Optimal Reserve Prices in Anonymous Asymmetric Auctions*

Sergei Izmalkov † Valery Topinsky ‡

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Abstract

We analyze a problem of seller maximizing the expected revenue in an asymmetric setting under a practically relevant constraint of relying on simple detailed-free selling mechanisms. We consider an asymmetric independent private value setting in which multiple possibly non-identical (commonly ranked) objects are sold to buyers who obtain value from at most one object and whose valuations come from two different distributions. The objects are sold by means of a generalized Vickrey-Clark-Groves mechanism with the reserve price being the only possibility for the seller to extract more revenue. We characterize optimal reserve prices under different assumptions about the seller's knowledge of asymmetries among the buyers. We consider four information treatments ranging from the detailed knowledge of every bidder's distribution to complete ignorance, that is, knowledge of the average distribution corresponding to the belief that all bidder's valuations come from the same distribution. We show that finer knowledge of distribution types of buyer's valuations helps the seller only if she also knows how many buyers have specific distributions. Any other knowledge beyond that, e.g. being able to assign a specific distribution to a specific buyer, is useless. Generally, the optimal reserve price is higher when more detailed knowledge is available.

1 Introduction

How to sell the good so as to extract the most revenue from potential buyers? Auction it off — this is the famous conclusion of Myerson (1981) and Riley and

*Preliminary and incomplete.

†New Economic School and Yandex LLC; sizmalkov@gmail.com

‡Yandex LLC and National Research University Higher School of Economics; topinsky@gmail.com

Samuelson (1981). When the seller does not distinguish among buyers and thinks that each buyer's valuation of the good comes from the same distribution — a so called symmetric setting — any common auction format with the appropriately chosen reserve price is optimal. When buyers are asymmetric, the optimal selling scheme is complicated, its allocation and payment rules depend on the finer details of the seller's knowledge of buyers' different distributions. It cannot be implemented by a simple auction format in a straightforward manner. Bidder asymmetries are commonplace in reality. Dealers, private art collectors, and museums in fine arts auctions; local and global firms in licence auctions; re-sellers and final customers in online auctions; different types of businesses in context advertising auctions — are all examples of likely asymmetric scenarios.

In practice, essentially the only effective tool for the seller to earn extra revenue is the reserve price. Of course, first the seller should attract all seriously interested buyers as an extra bidder effect on expected revenue is likely to dominate the effect of the reserve price (Bulow and Klemperer (1996)). But once all serious buyers are invited, and if there are only a few of them, reserve price matters for revenue.

Our goal for the paper is twofold. First, we want to determine what is the optimal reserve price in asymmetric settings and how does it depend on these asymmetries and on the details of the seller's knowledge about buyers' distributions. Second, we want to find what details of the seller's knowledge of buyers matter for revenue and quantify these effects.

We consider the Vickrey-Clark-Groves (VCG) mechanism and its equivalent mechanisms — the second-price, the uniform-price, and the generalized second-price (GSP) auctions — for simplicity of the analysis and to be able to cleanly trace the effects of different information the seller may possess about the buyers.[1]

Crucially, there is a dominant strategy equilibrium of the VCG auction in which all bidders bid their valuations, and the details of the buyers' knowledge about each other are irrelevant. In choosing these formats to analyze we are motivated by extensive use of uniform-price auctions and their extensions, notably for electronic commerce (eBay) and context advertisement auctions (Google, Yahoo, Yandex, and others).

Our overall setup is simple. It is an asymmetric independent private values setting with a single or multiple objects for sale. Each buyer can utilize at most one object; different objects are commonly ranked by the buyers, so that valuations are proportional to the value of the most preferred object. Valuations of buyers can come from two different distributions.

We consider four possible information treatments for the seller. First, a classic one, *detailed* knowledge: the seller knows the distribution of values for each

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[1] Vickrey (1961), Clarke (1971), Groves (1973) describe the efficient mechanism in quasilinear environments, Krishna and Perry (2000) generalize it.

buyer, so she can match the bidder to his distribution. Second, *average* knowledge: the seller is completely ignorant, she only knows the average distribution, as if believing that all bidders' valuations come from that distribution. Third, *probabilistic* knowledge: the seller knows that bidders' valuations come from different distributions, with a certain probability for each distribution. Forth, *anonymous* knowledge: the seller has correct aggregate information about how many bidders' valuations come from what distributions, but cannot distinguish bidder's identities and so does not know the exact distribution for each bidder.

We show that, in general, the optimal reserve price is between the reserve prices computed for each individual distribution (as if all the buyers' values were coming from that distribution). We show that what matters for the seller and for setting the optimal reserve price is the correct aggregate information about how many buyers come from which distributions. Specifically, the optimal reserve price and the expected revenues are the same in average and probabilistic treatments, and the same in detailed and anonymous treatments. Thus, in particular, additional knowledge of the two distributions and of their likelihoods is of no use to the seller if he does not possess better information on the likelihood of the current participants's values coming from specific distributions. At the same time, knowing only aggregate quantities of specific distribution types is sufficient for the seller to extract as much revenue as she can by the reserve price manipulation, the seller has no additional value in tying a particular buyer to a particular distribution.

With extra detailed information on quantities of specific distribution types, the reserve price is typically higher than without that knowledge. We show by examples that this extra information can be very valuable for the seller.

There is an extensive literature on asymmetric auctions, but most of it, following Maskin and Riley (2000), is focused on the revenue comparison between the first-price (FPA) and the second-price (SPA) sealed-bid auctions. Gayle and Richard (2008) and Marshall and Schulenberg (1998) compare these auction formats with reserve prices and show that, in contrast to most of the comparisons without the reserve price, SPA may dominate FPA. We do not consider FPAs as their equilibria also require the detailed specification of bidders' beliefs. There are many more possible information treatments depending on different variations of bidders' beliefs. A bidder, for instance, may or may not know his own distribution type, may believe that others are of the same type as he is, may believe that the seller knows a lot about the other bidders, etc. Given the complexity of computing equilibria in FPA for any given information treatment, it is simply infeasible to make comparisons of different information treatments for FPA.

A few papers compare symmetric and asymmetric auctions in terms of revenue. Gavious and Minchuk (2010), Kaplan and Zamir (2002), and Cantillon (2008) show that asymmetry weakens the competition and results in smaller revenue.

We are the first to our knowledge to model asymmetric auctions with anonymous bidders. Interestingly, Laurent (2012) considers a complementary to our paper empirical question of identifying asymmetric distributions from anonymous data. Suppose the seller observes the same bidders participating in repeated auctions, but cannot keep track of their identities. Laurent (2012) develops a technic with kernel-based estimators for identification of different distributions which turns out to be applicable even for small data sets.

Our results directly apply to the generalized-second price auctions used to allocate advertising positions on a search page (Varian (2007), Edelman, Ostrovsky and Schwarz (2007)). Here higher positions are more valuable to advertisers as users are more likely click on them. Ostrovsky and Schwarz (2009) describe the optimal reserve prices and the experiment involving them at Yahoo! in the symmetric (average in our terminology) setting. Our results suggest that due to inherent asymmetry in participants in these actions and possibilities to track these asymmetries by search engines, the computation of optimal reserve prices and of effects on revenue from any reserve price are likely to be biased due to asymmetries.

The rest of the paper is organized as follows. In Section 2 we describe our environment and introduce four information treatments. Section 3 contains our main results, in particular, the equivalence of two pairs of information treatments in terms of expected revenue, and comparisons of the optimal reserve prices under different information treatments. In Section 4 we present numerical examples of the reserve price and expected revenue comparisons. Section 5 concludes.

2 Preliminaries

There are K not necessarily identical goods for sale to $N \geq K$ buyers. The value of buyer i from obtaining good k only is

$$v_{ik} = \alpha_k v_i.$$

Parameters $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_K$ reflect the ordering of goods, e.g. in size or quality, common to all buyers. Value $v_i \in [0, \bar{v}]$ is private information of buyer i. The value of buyer i from having a subset of goods is equal to the value from the good with a higher rank. That is, for instance, $v_i \{2, 5, 7\} = v_{i2}$. The seller's valuation is normalized to be zero. The rest of the players' knowledge about the other players is going to be specified later.

The buyers and the seller have the usual quasilinear preferences: a player's payoff is the value from the good(s) minus the money payed. We are going to use personal pronouns *she* to refer to the seller and *he* to any buyer.

This setting encompasses both a multi-unit setting with identical goods in which each buyer demands or is restricted to buy at most 1 unit, with $\alpha_k = 1$ for all

$k = 1..K$, and a context advertisement setting, in which each buyer can be assigned no more than one advertising position, positions are commonly ranked according to their expected clickability, and α_k stands for the normalized click through rate of position k. For comparison with the multi-unit setting it is convenient to set $\alpha_1 = 1$.

In a multi-unit setting, the goods are sold by means of a uniform price auction with reserve price R. All the buyers submit non-negative real-valued bids, one per buyer. If less than K bids exceed or equal R, then less than K goods are sold, everyone who submitted a bid higher or equal to R wins one good and pays R. Otherwise, the buyers with K highest bids win one object each and pay the highest of the $(K+1)$st bid and R. The ties, if any, are broken at random.

The uniform-price auction has an equilibrium in dominant strategies, in which each buyer bids his true value, $\beta_i^* = v_i$. This is the equilibrium we are going to focus on in the paper. Our results also extend to an open ascending price format, the Ausubel auction, which has an equivalent equilibrium in dominant strategies.[2] We chose to consider the sealed-bid format instead of dynamic format — the uniform-price auction instead of Ausubel (2004) auction (and the second-price auction instead of the English auction for $K = 1$) for simplicity of formalization and notation.

In the context advertisement (multi-object) setting, the goods are sold by means of a generalized VCG mechanism with a reserve price R. Again, all the buyers report their valuations. Submitted reports are ranked from high to low, and the goods are allocated according to rank: good k is assigned to the buyer with the kth highest report provided the report exceeds R. The ties, if any, are broken at random. The buyer who gets object k pays the externality he imposes on others:

$$p_k = \sum_{j=k}^{K} (\alpha_j - \alpha_{j+1}) \max\{v_{(j+1)}, R\},$$

where $v_{(j)}$ is the jth highest reported valuation, and $\alpha_{K+1} = 0$. It is a dominant strategy for each buyer to report his valuation truthfully, and this is the equilibrium we will focus on.

[2] An open ascending price auction with a reserve price can be described as follows. The price starts from R and increases as long as at least K buyers are willing to buy one good each at the currently shown price. Buyers indicate their willingnesses to buy in some fashion, e.g. by pressing a button or holding a hand up, and can drop from bidding at any time. Dropping decisions are both public and irrevocable. Once the auction stops, K remaining active bidders are winners, they pay the price at which the auction stopped. If multiple bidders dropped out at the final price so that less than K bidders remain active, the rest of the goods are randomly allocated among those who exited last. It is trivial to show that staying active until the price reaches own value is a weakly dominant strategy for each buyer.

The context advertisement setting described here is exactly the same as the one introduced and analyzed in Varian (2007) and Edelman et al. (2007). Both papers analyze the generalized second-price auction (GSP). In GSP, buyers submit bids, goods are allocated according to ranked bids, and the payment of the buyer with the kth highest bid equals the $k+1$st highest bid. Edelman et al. (2007) show that the generalized second-price auction has an ex post equilibrium, in which buyers bid as if they know the bids (and valuations) of the others, and the efficient allocation obtains: buyers with higher values take higher (better) positions, and their payments correspond to the payments in the efficient dominant strategy equilibrium of the VCG mechanism. With reserve price R the equilibrium bids are:

$$\forall j \geq K,\ b_{(j)} = v_{(j)},$$
$$\forall j < K,\ b_{(j)} = \begin{cases} v_{(j)}, & \text{if } v_{(j)} < R; \\ \left(1 - \frac{\alpha_{j+1}}{\alpha_j}\right) v_{(j)} + \frac{\alpha_{j+1}}{\alpha_j} p_{j+1}, & \text{if } v_{(j)} \geq R; \end{cases}$$

where

$$\forall j = 1..K,\ p_j = \max\{b_{j+1}, R\}$$

is the price of position j (for $K = N$, $b_{K+1} = 0$).

Similarly to the multi-unit setting, our results would also extend to the GSP auction and to the open ascending price format equivalent to the VCG mechanism, the generalized English auction of Edelman et al. (2007).

2.1 Information treatments

We consider four information treatments representing different degrees of the seller's knowledge about the buyers. We are going to compare them against the benchmark (which is also one of the treatments) corresponding to a maximal or true knowledge of the environment. The benchmark is an asymmetric independent private values setting. For all i, the value of buyer i is independently drawn from a distribution with cumulative density function F_i. There are two possible distribution types, w and s, standing for "weak" and "strong." Thus, for all i, $F_i \in \{F_w, F_s\}$.

(D) *Detailed* knowledge (benchmark). The seller knows all the details, that is, the distribution function F_i for each buyer i.

(An) *Anonymous* knowledge. The seller knows that buyers valuations come from two different distributions F_w and F_s, knows the aggregate numbers of how many players have valuations drawn from each distribution type, but cannot associate a particular type to any bidder. Thus, the seller also knows n_w and n_s, respectively, the number of weak and strong types, $n_w, n_s \in \mathbb{N}$, $n_w + n_s = N$.

(Pr) *Probabilistic* knowledge. The seller knows that buyers valuations come from two different distributions F_w and F_s and does not have any aggregate knowledge. Instead the seller believes that each buyer, independently from the others, can be a strong one with probability p.

(Av) *Averaged* knowledge. The seller is ignorant of the fact that buyers valuations can come from two different distributions. She believes that each buyer's value is independently drawn from the same distribution F_{Av}.

These four information treatments represent different degrees of the seller's knowledge, ranked from the more to less detailed one. For a proper comparison of the the optimal reserve prices and expected revenues across treatments we set

$$\begin{aligned} p &= \frac{n_s}{n_s + n_w}; \\ F_{Av}(z) &= (1-p)\,F_w(z) + pF_s(z). \end{aligned}$$

For the distribution functions we are going to use the following definitions and notations. For a distribution function with cdf F and pdf f, we define

$$\text{hazard rate: } \lambda(x) = \frac{f(x)}{1-F(x)};$$
$$\text{virtual valuation: } \psi(x) = x - \frac{1}{\lambda(x)}.$$

We can compare two distributions in terms of first-order stochastic dominance, hazard rate dominance, and dominance in terms of the likelihood ratio:

$$F_1 \succcurlyeq_d F_2, \quad \text{if } \forall x \quad F_1(x) \le F_2(x), \tag{1}$$
$$F_1 \succcurlyeq_{hr} F_2, \quad \text{if } \forall x \quad \lambda_1(x) \le \lambda_2(x), \tag{2}$$
$$F_1 \succcurlyeq_{lr} F_2, \quad \text{if } \forall x\, \forall y > x \quad \frac{f_1(x)}{f_2(x)} \le \frac{f_1(y)}{f_2(y)}. \tag{3}$$

Clearly,

$$F_1 \succcurlyeq_{hr} F_2 \Longleftrightarrow \psi_2(x) \ge \psi_1(x).$$

It is also well-known (e.g. see Krishna (2009)) that

$$F_1 \succcurlyeq_{lr} F_2 \Longrightarrow F_1 \succcurlyeq_{hr} F_2.$$

For all of the results, except Theorem 1, we are going to assume that: pdfs f_s and f_w are strictly positive; distribution F_s dominates F_w in terms of the hazard rate, $\psi_s < \psi_w$; that ψ_w and ψ_s are strictly increasing (the regular case in Myerson (1981)) and that ψ_{Av} is strictly increasing as well.

Lemma 1. $\forall\, v \in (0, \bar{v})$,

$$\psi_w(v) > \psi_{Av}(v) > \psi_s(v).$$

Proof is in Appendix.

3 Reserve prices and revenues

3.1 The seller's problem

Following Myerson (1981) approach we can express the expected revenue to the seller as follows (for details see Krishna (2009, p.67))

$$\mathbb{E}[\mathcal{R}] = \sum_{i \in N} \int_R^{\bar{v}} \psi_i(v_i) f_i(v_i) q_i(v_i)\, dv_i, \tag{4}$$

where $q_i(v_i) = \mathbb{E}_{\mathbf{v}_{-i}} Q_i(v_i, \mathbf{v}_{-i})$ is the expected quantity allocated to bidder i when his reported valuation is v_i, $\mathbf{v}_{-i}$ is the profile of valuations of all the bidders but i, and $Q_i(v_i, \mathbf{v}_{-i})$ is the normalized quantity of goods allocated to bidder i given reported profile of valuations $\mathbf{v} = (v_i, \mathbf{v}_{-i})$. For the multi-unit setting and the uniform price auction the normalized quantity is the actual quantity, and for the multi-object (context advertising setting), $Q_i(\mathbf{v}) = \sum_{k=1}^{K} \alpha_k Q_i^k(\mathbf{v})$, where $Q_i^k(\mathbf{v})$ is the probability buyer i obtains good k given reports $\mathbf{v}$.

Therefore, the sellers' problem of choosing the optimal reserve price amounts to maximization of (4) with respect to R, with virtual valuations, pdfs, and expected quantities defined according to specifics of each setting and each information treatment.

It is important to note that while by rules of the auction bidders are not discriminated, individual expected probabilities of winning the good, $q_i(v)$, may differ across individuals. These differences are driven by differences in actual or expected identities of the opponents. Consider, for instance, a second-price auction with only two bidders in it, one strong and one weak, which is commonly known. Then, the strong bidder with value $v > R$ has a higher probability of winning than the weak bidder with the same value v as the strong bidder knows that he competes against the weak one, and the other way around: $q_s(v) = F_w(v) > q_w(v) = F_s(v)$.

We can rewrite (4) as

$$\mathbb{E}[\mathcal{R}] = \int_R^{\bar{v}} \sum_{i \in N} \psi_i(v) f_i(v) q_i(v)\, dv. \tag{5}$$

Due to anonymity (absence of discrimination) of the ex post allocation rule and random breaking of ties if any, $q_i(v) = q_j(v)$, $\psi_i(v) = \psi_j(v)$, and $f_i(v) = f_j(v)$ for any two players i and j from the same distribution type. Letting $q_s(\cdot)$ and $q_w(\cdot)$ denote the expected quantity functions for the strong and weak bidders, respectively, we can further express the expected revenue to the seller for treatments (D), (An), and (Pr) as

$$\mathbb{E}[\mathcal{R}] = \int_R^{\bar{v}} \hat{n}_s \psi_s(v) f_s(v) q_s(v) + \hat{n}_w \psi_w(v) f_w(v) q_w(v)\, dv, \tag{6}$$

where $\hat{n}_s$ and $\hat{n}_w$ are expected numbers of weak and strong bidders. Clearly, $\hat{n}_s = n_s$ and $\hat{n}_w = n_w$ for all three information treatments: for (D) and (An) expected numbers are actual numbers; and for (Pr), $\hat{n}_s = Np = n_s$. For treatment (Av), we can express the expected revenue as

$$\mathbb{E}[\mathcal{R}] = \int_R^{\bar{v}} N \psi_{Av}(v) f_{Av}(v) q_{Av}(v)\, dv, \tag{7}$$

We can state the following result.

Theorem 1. *The optimal reserve price and the expected revenue are the same in treatments (D) and (An),*

$$R_K^D = R_K^{An}, \qquad \mathbb{E}[\mathcal{R}^D] = \mathbb{E}[\mathcal{R}^{An}]$$

and in treatments (Pr) and (Av),

$$R_K^{Pr} = R_K^{Av}, \qquad \mathbb{E}[\mathcal{R}^{Pr}] = \mathbb{E}[\mathcal{R}^{Av}].$$

Proof. Equivalence of expected revenues and optimal reserve prices for treatments (D) and (An) trivially follows from (6) as the expected quantities $q_s(v)$ and $q_w(v)$ depend only on the composition of the opponents and so are the same for both treatments.

For treatment (Pr), the expectations of the opponents types are the same for strong and weak players, $q_s^{Pr}(v) = q_w^{Pr}(v) = q^{Pr}(v)$. Therefore, we can express

$$\mathbb{E}[\mathcal{R}^{Pr}] = \int_R^{\bar{v}} \left[n_s \psi_s(v) f_s(v) + n_w \psi_w(v) f_w(v)\right] q^{Pr}(v)\, dv. \tag{8}$$

Then note that $q^{Pr}(v) = q^{Av}(v)$ as for any given opponent the probability that that opponent has value below v is the same for both treatments, $pF_s(v)+(1-p)F_w(v) = F_{Av}(v)$. Thus, the probabilities of winning one of the K goods for the multi-unit setting or winning (any) good k for the multi-object setting are exactly the same for both (Pr) and (Av) treatments.

Finally, as $\psi_\cdot(v)f_\cdot(v) = vf_\cdot(v) - 1 + F_\cdot(v)$, and $n_sF_s(v) + n_wF_w(v) = NF_{Av}(v)$, $n_sf_s(v) + n_wf_w(v) = Nf_{Av}(v)$, we have

$$n_s\psi_s(v)f_s(v) + n_w\psi_w(v)f_w(v) = Nvf_{Av}(v) - N + NF_{Av}(v) = N\psi_{Av}(v)f_{Av}(v). \quad (9)$$

Thus, from (8) and (7) we obtain $\mathbb{E}[\mathcal{R}^{Pr}] = \mathbb{E}[\mathcal{R}^{Av}]$. □

While the theorem may be straightforward, its implications are substantial. It says that the knowledge of the two distributions per se does not increase the expected revenue to the seller unless the seller knows some extra details about the actual bidders she faces. The seller extract all possible expected revenue if she only knows the correct aggregate numbers of each bidder type. Any additional details, such as being able to associate a particular distribution to a particular buyer, are irrelevant.

3.2 Second-price auction

The comparison of the optimal reserve prices and expected revenues for the two pairs of information treatments is conceptually similar, but slightly different technically for single-unit, multi-unit, and multi-object settings. We start with the single-unit setting.

Theorem 2. *Let R^{An} and R^{Av} be the optimal reserve prices for the anonymous and average treatments. Let R_w and R_s be the optimal reserve prices for settings in which all the bidders are weak and strong, respectively. Then, the following comparison obtains*

$$R_w < R^{Av} < R^{An} < R_s.$$

Proof. First note that R_w and R_s are solutions to $\psi_w(v) = 0$ and $\psi_s(v) = 0$, respectfully (see Myerson (1981)). As $\psi_s(v) < \psi_w(v)$, $R_s > R_w$.

For the (Av) treatment, the optimization of the expected revenue (7) with respect to R, results in R^{Av} that solves $\psi_{Av}(v) = 0$. From (9) we can express

$$p\psi_s(v)f_s(v) + (1-p)\psi_w(v)f_w(v) = \psi_{Av}(v)f_{Av}(v).$$

At $v = R_w$ the left hand side is negative, as $\psi_s(R_w) < 0$ and $\psi_w(R_w) = 0$, while at $v = R_s$, the left hand side is positive. Thus, $R_w < R^{Av} < R_s$.

Now consider (An) treatment. From the perspective of a strong buyer there are $n_s - 1$ strong and n_w buyers among the other $n-1$ buyers. As identities of these buyers are irrelevant, one can think of the other buyers as belonging to two groups: $n-2$ buyers consisting of exactly $n_s - 1$ strong and $n_w - 1$ weak ones and a single weak buyer. Similarly, from a perspective of a weak buyer, there is the same group of $n-2$ buyers and a strong one. Let

$$q_{n-2}(v) = F_w^{n_w-1}(v)F_s^{n_s-1}(v)$$

be the probability of all the buyers in the group having value below v. Then, for $v \geq R$,

$$q_s^{An}(v) = q_{n-2}(v)F_w(v), \quad q_w^{An}(v) = q_{n-2}(v)F_s(v).$$

Thus,

$$n_s\psi_s(v)f_s(v)q_s^{An}(v) + n_w\psi_w(v)f_w(v)q_w^{An}(v) = \\ q_{n-2}\left[n_s\psi_s(v)f_s(v)F_w(v) + n_w\psi_w(v)f_w(v)F_s(v)\right]. \quad (10)$$

The expression in brackets we can further express as

$$n_s\psi_s(v)f_s(v)F_w(v) + n_w\psi_w(v)f_w(v)F_s(v) = \\ \left[n_s\psi_s(v)f_s(v) + n_w\psi_w(v)f_w(v)\right]F_{Av}(v) + \\ n_s\psi_s(v)f_s(v)\left[F_w(v) - F_{Av}(v)\right] - n_w\psi_w(v)f_w(v)\left[F_{Av}(v) - F_s(v)\right]. \quad (11)$$

From (9), we obtain

$$\left[n_s\psi_s(v)f_s(v) + n_w\psi_w(v)f_w(v)\right]F_{Av}(v) = N\psi_{Av}(v)f_{Av}(v)F_{Av}(v).$$

Altogether, starting with (6), we can express the expected revenue in (An) case as

$$\mathbb{E}[\mathcal{R}^{An}] = \int_R^{\bar{v}} N\psi_{Av}(v)f_{Av}(v)F_{Av}(v)q_{n-2}(v)\,dv \\ + \int_R^{\bar{v}} n_s\psi_s(v)f_s(v)\left[F_w(v) - F_{Av}(v)\right]q_{n-2}(v)\,dv \\ - \int_R^{\bar{v}} n_w\psi_w(v)f_w(v)\left[F_{Av}(v) - F_s(v)\right]q_{n-2}(v)\,dv. \quad (12)$$

Differentiating $\mathbb{E}[\mathcal{R}^{An}]$ (expression (4)) with respect to R, and using (10), we obtain

$$\frac{\partial}{\partial R}\mathbb{E}[\mathcal{R}^{An}] = -q_{n-2}\left[n_s\psi_s(v)f_s(v)F_w(v) + n_w\psi_w(v)f_w(v)F_s(v)\right]. \quad (13)$$

Clearly, at $v \leq R_w$ the expression in brackets is negative, while at $v \geq R_s$, it is positive. Thus, $R_w < R^{An} < R_s$.

Using (12) we can express

$$\frac{\partial}{\partial R}\mathbb{E}[\mathcal{R}^{An}] = -N\psi_{Av}(v)f_{Av}(v)F_{Av}(v)q_{n-2}(v)$$
$$- n_s\psi_s(v)f_s(v)\left[F_w(v) - F_{Av}(v)\right] + n_w\psi_w(v)f_w(v)\left[F_{Av}(v) - F_s(v)\right]q_{n-2}(v). \quad (14)$$

As both brackets are positive, $\psi_s(v)$ is negative, and $\psi_w(v)$ is positive on $v \in (R_w, R_s)$, we have that

$$\frac{\partial}{\partial R}\mathbb{E}[\mathcal{R}^{An}] > -N\psi_{Av}(v)f_{Av}(v)F_{Av}(v)q_{n-2}(v), \quad (15)$$

for $v \in (R_w, R_s)$. Therefore, $R^{An} > R^{Av}$. □

3.3 Uniform-price and VCG auctions

For the multi-unit and multi-object settings we can state the following two results.

Theorem 3. *Let R_K^{An} and R_K^{Av} be the optimal reserve prices for the anonymous and average treatments with K identical objects for sale. Then, for all $1 \le K \le N$, $R_K^{Av} = R^{Av}$, and for all $1 \le K \le N-1$,*

$$R_s > R_K^{An} > R_{K+1}^{An} > R_N^{An} = R^{Av} > R_w.$$

Theorem 4. *Let $R_{\alpha,K}^{Av}$ and $R_{\alpha,K}^{An}$ denote the optimal reserve prices in the VCG auction with K objects and quality vector $\alpha = \alpha_1, \ldots, \alpha_K$ for the (Av) and (An) treatments, respectively. Then, for all $1 \le K \le N$, $R_{\alpha,K}^{Av} = R^{Av}$, and for all $1 \le K \le N-1$,*

$$R_s > R_{\alpha,K}^{An} > R_{\alpha,K+1}^{An} > R_{\alpha,N}^{An} > R^{Av} > R_w.$$

Compared to optimal reserve prices for the uniform price auction, for any α with $\alpha_k > \alpha_{k+1}$ for some $k < K$,

$$R_{\alpha,K}^{An} > R_K^{An}.$$

The proofs of these two theorems are logically similar to Theorem 2. For (Av) treatment, the optimal reserve $R_{\cdot}^{Av}$ solves $\psi_{Av} = 0$ and is, in particular, independent of K or α. For (An) treatment, for a fixed K, the result is driven by differences in perceptions of strong and weak bidders about their competition. As in the single good case, one can attribute these differences to differences in beliefs about the type of a single competitor. Similarly to (11) one can then separate the common effect (as if both types had an average belief about that single competitor)

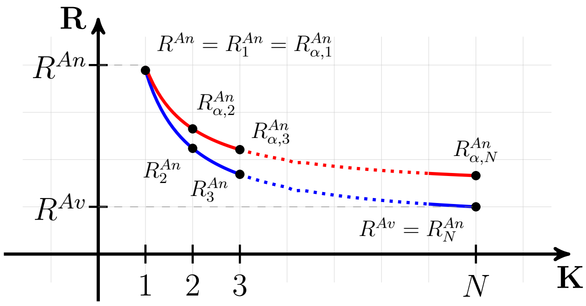


Figure 1: The optimal reserve prices for the uniform-price and VCG auctions depending on the number of objects for sale.

and the additional individual effects. If there were only the common effect, the optimal reserve would have been the same as in (Av) treatments, and the both individual effects pushing it up. When K increases these individual effects get smaller in magnitude. Similarly, when goods falling in relative quality ($\alpha_k > \alpha_{k+1}$ for some k) this is the opposite effect to an increase in K, and that causes the $R_{\cdot}^{An}$ to increase. The formal proofs of these theorems are in Appendix.

Intuitively, decreasing in K optimal reserve prices as well as higher reserve prices in VCG auction compared to the uniform price auction are due to the fact that on average strong bidders are more likely to win objects when there are fewer of them or win better objects if they are unequal.

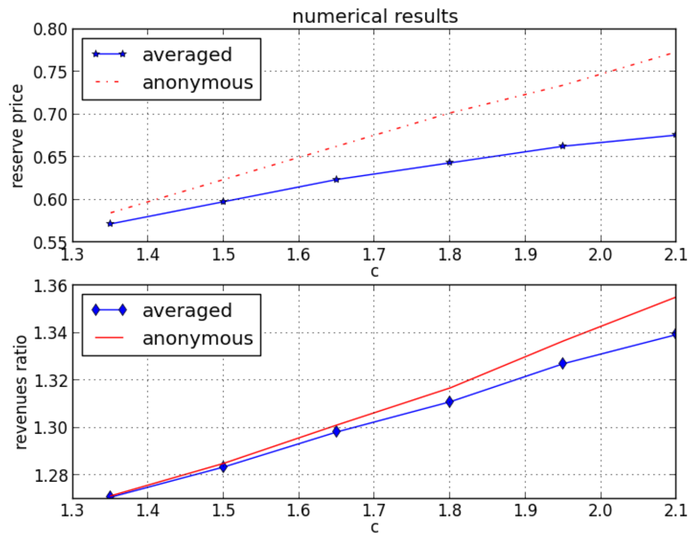
The reserve prices depending on K are presented and compared for the uniform-price and VCG auctions in Figure 1.

4 Numerical results

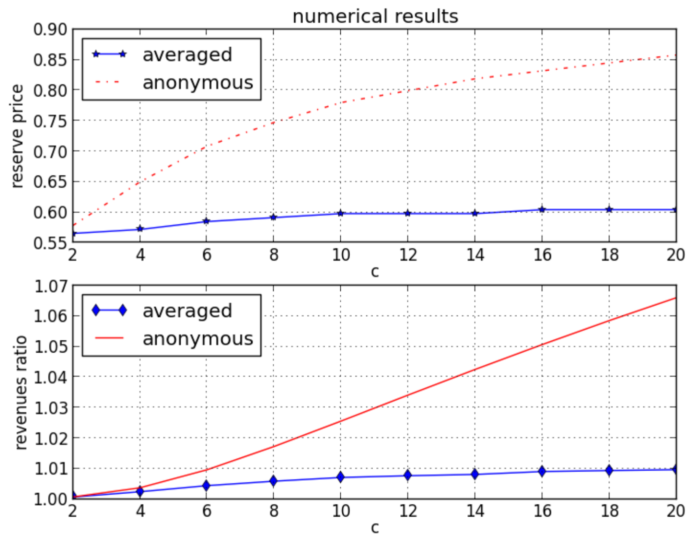
In this Section, we demonstrate the effects of asymmetric distributions and different information treatments on the optimal reserve price and expected revenue visually. We compute them numerically for the single good for sale, uniform distributions with different supports and for power distributions with the same support but different powers. We also vary buyers' composition (n_s, n_w).

Numerical results are presented on Figures 2a and 2b for the information treatments (An) and (Av). The revenue ratio is the ratio of the expected revenue with the optimal reserve price to the expected revenue without the reserve price.

We see that the optimal reserve prices grow with asymmetry, which is expected. The optimal reserves and the expected revenues are higher when more detailed knowledge about the buyers is available. The effects are more pronounced for power distributions, where such extra knowledge becomes crucial for revenue extraction. As we see, the effects from extra knowledge can be substantial.



(a) Uniform distributions: $F_w = U[0,1]$, $F_s = U[0,c]$, for $c > 1$. Here $n_s = n_w = 1$ — there are 2 asymmetric bidders in the auction.



(b) Power distributions: $F_w(x) = x^{1/c}$, $F_s(x) = x^c$, $x \in (0,1)$. Here $n_s = 1$, $n_w = 5$ — there are 6 bidders in the auction, 1 strong and 5 weak ones.

Figure 2: Numerical examples of revenues and reserve prices comparison.

5 Conclusions

Certainly, there is no surprise that the more detailed information the seller has about the buyers, the more revenue she can extract from them. But as collecting information is costly, what matters is if having extra information is actually profitable. We suppose that the seller is limited to using simple selling mechanisms with the only control option of a reserve price. We show that knowing that the buyers are asymmetric and knowing their different distributions does not increase profit if the seller can only say that a given buyer is of strong type with a certain probability independent from other buyers. If instead the seller knows that there are some strong buyers with certainty the knowledge about asymmetries is profitable.

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Appendix A

Proof of Lemma 1.
It is enough to show that $\lambda_w(v) > \lambda_{Av}(v) > \lambda_s(v)$. By definition of λ we have that

$$\lambda_{Av}(v) = \frac{f_{Av}(v)}{1 - F_{Av}(v)} = \frac{1}{\frac{\alpha(1-F_s(v))}{f_{Av}(v)} + \frac{(1-\alpha)(1-F_w(v))}{f_{Av}(v)}}$$

$$= \frac{1}{\frac{\alpha f_w(v)}{f_{Av}(v)}\frac{(1-F_w(v))}{f_w(v)} + \frac{(1-\alpha)f_s(v)}{f_{Av}(v)}\frac{(1-F_s(v))}{f_s(v)}} = \frac{1}{a_w(v)\frac{1}{\lambda_w(v)} + a_s(v)\frac{1}{\lambda_s(v)}},$$

where $a_w(v) + a_s(v) = 1$.

As

$$\lambda_w(v) > \frac{\lambda_w(v)\lambda_s(v)}{a_w(v)\lambda_w(v) + a_s(v)\lambda_s(v)} > \lambda_s(v)$$

we have the result.

□

Proof of Theorem 3.
Similarly to (11) we can separate the common effect and the additional individual effects. Formally,

$$q_s^{An} \equiv q_s^{An}[\mathcal{B}, w] = q_s^\star[\mathcal{B}, Av] + \underbrace{(q_s^{An}[\mathcal{B}, w] - q_s^\star[\mathcal{B}, Av])}_{=\Delta q_s,\ \text{individual effect}},$$
$$q_w^{An} \equiv q_s^{An}[\mathcal{B}, s] = q_w^\star[\mathcal{B}, Av] - \underbrace{(q_w^\star[\mathcal{B}, Av] - q_w^{An}[\mathcal{B}, s])}_{=\Delta q_w,\ \text{individual effect}},$$

where $\mathcal{B}$ is the same as in Theorem 2 set of $n-2$ buyers, and the brackets just show the structure of perceived competitors. Since $F_w(v) \geq F_{Av}(v) \geq F_s(v)$, we have

$$q_s^{An}(v) \leq \underbrace{q^\star(v)}_{q_s^\star(v) \equiv q_w^\star(v)} \leq q_w^{An}(v). \tag{16}$$

Hence both Δq_s and Δq_w are non-negative. Then, similarly to (12), we obtain

$$\begin{aligned}\mathbb{E}[\mathcal{R}_K^{An}] = \int_R^{\bar{v}} N\psi_{Av}(v) f_{Av}(v) q^\star(v)\, dv \\ + \int_R^{\bar{v}} n_s \psi_s(v) f_s(v) \Delta q_s(v)\, dv \\ - \int_R^{\bar{v}} n_w \psi_w(v) f_w(v) \Delta q_w(v)\, dv.\end{aligned} \tag{17}$$

Thus, $R_s > R_K^{An} > R_K^{Av} = R^{Av} > R_w$.

To obtain $R_K^{An} > R_{K+1}^{An} > R_N^{An} = R^{Av}$, it's enough to show that: (i) $q^\star$ increases with respect to K, which is straightforward as the expected quantity of goods won increases as the number of goods available increases; and (ii) both Δq_s and Δq_w decrease as K increases and vanish at $K = N$. Fact (ii) follows directly from (16) and Lemmas 2, 3 below (to have explicit reference to K and to ease the notation, we use $q_s^K(x)$ instead of $q_s^{An}(x)$ and $q_w^K(x)$ instead of $q_w^{An}(x)$). □

Proof of Theorem 4.
The proof is essentially the same as for Theorem 3.

It suffices to notice that for the relative quality vector α with $\alpha_k > \alpha_{k+1}$ for some k, the effect of lower quality of good $k+1$ is equivalent to destroying a fraction of a good $k+1$. Similarly defined $q^\star$ increases slower with respect to K than in case of identical goods, and this results in $R_{\alpha,K}^{An} > R_K^{An}$. □

Lemma 2. *Let there are two sets $\mathcal{N}$ and $\mathcal{N}'$ with the same number of bidders $n+1$. And let there are two classes of bidders 'f' and 'g' whose valuations are independent and distributed with respect to distribution functions $F(x)$ and $G(x)$ respectively.*

Moreover we claim that F stochastically dominate G (i.e. $F(x) \leq G(x)$) and our two sets $\mathcal{N}$ and $\mathcal{N}'$ are partitioned as follows: $\mathcal{N} = \mathcal{N}_f \sqcup \mathcal{N}_g \sqcup \{k\}$, $\mathcal{N}' = \mathcal{N}'_f \sqcup \mathcal{N}'_g \sqcup \{k\}$ with $n_f = |\mathcal{N}_f|$, $n_g = |\mathcal{N}_g|$, $n'_f = |\mathcal{N}'_f|$, $n'_g = |\mathcal{N}'_g|$ and $n_f < n'_f$.

If $\pi_i^{n+1}(x;\mathcal{N})$ is the conditional probability for the bidder k to have the i-th highest valuation amongst all from $\mathcal{N}$ given its own valuation equals x; then we have that $\pi_i^{n+1}(x;\mathcal{N}')$ dominates $\pi_i^{n+1}(x;\mathcal{N})$ in terms of the likelihood ratio as probability distributions with respect to $i \in (1, \ldots, n+1)$.

Proof. Because of $n = n_f + n_g = n'_f + n'_g$ and $n_f < n'_f$ we get $n_g > n'_g$. Without loss of generality we rewrite the partitions as follow

$$\mathcal{N} = \mathcal{N}_f \sqcup \mathcal{N}_g \sqcup \{k\} = \{k\} \sqcup \mathcal{N}_f \sqcup \mathcal{N}'_g \sqcup \mathcal{N}_g^{\Delta},$$
$$\mathcal{N}' = \mathcal{N}'_f \sqcup \mathcal{N}'_g \sqcup \{k\} = \{k\} \sqcup \mathcal{N}_f \sqcup \mathcal{N}'_g \sqcup \mathcal{N}_f^{\Delta},$$

where $\mathcal{N}_f^{\Delta}$ and $\mathcal{N}_g^{\Delta}$ are two sets of discriminating bidders with $|\mathcal{N}_f^{\Delta}| = |\mathcal{N}_g^{\Delta}| = \Delta$.

Now for the conditional probability we define the recurrent equality given a fixed order of bidders from $\mathcal{N}$ with $k = (1)$ as follow

$$\pi_i^m(x;\mathcal{N}) = \pi_i^{m-1}(x;\mathcal{N})F_{(m)}(x) + \pi_{i-1}^{m-1}(x;\mathcal{N})(1 - F_{(m)}(x)),$$

where $\pi_i^m(x;\mathcal{N})$ is the conditional probability for the bidder k to have the i-th highest valuation amongst first $m > 0$ bidders from $\mathcal{N}$ according to the given order; $F_{(m)}(x)$ is the distribution function of m-th bidder in the given oder. There are simple boundary conditions: $\pi_1^1(x;\mathcal{N}) = 1$, $\pi_0^m(x;\mathcal{N}) = 0$ and $\forall m < i$ we put $\pi_i^m(x;\mathcal{N}) = 0$.

Because of that the conditional probability $\pi_i^{n+1}(x;\mathcal{N})$ not depends on any order of bidders from $\mathcal{N}$ we can choose an order with the following conditions

$$k = (1) \text{ and } \forall b \in \mathcal{N}_{\cdot}^{\Delta}\ b \in \{(2), \ldots, (\Delta + 1)\}.$$

So let's choose the orders for $\mathcal{N}$ and $\mathcal{N}'$ with this conditions and with the same order for the rest bidders from $\mathcal{N}_f \sqcup \mathcal{N}'_g$. Now with the chosen orders of two bidders' sets we show that the ratio $\pi_i^m(x;\mathcal{N})/\pi_i^m(x;\mathcal{N}')$ is non-increasing in i by the induction with respect to m.

Now we show the initial statement for any $m \leq \Delta + 1$. It's simple to see that $\pi_i^m(x;\mathcal{N}) = C_{m-1}^{i-1}(1-G(x))^{i-1}G(x)^{m-i}$ and $\pi_i^m(x;\mathcal{N}') = C_{m-1}^{i-1}(1-F(x))^{i-1}F(x)^{m-i}$. Then we have

$$\frac{\pi_i^m(x;\mathcal{N})}{\pi_i^m(x;\mathcal{N}')} = \frac{(1-G(x))^{i-1}G(x)^{n-i}}{(1-F(x))^{i-1}F(x)^{n-i}} = \underbrace{\left(\frac{G(X)}{F(x)}\right)^n \frac{1-F(x)}{1-G(x)}}_{\geq 1} \left(\underbrace{\frac{1-G(x)}{G(x)} \frac{F(x)}{1-F(x)}}_{\leq 1}\right)^i.$$

And we get immediately that the ratio is non-increasing in i for any $m \le \Delta + 1$.

Now we assume that for some m the ratio is non-increasing in i and we prove it for $m+1$. It's easy to show if we use the followings obvious statements

$$\forall a, b, c, d > 0 \quad \frac{a+b}{c+d} \ge \frac{a}{c} \;\Leftrightarrow\; \frac{b}{d} \ge \frac{a}{c} \text{ and } \frac{a+b}{c+d} \le \frac{a}{c} \;\Leftrightarrow\; \frac{b}{d} \le \frac{a}{c}.$$

So now we immediately have that

$$\frac{\pi_i^{m+1}(x;\mathcal{N})}{\pi_i^{m+1}(x;\mathcal{N}')} = \frac{\pi_i^m(x;\mathcal{N})F_{(m)}(x) + \pi_{i-1}^m(x;\mathcal{N})(1-F_{(m)}(x))}{\pi_i^m(x;\mathcal{N}')F_{(m)}(x) + \pi_{i-1}^m(x;\mathcal{N}')(1-F_{(m)}(x))} \ge \frac{\pi_i^m(x;\mathcal{N})}{\pi_i^m(x;\mathcal{N}')} \ge$$

$$\ge \frac{\pi_{i+1}^m(x;\mathcal{N})F_{(m)}(x) + \pi_i^m(x;\mathcal{N})(1-F_{(m)}(x))}{\pi_{i+1}^m(x;\mathcal{N}')F_{(m)}(x) + \pi_i^m(x;\mathcal{N}')(1-F_{(m)}(x))} = \frac{\pi_{i+1}^{m+1}(x;\mathcal{N})}{\pi_{i+1}^{m+1}(x;\mathcal{N}')}.$$

So it's what we need, the ratio is non-increasing in i for $m+1$, and by means of the induction we get that $\pi_i^{n+1}(x;\mathcal{N}')$ l.r.-dominates $\pi_i^{n+1}(x;\mathcal{N})$. □

Lemma 3. *Let $\mathcal{N} = (1, \dots, N)$ is the set of bidders and moreover $\mathcal{N} = \mathcal{N}_w \sqcup \mathcal{N}_s$ with $n_w = |\mathcal{N}_w|, n_s = |\mathcal{N}_s|$ ($N = n_w + n_s$). Assume that valuation distribution function $F_s(x)$ for bidders from $\mathcal{N}_s$ stochastically dominates the distribution $F_w(x)$ for bidders from $\mathcal{N}_w$. Then the conditional probability q_s^K for a strong bidder to be amongst first K bidders with highest valuations given his own valuation equals x majorizes the analogous conditional probability q_w^K for a weak bidder as functions with respect to x, i.e. $\forall x \; q_s^K(x) \ge q_w^K(x)$; and moreover the ratio $q_s^K(x)/q_w^K(x)$ is non-increasing in K with $q_s^N(x)/q_w^N(x) = 1$.*

Proof. First of all we show that $q_s^N(x)/q_w^N(x) = 1$. It's obvious statement because in both the numerator and the denominator there are conditional probabilities for a bidder to be amongst all bidders given his own valuation equals x and each of these conditional probabilities equals one.

Now we prove the first statement about the majorization. For that let's for both cases note the picked bidder whose given valuation equals x as the bidder b. Then for the case when this bidder is from the strong class we have the following partition of the bidders' set $\mathcal{N}$:

$$\mathcal{N} = \{b \in \mathcal{N}_s\} \sqcup \mathcal{N}_w \sqcup \mathcal{N}_s'.$$

In the other hand for the case when bidder b is from the weak class we get the another partition:

$$\mathcal{N} = \{b \in \mathcal{N}_w\} \sqcup \mathcal{N}_w' \sqcup \mathcal{N}_s.$$

Here $\mathcal{N}_s'$ and $\mathcal{N}_w'$ are just the initial sets $\mathcal{N}_s$ and $\mathcal{N}_w$ without the bidder b respectively.

Now for using the result of Lemma 2 we rename the bidders' set $\mathcal{N}$ partitioned according to the case $b \in \mathcal{N}_w$ as $\mathcal{N}'$. It remains only to note that $q_s^K(x) = \sum_{i=1}^K \pi_i^N(x;\mathcal{N})$ and $q_w^K(x) = \sum_{i=1}^K \pi_i^N(x;\mathcal{N}')$. Hence $q_s^K(x)$ and $q_w^K(x)$ are the cumulative distribution functions for the discrete probability distributions $\pi_i^N(x;\mathcal{N})$ and $\pi_i^N(x;\mathcal{N}')$ respectively. And because of the last one dominates the former one in terms of the likelihood ratio we have that the last one stochastically dominates the former one Krishna (2009, p.278), i.e. $\forall K \in \{1,\dots,N\}\ \forall x\ \ q_w^K(x) \le q_s^K(x)$.

Now we show the non-increasing behavior of the ratio $q_s^K(x)/q_w^K(x)$ in K. We rewrite this ratio as follow

$$\frac{q_s^K(x)}{q_w^K(x)} = \frac{q_s^{K-1}(x) + \pi_K^N(x;\mathcal{N})}{q_w^{K-1}(x) + \pi_K^N(x;\mathcal{N}')}.$$

We use the following simple statement:

$$\forall a,b,c,d > 0 \quad \frac{a+b}{c+d} \le \frac{a}{c} \ \Leftrightarrow\ \frac{b}{d} \le \frac{a}{c} \ \Leftrightarrow\ \frac{b}{a+b} \le \frac{d}{c+d}.$$

So now we only have to show that $\pi_K^N(x;\mathcal{N})/q_s^K(x) \le \pi_K^N(x;\mathcal{N}')/q_w^K(x)$.

For that we assume the opposite, that $\pi_K^N(x;\mathcal{N})/q_s^K(x) > \pi_K^N(x;\mathcal{N}')/q_w^K(x)$. But from Lemma 2 we know that $\pi_i^N(x;\mathcal{N}')$ l.r.-dominates $\pi_i^N(x;\mathcal{N})$. That's why the conditional probability distribution, $\pi_i^{\le K}(x;\mathcal{N}') := \pi_i^N(x;\mathcal{N}')/q_w^K(x)$, to have the i-th highest valuation amongst $\mathcal{N}$ given your own valuation equals x and that you are amongst first K bidders with highest valuations; the conditional probability distribution $\pi_i^{\le K}(x;\mathcal{N}')$ dominates the $\pi_i^{\le K}(x;\mathcal{N}) := \pi_i^N(x;\mathcal{N})/q_s^K(x)$ in terms of the likelihood ratio too.

It means that the ratio $[\pi_i^N(x;\mathcal{N})/q_s^K(x)]/[\pi_i^N(x;\mathcal{N}')/q_w^K(x)]$ is the non-increasing in i. And if

$$\pi_K^N(x;\mathcal{N})/q_s^K(x) > \pi_K^N(x;\mathcal{N}')/q_w^K(x)$$

then $\pi_i^N(x;\mathcal{N})/q_s^K(x) > \pi_i^N(x;\mathcal{N}')/q_w^K(x)$ for all $i \in \{1,\dots,K\}$.
And we get the contradiction that

$$1 = \sum_{i=1}^K \pi_i^N(x;\mathcal{N})/q_s^K(x) > \sum_{i=1}^K \pi_i^N(x;\mathcal{N}')/q_w^K(x) = 1$$

. □

Corollary. *Now if we introduce some discriminative prizes $\alpha = (\alpha_1,\dots,\alpha_N)$ for different places in terms of an order place amongst bidders with highest valuations then we will get the conditional expectations for a strong bidder and a weak bidder to get a prize given their own valuations equal x as follow*

$$q_w^\alpha(x) = \sum_{i=1}^N \alpha_i \cdot \pi_i^N(x;\mathcal{N}') \ \textit{and}\ q_s^\alpha(x) = \sum_{i=1}^N \alpha_i \cdot \pi_i^N(x;\mathcal{N}).$$

If we have that $\alpha_1 \geq \cdots \geq \alpha_N$ (i.e. you gain more for a 'higher' place and you gain maximum for the first place) then $q_s^\alpha(x) \geq q_w^\alpha(x)$.

Moreover, if we define a slice $\alpha^{\leq K}$ as follow

$$\alpha^{\leq K} := (\alpha_1, \ldots, \alpha_K, 0, \ldots, 0),$$

i.e. for a position worse than Kth place you will gain nothing, then the ratio $q_s^{\alpha^{\leq K}}(x)/q_w^{\alpha^{\leq K}}(x)$ is non-increasing in K.

Proof. As we know from Lemma 3 the distribution $\pi_i^N(x;\mathcal{N}')$ l.r.-dominates $\pi_i^N(x;\mathcal{N})$. It's known Krishna (2009, p.275) that for an increasing sequence γ ($\gamma_i \leq \gamma_j\ \forall i < j$) the expectation $\sum_{i=1}^N \gamma_i \pi_i^N(x;\mathcal{N}')$ is no less than the expectation $\sum_{i=1}^N \gamma_i \pi_i^N(x;\mathcal{N})$. But we have the decreasing sequence α and that's why we have the opposite result, $q_s^\alpha(x) \geq q_w^\alpha(x)$.

For the last statement it's sufficient to show that

$$\frac{q_s^{\alpha^{\leq K}}(x)}{q_w^{\alpha^{\leq K}}(x)} \geq \frac{\pi_K^N(x;\mathcal{N})}{\pi_K^N(x;\mathcal{N}')}. \left[\text{Recall: } \forall a,b,c,d > 0 \;\; \frac{a}{c} \geq \frac{a+b}{c+d} \;\Leftrightarrow\; \frac{a}{c} \geq \frac{b}{d}\right]$$

But from Lemma 3 we know that the ratio $q_s^K(x)/q_w^K(x)$ is non-increasing in K, i.e.

$$\frac{q_s^K(x)}{q_w^K(x)} \geq \frac{\pi_K^N(x;\mathcal{N})}{\pi_K^N(x;\mathcal{N}')}.$$

So if we show that $q_s^{\alpha^{\leq K}}(x)/q_w^{\alpha^{\leq K}}(x) \geq q_s^K(x)/q_w^K(x)$ or equally $q_s^{\alpha^{\leq K}}(x)/q_s^K(x) \geq q_w^{\alpha^{\leq K}}(x)/q_w^K(x)$ then we will prove our statement.

But the last two terms $q_s^{\alpha^{\leq K}}(x)/q_s^K(x)$ and $q_w^{\alpha^{\leq K}}(x)/q_w^K(x)$ are just conditional expectations for the slice $\alpha^{\leq K}$ with respect to conditional probability distributions $\pi_i^{\leq K}(x;\mathcal{N})$ and $\pi_i^{\leq K}(x;\mathcal{N}')$ respectively. And because of $\pi_i^{\leq K}(x;\mathcal{N}')$ also l.r.-dominates $\pi_i^{\leq K}(x;\mathcal{N})$ we have that $q_s^{\alpha^{\leq K}}(x)/q_s^K(x) \geq q_w^{\alpha^{\leq K}}(x)/q_w^K(x)$. □