties. For any j, and any $i \neq j$: $V_j(\mathbf{0}) = 0$; $V_j(\mathbf{1}) < \infty$; V_j is continuously differentiable, $V_j \in C^1([0, 1]^N)$, $\frac{\partial V_j}{\partial s_j} > 0$. Value functions V_j for all j and distribution $f(\mathbf{s})$ are commonly known among the bidders.[8]

& Moldovanu (2001) and also Dasgupta & Maskin (2000) show that achieving efficiency is generically impossible.

[6] SC has to be satisfied for any pair of bidders with equal values, not only when their values are maximal. Thus, the strong SC and GSC are not comparable.

[7] Vectors and sets are denoted, respectively, by bold and calligraphic letters; $\mathbf{a} \gg \mathbf{b}$ ($\mathbf{a} \geqq \mathbf{b}$) means $a_i > b_i$ ($a_i \geq b_i$) in every component.

[8] These are fairly standard assumptions. The analysis can be easily extended to the unbounded supports as long as the valuations are bounded. The sufficiency proposition requires no assumption on f. The necessity proposition requires that signals have full

Definition 1. *For a given profile of signals* $\mathbf{s}$*, the* ***winners' circle*** $\mathcal{I}(\mathbf{s})$ *is the set of bidders with the highest values at* $\mathbf{s}$*. Formally,*

$$j \in \mathcal{I}(\mathbf{s}) \iff V_j(\mathbf{s}) = \max_{i \in \mathcal{N}} V_i(\mathbf{s}). \tag{1}$$

We require the value functions to be *regular*: at any $\mathbf{s}$ and for any subset $\mathcal{J} \subset \mathcal{I}(\mathbf{s})$, it is assumed $\det DV_{\mathcal{J}} \neq 0$, where $DV_{\mathcal{J}} = \left(\frac{\partial V_i(\mathbf{s})}{\partial s_j}\right)_{i,j \in \mathcal{J}}$.

Note that our specification allows the signal of one bidder to affect the values of the others. We allow for such cross effects to be negative but not arbitrarily large. Specifically, we require that at any $\mathbf{s}$, for any subset $\mathcal{J} \subset \mathcal{I}(\mathbf{s})$ and any direction $\mathbf{u} = (\mathbf{u}_{\mathcal{J}}, \mathbf{u}_{-\mathcal{J}})$, such that $\mathbf{u}_{\mathcal{J}} \gg \mathbf{0}$ and $\mathbf{u}_{-\mathcal{J}} = \mathbf{0}$, there exists $j \in \mathcal{J}$ with the directional derivative $D_{\mathbf{u}} V_j(\mathbf{s}) = \sum_{i=1}^{N} u_i \frac{\partial V_j(\mathbf{s})}{\partial s_i} > 0$. That is, if all $\mathbf{s}_{\mathcal{J}}$ are increased in the direction $\mathbf{u}$, the value of at least one bidder from $\mathcal{J}$ is increased.

This assumption is weaker than any of the comparable restrictions of the existing literature. In many papers, including Milgrom & Weber (1982), it is assumed that an increase in a bidder's signal has a non-negative effect on the other bidders' values. Krishna (2003) requires that such an increase, while possibly negatively affecting some bidders, has a positive total effect, which together with the average crossing condition implies the assumption imposed here. (Krishna's cyclical crossing condition is sufficient when the values are non-decreasing in the other bidders' signals.)

2.2 Generalized Single Crossing

Definition 2. *The* ***generalized single-crossing (GSC)*** *condition holds if at any* $\mathbf{s}$*, for any subset* $\mathcal{A} \subset \mathcal{I}(\mathbf{s})$ *and bidder* $k \in \mathcal{I}(\mathbf{s}) \setminus \mathcal{A}$*,*[9] *and any direction* $\mathbf{u}$*, such that* $u_j > 0$ *for all* $j \in \mathcal{A}$ *and* $u_j = 0$ *for all* $j \notin \mathcal{A}$*,*

$$D_{\mathbf{u}} V_k(\mathbf{s}) \leq \max_{j \in \mathcal{A}} \{D_{\mathbf{u}} V_j(\mathbf{s})\}. \tag{2}$$

support and for all j and s_j any subset of the other bidders' signals of positive measure has positive probability. The assumption of $f(\mathbf{s}) > 0$ is the simplest one guaranteeing these two properties. (For instance, if f is degenerate so that $\mathbf{s}$ is commonly known, then $\mathbf{V}(\mathbf{s})$ is known too and an efficient allocation can be achieved no matter what the properties of the value functions are.)

[9] By convention, GSC is satisfied at any $\mathbf{s}$ with $\#\mathcal{I}(\mathbf{s}) = 1$.

In words, select any group $\mathcal{A}$ of bidders from $\mathcal{I}(\mathbf{s})$, the bidders who have equal and maximal values and increase their signals. GSC requires that the increments to the values of the bidders from $\mathcal{I}(\mathbf{s})\backslash\mathcal{A}$ are at most as high as the highest increment among the bidders from $\mathcal{A}$. Or, stated differently, at least one bidder from $\mathcal{A}$ should be in the resulting winners' circle. *Single-crossing condition (SC)* is GSC required for subsets $\mathcal{A}$ consisting of one bidder.

GSC is *violated* at signal profile $\mathbf{s}$ for subset $\mathcal{A} \subset \mathcal{I}(\mathbf{s})$ and bidder $k \in \mathcal{I}(\mathbf{s}) \setminus \mathcal{A}$ if there exists vector $\mathbf{u}$, with $u_j > 0$ for all $j \in \mathcal{A}$, $u_j = 0$ for all $j \notin \mathcal{A}$, such that $D_{\mathbf{u}}V_k(\mathbf{s}) > \max_{i\in\mathcal{A}}\{D_{\mathbf{u}}V_i(\mathbf{s})\}$. In what follows, whenever we say that GSC is violated it means that there exist such $\mathbf{s}$, $\mathcal{A}$, and k.

2.3 The English Auction

Following Milgrom & Weber (1982), we consider the standard model for the analysis of the English auction. Specifically, the price of the object rises continuously, and the bidders indicate whether they are willing to buy the object at that price or not. A bidder who is willing to buy at the current price is said to be an active bidder. At a price of 0 all the bidders are active, and, as the price rises, bidders can choose to drop out of the auction. The decision to drop out is both public and irrevocable. Thus, if bidder j drops out at price p_j, both her identity and the exiting price p_j are observed by all the bidders. Furthermore, once bidder j drops out she cannot "re-enter" the auction at a higher price. The auction ends when the second last bidder drops out. The clock stops, the only remaining bidder is the winner. If no bidders remain active the winner is chosen at random among those who exited last. The winner is obliged to pay the price shown on the clock. If two or more bidders decide to remain active forever then the auction continues indefinitely. We assign to every such bidder a payoff of $-\infty$.

Since all drop-out decisions are public, the *public history* $H(p)$ can be effectively summarized as the sequence of prices at which the bidders, inactive at p, have exited, $H(p) = \mathbf{p}_{-\mathcal{M}}$, where $\mathcal{M}$ is the set of the bidders active just before p. If no bidder exits at $p \in [p', p'')$, then $H(p') = H(p'')$. Denote with $\bar{H}(p)$ the public history $H(p)$ together with all the exits that happen at p. Therefore, if $\bar{H}(p) \neq H(p)$, then there exists a bidder who exited at p. All the bidders are assumed to be active just before the clock starts at $p = 0$, so $H(0) = \varnothing$.

In the English auction a bidder's strategy determines the price at which she would drop out given the public history provided no other bidder drops out first. Formally, following Krishna (2003), bidder j's bidding strategy is a collection of functions $\beta_j^{\mathcal{M}} : [0,1] \times \mathbb{R}_+^{N-M} \longrightarrow \mathbb{R}_+$, for each $\mathcal{M} \subset \mathcal{N}$, with $j \in \mathcal{M}$ and $M = \#\mathcal{M} > 1$. Function $\beta_j^{\mathcal{M}}$ determines the price $\beta_j^{\mathcal{M}}(s_j; H(p))$ at which bidder j with signal s_j will drop out when the set of active bidders is $\mathcal{M}$ and the bidders $\mathcal{N} \setminus \mathcal{M}$ dropped out at prices $H(p) = \mathbf{p}_{-\mathcal{M}} = \{p_j\}_{j \in \mathcal{N} \setminus \mathcal{M}}$. If $\beta_j^{\mathcal{M}}(s_j; H(p)) \leq p$ at some p, the strategy says that bidder j must exit at p. If active bidders are able to infer true signals $\mathbf{s}_{-\mathcal{M}}$ of inactive bidders from their exit prices $\mathbf{p}_{-\mathcal{M}}$, the strategies can be equivalently written as $\beta_j^{\mathcal{M}}(s_j; \mathbf{s}_{-\mathcal{M}})$. Since the public history uniquely determines the set of active bidders $\mathcal{M}$, we simply use $\beta_j(s_j; H(p))$ in place of $\beta_j^{\mathcal{M}}(s_j; H(p))$ in the rest of the paper.

The equilibrium concept we use is a Bayesian-Nash equilibrium. The equilibrium we present in Section 3 is also ex-post and efficient.

Definition 3. *An **ex-post equilibrium** is a Bayesian-Nash equilibrium $\boldsymbol{\beta}$ that remains a Nash equilibrium even if the signals $(s_1, s_2, ..., s_N)$ are commonly known. An equilibrium is **efficient** if the object is allocated to the bidder with the highest value at every realization of signals $(s_1, s_2, ..., s_N)$.*

Example 1 (Perry and Reny)**.** *Consider the following value functions*

$$\begin{aligned} V_1 &= s_1 + s_2 s_3, \\ V_2 &= \tfrac{1}{2} s_1 + s_2, \\ V_3 &= s_3. \end{aligned}$$

There exists no efficient equilibrium of the English auction.

Perry & Reny (2005) prove that no efficient equilibrium exists. Observe that GSC is violated at $\mathbf{s} = (.3, .6, .75)$ for $\mathcal{A} = \{2, 3\}$, bidder 1, and vector $\mathbf{u} = (0, 1, 1)$. Indeed, $D_{\mathbf{u}} V_2 = D_{\mathbf{u}} V_3 = 1$, while $D_{\mathbf{u}} V_1 = \frac{\partial V_1(\mathbf{s})}{\partial s_2} + \frac{\partial V_1(\mathbf{s})}{\partial s_3} = 1.35 > 1$.

3 Sufficiency

In this section we show that GSC is sufficient for the existence of an efficient equilibrium of the N-bidder English auction.[10]

[10] We build upon the existing constructions. Milgrom & Weber (1982) present an efficient equilibrium of the English auction with symmetric bidders; Maskin (1992) extends it to the case of two asymmetric bidders, and Krishna (2003) generalizes it to the case of N asymmetric bidders.

Proposition 1 (Sufficiency)**.** *Suppose the value functions satisfy GSC. Then, there exists an* efficient *ex post equilibrium of the N-bidder English auction.*

The key component of the proposed efficient equilibrium is the weakly-increasing profile of *inferences*, or *inverse bidding functions*, $\boldsymbol{\sigma}(p, H(p))$, computed at any p for any given public history $H(p)$. Then, the equilibrium strategy of each active bidder j prescribes her to remain active as long as $\sigma_j(p, H(p))$ is lower than s_j, and exit at the lowest p_j at which $s_j = \sigma_j(p_j, H(p))$.

Formally, we define the (suggested equilibrium) strategies as follows. Suppose there exists a profile of functions $\boldsymbol{\sigma}(p, H(p))$, such that, letting $\mathcal{M}$ be the set of active bidders given $H(p)$, for any $p \geq \max_{i \notin \mathcal{M}} p_i$:

1. for any (inactive) bidder $i \notin \mathcal{M}$, $\sigma_i(p) = \sigma_i(p_i, H(p_i))$,[11] that is, $\sigma_i(p)$ is fixed after bidder i exits at p_i;

2. for any bidder $j \in \mathcal{M}$, $\sigma_j(p) \in [0, 1]$ solves $V_j(\sigma_j(p), \boldsymbol{\sigma}_{-j}(p)) = p$ if such a solution exists with $\sigma_j(p) \leq 1$, else $\sigma_j(p) = 1$ and $V_j(\sigma_j(p), \boldsymbol{\sigma}_{-j}(p)) < p$.

That is, for all active bidders, $\boldsymbol{\sigma}_{\mathcal{M}}(p)$ are determined simultaneously as a solution to

$$\begin{gathered} \mathbf{V}_{\mathcal{M}}(\boldsymbol{\sigma}_{\mathcal{M}}(p), \boldsymbol{\sigma}_{-\mathcal{M}}(p)) \leqq p\, \mathbf{1}_{\mathcal{M}}, \quad \boldsymbol{\sigma}_{\mathcal{M}}(p) \leqq \mathbf{1}_{\mathcal{M}}, \\ \forall j : (V_j(\boldsymbol{\sigma}(p)) - p)(\sigma_j(p) - 1) = 0. \end{gathered} \tag{3}$$

Then, for bidder $j \in \mathcal{M}$ *strategy* $\beta_j^{\mathcal{M}} : (s_j, H(p)) \longrightarrow \mathbb{R}_+$ is

$$\beta_j^{\mathcal{M}}(s_j; \mathbf{p}_{-\mathcal{M}}) = \arg\min_p \left\{ \sigma_j(p) \geq s_j \right\}. \tag{4}$$

Strategy β_j can be interpreted as follows. Given the public history $H(p) = \mathbf{p}_{-\mathcal{M}}$, an active bidder j is supposed to exit the auction at $p_j = \beta_j^{\mathcal{M}}(s_j; \mathbf{p}_{-\mathcal{M}})$, provided no other bidder exits before. If the current price $p < \beta_j^{\mathcal{M}}(s_j; \mathbf{p}_{-\mathcal{M}})$, bidder j is suggested to remain active; if $p \geq \beta_j^{\mathcal{M}}(s_j; \mathbf{p}_{-\mathcal{M}})$ bidder j is suggested to exit at p. Once bidder j exits at p_j, the other bidders update the public history and, expecing bidder j to follow (4), infer $s_j^* = \sigma_j(p_j)$. If $\sigma_j(\cdot)$ is non-decreasing the inferred s_j^* is unique and coincides with true signal s_j.

[11] To shorten the notation we are omitting $H(p)$ from the arguments, whenever the public history is explicitly mentioned or implied by the context.

The strategies can then be reformulated as the functions of the own and inferred signals of inactive bidders, $\beta_j^{\mathcal{M}}(s_j; \mathbf{s}_{-\mathcal{M}}) = \beta_j^{\mathcal{M}}(s_j; \mathbf{p}_{-\mathcal{M}})$.

The proof of Proposition 1 is based on the following three lemmas.

Lemma 1. *GSC holds if and only if at any* $\mathbf{s}$ *with* $\#\mathcal{I}(\mathbf{s}) \geq 2$*, for any* $\mathcal{A} \subset \mathcal{I}(\mathbf{s})$ *and* $k \in \mathcal{I}(\mathbf{s}) \setminus \mathcal{A}$*, for vector* $\mathbf{u}^{\mathcal{A}} = (\mathbf{u}^{\mathcal{A}}_{\mathcal{A}}, \mathbf{0}_{-\mathcal{A}})$ *that solves* $D_{\mathbf{u}^{\mathcal{A}}} \mathbf{V}_{\mathcal{A}}(\mathbf{s}) = \mathbf{1}$*,*

$$D_{\mathbf{u}^{\mathcal{A}}} V_k(\mathbf{s}) \leq 1. \tag{5}$$

In addition, if GSC is satisfied, then for any $\mathcal{A} \subset \mathcal{I}(\mathbf{s})$*,* $\mathbf{u}^{\mathcal{A}}_{\mathcal{A}} \geqq \mathbf{0}$*.*

We will refer to $\mathbf{u}^{\mathcal{A}}$ as *the equal increments vector* corresponding to subset $\mathcal{A}$.

Consider increasing the values of bidders $\mathcal{A} \subset \mathcal{I}(\mathbf{s})$ uniformly while keeping the signals of the others fixed. Lemma 1 states that if GSC is satisfied then, as a result, the values of bidders $\mathcal{A} \subset \mathcal{I}(\mathbf{s})$ remain maximal and their signals do not decrease in the process. Note also that Lemma 1 provides a relatively simple way of verifying GSC. Indeed, at any $\mathbf{s}$ and for a given $DV_{\mathcal{I}(\mathbf{s})}$, it suffices to check whether linear inequalities (5) are jointly satisfied.

Lemma 2. *Suppose GSC holds. Then there exist inferences* $\boldsymbol{\sigma}(p, H(p))$*, such that each* $\sigma_j(\cdot, H(p))$ *is continuous and non-decreasing for any* $H(p)$*, and* $\sigma_j(p, \bar{H}(p)) = \sigma_j(p, H(p))$ *for all* p *such that* $\bar{H}(p) \neq H(p)$*. For any active at* $H(p)$ *bidder* j*,* $\sigma_j(p, H(p)) < 1$ *implies* $j \in \mathcal{I}(\boldsymbol{\sigma}(p))$*.*

This Lemma establishes the existence of the inferences satisfying (3). The proofs of Lemmata 1 and 2 are in Appendix A.1.

Lemma 3. *Suppose GSC holds. Then the strategies defined by (4) constitute an* efficient ex-post *equilibrium of the* N*-bidder English auction.*

Proof. We first show that $\boldsymbol{\beta}$ are well-defined. For any bidder j, arbitrarily fix the exit prices of the other bidders, $\mathbf{p}_{-j}$, possibly with $p_i = \infty$ for some bidders. Then one can obtain $\sigma_j(p)$ defined for any $p \geq 0$ as $\sigma_j(p) = \sigma_j(p, H(p))$, where $H(p) = \cup_{p_i < p}\{p_i\}$. Lemma 2 shows that $\sigma_j(p)$ is continuous and non-decreasing for any given $\mathbf{p}_{-j}$. Therefore, $p_j = \arg\min_p \{\sigma_j(p) \geq s_j\}$ is unique, so $\beta_j(s_j; \cdot)$ is well defined.

Next, we show that when all the bidders follow strategies (4), the object is allocated efficiently. Suppose bidder j wins the object at price p^*. Then, for any $i \neq j$, $\sigma_i(p^*) = \sigma_i(p_i) = s_i$, while $\sigma_j(p^*) \leq s_j$, and, according to Lemma 2,

$$V_j(\sigma_j(p^*), \mathbf{s}_{-j}) = \max_{i \neq j} V_i(\sigma_j(p^*), \mathbf{s}_{-j}) = p^*. \tag{6}$$

The pairwise single-crossing, $\sigma_j(p^*) \leq s_j$, and equation (6) imply that

$$V_j(\mathbf{s}) \geq \max_{i \neq j} V_i(\mathbf{s}) \geq p^*, \tag{7}$$

so bidder j is (one of) the bidder(s) with the highest value, and the price p^* that bidder j has to pay does not depend on the signal of bidder j.

Finally, we show that $\boldsymbol{\beta}$ form an ex-post equilibrium. Suppose every bidder other than bidder j follows the proposed strategy and bidder j deviates. The payoff of bidder j can change only if the deviation affects whether bidder j obtains the object. If bidder j wins the object as a result of the deviation, she has to pay $p_j^* = \max_{i \neq j} V_i(\sigma_j(p_j^*), \mathbf{s}_{-j})$. If bidder j is not the winner in the equilibrium, then $\sigma_j(p_j^*) \geq s_j$ since $\sigma_j(p)$ is non-decreasing, so $V_j(\mathbf{s}) \leq p_j^*$, and the deviation is not profitable. If, as a result of the deviation, bidder j is not the winner while she is in the equilibrium, she is possibly forfeiting positive profits according to (7). Thus, no profitable deviation exists. The above arguments are valid even if signals $\mathbf{s}$ are commonly known, hence the presented equilibrium is ex-post. $\square$

4 Necessity

In this section we establish that GSC is essentially necessary for the existence of an efficient equilibrium of the N-bidder English auction.

There are three kinds of settings in which an efficient equilibrium may exist despite the violation of GSC. Firstly, if GSC is violated only on the boundary, one of the bidders may have the highest value at all signals' realizations and follow "remain active forever" strategy in an efficient equilibrium. For example, this occurs in a 2-bidder auction with $V_1 = s_1$ and $V_2 = 2s_1 + s_2$. In general, if GSC is violated only on the boundary, the existence of an efficient equilibrium is not precluded and may depend on the properties of the values that are not the highest. While we do expect that a condition qualitatively similar to GSC ultimately determines whether an efficient equilibrium exists or not in such settings, we choose to abstain from analyzing them since it represents significant technical challenge without much additional insight.

The following two examples demonstrate that, contrary to the existing perceptions, an efficient equilibrium may exist while SC is violated in the interior.

Example 2. *Consider the following value functions*

$$\begin{aligned} V_1 &= s_1 + 2s_2 + 4s_3, \\ V_2 &= 2s_1 + s_2 + 4s_3, \\ V_3 &= s_3. \end{aligned}$$

Suppose that $s_3 = 1 - (s_1 + s_2)/2$. *There exists an efficient equilibrium of the English auction.*

Note that bidder 3 here is "irrelevant: " her value is never the highest and her signal does not affect the efficient allocation. In an efficient equilibrium she exits at 0 for all s_3. Given this, the expected values of bidders 1 and 2 are: $W_1(s_1, s_2) = E_{s_3|s_1,s_2} V_1(s_1, s_2, s_3) = 4 - s_1$ and $W_2(s_1, s_2) = 4 - s_2$. Let $t_1 = 4 - s_1$ and $t_2 = 4 - s_2$, then $W_1(t_1, t_2)$ and $W_2(t_1, t_2)$ satisfy SC, and $\beta_i(t_i) = t_i$ for $i = 1, 2$ ensure efficiency. The degeneracy of the signals distribution is not consequential. A similar equilibrium can be constructed for any non-degenerate f for which $E(s_3|s_1, s_2) = A - (2A - 1)(s_1 + s_2)/2$ with A less than but close to 1. □

Example 3. *Consider the following value functions*

$$\begin{aligned} V_1 &= Bs_1 + 2s_2 - \tfrac{1}{2}, \\ V_2 &= Bs_1 + s_2, \end{aligned}$$

and the conditional density (with $|A| < 4$*):* $f(s_1|s_2) = 1 - A\left(s_1 - \frac{1}{2}\right)\left(s_2 - \frac{1}{2}\right)$. *If* $AB > 12$, *there exists an efficient equilibrium of the English auction.*[12]

Here, SC is violated at $\left(s_1, \frac{1}{2}\right)$ for any s_1, and $\frac{\partial V_1(\mathbf{s})}{\partial s_1} = \frac{\partial V_2(\mathbf{s})}{\partial s_1}$ for all $\mathbf{s}$. Thus, who has the highest value is independent of s_1: bidder 1 does when $s_2 > \frac{1}{2}$ and bidder 2 does when $s_2 < \frac{1}{2}$. It is also crucial that bidder 2 with a low (high) signal is relatively optimistic (pessimistic) about bidder 1's signal. The following pair of strategies constitutes an efficient equilibrium. Bidder 1

[12] The p.d.f. $f(\mathbf{s})$ is strictly positive whenever the marginal distribution of s_2 is strictly positive ($|A| < 4$ gives $f(s_1|s_2) > 0$). It is not important that $\mathbf{V}(\mathbf{0}) \neq \mathbf{0}$, as we can redefine $V_1 = Bs_1 + 2\left(\frac{1}{2} - s_2\right)^2 + 2s_2 - \frac{1}{2}$ and construct a similar equilibrium.

with any s_1 bids $\beta_1 = \frac{B}{2} + \frac{1}{2}$, while bidder 2 with $s_2 \leq \frac{1}{2}$ bids $\beta_2 = \infty$ and with $s_2 > \frac{1}{2}$ bids $\beta_2 = 0$. Bidder 1 pays 0 whenever she wins and cannot change her bid to increase her payoff. Bidder 2 with $s_2 > \frac{1}{2}$ loses to $\beta_1 = \frac{B}{2} + \frac{1}{2}$ and has a payoff of 0. She cannot improve since by bidding more than β_1 and winning she obtains

$$E\left(V_2 - \beta_1 | s_2\right) = \left(1 - \tfrac{1}{12}AB\right)\left(s_2 - \tfrac{1}{2}\right),$$

which is negative if $AB > 12$. Similarly, bidder 2 with $s_2 \leq \frac{1}{2}$ wins and does not regret paying $\frac{B}{2} + \frac{1}{2}$ since her expected payoff is positive. Note that a fully revealing strategy $\beta_1(s_1) = \varepsilon s_1 + \frac{B-\varepsilon}{2} + \frac{1}{2}$ for a sufficiently small $\varepsilon > 0$ and the same β_2 also form an efficient equilibrium. $\square$

These examples share a common feature: the signal of one of the bidders is irrelevant for efficiency and correlated with the signals of the others. Paradoxically, the presence of this bidder allows us to achieve efficiency. Different types of this bidder are "pooled" to affect the beliefs and the strategies of the others. We consider these settings to be non-generic and impose two mild assumptions, under which they do not arise.

We assume that for any $\mathbf{s}$, any bidder $k \in \mathcal{I}(\mathbf{s})$ and subset $\mathcal{A} = \mathcal{I}(\mathbf{s}) \setminus \{k\}$, if for all $j \in \mathcal{A}$, $\frac{\partial V_j(\mathbf{s})}{\partial s_k} = \frac{\partial V_k(\mathbf{s})}{\partial s_k}$, then for any $\varepsilon > 0$, there exists $\mathbf{s}'$, such that $\|\mathbf{s}' - \mathbf{s}\| < \varepsilon$, $\mathcal{I}(\mathbf{s}') = \mathcal{I}(\mathbf{s})$, and for some $j \in \mathcal{A}$, $\frac{\partial V_j(\mathbf{s}')}{\partial s_k} \neq \frac{\partial V_k(\mathbf{s}')}{\partial s_k}$. For auctions with more than two bidders, we assume that for any $\mathbf{s}$ with $\#\mathcal{I}(\mathbf{s}) = 2$ and, letting $\mathcal{A} = \mathcal{I}(\mathbf{s})$ and $\mathcal{B} = \mathcal{N} \setminus \mathcal{A}$, for any $\mathbf{s}'_{\mathcal{B}} \neq \mathbf{s}_{\mathcal{B}}$, there exists an interior $\mathbf{s}'_{\mathcal{A}}$ such that either $\mathcal{I}(\mathbf{s}'_{\mathcal{A}}, \mathbf{s}_{\mathcal{B}}) = \mathcal{A} \neq \mathcal{I}(\mathbf{s}'_{\mathcal{A}}, \mathbf{s}'_{\mathcal{B}})$ or $\mathcal{I}(\mathbf{s}'_{\mathcal{A}}, \mathbf{s}_{\mathcal{B}}) \neq \mathcal{A} = \mathcal{I}(\mathbf{s}'_{\mathcal{A}}, \mathbf{s}'_{\mathcal{B}})$.

The first assumption explicitly bars the settings as in Example 3. The second assumption requires that, unlike in Example 2, the signals of the "non-winners" are weakly relevant in deciding who should receive the good. Consider two different signal profiles $\mathbf{s}_{\mathcal{B}}$ and $\mathbf{s}'_{\mathcal{B}}$, and the two sets of profiles of $\mathcal{A}$'s signals for which the winner's circle is $\mathcal{A}$ given $\mathbf{s}_{\mathcal{B}}$ and $\mathbf{s}'_{\mathcal{B}}$, respectively. We require that these two sets are different. Thus, whether bidders $\mathcal{B}$ have signals $\mathbf{s}_{\mathcal{B}}$ or $\mathbf{s}'_{\mathcal{B}}$ is relevant for efficiency. Because the assumption is not imposed on $\mathbf{s}$ with $\#\mathcal{I}(\mathbf{s}) = 1$, it does not eliminate "waiting" efficient equilibria in which one bidder always has the highest value and "remains active forever."

Proposition 2 (Necessity)**.** *Suppose GSC is violated at an interior* $\mathbf{s}^*$*, then no efficient equilibrium exists in the* N*-bidder English auction.*

Proof. Suppose GSC is violated at $\mathbf{s}^*$ for $\mathcal{A}$ and k. Without loss of any generality we can assume that: (i) $\mathcal{I}(\mathbf{s}^*) = \mathcal{A} \cup \{k\}$; (ii) GSC holds at any interior $\mathbf{s}$ for any $\mathcal{B} \subset \mathcal{I}(\mathbf{s})$ with $\#\mathcal{B} < \#\mathcal{A}$ and any $i \in \mathcal{I}(\mathbf{s}) \setminus \mathcal{B}$; and (iii) there exists $j \in \mathcal{A}$, $\frac{\partial V_j}{\partial s_k} \neq \frac{\partial V_k}{\partial s_k}$. Indeed, if (i) is false, then letting $\mathcal{C} = \mathcal{A} \cup \{k\}$ one can slightly reduce the values of all the bidders $\mathcal{I}(\mathbf{s}^*) \setminus \mathcal{C}$ while keeping fixed the values of bidders $\mathcal{C}$ and the signals of the bidders not in $\mathcal{I}(\mathbf{s}^*)$. If the reduction in the values is sufficiently small, the resulting profile $\mathbf{s}'$ is interior, and by continuity GSC is violated at $\mathbf{s}'$ for $\mathcal{A}$ and k. If (ii) is false, then one can start with such $\mathbf{s}$, $\mathcal{B}$ and i. Finally, if (iii) is false, that is for all $j \in \mathcal{A}$, $\frac{\partial V_j}{\partial s_k} = \frac{\partial V_k}{\partial s_k}$, then, by assumption, one can find $\mathbf{s}'$ close to $\mathbf{s}^*$ for which (iii) holds.

We proceed from the contrary, assuming that an efficient equilibrium exists. Fix an efficient equilibrium $\boldsymbol{\beta}$. If any of the bidders follows a mixed strategy, select an arbitrary strategy in its support. No restrictions on the strategies are imposed, they need not be monotonic and can be discontinuous everywhere.

At any step of the proof, it will be clear in the bidding of what subset of bidders $\mathcal{C}$ we are interested in and the signals of the others, $\mathbf{s}_{-\mathcal{C}}$, will be fixed. To keep track of the histories, and, thus, different parts of the strategies, we propose the following notational convention. For any bidder $j \in \mathcal{C}$ and signal s_j we define $\hat{\beta}_j(s_j)$ as the price at which she exits according to her equilibrium strategy if the other bidders from $\mathcal{C}$ *remain active forever* and all the bidders $\mathcal{N} \setminus \mathcal{C}$ follow their equilibrium strategies given their signals. It is possible that $\hat{\beta}_j(s_j) = \infty$ for some j and s_j. Note that the actual price at which the first bidder from $\mathcal{C}$ drops out is equal to $\min_{i \in \mathcal{C}} \hat{\beta}_i(s_i)$. In the proof we will focus on *bidding functions* $\hat{\boldsymbol{\beta}}$. In particular, if bidder j has strictly the highest value at $\mathbf{s}$, she must not be the first to drop out, and so

$$\hat{\beta}_j(s_j) > \min_{i \in \mathcal{C}} \hat{\beta}_i(s_i).$$

To avoid excessive notation, we will simply write $\beta_j(s_j)$ in place of $\hat{\beta}_j(s_j)$. With $\mathcal{C}$ being a subset of interest, we will omit the signals of bidders $\mathcal{N} \setminus \mathcal{C}$, as they are going to be fixed, and write $\mathbf{s}$ in place of $\mathbf{s}_{\mathcal{C}}$. We will write $\mathbf{s}_{\mathcal{N}}$ to refer to the full profile of signals.

When more than two bidders are active, the exit decision of any active bidder depends on a calculation of what is going to happen if she stays longer and somebody else exits first. As GSC holds for any smaller than $\#\mathcal{I}(\mathbf{s}^*)$ number of bidders, we are going to derive the implications of GSC on the bidding strategies for any number of bidders in the winners' circle, both when GSC is satisfied and when it is violated.

First, we consider the case $\#\mathcal{I}(\mathbf{s}) = 2$. Let $\mathcal{A} = \{j\}$. We start with an interior $\mathbf{s} = (s_j, s_k)$ such that $\frac{\partial V_j}{\partial s_j} \neq \frac{\partial V_k}{\partial s_j}$, $\frac{\partial V_j}{\partial s_k} \neq \frac{\partial V_k}{\partial s_k}$, and $\mathcal{I}(\mathbf{s}) = \{j, k\}$. We will call the profiles satisfying these properties *strictly competitive for* $\{j, k\}$.

Step 1. Here we establish that there exists a neighborhood $U^{\mathbf{s}} = U_j^{\mathbf{s}} \times U_k^{\mathbf{s}}$ of $\mathbf{s}$ so that: (1) bidding functions of j and k are monotonic over $U_j^{\mathbf{s}}$ and $U_k^{\mathbf{s}}$: $\beta_j(s_j)$ is increasing (decreasing) whenever $\frac{\partial V_j}{\partial s_j} > \frac{\partial V_k}{\partial s_j}$ $\left(\frac{\partial V_j}{\partial s_j} < \frac{\partial V_k}{\partial s_j}\right)$ and (2) for each $s'_j \in U_j^{\mathbf{s}}$ there exists $s'_k \in U_k^{\mathbf{s}}$ so that $\{j, k\} = \mathcal{I}(s'_j, s'_k)$ and if β_j is continuous at s'_j, then β_k is continuous at s'_k and $\beta_j(s'_j) = \beta_k(s'_k)$.

Consider trajectory $\mathbf{s}(\tau)$ such that $V_j(\mathbf{s}(\tau)) = V_k(\mathbf{s}(\tau)) = V_j(\mathbf{s}) + \tau$. We can find a sufficiently small neighborhood (τ_-, τ_+) of 0 and the corresponding $U^{\mathbf{s}} = s_j(\tau_-, \tau_+) \times s_k(\tau_-, \tau_+)$,[13] so that: (i) $s_j(\tau)$ and $s_k(\tau)$ are strictly monotonic on (τ_-, τ_+); (ii) $U^{\mathbf{s}} \in (0,1) \times (0,1)$; (iii) the single-crossing inequalities are of the same sign at all $\mathbf{s}' \in U^{\mathbf{s}}$; and (iv) for all $(s'_j, s'_k) \in U^{\mathbf{s}}$, $\mathcal{I}(s'_j, s'_k) = \{j, k\} \Leftrightarrow (s'_j, s'_k) = \mathbf{s}(\tau)$ for some $\tau \in (\tau_-, \tau_+)$. Pick any $\tau, \tau', \tau'' \in (\tau_-, \tau_+)$ so that $s_j(\tau) < s_j(\tau') < s_j(\tau'')$. Suppose that $\frac{\partial V_j}{\partial s_j} < \frac{\partial V_k}{\partial s_j}$, then at $(s_j(\tau), s_k(\tau'))$ bidder j has the highest value, while at $(s_j(\tau''), s_k(\tau'))$ bidder k does. Thus,

$$\beta_j(s_j(\tau)) > \beta_k(s_k(\tau')) > \beta_j(s_j(\tau'')). \tag{8}$$

Similarly, if $\frac{\partial V_j}{\partial s_j} > \frac{\partial V_k}{\partial s_j}$, β_j is increasing. The same holds for bidder k. As a monotonic function is continuous almost anywhere, if β_j is continuous at $s_j(\tau)$, then $\beta_j(s_j(\tau)) = \beta_k(s_k(\tau))$, else β_k is discontinuous at $s_k(\tau)$ as well.

Step 2. An immediate corollary to Step 1 is that when $\#\mathcal{N} > 2$, for all $\mathbf{s}' \in U^{\mathbf{s}}$ all the other bidders drop out before both j and k (recall that $\mathbf{s}_{-jk}$ is fixed),

$$\max_{i \neq j,k} \beta_i(s_i) \leq \inf_{(s'_j, s'_k) \in U^{\mathbf{s}}} \{\beta_j(s'_j), \beta_k(s'_k)\}.$$

In addition, $\mathbf{s}_{-jk}$ is uniquely identified by the history.

Indeed, if $\exists i \neq j, k$ with $\beta_i(s_i) > \inf_{\mathbf{s}'_{jk} \in U^{\mathbf{s}}} \{\beta_j(s'_j), \beta_k(s'_k)\}$, then $\exists \tau$, such that $\beta_j(s_j(\tau)) = \beta_k(s_k(\tau)) < \beta_i(s_i)$, which contradicts efficiency. If the history of exits for some $\mathbf{s}'_{-jk} \neq \mathbf{s}_{-jk}$ is the same, then by assumption, there exists interior $\mathbf{s}'_{jk}$ for which only one of $\mathcal{I}(\mathbf{s}'_{jk}, \mathbf{s}_{-jk})$ and $\mathcal{I}(\mathbf{s}'_{jk}, \mathbf{s}'_{-jk})$ equals $\{j, k\}$. Without loss of any generality we can suppose that $\mathbf{s}'_{jk}$ is strictly

[13] For any function $f : X \to Y$ and subsets $S \subset X$ and $T \subset Y$, we define $f(S) = \{f(x) : x \in S\}$ and $f^{-1}(T) = \{x : f(x) \in T\}$.

competitive for j and k (it can be slightly perturbed if necessary while keeping all the other relevant properties intact); β_j and β_k are continuous at s'_j and s'_k, respectively (one can apply Step 1 for $\mathbf{s}'_{jk}$ and, if necessary, choose a continuity point along the trajectory and sufficiently close to $\mathbf{s}'_{jk}$); and $\mathcal{I}(\mathbf{s}'_{jk}, \mathbf{s}_{-jk}) = \{j, k\}$, while $j \notin \mathcal{I}(\mathbf{s}'_{jk}, \mathbf{s}'_{-jk})$. Now, for s''_j sufficiently close to s'_j and such that $\mathcal{I}(s''_j, s'_k, \mathbf{s}_{-jk}) = \{j\}$ while still $j \notin \mathcal{I}(s''_j, s'_k, \mathbf{s}'_{-jk})$ we have $\beta_j(s''_j) > \beta_k(s'_k)$. Thus, as all the other bidders exit before j and k do and with the same history, bidder j wins at both $(s''_j, s'_k, \mathbf{s}_{-jk})$ and $(s''_j, s'_k, \mathbf{s}'_{-jk})$, which contradicts efficiency.

Step 3. Now we connect the bidding functions of bidders j and k to their valuations. There are two cases to consider: (1) for each $s'_j \in U^{\mathbf{s}}_j$ there is a unique s'_k such that $\beta_j(s'_j) = \beta_k(s'_k)$; and (2) there are more than one such signal for some $s'_j \in U^{\mathbf{s}}_j$. In Case 1 $\beta_j(s'_j) = V_j(\mathbf{s}')$. In Appendix A.3 we show that Case 2 is incompatible with efficiency.

Let $CU^{\mathbf{s}}_j \subset U^{\mathbf{s}}_j$ and $CU^{\mathbf{s}}_k \subset U^{\mathbf{s}}_k$ be the sets over which, respectively, β_j and β_k are continuous. In Case 1, $U^{\mathbf{s}}_k \supset \beta_k^{-1}(\beta_j(CU^{\mathbf{s}}_j))$. Consider any $s'_j \in CU^{\mathbf{s}}_j$ and s'_k such that $\beta_j(s'_j) = \beta_k(s'_k)$. Suppose $\beta_j(s'_j) < V_j(\mathbf{s}')$. If bidder j with s'_j increases her bid by a small $\varepsilon > 0$, she also wins when $\beta_k(s''_k) \in (\beta_j(s'_j), \beta_j(s'_j) + \varepsilon)$, pays $\beta_k(s''_k)$, and obtains value $V_j(s'_j, s''_k)$ close to $V_j(s'_j, s'_k)$. For sufficiently small ε, $\beta_k(s''_k) < \beta_j(s'_j) + \varepsilon < V_j(s'_j, s''_k)$, and thus the deviation is profitable. Since V_j and V_k are continuous, β_j and β_k are continuous on, respectively, $U^{\mathbf{s}}_j$ and $U^{\mathbf{s}}_k$.

Step 4. Finally, if SC is violated at $\mathbf{s} = \mathbf{s}^*$ for $\mathcal{A} = \{j\}$ and k, then bidder j has a profitable deviation and so efficiency cannot be achieved.

As follows from the previous steps, β_j is decreasing and $\beta_j(s_j(\tau)) = \beta_k(s_k(\tau)) = V_j(\mathbf{s}(\tau)) = V_k(\mathbf{s}(\tau))$ for all $\tau \in (\tau_-, \tau_+)$. Since the values are increasing with τ, $s_j(\tau)$ and $s_k(\tau)$ cannot be both decreasing or both increasing (as then β_k is also decreasing and so cannot be equal to V_k). If $s_j(\tau)$ is decreasing, then $s_k(\tau)$ is increasing, and so $\frac{\partial V_k(\mathbf{s}(\tau))}{\partial s_k} > \frac{\partial V_j(\mathbf{s}(\tau)}{\partial s_k}$. Bidder j with $s_j(\tau)$ can improve by staying longer: by bidding $\beta_j(s_j(t))$ for $t \in (\tau, \tau_+)$ she also wins against all $s_k(t')$ for $t' \in (\tau, t)$ and pays $V_j\left(s_j(t'), s_k(t')\right) < V_j\left(s_j(\tau), s_k(t')\right)$. If $s_j(\tau)$ is increasing, then $s_k(\tau)$ is decreasing and $\frac{\partial V_k(\mathbf{s}(\tau))}{\partial s_k} > \frac{\partial V_j(\mathbf{s}(\tau))}{\partial s_k}$. Again, bidder j with $s_j(\tau)$ improves by bidding $\beta_j(s_j(t))$ for $t \in (\tau_-, \tau)$, since then she also wins against $s_k(t')$ for all $t' \in (t, \tau)$ and pays $V_j\left(s_j(t'), s_k(t')\right) < V_j\left(s_j(\tau), s_k(t')\right)$.

Now, suppose $\#\mathcal{A} \geq 2$. For convenience, we relabel bidder k as bidder 1.

Step 5. Consider trajectory $\mathbf{s}(t)$ that for each t solves

$$V_j(\mathbf{s}(t)) = V_j(\mathbf{s}) + t, \text{ for all } j \in \mathcal{C} = \mathcal{A} \cup \{1\}.$$

Such a trajectory exists and is unique, since it solves

$$\frac{d\mathbf{s}}{dt} = (DV(\mathbf{s}))^{-1} \cdot \mathbf{1}. \tag{9}$$

By continuity of the value functions and their first derivatives, there exists an open neighborhood U_t^0 of $t = 0$, such that for all $t \in U_t^0$: $\mathcal{C} = \mathcal{I}(\mathbf{s}_{\mathcal{N}}(t))$, GSC is violated at $\mathbf{s}_{\mathcal{N}}(t)$ for $\mathcal{A}$ and 1, and $\frac{\partial V_j(\mathbf{s}_{\mathcal{N}}(t))}{\partial s_1} \neq \frac{\partial V_1(\mathbf{s}_{\mathcal{N}}(t))}{\partial s_1}$ for some $j \in \mathcal{A}$.

Step 6. Consider $\mathbf{s}' = \mathbf{s}(t)$ for an arbitrary $t \in U_t^0$ and let, for any $j \in \mathcal{A}$, $b_j(s'_j) \equiv \lim_{s_j \downarrow s'_j} \inf \beta_j(s_j)$. Lemma 6 in Appendix A.3 shows that these limits are equal: for any $j \in \mathcal{A}$, $b_j(s'_j) = b(t) < \infty$. In addition, for any $j \in \mathcal{A}$ and $s_j > s'_j$ sufficiently close to s'_j, $\beta_j(s_j) \geq b(t)$, and for bidder 1, $\beta_1(s'_1) > b(t)$.

Step 7. Corollary 1 in Appendix A.2 shows that for $t' > t$ either: (i) $s_1(t') < s_1(t)$ and, for all $j \in \mathcal{A}$, $s_j(t') > s_j(t)$; or (ii) $s_1(t') > s_1(t)$ and, for all $j \in \mathcal{A}$, $s_j(t') < s_j(t)$. This, together with Step 6, implies that $b(t)$ is (weakly) monotonic in t. In Case (i) it is non-decreasing, in Case (ii) it is non-increasing.

Step 8. Corollary 2 in Appendix A.3 shows that if for some bidder $j \in \mathcal{A}$, $\beta_j(s_j(t)) \neq b(t)$, then t has to be a discontinuity point for $b(t)$. Since $b(t)$ is monotonic it has no more than a countable number of discontinuity points. Hence for almost all $t \in U_t^0$, $\beta_j(s_j(t)) = b(t)$ for every $j \in \mathcal{A}$. That is, when the signals of the bidders from $\mathcal{A}$ belong to trajectory $\mathbf{s}(t)$, they almost always exit simultaneously.

Step 9. Consider two continuity points for $b(t)$, t and t', such that $b(t') \geq b(t)$. In Case (i), $t' > t$; in Case (ii), $t' < t$. Then, $s_1(t') < s_1(t)$, and $\beta_1(s_1(t')) > b(t') \geq b(t) = \beta_j(s_j(t))$ for all $j \in \mathcal{A}$.

Step 10. By construction, at t, $\mathcal{I}(s_1(t), \mathbf{s}_{\mathcal{A}}(t)) = \mathcal{C}$. We have

$$\frac{\partial V_1(s_1, \mathbf{s}_{\mathcal{A}}(t))}{\partial s_1} > \min_{j \in \mathcal{A}} \frac{\partial V_j(s_1, \mathbf{s}_{\mathcal{A}}(t))}{\partial s_1}. \tag{10}$$

Indeed, SC holds since $\#\mathcal{A} > 1$, and the equality is prevented by the uniqueness of the solution to (9). Thus, if starting from $\mathbf{s}(t)$ we slightly decrease the signal of bidder 1, she can no longer possess the highest value.

Then, we can find t' sufficiently close to t such that t' is a continuity point for $b(t)$, $s_1(t') < s_1(t)$, and all the bidders with the highest value at $(s_1(t'), \mathbf{s}_{\mathcal{A}}(t))$ belong to $\mathcal{A}$. Then by the results of Step 8, at the profile $(s_1(t'), \mathbf{s}_{\mathcal{A}}(t))$, the bidders from $\mathcal{A}$ drop out simultaneously at $b(t)$ while bidder 1 stays longer. Thus, efficiency is not achieved—a contradiction. $\square$

A Appendix

A.1 Proof of Proposition 1

Proof of Lemma 1. Suppose inequalities (2) are strict for all $\mathcal{A}$. By induction on $\#\mathcal{A}$, we show $\mathbf{u}^{\mathcal{A}}_{\mathcal{A}} \gg \mathbf{0}$. For $\#\mathcal{A} = 1$, $\mathbf{u}^{\mathcal{A}}_{\mathcal{A}} = \left(\frac{\partial V_{\mathcal{A}}}{\partial s_{\mathcal{A}}}\right)^{-1} > 0$. Suppose, on the contrary, there exists $\mathcal{A} \subset \mathcal{I}(\mathbf{s})$ with $\#\mathcal{A} > 1$, such that $\mathbf{u}^{\mathcal{A}}_{\mathcal{A}} \not\gg \mathbf{0}$ while $\mathbf{u}^{\mathcal{B}}_{\mathcal{B}} \gg \mathbf{0}$ for all $\mathcal{B} \subset \mathcal{I}(\mathbf{s})$ with $\#\mathcal{B} < \#\mathcal{A}$. Partition $\mathcal{A} = \mathcal{B} \sqcup \mathcal{C} \sqcup \mathcal{D}$, so that $\mathbf{u}^{\mathcal{A}}_{\mathcal{B}} \ll \mathbf{0}$, $\mathbf{u}^{\mathcal{A}}_{\mathcal{C}} = \mathbf{0}$, and $\mathbf{u}^{\mathcal{A}}_{\mathcal{D}} \gg \mathbf{0}$. Clearly, $\mathcal{D} \neq \varnothing$ and also $\mathcal{B} \neq \varnothing$, since the inequalities are strict. Consider $\mathbf{u} \equiv (\mathbf{0}_{\mathcal{B}}, \mathbf{u}^{\mathcal{A}}_{-\mathcal{B}})$ and $\mathbf{u}' \equiv (-\mathbf{u}^{\mathcal{A}}_{\mathcal{B}}, \mathbf{0}_{-\mathcal{B}})$. Note that $\mathbf{u}_{\mathcal{D}} \geqq \mathbf{0}$, $\mathbf{u}_{-\mathcal{D}} = \mathbf{0}$, $\mathbf{u}'_{\mathcal{B}} \gg \mathbf{0}$, $\mathbf{u}'_{-\mathcal{B}} = \mathbf{0}$, and $\mathbf{u}^{\mathcal{A}} = \mathbf{u} - \mathbf{u}'$. Let $i \in \arg\max_{l \in \mathcal{B}} D_{\mathbf{u}'} V_l$ and $j \in \arg\max_{l \in \mathcal{D}} D_{\mathbf{u}'} V_l$. GSC for subsets $\mathcal{B}$ and $\mathcal{D}$ respectively, implies $D_{\mathbf{u}'} V_i > D_{\mathbf{u}'} V_j$, $D_{\mathbf{u}} V_i < D_{\mathbf{u}} V_j$, and $D_{\mathbf{u}^{\mathcal{A}}} V_i < D_{\mathbf{u}^{\mathcal{A}}} V_j$. But by the definition of $\mathbf{u}^{\mathcal{A}}$, $D_{\mathbf{u}^{\mathcal{A}}} V_i = D_{\mathbf{u}^{\mathcal{A}}} V_j = 1$; thus $\mathbf{u}^{\mathcal{A}}_{\mathcal{A}} \gg \mathbf{0}$. If some inequalities in (2) are not strict, perturb $DV_{\mathcal{I}}(\mathbf{s})$ by adding $\varepsilon > 0$ to every diagonal element; so that $DV'_{\mathcal{I}}(\mathbf{s}) = DV_{\mathcal{I}}(\mathbf{s}) + \varepsilon I_{\#\mathcal{I}}$. By continuity, $\mathbf{u}^{\mathcal{A}}_{\mathcal{A}} = \lim_{\varepsilon \to 0} \mathbf{u}^{\mathcal{A}}_{\mathcal{A}}(\varepsilon) \geqq \mathbf{0}$ for all subsets $\mathcal{A} \subset \mathcal{I}(\mathbf{s})$. Finally, (5) follows from GSC verified on $\mathbf{u}^{\mathcal{A}}$.

Next, suppose (5) holds. Suppose GSC holds at $\mathbf{s}$ for all $\mathcal{B} \subset \mathcal{I}(\mathbf{s})$ with $\#\mathcal{B} < n - 1$. We show that then it also holds for all $\mathcal{A} \subset \mathcal{I}(\mathbf{s})$ with $\#\mathcal{A} = n > 1$. (GSC trivially holds when $\#\mathcal{A} = 1$.) Suppose, on the contrary, there exists $\mathcal{A} \subset \mathcal{I}(\mathbf{s})$ with $\#\mathcal{A} > 1$, $k \in \mathcal{I}(\mathbf{s}) \setminus \mathcal{A}$, and $\mathbf{u}$ with $\mathbf{u}_{\mathcal{A}} \gg \mathbf{0}$ and $\mathbf{u}_{-\mathcal{A}} = \mathbf{0}$, such that $D_{\mathbf{u}} V_k > \max_{j \in \mathcal{A}} D_{\mathbf{u}} V_j$. Let $\mathcal{B} = \arg\max_{j \in \mathcal{A}} D_{\mathbf{u}} V_j$. Since $\mathbf{u} \neq \mathbf{u}^{\mathcal{A}}$, $\mathcal{B} \subsetneq \mathcal{A}$, $\mathbf{u}_{\mathcal{B}} \gg \mathbf{0}$ and for vector $\mathbf{u}^{\mathcal{B}}$, $\mathbf{u}^{\mathcal{B}}_{\mathcal{B}} \geqq \mathbf{0}$ by the argument above. Consider vector $\mathbf{w}(t) = \mathbf{u} - t\mathbf{u}^{\mathcal{B}}$. When t increases, only coordinates $\mathbf{w}_{\mathcal{B}}(t)$ are weakly decreasing, and as GSC is satisfied for $\mathcal{B}$, $\mathcal{B} = \arg\max_{j \in \mathcal{A}} D_{\mathbf{w}(t)} V_j$ and $\max_{j \in \mathcal{A}} D_{\mathbf{w}(t)} V_j < D_{\mathbf{w}(t)} V_k$. Then there exist the smallest $t^* > 0$ and $i \in \mathcal{A} \backslash \mathcal{B}$, such that either $\mathbf{w}_i(t^*) = 0$ or $i \in \arg\max_{j \in \mathcal{A}} D_{\mathbf{w}(t)} V_j$ as well. In the first case GSC is violated for $\mathcal{A} \setminus \{i\}$ and k for vector $\mathbf{w}(t^*)$ which contradicts the induction presumption. In the second case, repeat the procedure starting with $\mathbf{u} = \mathbf{w}(t^*)$. As $\#\mathcal{A}$ is finite, the first case applies eventually. $\square$

Proof of Lemma 2. Suppose that at p^0 with $H(p^0) = \bar{H}(p^0)$ there exists a profile $\boldsymbol{\sigma}^0(p^0)$ satisfying (3). Let $\mathcal{A}$ be all $j \in \mathcal{M}$ with $\sigma_j(p, H(p)) < 1$. Fix $\boldsymbol{\sigma}_{-\mathcal{A}}(p) = \boldsymbol{\sigma}^0_{-\mathcal{A}}(p^0)$ for $p \geq p^0$. To find $\boldsymbol{\sigma}_{\mathcal{A}}(p)$ that solves

$$\mathbf{V}_{\mathcal{A}}(\boldsymbol{\sigma}_{\mathcal{A}}(p), \boldsymbol{\sigma}^0_{-\mathcal{A}}(p^0)) = p\mathbf{1}_{\mathcal{A}}, \tag{11}$$

it suffices to solve the system of differential equations

$$\frac{d\boldsymbol{\sigma}_{\mathcal{A}}}{dp} = (DV_{\mathcal{A}})^{-1}\mathbf{1}_{\mathcal{A}}. \tag{12}$$

By the Cauchy-Peano theorem, given $\boldsymbol{\sigma}_{\mathcal{A}}(p^0) = \boldsymbol{\sigma}^0_{\mathcal{A}}(p^0)$, there exists a unique continuous solution $\boldsymbol{\sigma}_{\mathcal{A}}(p)$ to (12) on $p \in [p^0, p^*]$, for some $p^* > p^0$.

Suppose GSC is satisfied. As long as $\mathcal{A} \subset \mathcal{I}(\boldsymbol{\sigma}(p))$, $\frac{d\boldsymbol{\sigma}_{\mathcal{A}}}{dp} = \mathbf{u}^{\mathcal{A}}_{\mathcal{A}} \geqq \mathbf{0}$ (from the proof of Lemma 1) and $\frac{\partial V_i}{\partial \boldsymbol{\sigma}_{\mathcal{A}}} \frac{d\boldsymbol{\sigma}_{\mathcal{A}}}{dp} \leq 1$ for any $i \in \mathcal{I}(\boldsymbol{\sigma}(p)) \setminus \mathcal{A}$ by (5). As a result, $\boldsymbol{\sigma}(p)$ extends to all $p \leq p^*_{\mathcal{A}}$, where $p^*_{\mathcal{A}}$ is the lowest price at which $\sigma_j(p^*_{\mathcal{A}}) = 1$ for some $j \in \mathcal{A}$. To extend $\boldsymbol{\sigma}(p, H(p))$ beyond $p^*_{\mathcal{A}}$, a new system (11) is solved for $\mathcal{A}' = \mathcal{A}(\sigma(p^*_{\mathcal{A}}, H(p^*_{\mathcal{A}}))) \subsetneq \mathcal{A}$ with the initial condition $\boldsymbol{\sigma}^0_{\mathcal{A}'}(p^*_{\mathcal{A}}) = \boldsymbol{\sigma}_{\mathcal{A}}(p^*_{\mathcal{A}})$. This is repeated until there is no j with $\sigma_j(p) < 1$, after which $\boldsymbol{\sigma}(p)$ is fixed.

To provide $\boldsymbol{\sigma}(p, H(p))$ for all prices and histories we need to specify for each $H(p)$ the initial $\boldsymbol{\sigma}^0(p^0)$ for $p^0 = \max_{p_j \in H(p)} p_j$. Set $\boldsymbol{\sigma}^0(0, \varnothing) = \mathbf{0}$ and compute $\boldsymbol{\sigma}(p)$ for all $p > 0$, starting with $\mathcal{A} = \mathcal{N}$. At p^* such that $H(p^*) \neq \bar{H}(p^*)$, define $\boldsymbol{\sigma}^0(p^*, \bar{H}(p^*)) = \boldsymbol{\sigma}(p^*, H(p^*))$ (which maintains continuity of inferences), and compute $\boldsymbol{\sigma}(p, \bar{H}(p^*))$ for all $p > p^*$. $\square$

A.2 Supporting results for the value functions

Corollary 1. *Consider interior* $\mathbf{s}$ *with* $\#\mathcal{I}(\mathbf{s}) > 2$, *any* $\mathcal{A} \subsetneq \mathcal{I}(\mathbf{s})$ *with* $\#\mathcal{A} = n \geq 2$ *and bidder* $k \in \mathcal{I}(\mathbf{s}) \setminus \mathcal{A}$. *Suppose GSC is satisfied at* $\mathbf{s}$ *for any* $\mathcal{B} \subset \mathcal{I}(\mathbf{s})$ *with* $\#\mathcal{B} < n$. *Suppose also that for some* $j \in \mathcal{A}$, $\frac{\partial V_j(\mathbf{s})}{\partial s_k} \neq \frac{\partial V_k(\mathbf{s})}{\partial s_k}$. *Then, GSC is violated at* $\mathbf{s}$ *for* $\mathcal{A}$ *and* k *if and only if (1)* $u^{\mathcal{C}}_k < 0$ *and* $\forall j \in \mathcal{A}$, $u^{\mathcal{C}}_j > 0$, *or (2)* $u^{\mathcal{C}}_k > 0$ *and* $\forall j \in \mathcal{A}$, $u^{\mathcal{C}}_j < 0$, *where* $\mathcal{C} \equiv \mathcal{A} \cup \{k\}$ *and* $\mathbf{u}^{\mathcal{C}}$ *is an equal increment vector for the subset* $\mathcal{C}$.

Proof. First note that $\mathbf{u}^{\mathcal{A}}_{\mathcal{A}} \geqq \mathbf{0}$ and $\mathbf{u}^{\mathcal{B}}_{\mathcal{B}} \geqq \mathbf{0}$ for all $\mathcal{B} \subset \mathcal{I}(\mathbf{s})$ with $\#B < n$ (this follows from the proof of Lemma 1).

($\Longleftarrow$) Suppose that $u^{\mathcal{C}}_k < 0$, while $\mathbf{u}^{\mathcal{C}}_{\mathcal{A}} \gg \mathbf{0}$ (the argument for the other case is similar), and define $\mathbf{u}' = (-u^{\mathcal{C}}_k, \mathbf{0}_{-k})$. Since GSC is satisfied for $\mathcal{B} =$

$\{k\}$, $D_{\mathbf{u}'}V_k \geq \max_{i\in\mathcal{A}} D_{\mathbf{u}'}V_i$. If $D_{\mathbf{u}'}V_k > \max_{i\in\mathcal{A}} D_{\mathbf{u}'}V_i$, then GSC is violated for k, $\mathcal{A}$, and $\mathbf{u} \equiv \mathbf{u}^{\mathcal{C}} + \mathbf{u}'$. Otherwise, define $\mathcal{D} = \arg\max_{i\in\mathcal{A}} D_{\mathbf{u}'}V_i$. We have $\mathcal{D} \subsetneq \mathcal{A}$ (since $\mathbf{u}' \neq \mathbf{u}^{\mathcal{C}}$), $\mathbf{u}^{\mathcal{D}}_{\mathcal{D}} \geqq \mathbf{0}$, and for $i \in \mathcal{D}$, $D_{\mathbf{u}^{\mathcal{D}}}V_k < D_{\mathbf{u}^{\mathcal{D}}}V_i$ (since $\mathbf{u}' \neq \mathbf{u}^{\mathcal{D}}$). Then, for sufficiently small $\varepsilon > 0$ and vector $\mathbf{u} \equiv \mathbf{u}^{\mathcal{C}} + \mathbf{u}' - \varepsilon\mathbf{u}^{\mathcal{D}}$ we have: (i) $\mathbf{u}_{\mathcal{A}} \gg \mathbf{0}$, $\mathbf{u}_{-\mathcal{A}} = \mathbf{0}$; (ii) for each $i \in \mathcal{D}$ and $j \in \mathcal{A}\backslash\mathcal{D}$, $D_{\mathbf{u}'}V_i > D_{\mathbf{u}'}V_j$; (iii) $D_{\mathbf{u}}V_k > \max_{i\in\mathcal{A}} D_{\mathbf{u}}V_i$, that is GSC is violated for k, $\mathcal{A}$, and $\mathbf{u}$.

($\Longrightarrow$) First note that $u^{\mathcal{C}}_k \neq 0$ (else, $\mathbf{u}^{\mathcal{C}} = \mathbf{u}^{\mathcal{A}}$ and GSC is satisfied with equality for bidder k and $\mathcal{A}$) and for all $i \in \mathcal{A}$, $u^{\mathcal{C}}_i \neq 0$ (else, GSC is violated for a subset with less than n bidders). We can suppose that all the inequalities in (5) are strict. If not, we can first perturb the Jacobian as we did in Step 2 of the proof of Lemma 1 to eliminate all equalities.

We show that $\mathbf{u}^{\mathcal{C}}_{\mathcal{C}}$ can have either 1 or n positive components. Partition $\mathcal{C} = \mathcal{B} \sqcup \mathcal{D}$, where $i \in \mathcal{B}$ ($i \in \mathcal{D}$) if $u^{\mathcal{C}}_i < 0$ ($u^{\mathcal{C}}_i > 0$). Clearly $\mathcal{D} \neq \varnothing$. Suppose that $\#\mathcal{D} \neq 1, n$ and consider vector $\mathbf{w}_1(t) = \mathbf{u}^{\mathcal{C}} - t\mathbf{u}^{\mathcal{D}}$. Since $\#\mathcal{D} < n$, we have $\mathbf{u}^{\mathcal{D}}_{\mathcal{D}} \gg \mathbf{0}$ and $\forall j \in \mathcal{D}, \forall i \in \mathcal{B}$, $D_{\mathbf{u}^{\mathcal{D}}}V_j > D_{\mathbf{u}^{\mathcal{D}}}V_i$, thus $D_{\mathbf{w}_1(t)}V_j < D_{\mathbf{w}_1(t)}V_i$ for all $t > 0$. Let t_1 be the minimal $t > 0$ such that $\mathbf{w}_{1j}(t) = 0$ for some $j \in \mathcal{D}$. Let $\mathcal{E}$ consist of all $l \in \mathcal{D}$ with $w_{1l}(t) > 0$ and consider vector $\mathbf{w}_2(t) = \mathbf{w}_1(t_1) - t\mathbf{u}^{\mathcal{E}}$. Let t_2 be the minimal $t > 0$ such that for some bidder $j \in \mathcal{E}$, $w_{2j}(t) = 0$. Again, $\forall j \in \mathcal{E}, \forall i \in \mathcal{B}$, $D_{\mathbf{w}_2(t_2)}V_j < D_{\mathbf{w}_2(t_2)}V_i$. By continuing in this manner we eventually obtain vector $\mathbf{w}_m(t_m)$ such that for all $j \in \mathcal{D}$, $w_{mj}(t_m) = 0$. Fix $j \in \mathcal{D}$ with $w_{mj}(0) > 0$, thus $D_{\mathbf{w}_m(t_m)}V_j < D_{\mathbf{w}_m(t_m)}V_i$ for all $i \in \mathcal{B}$. Consider $\mathbf{w} \equiv -\mathbf{w}_m(t_m)$ and note that $\mathbf{w}_{\mathcal{B}} = -\mathbf{u}^{\mathcal{C}}_{\mathcal{B}} \gg 0$, $\mathbf{w}_{-\mathcal{B}} = \mathbf{0}$. Therefore, GSC is violated for $\mathcal{B}$, j, and $\mathbf{w}$, which is a contradiction since $\#\mathcal{B} < n$.

Suppose $u^{\mathcal{C}}_k > 0$ (the proof for $u^{\mathcal{C}}_k < 0$ is similar) and $\mathbf{u}^{\mathcal{C}}_{\mathcal{A}} \not\ll \mathbf{0}$. Consider $\mathbf{u}' = (0_k, \mathbf{u}^{\mathcal{C}}_{-k})$. Clearly, $\mathbf{u}' \neq \mathbf{u}^{\mathcal{A}}$. Since $D_{\mathbf{u}^{\mathcal{C}}}V_k = D_{\mathbf{u}^{\mathcal{C}}}V_i = 1$ for all $i \in \mathcal{A}$ and GSC is (strictly) satisfied for $\mathcal{B} = \{k\}$, we have $D_{\mathbf{u}'}V_k < \min_{i\in\mathcal{A}} D_{\mathbf{u}'}V_i$. Since GSC is violated for $\mathcal{A}$ and k, there exists $\mathbf{u}$, with $\mathbf{u}_{\mathcal{A}} \gg \mathbf{0}$ and $\mathbf{u}_{-\mathcal{A}} = \mathbf{0}$, such that $D_{\mathbf{u}}V_k > \max_{i\in\mathcal{A}} D_{\mathbf{u}}V_i$. Consider $\mathbf{w}(t) = \mathbf{u} - t\mathbf{u}'$ and note that $D_{\mathbf{w}(t)}V_k > \max_{i\in\mathcal{A}} D_{\mathbf{w}(t)}V_i$ for $t \geq 0$, $\mathbf{w}_{-\mathcal{C}}(t) = \mathbf{0}$ for any t, and $\mathbf{w}_{\mathcal{A}}(0) \gg \mathbf{0}$. Since $u'_i = u^{\mathcal{C}}_i > 0$ for some $i \in \mathcal{A}$, there exist the minimal $t' > 0$ such that $w_i(t') = 0$ for some $i \in \mathcal{A}$. Then, GSC is violated for $\mathcal{B} = \mathcal{A} \setminus \{i\}$, k, and $\mathbf{w}(t')$, which is a contradiction since $\#\mathcal{B} < n$. $\square$

Definition 4. *For a given* $\mathbf{s}'$, $\mathcal{B} \subset \mathcal{I}(\mathbf{s}')$, *and vector* $\mathbf{x}$ *with* $\mathbf{x}_{-\mathcal{B}} = \mathbf{0}$, *define* $\mathbf{y}_{\mathcal{B}} \equiv D_x\mathbf{V}_{\mathcal{B}}(\mathbf{s}')$. *Define trajectory* $\mathbf{s}^{\mathbf{x}\mathcal{B}}(\tau)$ *with* $\mathbf{s}^{\mathbf{x}\mathcal{B}}(0) = \mathbf{s}'$ *and* $\mathbf{s}^{\mathbf{x}\mathcal{B}}_{-\mathcal{B}}(\tau) = \mathbf{s}'_{-\mathcal{B}}$ *as a solution to the system*

$$\mathbf{V}_{\mathcal{B}}(\mathbf{s}^{\mathbf{x}\mathcal{B}}(\tau)) = V\mathbf{1}_{\mathcal{B}} + \tau\mathbf{y}_{\mathcal{B}},$$

where $V = \max_{j\in\mathcal{N}} V_j(\mathbf{s}')$. *Clearly,* $\left.\frac{d\mathbf{s}_{\mathcal{B}}^{\mathbf{x}_{\mathcal{B}}}(\tau)}{d\tau}\right|_{\tau=0} = \mathbf{x}_{\mathcal{B}}$.

Lemma 4. *For any proper subset* $\mathcal{B} \subsetneq \mathcal{A}$ *there exists* $\mathcal{D} = \mathcal{D}(\mathcal{B})$, $\mathcal{B} \subseteq \mathcal{D} \subsetneq \mathcal{A}$, *such that for any* $k \in \mathcal{A}\setminus\mathcal{D}$, $D_{\mathbf{u}^{\mathcal{D}}}V_k(\mathbf{s}) < 1$. *Also, for any* $\varepsilon > 0$, *there exists vector* $\mathbf{v}^{\mathcal{B}} \geqq \mathbf{0}$ *such that* $\left\|\mathbf{v}^{\mathcal{B}} - \mathbf{u}^{\mathcal{B}}\right\| < \varepsilon$, $\mathbf{v}_k^{\mathcal{B}} = 0$ *for any* $k \in \mathcal{A}\setminus\mathcal{D}$, $D_{\mathbf{v}^{\mathcal{B}}}V_i(\mathbf{s}) < 1$ *for any* $i \in \mathcal{A}\setminus\mathcal{B}$, *and* $D_{\mathbf{v}^{\mathcal{B}}}V_j(\mathbf{s}) = 1$ *for all* $j \in \mathcal{B}$.

Proof. The proof is by induction on the number of bidders in $\mathcal{B}$. Define $\mathcal{C}$ as the set of bidders $k \in \mathcal{A}\backslash\mathcal{B}$ such that $D_{\mathbf{u}^{\mathcal{B}}}V_k(\mathbf{s}) = 1$. Since GSC is satisfied for $\mathcal{B}$, $\mathbf{u}^{\mathcal{B}} \neq \mathbf{u}^{\mathcal{A}}$ and $\mathcal{C} \neq \mathcal{A}\backslash\mathcal{B}$. If $\mathcal{C} = \varnothing$, then set $\mathcal{D} \equiv \mathcal{B}$, and $\mathbf{v}^{\mathcal{B}} \equiv \mathbf{u}^{\mathcal{B}}$. Thus, lemma is true for $\#\mathcal{B} = \#\mathcal{A} - 1 = n - 1$. Suppose now that lemma is true for all $\mathcal{B}'$, with $\#\mathcal{B}' > \#\mathcal{B}$. If $\mathcal{C} \neq \varnothing$ define $\mathcal{B}' = \mathcal{A}\setminus\mathcal{C}$, then $D_{\mathbf{u}^{\mathcal{B}}}V_k(\mathbf{s}) < 1$ for any $k \in \mathcal{B}'\setminus\mathcal{B}$. Pick $\mathcal{D} \equiv \mathcal{D}(\mathcal{B}')$. Consider $\mathbf{v}^{\mathcal{B}} = \lambda_1\mathbf{u}^{\mathcal{B}} + (1-\lambda_1)\,\mathbf{v}^{\mathcal{B}'}$ with $\lambda_1 \in (0,1)$. When $\lambda_1 \to 1$, $\mathbf{v}^{\mathcal{B}} \to \mathbf{u}^{\mathcal{B}}$. By induction, for all $j \in \mathcal{A}\setminus\mathcal{B}'$, $D_{\mathbf{v}^{\mathcal{B}'}}V_i(\mathbf{s}) < 1$. Thus, for all $j \in \mathcal{A}\backslash\mathcal{B}$, $D_{\mathbf{v}^{\mathcal{B}}}V_i(\mathbf{s}) < 1$ as long as $\lambda_1 \in (0,1)$. □

Remark 1. *As follows from the proof of Lemma 4 we can find a finite sequence* $\mathcal{B} \subsetneq \mathcal{B}' \subsetneq \mathcal{B}'' \subsetneq \ \ldots\ \subsetneq \mathcal{A}$, *such that* $\mathbf{v}^{\mathcal{B}}(\boldsymbol{\lambda}) = \lambda_1\mathbf{u}^{\mathcal{B}} + \lambda_2\mathbf{u}^{\mathcal{B}'} + \lambda_3\mathbf{u}^{\mathcal{B}''} + \ldots$, *where* $\sum_i \lambda_i = 1$, $\lambda_i \in (0,1)$ *for all* i, *and* λ_1 *is arbitrarily close to* 1.

Lemma 5. *For any* $\mathcal{B} \subsetneq \mathcal{A}$ *and* $\mathbf{v}^{\mathcal{B}} \equiv \mathbf{v}^{\mathcal{B}}(\boldsymbol{\lambda})$ *with* λ_1 *sufficiently close to* 1, *it is either* $D_{\mathbf{v}^{\mathcal{B}}}V_1(\mathbf{s}) < \max_{j\in\mathcal{B}} D_{\mathbf{v}^{\mathcal{B}}}V_j(\mathbf{s})$ *or* $\frac{\partial V_1}{\partial s_1}(\mathbf{s}') > \min_{j\in\mathcal{B}} \frac{\partial V_j}{\partial s_1}(\mathbf{s}')$.

Proof. If $D_{\mathbf{u}^{\mathcal{B}}}V_1(\mathbf{s}) < 1 = \max_{j\in\mathcal{B}} D_{\mathbf{u}^{\mathcal{B}}}V_j(\mathbf{s})$, then $D_{\mathbf{v}^{\mathcal{B}}}V_1(\mathbf{s}) < \max_{j\in\mathcal{B}} D_{\mathbf{v}^{\mathcal{B}}}V_j(\mathbf{s})$ for λ_1 sufficiently close to 1. Else, as $\#\mathcal{B} < n$, $D_{\mathbf{u}^{\mathcal{B}}}V_1(\mathbf{s}') = 1$ or $DV_{\mathcal{E}}\mathbf{u}_{\mathcal{E}}^{\mathcal{B}} = \mathbf{1}_{\mathcal{E}}$, where $\mathcal{E} = \mathcal{B}\cup\{1\}$. SC and regularity imply $\frac{\partial V_1(\mathbf{s}')}{\partial s_1} > \min_{j\in\mathcal{B}} \frac{\partial V_j(\mathbf{s}')}{\partial s_1}$. □

A.3 Proof of Proposition 2

Case 2 of Step 3. For each $\tau \in (\tau_-, \tau_+)$, define $\Theta_k(\tau) = \{s_k' : \beta_j(s_j(\tau)) = \beta_k(s_k')\}$ and $\Theta_j(\tau) = \{s_j' : \beta_j(s_j(\tau)) = \beta_j(s_j')\}$. Then, $\{j,k\} \in \mathcal{I}(s_j(\tau), s_k')$ for all $s_k' \in \Theta_k(\tau)$. Indeed, j and k are the last to drop out, and if, for instance, $V_k\left(s_j(\tau), s_k'\right) < V_j\left(s_j(\tau), s_k'\right)$, then one can select s_j'' sufficiently close to $s_j(\tau)$, so that $\beta_j(s_j'') < \beta_j(s_j(\tau)) = \beta_k(s_k')$ but $V_k(s_j'', s_k') < V_j(s_j'', s_k')$, which contradicts efficiency.

First, we establish that generically for all $s_k' \in \Theta_k(\tau)$, profile $(s_j(\tau), s_k')$ is strictly competitive for $\{j,k\}$ and set $\Theta_k(\tau)$ is finite. Indeed, let $SS_j \subset U_j^{\mathbf{s}}$ consist of all $s_j(\tau)$ for which $\exists s_k' \in \Theta_k(\tau)$ such that $\frac{\partial V_j(s_j(\tau), s_k')}{\partial s_k} = \frac{\partial V_k(s_j(\tau), s_k')}{\partial s_k}$. Set SS_j is finite. (Otherwise, select $(s_j^n, s_k^n) \to_{n\to\infty} (s_j^*, s_k^*)$ such that $s_j^n =$

$s_j(\tau^n) \in SS_j$, $s_j^n \neq s_j^m$ for $n \neq m$, and $s_k^n \in \Theta_k(\tau^n)$. Since the value functions are C^1 and regular, $\frac{\partial V_j(s_j^*, s_k^*)}{\partial s_k} = \frac{\partial V_k(s_j^*, s_k^*)}{\partial s_k}$, $\{j,k\} \in \mathcal{I}(s_j^*, s_k^*)$, and $\frac{\partial V_j(s_j^*, s_k^*)}{\partial s_j} \neq \frac{\partial V_k(s_j^*, s_k^*)}{\partial s_j}$. There exists M, so that for all $n > M$, (s_j^n, s_k^n) belongs to the unique trajectory defined by $V_j = V_k$ and passing through (s_j^*, s_k^*). But then, $\exists M_1 > M$, so that: either $\frac{\partial V_j(s_j^n, s_k^n)}{\partial s_k} = \frac{\partial V_k(s_j^n, s_k^n)}{\partial s_k}$ and so $\forall n, m > M_1$, $s_j^n = s_j^m$; or $\frac{\partial V_j(s_j^n, s_k^n)}{\partial s_k} \neq \frac{\partial V_k(s_j^n, s_k^n)}{\partial s_k}$ and so $\forall n > M_1$, $s_j^n \notin SS_j$.) Then, for all τ with $s_j(\tau) \notin SS_j$, set $\Theta_k(\tau)$ is finite. (If not, select converging to some s_k^* sequence $s_k^n \in \Theta_k(\tau)$. In the limit, $\frac{\partial V_k(s_j(\tau), s_k^*)}{\partial s_k} = \frac{\partial V_j(s_j(\tau), s_k^*)}{\partial s_k}$, which contradicts $s_j(\tau) \notin SS_j$.) By similar arguments, the number of τ for which $\exists s_k' \in \Theta_k(\tau)$, $\frac{\partial V_j(s_j(\tau), s_k')}{\partial s_j} = \frac{\partial V_k(s_j(\tau), s_k')}{\partial s_j}$ is finite. Finally, if for some $s_k' \in \Theta_k(0)$, $\mathcal{I}(s_j(0), s_k') \neq \{j,k\}$, we can, for each such s_k', perturb slightly the initial $\mathbf{s}^*$ so that the values of $\mathcal{D} = \mathcal{I}(s_j(0), s_k') \setminus \{j,k\}$ are uniformly reduced, the values of $\{j,k\}$ remain equal and maximal, and all GSC inequalities remain of the same sign. Therefore the arguments made so far can be repeated for this profile. Once there is no such s_k', by continuity and wlog we can assume that $\forall \tau \in (\tau_-, \tau_+)$, $\mathcal{I}(s_j(\tau), s_k') = \{j,k\}$.

Next, select $\tau^* \in (\tau_-, \tau_+)$, such that $\#\Theta_k(\tau) > 1$ and $\forall s_k' \in \Theta_k(\tau^*)$, $\forall s_j' \in \Theta_j(\tau^*)$, profile (s_j', s_k') is strictly competitive for $\{j,k\}$. (This can be done generically by the same argument as above. If τ^* does not exist, then $\forall \tau$ with $s_j(\tau) \in CU_j^{\mathbf{s}}$, $\#\Theta_k(\tau) = 1$, and as in Case 1, $\beta_k(s_j(\tau), s_k(\tau)) = V_k(s_j(\tau), s_k(\tau))$ for all $\tau \in (\tau_-, \tau_+)$. In turn, for τ with $\#\Theta_k(\tau) > 1$, for $s_k' \in \Theta_k(\tau)$ and $s_k' \neq s_k(\tau)$, bidder k has a profitable deviation.) Thus, instead of one trajectory $\mathbf{s}(\tau)$ as in Case 1, we have $\#\Theta_k(\tau^*) \times \#\Theta_j(\tau^*)$ trajectories along which the values of j and k and their bids are equal. By the argument similar to the one in Case 1, we have $\beta_j(s_j') = E_{s_k \in \Theta_k(\tau^*)} V_j(s_j', s_k) = E_{s_j \in \Theta_j(\tau^*)} V_k(s_j, s_k') = \beta_k(s_k')$. Let $\bar{s}_k = \max \Theta_k(t')$. Then for each $s_j' \in \Theta_j(\tau^*)$, $\beta_k(\bar{s}_k) = E_{s_k \in \Theta_k(\tau^*)} V_j(s_j', s_k) = E_{s_k \in \Theta_k(\tau^*)} V_k(s_j', s_k) < V_k(s_j', \bar{s}_k)$. In turn, $\beta_k(\bar{s}_k) < E_{s_j \in \Theta_j(\tau^*)} V_k(s_j', \bar{s}_k)$, a contradiction.

Lemma 6. *Consider $\mathbf{s}' = \mathbf{s}(t)$ for an arbitrary $t \in U_t^0$, $\mathbf{s}(t)$ is the trajectory defined in Step 5. For any $j \in \mathcal{A}$, there exists $b_j(s_j') \equiv \lim_{s_j \downarrow s_j'} \inf \beta_j(s_j)$, and these limits are equal: $b_j(s_j') \equiv b < \infty$. In addition, $\beta_1(s_1') > b$ and for any $j \in \mathcal{A}$ and $s_j > s_j'$ sufficiently close to s_j', $\beta_j(s_j) \geq b$.*

Proof. The proof relies on some supporting results about the value functions presented in Appendix A.2 above. Consider trajectory $\mathbf{s}^{\mathcal{A}}(\tau) \equiv$

$\mathbf{s}^{\mathbf{u}^{\mathcal{A}}\mathcal{A}}(\tau)$ with $\mathbf{s}^{\mathcal{A}}(0) = \mathbf{s}'$ along which the values of bidders $\mathcal{A}$ remain equal and s'_1 is fixed (see Definition 4). Because GSC holds for any subset smaller than $\mathcal{A}$, $\mathbf{u}^{\mathcal{A}}_{\mathcal{A}} \gg \mathbf{0}$ and thus, for any $j \in \mathcal{A}$, $s^{\mathcal{A}}_j(\tau)$ is strictly increasing (for τ close to 0). Since GSC is violated for $\mathcal{A}$ and 1, $V_1(\mathbf{s}^{\mathcal{A}}(\tau)) > \max_{j\in\mathcal{A}} V_j(\mathbf{s}^{\mathcal{A}}(\tau))$ for any sufficiently small $\tau > 0$. Therefore,

$$\beta_1(s'_1) > \min_{j\in\mathcal{A}} \beta_j(s^{\mathcal{A}}_j(\tau)). \tag{13}$$

In turn, by continuity of $\mathbf{V}$, for any s_1 sufficiently close to s'_1 and $s_1 < s'_1$,

$$\beta_1(s_1) > \min_{j\in\mathcal{A}} \beta_j(s^{\mathcal{A}}_j(\tau)). \tag{14}$$

Define $b_{\mathcal{B}} \equiv \min_{i\in\mathcal{A}} b_i(s'_i)$, $\mathcal{B} = \arg\min_{i\in\mathcal{A}} b_i(s'_i)$, and $b_{-\mathcal{B}} \equiv \min_{i\in\mathcal{A}\setminus\mathcal{B}} b_i(s'_i)$. (It is possible that $b_{\mathcal{B}} = \infty$ and/or $b_{-\mathcal{B}} = \infty$. In what follows, $a > b$ together with $b = \infty$ implies $a = \infty$.) By definition, $b_{\mathcal{B}} < \infty$ when $\#\mathcal{B} < n = \#\mathcal{A}$. We show by induction on $\#\mathcal{B}$ that unless $\#\mathcal{B} = n$ efficiency is violated.

[1]. Suppose $\#\mathcal{B} = 1$, let $\mathcal{B} = \{j\}$ and fix $\varepsilon = (b_{-j} - b_j)/2$. There exists $\delta_\tau > 0$, such that for all $\tau \in (0, \delta_\tau)$ and $i \in \mathcal{A} \setminus \{j\}$, $\beta_i(s^{\mathcal{A}}_i(\tau)) > b_{-j} - \varepsilon$ and (13) holds. Consider trajectory $\mathbf{s}^*(r) = \mathbf{s}' + r\mathbf{v}^j$, where $\mathbf{v}^j \equiv \mathbf{v}^j(\boldsymbol{\lambda})$ is defined in Lemma 4 (see also Remark 1), and λ_1 is sufficiently close to 1 so that Lemma 5 holds. Then there exists $\delta_r > 0$, so that for all $r \in (0, \delta_r)$ and for all $i \in \mathcal{A}$, $s^*_i(r) \in s^{\mathcal{A}}_j[0, \delta_\tau)$ (that is, $s^*_i(r) = s^{\mathcal{A}}_i(\tau)$ for some $\tau \in [0, \delta_\tau)$).

Consider a sequence $s_{jm} \downarrow s'_j$, for which $\beta_j(s_{jm}) \to b_j(s'_j)$ and, for all m, $\beta_j(s_{jm}) < b_j + \varepsilon$ and $r_m \in (0, \delta_r)$, where $s^*_j(r_m) = s_{jm}$. Then, by (13),

$$\beta_1(s'_1) > \beta_j(s^*_j(r_m)) = \min_{j\in\mathcal{A}} \beta_j(s^{\mathcal{A}}_j(\tau_m)), \tag{15}$$

where τ_m is defined by $s^{\mathcal{A}}_j(\tau_m) = s^*_j(r_m)$. Also, similarly to (14), there exists $\delta_1 > 0$ such that for any $s_1 \in (s'_1 - \delta_1, s'_1)$,

$$\beta_1(s_1) > \beta_j(s_j(r_m)). \tag{16}$$

Since $\#\mathcal{B} < n$ by continuity and Lemma 4, for sufficiently large m (and so for sufficiently small $r_m > 0$), $\{j\} \subset \mathcal{I}(\mathbf{s}^*(r_m)) \subset \{j, 1\}$. By continuity and Lemma 5, for sufficiently small $r_m > 0$, there exists $\delta'_1(r_m) > 0$ such that for any $s_1 \in (s'_1 - \delta'_1(r_m), s'_1)$,

$$\{j\} = \mathcal{I}(s_1, \mathbf{s}^*_{\mathcal{A}}(r_m)). \tag{17}$$

Pick s_1 such that both (16) and (17) hold. Then even if we slightly increase the signals of those $i \in \mathcal{A}$ for whom $s_i^*(r_m) = s_i'$, by continuity $\{j\} = \mathcal{I}(\mathbf{s})$ at the so obtained profile $\mathbf{s}$. Since $\beta_i(s_i) > b_{-j} - \varepsilon$ for any $s_i \in (s_i', s_i^{\mathcal{A}}(\delta_\tau))$, $\beta_j(s_{jm}) = \beta_j(s_j) = \min_{i \in \mathcal{A}^{+1}} \beta_i(s_i)$. Thus, we have reached a contradiction as j has the highest value but drops out the first.

[2]. Here we show that for any $j \in \mathcal{A}$ with $s_j > s_j'$ and sufficiently close to s_j', $\beta_j(s_j) \geq b_{\mathcal{B}}$. For each $j \in \mathcal{A}$ pick a trajectory $\mathbf{s}^j(r) = \mathbf{s}' + r\mathbf{v}^j(\boldsymbol{\lambda}^j)$, where $\mathbf{v}^j(\boldsymbol{\lambda}^j)$ satisfies Lemmata 4 and 5 (see also Remark 1), and so that $\forall j \in \mathcal{A}$, $\lambda_1^j \in (1 - \delta_\lambda, 1)$, where $\delta_\lambda > 0$ and is arbitrarily close to 0. Then, there exists $\delta_r > 0$, so that for all $r \in (0, \delta_r)$ and for all $i, j \in \mathcal{A}$ we have: (1) $s_i^j(r) = s_i^{\mathcal{A}}(\tau)$ for some $\tau \in [0, \delta_\tau)$; and (2) $\{j, 1\} \supset \mathcal{I}(\mathbf{s}^j(r))$ and $\{j\} = \mathcal{I}(s_1, \mathbf{s}_{\mathcal{A}}^j(r))$ for any $s_1 \leq s_1'$ sufficiently close to s_1' (the latter may depend on particular r and j).

Suppose that there exists $j \in \mathcal{A}$ with $\beta_j(s_j) < b_{\mathcal{B}}$ for $s_j > s_j'$ arbitrarily close to s_j'. Pick such s_j so that $s_j = s_j^j(r_j)$ and $r_j \in (0, \delta_r)$, let $\varepsilon = b_{\mathcal{B}} - \beta_j(s_j) > 0$. Then $\{j, 1\} \supset \mathcal{I}(\mathbf{s}^j(r))$ and, as in [1], by slightly reducing s_1' and slightly increasing the signals of all $i \in \mathcal{A}$ with $s_i^j(r_j) = s_i'$ we obtain profile $\mathbf{s}$ with $\{j\} = \mathcal{I}(\mathbf{s})$. Thus there exists $i \in \mathcal{C}$ with $\beta_i(s_i) < \beta_j(s_j)$. If $\beta_1(s_1)$ is always the smallest no matter how slight the decrease in s_1 is, then $\lim_{s_1 \uparrow s_1'} \sup \beta_1(s_1) \leq b_{\mathcal{B}} - \varepsilon$, contradicting (14). Thus there exists $i \in \mathcal{A}$ with $s_i > s_i'$, such that $\beta_i(s_i) < \beta_j(s_j) = b_{\mathcal{B}} - \varepsilon$ and r_i, defined as $s_i^i(r) = s_i$, satisfies $r_i < \kappa r_j$. By choosing δ_λ as close to 1 as necessary, we can make κ as close to zero as necessary. It suffices to have $\kappa < 1$. By repeating this procedure we either find a contradiction that involves bidder 1 or find $i \in \mathcal{A}$ and a converging sequence of $r_{im} \downarrow 0$, such that $s_{im} = s_i^i(r_{im}) \downarrow s_i'$ and $\beta_i(s_{im}) < b_{\mathcal{B}} - \varepsilon$ for any m. But then, $b_i(s_i') \leq b_{\mathcal{B}} - \varepsilon$.

[3]. Suppose now that $\#\mathcal{B} = k \geq 2$. We show that we can find a trajectory $\mathbf{s}(\rho)$ along which bidders from $\mathcal{B}$ have the highest value and are dropping simultaneously for almost all ρ. Then, as in [1], even after slightly increasing the signals of the bidders from $\mathcal{A} \setminus \mathcal{B}$, there exists $i \in \mathcal{A} \setminus \mathcal{B}$ who drops earlier. Thus, for some $i \in \mathcal{A} \setminus \mathcal{B}$, $b_i(s_i') \leq b_{\mathcal{B}}$.

Formally, consider trajectory $\mathbf{s}^{\mathcal{B}}(\rho) = \mathbf{s}^{\mathbf{v}^{\mathcal{B}}\mathcal{D}}(\rho)$, for $\rho \geq 0$ and $\mathbf{s}^{\mathcal{B}}(0) = \mathbf{s}'$, defined for subset $\mathcal{D}$ from Lemma 4. (It is possible that $v_j^{\mathcal{B}} = 0$ for some $j \in \mathcal{B}$, thus $s_j^{\mathcal{B}}(\rho)$ is not necessarily increasing.) Define $b_j(\rho) \equiv b_j(s_j^{\mathcal{B}}(\rho))$. Clearly, $\lim_{\rho \to 0} b_j(\rho) = b_j(s_j')$ for all $j \in \mathcal{B}$. By construction, since GSC is satisfied for $\mathcal{D}$ and 1, and by Lemmata 4 and 5, for sufficiently small $\rho > 0$, $\mathcal{I}(\mathbf{s}^{\mathcal{B}}(\rho)) \subset \mathcal{B} \cup \{1\}$. Once we slightly decrease the signal of bidder 1, all the bidders with the highest value belong to $\mathcal{B}$. Also, for sufficiently small ρ,

$\max_{j\in\mathcal{B}} b_j(\rho) < b_{-\mathcal{B}}$.

Proceeding from the contrary; by the arguments similar to the one in [1] applied to $\mathbf{s} = \mathbf{s}^{\mathcal{B}}(\rho)$ for a sufficiently small $\rho > 0$, and by induction (on the size of a subset of $\mathcal{B}$), we have that $b_j(\rho) = b_{\mathcal{B}}(\rho)$ for all $j \in \mathcal{B}$. In general, as long as we stay sufficiently close to $\mathbf{s}'$, by slightly moving in appropriate directions away from $\mathbf{s}'$ and possibly in several steps, we can separate bidders $\mathcal{A}$ in any given order. Thus, whenever two or more bidders from $\mathcal{A}$ have equal and maximal values, the limits of their bids from the right have to be equal.

Similarly, by the argument in [2], $\beta_j(s_j) \geq b_{\mathcal{B}}(\rho)$ for any $j \in \mathcal{A}$ with $s_j > s_j^{\mathcal{B}}(\rho)$ and sufficiently close to $s_j^{\mathcal{B}}(\rho)$. Therefore, since $s_j^{\mathcal{B}}(\rho)$ is strictly increasing for some $j \in \mathcal{B}$, $b_{\mathcal{B}}(\rho)$ is weakly increasing.

There exists $\delta_\rho > 0$, so that for all $\rho \in (0, \delta_\rho)$ and any $j \in \mathcal{B}$, $\beta_j(s_j^{\mathcal{B}}(\rho)) \leq b_{\mathcal{B}}(\rho)$. (Otherwise, for any $\delta_\rho > 0$ we can find $\rho \in (0, \delta_\rho)$ and $j \in \mathcal{B}$, such that $\beta_j(s_j^{\mathcal{B}}(\rho)) > b_{\mathcal{B}}(\rho)$. Fix $s_j^{\mathcal{B}}(\rho)$ and $\beta_j(s_j^{\mathcal{B}}(\rho))$. By induction and repeating the above arguments for $\mathcal{B}' = \mathcal{B}\setminus\{j\}$ and $\mathcal{A}' = \mathcal{A}\setminus\{j\}$, we have a contradiction.) In turn, since $b_{\mathcal{B}}(\rho)$ is monotonic and $\forall j$, $\beta_j(s_j) \geq b_{\mathcal{B}}(\rho)$ for any $s_j > s_j^{\mathcal{B}}(\rho)$ (locally), ρ is a discontinuity point for $b_{\mathcal{B}}(\rho)$ if $\beta_j(s_j^{\mathcal{B}}(\rho)) < b_{\mathcal{B}}(\rho)$ for some $j \in \mathcal{B}$. Since a monotonic function can have only a countable number of discontinuity points, we have that for almost all ρ, for all j, $\beta_j(s_j^{\mathcal{B}}(\rho)) = b_{\mathcal{B}}(\rho)$.

Now we add bidder 1 into the picture. First, suppose that for some $j \in \mathcal{B}$ there exists $\delta_\rho > 0$, such that $s_j^{\mathcal{B}}(\rho) = s_j'$ for $\rho \in (0, \delta_\rho)$. Then, $b_{\mathcal{B}}(\rho) = b_{\mathcal{B}}$, and so for any $i \in \mathcal{B}$ with $s_i^{\mathcal{B}}(\rho)$ strictly increasing, $\beta_i(s_i^{\mathcal{B}}(\rho)) = b_{\mathcal{B}}$ for all $\rho \in (0, \delta_\rho)$. Consider $\mathbf{s}^{\mathcal{A}}(\tau)$. If $\tau > 0$ is sufficiently small, the bids of bidders $\mathcal{A}\setminus\mathcal{B}$ are separated away from $b_{\mathcal{B}}$, for each $j \in \mathcal{B}$, $\beta_j(s_j^{\mathcal{A}}(\tau)) \geq b_{\mathcal{B}}$ and for some $i \in \mathcal{B}$, $\beta_i(s_i^{\mathcal{A}}(\tau)) = b_{\mathcal{B}}$. Therefore from (13), $\beta_1(s_1') > b_{\mathcal{B}} = \min_{j\in\mathcal{A}} \beta_j(s_j^{\mathcal{A}}(\tau))$. Let $\mathcal{B}'$ consist of $i \in \mathcal{B}$, for whom $\beta_i(s_i) = b_{\mathcal{B}}$ in the right-hand neighborhood of s_i'. Consider a trajectory $\mathbf{s}^*(r) = \mathbf{s}' + r\mathbf{v}^{\mathcal{B}'}$. Along this trajectory, the set of bidders with the highest value is a subset of $\mathcal{B}' \cup \{1\}$. By continuity and Lemma 5, for a sufficiently small $r > 0$, (14) holds as well, and once s_1' is slightly reduced, all the bidders with the highest value belong to $\mathcal{B}'$. After slightly increasing the signal of each $j \in \mathcal{A}$ with $s_j^*(r) = s_j'$, we obtain profile $\mathbf{s}$, at which bidders $\mathcal{B}'$ drop out simultaneously at $b_{\mathcal{B}} = \min_{j\in\mathcal{A}^{+1}} \beta_j(s_j)$—a contradiction.

In the remaining case, for all $j \in \mathcal{B}$, $s_j^{\mathcal{B}}(\rho)$ is strictly increasing and so β_j is monotonic in the right-hand neighborhood of s_j'. Since $\mathbf{u}_{\mathcal{A}}^{\mathcal{A}} \gg \mathbf{0}$, for any small $\tau > 0$ and for each $j \in \mathcal{B}$, let ρ_j be the solution to $s_j^{\mathcal{B}}(\rho_j) = s_j^{\mathcal{A}}(\tau)$ and $\rho' \equiv \min_{j\in\mathcal{B}} \rho_j$. For any $\varepsilon > 0$ there exists $\delta_\tau > 0$, such that for any $\tau \in (0, \delta_\tau)$ we have: (i) for any $i \in \mathcal{A} \setminus \mathcal{B}$, $\beta_i(s_i^{\mathcal{A}}(\tau)) > b_{-\mathcal{B}}(\mathbf{s}') - \varepsilon/2$; (ii) for all $j \in \mathcal{B}$, ρ_j is sufficiently small so that $|b_{\mathcal{B}}(\rho) - b_{\mathcal{B}}| < \varepsilon/2$ and the above results hold. (That is, in particular: (1) the bidders from $\mathcal{B}$ have the highest value at $\mathbf{s}^{\mathcal{B}}(\rho)$, (2) $b_{\mathcal{B}}(\rho)$ is weakly increasing and, for all j, $\beta_j(s_j^{\mathcal{B}}(\rho)) \leq b_{\mathcal{B}}(\rho)$; and (3) starting from $\mathbf{s}^{\mathcal{B}}(\rho')$, once s_1' is slightly reduced, all the bidders with the highest value belong to $\mathcal{B}$.

Pick any $\tau \in (0, \delta_\tau)$ such that $b_{\mathcal{B}}(\rho)$ is continuous at ρ'. Then, consider $i \in \mathcal{B}$ with $\rho_i = \rho'$. From (14), for any $s_1 < s_1'$ and sufficiently close to s_1', we have

$$\beta_1(s_1) > \min_{j \in \mathcal{B}} \beta_j(s_j^{\mathcal{A}}(\tau)) = \min_{j \in \mathcal{B}} \beta_j(s_j^{\mathcal{B}}(\rho_j)) = \beta_i(s_i^{\mathcal{B}}(\rho')) = b_{\mathcal{B}}(\rho').$$

Then, starting from $\mathbf{s}^{\mathcal{B}}(\rho')$, by reducing slightly s_1' and increasing slightly s_j' for each $j \in \mathcal{A}$ with $s_j^{\mathcal{B}}(\rho') = s_j'$, we obtain profile $\mathbf{s}$, at which $\mathcal{I}(\mathbf{s}) \subset \mathcal{B}$, but bidders $\mathcal{B}$ exit first simultaneously at $b_{\mathcal{B}}(\rho')$—a contradiction.

[4]. We have shown that $\#\mathcal{B} = n$, and so $\mathcal{B} = \mathcal{A}$. Let $b \equiv b_{\mathcal{A}}$. Since for all $j \in \mathcal{A}$, $\beta_j(s_j) \geq b(\mathbf{s}')$ for all $s_j > s_j'$ close to s_j', from (13) we have $\beta_1(s_1') > b$.

It remains to be shown that $b < \infty$. If $b = \infty$, then for each $j \in \mathcal{A}^{+1}$ there exists an interval of signals with $\beta_j(s_j) = \infty$. Then, at a profile with such signals, each bidder's payoff is $-\infty$, which cannot happen in equilibrium since instead each bidder can exit at $p = 0$ and assure herself the payoff of 0. □

Corollary 2. *If for some $j \in \mathcal{A}$, $\beta_j(s_j(t)) \neq b(t)$, then t is a discontinuity point for $b(t)$.*

Proof. If for some $j \in \mathcal{A}$, $\beta_j(s_j(t)) > b(t)$, then by the argument similar to the one in [3] of the proof of Lemma 6, considering $\mathcal{A}' = \mathcal{A} \setminus \{j\}$, we can find a profile $\mathbf{s}$, at which all the bidders from $\mathcal{A}'$ exit simultaneously prior to bidder 1 and j, while all the bidders with the highest value belong to $\mathcal{A}'$.

Monotonicity of $b(t)$ is established in Step 7 of the proof of Proposition 2. From Lemma 6 it follows that for all $j \in \mathcal{A}$, whenever $s_j(t') > s_j(t)$, $\beta_j(s_j(t')) \geq b(t)$, for t and t' from the considered neighborhood U_t^0. Therefore, if for some $j \in \mathcal{A}$, $\beta_j(s_j(t)) < b(t)$, then $b(t'') \leq \beta_j(s_j(t))$ whenever $s_j(t'') < s_j(t)$, so t is a discontinuity point for $b(t)$. □

References

Dasgupta, P. S. & Maskin, E. (2000), 'Efficient auctions', *Quarterly Journal of Economics* **115**, 341–388.

Dubra, J., Echenique, F. & Manelli, A. M. (2009), 'English auctions and the stolper-samuelson theorem', *Journal of Economic Theory* **144**(2), 825–849.

Izmalkov, S. (2003), 'English auctions with reentry', MIT, mimeo.

Jehiel, P. & Moldovanu, B. (2001), 'Efficient design with interdependent valuations', *Econometrica* **69**, 1237–1259.

Kirchkamp, O. & Moldovanu, B. (2004), 'An experimental analysis of auctions with interdependent valuations', *Games and Economic Behavior* **48**(1), 54–85.

Krishna, V. (2003), 'Asymmetric English auctions', *Journal of Economic Theory* **112**, 261–288.

Maskin, E. (1992), Auctions and privatization, *in* H. Siebert, ed., 'Privatization: Symposium in Honor of Herbert Giersch', Institut für Weltwirtschaft an der Univarsität Kiel, pp. 115–136.

Milgrom, P. R. & Weber, R. J. (1982), 'A theory of auctions and competitive bidding', *Econometrica* **50**, 1089–1122.

Perry, M. & Reny, P. J. (2002), 'An efficient auction', *Econometrica* **70**, 1199–1212.

Perry, M. & Reny, P. J. (2005), 'An ex-post efficient ascending auction', *The Review of Economic Studies* **72**, 567–592.

Porter, R. H. (1995), 'The role of information in U.S. offshore oil and gas lease auctions', *Econometrica* **63**, 1–27.