Informed seller in a Hotelling market*

Filippo Balestrieri
HP Labs

Sergei Izmalkov†
New Economic School

Abstract

We derive the optimal selling mechanism for a monopolist who is privately informed about the attributes of a horizontally differentiable good. To do so, we set up an informed principal problem and offer a way to measure the information transmitted within a mechanism. The optimal mechanism can take variety of shapes, depending on how sensitive the buyer's utility is to the seller's information. Unlike full unraveling results in vertically differentiated settings, in our setting the seller keeps her information private when such sensitivity is low and discloses different information to different buyer's types when it is intermediate. In the latter case, the optimal mechanism can be often implemented by a two-item menu: (i) an opaque good or (ii) information with an option to buy the good; alternatively, by selling the good with and without return option. Sometimes, the optimal solution involves partial (probabilistic) allocation to some of the buyer's types.

Keywords: informed principal, information discrimination, horizontal differentiation, lotteries, optimal mechanism.

1 Introduction

It is a common characteristic of several business negotiations that the interaction between sellers and buyers originally begins in a particular condition of asymmetric information: the buyers are uncertain about their valuation for the good on sale, the sellers own private information that would help the buyers to resolve their uncertainty. For example, a seller may be better informed about attributes of the good that are not immediately visible to the buyer: the ingredients used to cook a dish, the internal components of a technological device, the details of a financial investment plan. In

*We are grateful to Martin Hellwig, Bengt Holmstrom, Vijay Krishna, Eric Maskin, Leslie Marx, Steven Matthews, Roger Myerson, Timofiy Mylovanov, Alessandro Pavan, Michael Riordan, Fereydoon Safai, Tomas Sjöström, Vasiliki Skreta, Jacques Thisse, and to participants of conferences and seminars at which earlier drafts of the paper were presented for their valuable comments and suggestions.

†Emails: filippo.balestrieri@hp.com and sizmalkov@nes.ru.

these situations each seller may try to use her information strategically: revealing or hiding what she knows in order to maximize her profits.

When is profitable for the seller to disclose her private information? How much to reveal? To whom?

We consider the problem of a profit maximizing seller facing a buyer in an horizontally differentiated market. Both the seller and the buyer have private information. The seller's type corresponds to the actual specifications of the good with respect to some critical dimension. The buyer's type represents his preferences over such dimension so that his utility from consuming the good is a base consumption value minus a cost depending on the difference between his ideal specifications and the actual ones. Such a market is represented via a standard Hotelling (1929) model, where the seller's and the buyer's types can be thought of as locations on the Hotelling line.

We derive the optimal (revenue maximizing) mechanism and offer ways to implement it. We show that for low base consumption values the seller reveals her information, for high values she reveals nothing, and for intermediate values the seller gains by price discriminating the buyer's types over their value for information, revealing different information to different types of the buyer.

Our analysis inherits the complexity of the informed principal problem (Myerson (1983), Maskin & Tirole (1990), Maskin & Tirole (1992), Mylovanov & Tröger (2012)). After the seller (the principal) learns her type, she designs the mechanism to sell the good and announces it to the buyer (the agent). Any choice she makes affects the buyer's beliefs over her types, and, thus, the buyer's willingness to pay for the good. However, the seller cannot manipulate arbitrarily the buyer's beliefs. Any signal the seller sends to the buyer has to pass through a credibility test. Whenever the seller wants to induce some probability distribution over her types, these probabilities should be consistent with what is indeed in the interest of each of the seller's types from the perspective of the buyer. In this sense, the actual seller (i.e. the true type of the seller) needs to consider what the other seller's type would do, if she wants to credibly reveal or conceal her identity.

We consider an environment in which the good can be located only at the two extremes of the line with equal probabilities. We also assume that the seller's types are unverifiable so every type can pretend to be the other one. Thus, our setting is one of pure horizontal differentiation: there exists no "better" or "high value" type of the seller who finds advantageous to always reveal her identity.[1]

In deriving the optimal mechanism, we rely on the Inscrutability Principle (Myerson (1983)): without loss of generality we can restrict our attention to mechanisms in which no information about the seller's type is revealed to any type of buyer. It is important to notice that this does not mean that information cannot be transmitted

[1] The informed principal literature (see Yilankaya (1999), Mylovanov & Tröger (2008)) considers several environments where the presence of high and low types of the principal hinders any strategic use of private information by the principal and supports a complete unraveling of information (i.e. full disclosure) as the unique equilibrium.

within an inscrutable mechanism. Indeed, the inscrutable mechanism can be such that different types of the buyer know in advance that they will receive the good only if the seller is of a certain type. In that way, an inscrutable mechanism can always replicate more complex mechanisms in which the seller actually communicates information about her type to the buyer and, in light of such information, the buyer decides whether to buy the good.

We formally define the concept of within mechanism information transmission in terms of allocation probabilities. In a nutshell, considering a given type of the buyer, a higher level of within mechanism information transmission is equivalent to a lower probability of receiving his undesirable good. By modeling the informed principal problem only in terms of allocation probabilities, in our analysis incentive compatibility and individual rationality constraints automatically take care of all restrictions related to information transmission as well (e.g. revealing information to one type may preclude the option to keep information hidden to some other types).

For intermediate base consumption values, a large variety of mechanisms are optimal under different conditions. Some entail particularly interesting features. For example, when the cost function is linear or concave the optimal mechanism is characterized by full within mechanism information transmission to the buyer's types at the extremes of the Hotelling line (i.e. zero probability of receiving the undesirable good) and no information transmission to the types in the middle (i.e. probability one of receiving the good, no matter the seller's type). Such solution can be simply implemented as a two-item menu. The buyer is offered the choice between either buying the good with no information attached — a product we may refer to as *opaque good*[2] — or buying the information about the good first and then having an option to purchase the good at a predetermined exercise price. The buyer's types that are almost indifferent between the two types of the seller choose to buy the opaque good. The buyer's types who have strong biases in their preferences are the ones who value information the most and buy the option. Out of this latter group, the ones that learn "positive" information (i.e. the actual good is the one that they like the most) buy the good at the exercise price of the option. The ones who learn "negative" information do not exercise the option. Alternatively, the solution can be implemented by the sale of the opaque good with and without return rights.

Under certain conditions, when the costs are concave or convex (not linear), the optimal mechanism involves random allocation: different types of the buyer receive the good with some positive (but less than one) type-dependent probabilities. These probabilities reflect different within mechanism information transmission to different types. To the best of our knowledge, our analysis is the first to determine the optimality of probabilistic allocation for a monopolist selling a single good.[3] This format

[2] We borrow the opaque good term from the marketing literature. Opaque are goods that are offered on sale purposely without disclosing relevant information about their attributes.

[3] Riley & Zeckhauser (1983) were the first to examine the opportunity for a single-good monopolist to increase profits by selling contracts with a random delivery of the good. They show that the

of the optimal mechanism can be implemented by menus of buying options that may include lotteries or return options. Depending on the characteristics of the setting, the optimal lotteries may be *opaque* or *mixed (conditional)*. An opaque lottery is such that the prize is opaque, that is, it is not known at the time of purchase. The winning probability is fixed, the buyer is informed about the exact characteristics of the good only after buying the lottery and can decide whether to participate in the draft of the lottery or not. A mixed (conditional) lottery is such that it may specify different winning probabilities for different possible prizes. A buyer who buys a mixed lottery does not know the type of the good awarded at the time of purchase and cannot opt out of the draft after learning.

We contribute to the informed principal literature (Myerson (1983), Maskin & Tirole (1990), Maskin & Tirole (1992), Yilankaya (1999), Skreta (2011), Mylovanov & Tröger (2008), Balestrieri (2008)) by setting up and solving the informed principal problem in an horizontally differentiated market.[4] We introduce the concept of within mechanism information transmission and offer a tractable framework that allows us to derive the optimal mechanism in the setting considered.

Our analysis is closely related to the literature on optimal mechanisms for a multi-product monopolist (McAfee & McMillan (1988), Thanassoulis (2004), Pavlov (2006), Balestrieri, Izmalkov & Leão (2013)). The uncertainty about the two types of the seller embedded in our environment happens to provide a natural link between our work and the ones that solve the profit maximization problem of a two-good monopolist, whose goods are horizontally differentiable. Both in Pavlov (2006) and in Balestrieri et al. (2013) the optimal mechanism entails the sale of opaque goods. There, the uncertainty is a feature of the mechanism as opposed of the environment. The buyers are price discriminated on the base of their degree of indifference between the two goods. In our setting, information has value per se: the problem of each type of the seller is how to extract revenue from the buyers who prefer the other type and the solution is to sell them only information alone.

More broadly, our work adds to a vast literature that analyzes information disclosure policies in settings in which an informed player wants to induce other players to take specific actions. Relevant works are also the ones that study the disclosure of information by an informed auctioneer,[5] the optimal behavior of sellers who do not own private information but can strategically manipulate the buyers' access to infor-

optimal selling strategy does not include lotteries.

[4] In the jargon of Maskin & Tirole (1992), we consider an informed seller in a *common value* environment: the seller's type enters directly into the utility function of the buyers.

[5] Milgrom & Weber (1982) raise the question of whether the seller should reveal her information and answer it positively for a general affiliated values setting. Ganuza & Penalva (2010) consider the incentives of an auctioneer to provide her private information to the bidders in a private value setting and compare different definitions of signal's precision in a second price auction. Competition and precision appear to be complementary factors to maximize the auctioneer's profits.

mation,[6] and persuasion games.[7] Most of these works consider settings with vertical differentiation.[8]

In none of these studies, a player has both private information and full bargaining power to design the mechanism (i.e. the problem is not set up as an informed principal problem). The informed seller selects full, no, or partial public disclosure of information in different contexts. We differ from previous literature as we allow the seller to tailor different disclosure policies to different buyer types and set prices on different amounts of information. Under some parameter values the seller does informationally discriminate the buyers in the optimal mechanism. This means that limiting attention to public communication disclosure policies is with the loss of generality.

Price discrimination across buyers in terms of their valuation for information is analyzed in models of mechanisms with refunds.[9] We show that, under some parameters, refunds or options to return the good are used to implement the optimal mechanism for an informed seller.

The rest of the paper is organized as follows. The model is set up in Section 2. In Section 3 we describe the revenue maximizing mechanism in settings in which the principal is constrained to specific disclosure policies. In Section 4 we set up

[6]In industrial organization literature, for example, Lewis & Sappington (1994) consider the trade-off faced by a seller who can control the accuracy with which buyers learn their valuation of the good on sale: more precise information enhances price discrimination opportunities to the seller, but leaves higher informational rents to the buyers. They characterize settings in which extreme disclosure policies (i.e., maximum precision, maximum noise) are optimal. More generally, Johnson & Myatt (2006) show that when the monopolist can vary the precision with which the buyers learn their valuation for the good on sale, her profits are a U-shaped function of the dispersion of the buyer's valuation. Thus, the optimal strategy for the seller is either a niche-market strategy: provide full information access in order to identify the high valuation buyers and charge them a high price; or a mass-market strategy: hide the information and charge low price to a large number of buyers.

Similarly, in auction theory literature, Bergemann & Pesendorfer (2007) and Eső & Szentes (2007) derive the optimal auction in environments in which the auctioneer controls how precisely the bidders learn their valuations. In both these works the degree of uncertainty left to the buyers is an endogenous variable, and in Eső & Szentes (2007) the optimal mechanism design entails selling access to more precise information.

[7]Seminal contributions to this literature are Grossman & Hart (1980), Grossman (1981), Milgrom & Roberts (1986), Rayo & Segal (2010), Kamenica & Gentzkow (2011). See Milgrom (2008) for a survey.

[8]Exceptions are Anderson & Renault (2006) and Koessler & Renault (2012) in the persuasion game literature; Ganuza (2004), Board (2009) and Jewitt & Li (2012) in auction theory; Sun (2011) and Celik (2012) in industrial organization. Ganuza (2004), Sun (2011), Celik (2012) and Jewitt & Li (2012) model horizontal differentiation by using Hotelling or Salop models with linear or quadratic transportation costs.

[9]Courty & Li (2000) characterize the optimal selling mechanisms with refunds in settings where each buyer's type is associated with a conditional probability distribution over valuations and the ranking of the types corresponds to a ranking of the conditional distributions in terms of first order stochastic dominance or mean-preserving spread. Zhang (2008) considers an optimal auction problem with refunds.

the informed principal problem and characterize individual rationality and incentive compatibility constraints for principal and agent. Then, in Section 5 we solve for a general optimal selling scheme and provide ways to implement the optimal mechanism for the specific cost functions. Section 6 concludes.

2 The model

There are two players, the principal (or the seller) and the agent (the buyer). The principal sells a good or service which the agent wants to purchase. There are two sources of incomplete information. First, the exact characteristics of the good are known to the principal but not fully known to the agent. Second, the agent's consumption utility (conditional on specifics of the good) is not fully known to the principal. In addition, the nature of the principal's information is such that no possible realization of such information (the principal's type) is a priori better than some other realization for all possible types of the agent. That is, the types of the principal are horizontally differentiated (e.g., by taste) rather than vertically (e.g., by quality). Such an environment can be conveniently modeled by a Hotelling-like model. For convenience, we would refer to the principal as she and to the agent as he.

The information of both the principal and the agent can be represented as a location on a line, with s denoting the location of the principal (the seller), and x – the one of the agent. Both the principal and the agent are risk neutral. The utility of the agent located at x from purchasing the good from principal s at price p is

$$U_A(x, s) = V - c(|x - s|) - p,$$

where V is the base value of the transaction, p is the price, and $c(\cdot)$ is the cost function specifying the loss of consumption value to the agent from the difference in the ideal and the actual characteristics of the good, $c(0) = 0$, and c is strictly increasing. Without loss of generality the cost of producing the good to the principal is set to be 0, and so the principal's utility from the transaction equals p. The outside participation values for both the principal and the agent are assumed to be 0.

We are going to assume that the principal's type can take one of the two values $s \in \{0, 1\}$ with equal probabilities, while the agent's type can take a continuum of values $x \in [0, 1]$ and is drawn from a uniform distribution. Accordingly, one can interpret this setup as the principal facing a continuum of agents of a unit mass, uniformly located over the segment. Thus, each of the players knows his own type, while types are independently drawn and the distributions are commonly known.

The interaction between the principal and the agent proceeds as follows. The principal offers a mechanism. The agent decides weather to participate in it. If he participates, both players play by the rules of the mechanism to generate an outcome (q, p) consisting of the quantity q to be transacted (possibly random) and the transfer p from the agent to the principal. If the agent refuses the mechanism, no transaction will take place and the utility of each player is 0. The equilibrium notion we use throughout the paper is Bayesian-Nash equilibrium.

We are going to assume that the nature of the principal's information is such that it is not verifiable to the agent, so that each type of the principal can pretend to be any other type. In particular, no type of the principal has actions available solely to her that can be used to credibly convey her information to the agent without regard to incentive constraints, and the agent cannot learn the value of the good to him ex interim. Any credible information communication must be incentive compatible. Assuming that the information is not verifiable ensures that the same set of mechanisms is available to each type of the principal.

3 Revenue maximizing mechanisms under specific information structures

In this section we derive revenue maximizing mechanisms for the principal under restriction that she has only two information revelation disclosure options available: fully reveal her private information or not reveal any information at all. We do so to establish a benchmark against which to compare optimal mechanism. We will show that often the optimal mechanism implies an intermediate disclosure policy, directed at specific types of the agent and so non-public. This benchmark is also natural as these two extreme disclosure policies show prominently in existing literature on regulation and information disclosure.

Mathematically, we are going to solve the revenue maximization problem for the principal from the ex ante perspective assuming that the principal can commit to and exercise only the two extreme disclosure policies and then, given the chosen disclosure policy, can offer a selling mechanism to the agent without any restrictions. We bypass ex interim incentive constraints for the principal completely, as any careful treatment of them would require a formal setup of the informed principal problem, which is done in the next section.[10]

Under full and truthful revelation of the principal's type we have the classic bilateral trade setting with one-sided incomplete information. Following Myerson (1981) approach (see also Krishna (2002)), by the Revelation principle it suffices to limit the search for the optimal selling mechanism to incentive compatible direct mechanisms. We obtain

[10] One may think that a more natural benchmark would be to consider selling mechanisms with general public disclosure policies, for instance, considering games in which the principal first makes an announcement and then offers a mechanism. Without any restrictions on the mechanisms offered, such a problem is not a well-defined one, as information may also be communicated via mechanisms offered. The formal setup of the optimization problem under public information disclosure restriction would require formalization of the public disclosure constraint on the offered mechanism, and the informed principal problem structure overall.

Proposition 1 *When the principal's type $s = 0$ is common knowledge and the virtual valuation function $\Psi(x) = V - c(x) - c'(x)x$ is strictly increasing, the optimal mechanism is*

$$\mu^{ck} = \begin{cases} q(x) = 1, \ p(x) = V - c(x^*), & \text{for } x \in [0, x^*], \\ q(x) = 0, \ p(x) = 0, & \text{for } x \in (x^*, 1], \end{cases}$$

where x^* is such that $\Psi(x^*) = 0$. If $\Psi(1) \geq 0$, then $q(x) = 1, \ p(x) = V - c(1)$ for all x.

Proof. For any incentive compatible direct mechanism μ consisting of a pair of functions (q, p), where $q(z)$ is the probability of the sale (the allocation function) and $p(z)$ is the expected payment of the agent given the agent's report z, we can express its expected revenue to the seller as

$$ER = -U(1) + \int_0^1 q(x)\Psi(x)dx. \quad (1)$$

It is optimal to set $q(x) = 1$ whenever $\Psi(x) \geq 0$, and $q(x) = 0$, otherwise. ■

Thus, the optimal mechanism under full revelation disclosure policy, μ^r, is the combination of μ^{ck} for $s = 0$ and its mirror image for $s = 1$. It can be implemented through the sale of the good at a posted price $P^r = V - c(x^*)$.

Certainly, the function $c(x)$ can be such that $\Psi(x)$ is not monotone, can cross 0 several times, and so $q(x)$ defined in the proof may not be non-increasing. But in this case (which corresponds to the case where SOC condition for maximization of $x(V - c(x))$ does not hold globally), the familiar ironing technique should be used (or the appropriate global maximum should be chosen), see Myerson (1981).

The simplest way to solve for the optimal selling scheme under non-disclosure is to reorder the agent's types according to their expected costs. For each type x of the agent we can assign type $y(x) = \frac{1}{2}c(x) + \frac{1}{2}c(1 - x)$. Letting $y_{\min} = \min_{x\in[0,1]} \left\{\frac{1}{2}c(x) + \frac{1}{2}c(1 - x)\right\}$ and $y_{\max} = \max_{x\in[0,1]} \left\{\frac{1}{2}c(x) + \frac{1}{2}c(1 - x)\right\}$, we have that the new types of the agent belong to $[y_{\min}, y_{\max}]$ (ergo $Y \sim [y_{\min}, y_{\max}]$), the utility from purchasing the good to the agent of type y net of the price is $V - y$, and the distribution of the types Y, F_Y, is given by $F_Y(y) = \Pr(Y < y) = \Pr\left(\frac{1}{2}c(x) + \frac{1}{2}c(1 - x) < y\right)$.

Again, by the Revelation principle we can limit our attention to direct mechanisms. Let $\hat{q}^{nr}(y) = q^{nr}(x)$ to be the probability that type $y = y(x)$ is getting the good and $\hat{p}^{nr}(y) = p^{nr}(x)$ be the payment of type y.

Proposition 2 *When the principal's type is not revealed and $\widehat{\Psi}(y) = V - y - \frac{F_Y(y)}{f_Y(y)}$ is strictly monotonic, the optimal mechanism is*

$$\mu^{nr} = \begin{cases} \hat{q}(y) = 1, \ \hat{p}(y) = V - y^*, & \text{for } y \in [y_{\min}, y^*], \\ \hat{q}(y) = 0, \ \hat{p}(y) = 0, & \text{for } y \in (y^*, y_{\max}], \end{cases}$$

where y^* is such that $\widehat{\Psi}(y^*) = 0$. If $\widehat{\Psi}(y_{\max}) \geq 0$, then $\hat{q}(y) = 1, \ \hat{p}(y) = V - y_{\max}$ for all y.

Proof. An equivalent of (1) is

$$ER = -U(y_{\max}) + \int_{y_{\min}}^{y_{\max}} \hat{q}(y)\widehat{\Psi}(y) f_y(y)dy. \tag{2}$$

Expected revenue is maximized by setting $\hat{q}(y) = 1$ whenever $\widehat{\Psi}(y) \geq 0$. ■

Mechanism μ^{nr} can be implemented by selling the good at a posted price $P^{nr} = V - y^*$.

We now compare the optimal mechanisms under full and no revelation disclosure policies under different assumptions regarding the cost function and, for each case, we determine when it is optimal for the principal to fully reveal her type and when it is not.

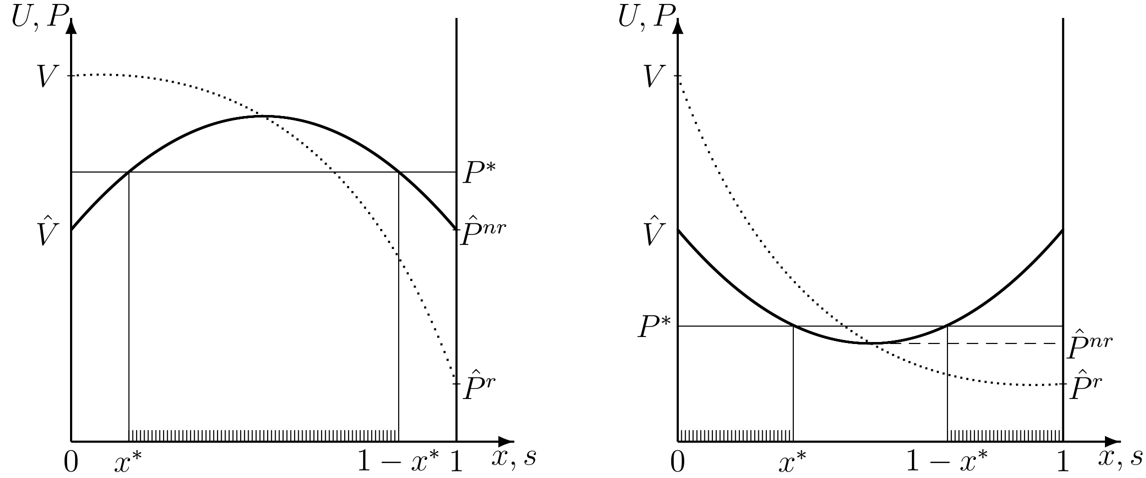
Lemma 1 *If the cost function is linear $c(x) = cx$, then the revenue maximizing mechanism for the principal is μ^{nr} when $V > 2c - c\sqrt{2}$, and μ^r otherwise.*

Proof. If no information is revealed, as $\frac{1}{2}c(x) + \frac{1}{2}c(1-x) = \frac{c}{2}$, all the agents have the same expected utility from the good: the variable Y has a degenerate density distribution f_Y with probability one on the value $y = \frac{c}{2}$. Thus, as long as $V - \frac{c}{2} > 0$, it is optimal for the principal to set $P^{nr}(x) = V - \frac{c}{2}$ and serve the whole market for a profit of $\pi^{nr} = V - \frac{c}{2}$ to each type of the principal. If the principal reveals her private information, then the optimal cut-off type is $x^* = \frac{V}{2c}$ or $x^* = 1$ if $V > 2c$. The price and the profit are $P^r = V - cx^* = \frac{V}{2}$ and $\pi^r = \frac{V^2}{4c}$ if $V \leq 2c$, or $P^r = \pi^r = V - c$ if $V > 2c$. By comparing $\frac{V^2}{4c}$ and $V - \frac{c}{2}$, we obtain the result. ■

For the case of convex costs, i.e. strictly increasing $c'(x)$, we have $y(x) = \frac{1}{2}c(x) + \frac{1}{2}c(1-x)$ is decreasing on $x \in \left[0, \frac{1}{2}\right]$ as $y'(x) = \frac{1}{2}\left(c'(x) - c'(1-x)\right) < 0$. Thus, $y_{\min} = c\left(\frac{1}{2}\right)$ and $y_{\max} = \frac{1}{2}c(1)$. If $V > y_{\min}$, then $\widehat{\Psi}(y_{\min}) > 0$, and so a positive revenue can be earned if the principal reveals no information. In the optimal mechanism under no revelation disclosure policy, the principal sets up a price $P^{nr} > V - y_{\min}$, in which case agent's types in the middle of the segment (with $y(x)$ close to $y_{\min}$) purchase the good, while those who are at the edges may be left out.

For the case of concave costs, i.e. strictly decreasing $c'(x)$, we have $y(x)$ is increasing on $x \in \left[0, \frac{1}{2}\right]$, $y_{\min} = \frac{1}{2}c(1)$, and $y_{\max} = c\left(\frac{1}{2}\right)$. The difference with the case of convex costs is that for the intermediate values of V — when the optimal price is in between $y_{\min}$ and $y_{\max}$ — only the agent's types at the edges, in the intervals $[0, x^*]$ and $[1 - x^*, 1]$, buy the good.

The expected valuations of the agent, and possible optimal selling schemes for the concave and convex costs cases are shown in Figure 1. Here the solid curve is the expected value of the agent under no revelation disclosure policy. The dotted curve is the expected value of the agent if the type of the seller is $s = 0$ and known. If the base value V is in the intermediate range, the typical optimal solution under no revelation is to set price $P^{nr} = P^*$, $\hat{V} = V - \frac{1}{2}c(1) < P^* < V - c\left(\frac{1}{2}\right)$, in which case the agents of types in spiked regions buy the good. Note that if V is sufficiently



a) Convex costs b) Concave costs

Figure 1: Revenue maximizing mechanisms under no revelation disclosure policy

high, then $\widehat{\Psi}(y) > 0$ for all $y \in [y_0, y_1]$ and $\Psi(x) > 0$ for all $x \in [0, 1]$, in which case in the optimal mechanism all the agents buy the good, no matter which mechanism, with full or no revelation, is selected. The profit from not revealing any information, $\pi^{nr} = P^{nr} = V - y_{\max} = V - \frac{1}{2}c(1)$, is higher than the one obtained when the information is fully revealed, $\pi^r = P^r = V - c(1)$. Prices $\hat{P}^{nr}$ and $\hat{P}^r$ in Figure 1. are the optimal prices under no revelation and under full revelation disclosure policies, respectively, when V is sufficiently high.

Clearly, for both concave and convex costs cases, if V is relatively small, and in particular if $V \leqslant y_{\min}$, it is better to reveal the information, and if V is sufficiently high, it is better not to reveal any information. We can state a general result for arbitrary cost functions.

Proposition 3 *For any cost function $c(x)$ there exists a threshold value V^*, such that if the base consumption value V is low, $V < V^*$, then it is better for the principal to reveal her information, and if the base consumption value is high, $V > V^*$, then it is better not to reveal anything.*

Proof. See Appendix A.1. ■

4 The Informed Principal Problem

4.1 Inscrutability Principle

The characterization of the optimal mechanism under incomplete information and without restrictions on disclosure policy is a complex exercise, as we have multiple types of the principal and assume that the principal selects the mechanism after learning her true type. Indeed, each type of the principal wants to maximize her own revenue. By offering a specific mechanism the principal may try to influence beliefs of the agent about her type in the way that is more profitable for her. On the other hand, given the mechanism offered, the agent may reason about which type of the principal have offered the mechanism and adjust his behavior accordingly. The principal can still commit to the rules of the mechanism she is offering but cannot commit or force the agent to believe that she is of a specific type or that she would have offered a specific mechanism if she were some other type.

How to deal with the mechanism selection issue and alignment of simultaneous objectives of all of the principal's types is the heart of the informed principal problem.

By the Inscrutability Principle (see Myerson (1983)) we can always represent the menu of the mechanisms offered by the principal (depending on her type) as a single inscrutable mechanism, in which agents infer nothing about the type of the principal when the mechanism (and its equilibrium) is offered. The rationale behind the Inscrutability Principle is based on the observation that any information transmitted by the principal through the selection of a specific mechanism can be conveyed through the application of specific rules inside a more general mechanism. By the Revelation Principle, for any such inscrutable mechanism and its equilibrium, there exists a direct inscrutable mechanism with truthtelling as an equilibrium.

Accordingly, we can limit our search of the optimal incentive scheme for all types of the principal to the set of inscrutable direct mechanisms. An inscrutable direct mechanism is a function $\mu : (s, x) \rightarrow (q, p)$ that maps a report s from the principal and a report x from the agent into a tuple composed of the (possibly random) traded quantity q and the transfer p paid by the agent to the principal. Direct inscrutable mechanism μ is *interim incentive compatible (IC)* if each type of each player is willing to report her or his type truthfully given that the other player reports his or her type truthfully and *interim individually rational (IR)* if each type of each player is willing to participate in it. Notice that, in particular, interim IC and IR constraints have to hold for all types of the principal for the mechanism to be inscrutable: a mechanism cannot be expected to be selected by all the principal's types if it is not individually rational or incentive compatible for some of them. In our setup an inscrutable direct mechanism can be represented by a collection of functions $(Q_0, P_0; Q_1, P_1)$, where for all $s \in \{0, 1\}$, $Q_s(x)$ is the probability of sale and $P_s(x)$ is the expected payment of the agent reporting x when the reported type of the principal is s. Due to feasibility, it has to be that $0 \leq Q_s(x) \leq 1$ for any x and s.

4.2 IC and IR constraints of the agent

Letting $U(z|x)$ stand for the expected utility of the agent of type x reporting z and assuming all other players' types report their types truthfully, we have: $\forall x, z \in [0,1]$,

$$\begin{aligned} U(x) &\triangleq U(x|x) = \frac{1}{2}q_0(x)(V - c(x)) + \frac{1}{2}q_1(x)(V - c(1-x)) - p(x) \\ &\geq U(z|x) = \frac{1}{2}q_0(z)(V - c(x)) + \frac{1}{2}q_1(z)(V - c(1-x)) - p(z); \qquad (3) \\ U(x) &\geq 0, \qquad (4) \end{aligned}$$

where $q_s(x) = Q_s(x)$ and $p(x) = \frac{1}{2}P_0(x) + \frac{1}{2}P_1(x)$ for any x and any s (always assuming that s is a truthful report). Notice that, due to the inscrutability of the mechanism, the agent cannot distinguish principal's types and does not receive any information that allow him to update its prior over the principal's types. This implies that, from the agent prospective, $\Pr(s=0) = \Pr(s=1) = \frac{1}{2}$. Combining IC constraints for x and z, we can write $[U(x) - U(x|z)] \geq [U(z|x) - U(z)]$ and obtain

$$(q_0(x) - q_0(z))(c(z) - c(x)) + (q_1(x) - q_1(z))(c(1-z) - c(1-x)) \geq 0. \qquad (5)$$

Unlike the complete information case, one cannot establish monotonicity of $q(x)$, since for $z > x$, $c(z) > c(x)$ and $c(1-z) < c(1-x)$. However, if $q_0(x)$ is constant at around some x, then $q_1(x)$ is increasing; and if $q_1(x)$ is constant at around x, then $q_0(x)$ is decreasing. Intuitively, in any incentive compatible mechanism there is a pressure to sell the good from principal $s=0$ (or $s=1$) more often to agents closer to 0 (or 1).

As in the classic complete information case we obtain that U is differentiable almost everywhere and, when at x the derivative exists,

$$U'(x) = -\frac{1}{2}q_0(x)c'(x) + \frac{1}{2}q_1(x)c'(1-x). \qquad (6)$$

In turn, we can express

$$\begin{aligned} U(x) &= U(0) - \frac{1}{2}\int_0^x q_0(t)c'(t)dt + \frac{1}{2}\int_0^x q_1(t)c'(1-t)dt, \qquad (7) \\ p(x) &= \frac{1}{2}q_0(x)(V - c(x)) + \frac{1}{2}q_1(x)(V - c(1-x)) \\ &\quad - U(0) + \frac{1}{2}\int_0^x q_0(t)c'(t)dt - \frac{1}{2}\int_0^x q_1(t)c'(1-t)dt. \qquad (8) \end{aligned}$$

4.3 IC and IR of the principal

To describe the incentive constraints for the principal It is convenient to view the strategic situation between principal and agent as happening on two levels: a *game* and a *meta-game* ones.

The *game level* represents the interaction inside a given mechanism. A direct inscrutable mechanism μ is incentive compatible for the principal if she is better off by reporting her true type s than by reporting some alternative s', given all the agents report their type truthfully. Letting $ER_s(\mu; s')$ to denote expected revenue of the principal of type s submitting a report s' in a direct inscrutable mechanism μ that is IC and IR for the agent, incentive compatibility condition for the principal takes the form

$$\text{IC} \qquad \forall s, s' \neq s, \quad ER_s(\mu) \triangleq ER_s(\mu; s) \geq ER_s(\mu; s').$$

The *meta-game level* formalizes the principal's choice over different mechanisms. The principal's ability to choose over different mechanisms entails a new specification for her individual rationality constraint. Traditionally, the individual rationality constraint captures the choice of a player to "opt out of the game." In a conventional setting, the principal is committed to a mechanism, and her individual rationality constraint dictates that the utility she is expected to receive from that mechanism has to be higher than the utility associated with an exogenously defined outside option. This applies to an informed principal setting too. In our context, however, the principal can choose among many mechanisms, and so her choice of the specific mechanism means that she believes that she cannot obtain a higher utility from offering any other mechanism.

Accordingly, we specify the IR constraint of the principal considering two different utility levels, and we require that any solution to the informed principal problem should give each type of the principal no less than the highest of the two. The first utility level is the one associated with the outside option, as in standard principal-agent models. We assume that it is exogenous and we allow it to be type-specific. The second utility level is the maximal payoff each type of the principal can obtain by offering a different inscrutable mechanism.

Letting $\overline{U}_s$ stand for the outside option utility of the principal of type $s = 0, 1$, the individual rationality of the principal for the direct interim incentive compatible inscrutable mechanism μ requires

$$\begin{aligned} &\text{IR(i)} && \forall s : ER_s(\mu) \geq \overline{U}_s, \\ &\text{IR(ii)} && \nexists \mu' \in \mathcal{M},\ \forall s : ER_s(\mu') > ER_s(\mu), \end{aligned}$$

where $\mathcal{M}$ is the set of direct inscrutable mechanisms that are interim IC and IR for the agent and interim IC for the principal.

Our way to set up the principal's individual rationality constraints is different from previous literature because of the presence of the second type of the constraint. By its nature, such constraint is endogenous: the maximum expected revenue that each type of the principal can guarantee herself at the meta-game level depends on the set of mechanisms that can be credibly selected by all the types of the principal; what can be credibly selected by the different types of the principal is in turn determined by what is each type's equilibrium strategy. The role of IR(ii) is to make sure that the solution of the informed principal problem is undominated in the sense of Myerson (1983): there is no alternative feasible inscrutable mechanism that all types of the principal would prefer instead.[11]

The set $\mathcal{M}$ is non-empty as it includes mechanisms μ^r and μ^{nr} introduced in Section 3 and presented as inscrutable mechanisms: with allocation and payment functions conditional on the principal's type. It is trivial to see that IC constraint for the principal holds for these two mechanisms.

In our symmetric setting, the incentive compatibility for the principal implies that unless one type of the principal chooses her outside option both types obtain the same payoff. In what follows we are going to assume that the revenue for the outside option $\overline{U}_s$ for each type s does not exceed $ER_s(\mu^r)$, and so the constraint IR(i) does not bind for all types of the principal as $\mu^r \in \mathcal{M}$.

4.4 The Informed Principal problem

Once defined the set of feasible inscrutable mechanisms in terms of incentive compatibility and individual rationality of both the agent and the principal, we set up the informed principal problem. After learning her type, the principal selects a mechanism in order to maximize her revenues (i.e. ex-interim optimal mechanism). As such, our solution is a collection of mechanisms: one for each type of the principal. Mechanism $\mu_s^* \in \mathcal{M}$ is the type s-optimal solution to the informed principal problem if it solves the following constrained maximization

$$\begin{aligned} &\max_{\mu \in \mathcal{M}} ER_s(\mu). \\ \text{s.t. } \nexists \mu' \in \ &\mathcal{M},\ \forall s : ER_s(\mu') > ER_s(\mu). \end{aligned} \tag{9}$$

It should be noted that Myerson (1983), Maskin & Tirole (1990), and Maskin & Tirole (1992) when defining what constitutes a solution to the informed principal problem set up the problem first and then define characteristics of specific kinds of solution, e.g. as being safe and undominated in Myerson (1983), weakly-interim efficient and/or Rothschild-Stiglitz-Wilson allocation in Maskin & Tirole (1992). Thus, it may happen that there exists a solution to the informed principal problem that does not have these characteristics. Sometimes, these additional characteristics are at odds with the way the problem is set up. For instance, a strong solution of Myerson (1983) is undominated and safe, where safe means that the agent would not change his action if he actually knew the true type of the principal.[12] The solution is still

[11] The same property defines the notion of durable decision rules in Holmström & Myerson (1983), and of renegotiation-proof allocation in Maskin & Tirole (1992).

[12] Notice that we can still use Myerson's solution concepts to interpret our solutions. For example, whenever μ^r is the optimal mechanism, such solution is a *strong solution*; mechanism μ^{nr}, whenever optimal, is a *core mechanism equilibrium.*

an inscrutable mechanism, but, in order to be characterized, its optimality is verified against agent's beliefs that would make the mechanism scrutable. In Maskin & Tirole (1992), when multiple equilibria arise,[13] the selection is supported by off-equilibrium beliefs that are punishing the principal in expectation following possible deviations, a conventional trick from signaling games to support equilibria by specifying adverse off-equilibrium beliefs. This is at odds with the ability of the principal to offer a mechanism and manipulate the agents' beliefs at the least to the extent of coordinating them on a specific equilibrium.

In our approach we formulate the informed principal problem by directly incorporating constraints into the solution concept, recognizing the principal's ability to manipulate beliefs and select equilibria on and off equilibrium path naturally constraining it by the incentive compatibility constrain of the principal. Thus, Pareto optimality of the solution becomes an implication of the individual rationality constraints of the principal (in a quasi-linear environment).

In order to solve the constrained maximization, we take advantage of the specific characteristics of the environment we consider. Consider the optimization problem

$$\max_{\mu \in \mathcal{M}} ER(\mu) = \sum_{s \in S} \Pr(s) ER_s(\mu), \tag{10}$$

where $\Pr(s)$ is the prior probability of type s, and S is the set of the principal's types. In our setting, $S = \{0, 1\}$ and $\Pr(0) = \Pr(1) = \frac{1}{2}$. We show that, in an environment with two symmetric and unverifiable types of the principal, the solution to (9) and the solution to (10) coincide.

Proposition 4 *Let μ_s^* be the solution to (9) and μ^* be the solution to (10). Then,*

$$\forall t \in S, \qquad ER_s(\mu_s^*) = ER(\mu_s^*) = ER(\mu^*).$$

Proof. IC constraint for the principal implies: $ER_s(\mu_s^*) = ER_t(\mu_s^*)$ and $ER_s(\mu^*) = ER_t(\mu^*)$ for all $t, s \in S$. But then, if $ER(\mu^*) > ER(\mu_s^*)$ it would contradict s-optimality of μ_s^* as μ^* is also available for s to choose. ■

Proposition 5 *The solution to problem (10) exists.*

Proof. The proof is rather straightforward. Set $\mathcal{M}$ is convex, closed, and non-empty. Accordingly, the set of feasible payoffs available to different types of the principal is bounded from above and is a compact. Thus, there exists a maximum. ■

The two propositions together imply that the same mechanism is s-optimal for all types of the principal and that it is the solution to the maximization problem (10).

[13]This happens when RSW allocation is not interim efficient, Maskin & Tirole (1992, p. 92).

4.5 Information transmission within a mechanism

What information is revealed or has to be revealed by the principal in the optimal selling mechanism? The solution is derived as an inscrutable mechanism that does not reveal any information to the agent when proposed by the principal. But this does not mean that no information is revealed to the agent *within the mechanism.*

One way to formalize the analysis of information transmission within a mechanism is to measure the information embedded into the allocation function of that mechanism. In other words, we want to determine how much the agent learns about the type of the principal, in the moment in which he is communicated that the good is or is not allocated to him.

Specifically, let ξ^0 be the prior probability distribution over the principal's types from some general set of types S (in our case $S = \{0, 1\}$), so ξ_s^0 is the probability that the principal is of type $s \in S$ (in our case $\xi_s^0 = \frac{1}{2}$ for each s). Then, for a mechanism with allocation functions $\{Q_s(x)\}_{s \in S}$, for any agent x, such that $\max_{s \in S} Q_s(x) > 0$, define the posterior probability distributions $\xi(x)$ and $\zeta(x)$, as

$$\forall s \in S, \qquad \xi_s(x) = \frac{\xi_s^0 Q_s(x)}{\sum_{j \in S} \xi_j^0 Q_j(x)}, \tag{11}$$

$$\forall s \in S, \qquad \zeta_s(x) = \frac{\xi_s^0 (1 - Q_s(x))}{\sum_{j \in S} \xi_j^0 (1 - Q_j(x))}. \tag{12}$$

Let $\xi(x) = \bar{\xi}$ and $\zeta(x) = \bar{\zeta}$ — special symbols standing for indeterminate distributions — for all agents who do not buy anything, that is, for x with $\max_{s \in S} Q_s(x) = 0$. Then $\xi(x)$ captures what agent x learns or, equivalently, in comparison to ξ^0, what information is being revealed to agent x within the mechanism when he is allocated the good, and $\zeta(x)$ captures the information revealed to agent x within the mechanism when he is not allocated the good.

If $\xi(x)$ is concentrated at some s^*, that is, $\xi_{s^*}(x) = 1$ and $\xi_s(x) = 0$ for $s \neq s^*$, then, as soon the agent of type x is announced the allocation of the good, he knows with certainty that the good is from the principal of type s^*. If $\xi(x) = \xi^0$, the agent learns nothing. If $\xi(x) = \bar{\xi}$, what the agent does learn is irrelevant, as he does not consume anything. Then, if there exist a type x, such that $\xi(x) \neq \xi^0$ and $\xi(x) \neq \bar{\xi}$, we can say that some information is being revealed within the mechanism when the good is allocated to x. Similar logic applies to $\zeta(x)$. Moreover, if, for $x \neq x'$, $\xi(x) \neq \xi(x')$ with $\xi(x) \neq \bar{\xi}$, $\xi(x') \neq \bar{\xi}$ and/or $\zeta(x) \neq \zeta(x')$ with $\zeta(x) \neq \bar{\zeta}$, $\zeta(x') \neq \bar{\zeta}$, we can say that different information is being communicated to agents x and x'.

This notion of *information transmitted within the mechanism* allows us to describe mechanisms and characterize the solution to the informed principal problem in a different dimension compared to the previous literature. Inscrutability principle is crucial to set up and solve the informed principal problem but it also completely shuts down the conventional information transmission channel: the agent's beliefs must be unaffected when mechanism is offered in equilibrium. In our approach, inscrutability does not preclude information transmission, instead different inscrutable mechanisms are characterized by different within the mechanism information transmission. By solving for the optimal mechanism we show that the environment considered (e.g. the value of V, the shape of the cost function) determines which information can be communicated to different types of the agent within an inscrutable mechanism. In each case we determine which feasibility constraints arise. For example, under certain conditions, it may be that some information, when communicated (within the mechanism) to some types of the agents, cannot be hidden from other types. The solution to the informed principal problem becomes the mechanism that maximizes (10) through the optimal within the mechanism information transmission (out of the set of feasible alternatives).

5 Optimal mechanism

In this section we solve for the optimal mechanism. For any mechanism μ,

$$ER(\mu) = \sum_{s \in S} \Pr(s) ER_s(\mu) = \frac{1}{2} ER_{s=0}(\mu) + \frac{1}{2} ER_{s=1}(\mu) = \int_0^1 p(x)dx.$$

The expected revenue collected from the agent in the incentive compatible mechanism $q_0(x)$, $q_1(x)$, $p(x)$ is

$$\begin{aligned} ER &= \int_0^1 p(x)dx \\ &= -U(0) + \frac{1}{2}\int_0^1 q_0(x)(V - c(x))dx + \frac{1}{2}\int_0^1 q_1(x)(V - c(1-x))dx \\ &\quad + \frac{1}{2}\int_0^1 \int_0^x q_0(t)c'(t)dtdx - \frac{1}{2}\int_0^1 \int_0^x q_1(t)c'(1-t)dtdx. \end{aligned} \tag{13}$$

By changing the order of integration, we obtain

$$\begin{aligned} ER &= -U(0) + \frac{1}{2}\int_0^1 q_0(x)\left[V - c(x) + c'(x)(1-x)\right]dx + \\ &\quad + \frac{1}{2}\int_0^1 q_1(x)\left[V - c(1-x) - c'(1-x)(1-x)\right]dx. \end{aligned} \tag{14}$$

Letting x^* be a type with the lowest utility from the mechanism and using (7), we can express

$$U(0) = U(x^*) + \frac{1}{2}\int_0^{x^*} q_0(t)c'(t)dt - \frac{1}{2}\int_0^{x^*} q_1(t)c'(1-t)dt. \tag{15}$$

Therefore, finding the optimal mechanism boils down to maximizing

$$ER = -U(x^*) + \frac{1}{2}\int_0^{x^*} q_0(x)A(x) + q_1(x)C(x)\ dx + \frac{1}{2}\int_{x^*}^1 q_0(x)B(x) + q_1(x)D(x)\ dx \tag{16}$$

over x^*, $q_0(x)$, and $q_1(x)$ for all $x \in [0,1]$, subject to the agent's IR and IC constraints (5), and feasibility constraints $q_0(x) \in [0,1]$, $q_1(x) \in [0,1]$ for all $x \in [0,1]$, where[14]

$$\begin{array}{ll} A(x) = V - c(x) - c'(x)x, & C(x) = V - c(1-x) + c'(1-x)x, \\ B(x) = A(x) + c'(x), & D(x) = C(x) - c'(1-x). \end{array} \tag{17}$$

Note also that

$$A(x) = D(1-x), \quad C(x) = B(1-x). \tag{18}$$

Once $q_0(x)$ and $q_1(x)$ are set for all x, $p(x)$ is determined by (8) and the IR constraint $U(x^*) = 0$, the actual payments $P_0(x)$ and $P_1(x)$ can be chosen in arbitrary fashion as long as $p(x) = \frac{1}{2}(P_0(x) + P_1(x))$ and IC constraint for the principal holds. This can always be done, as one can set $P_0(x) = P_1(x) = p(x)$.

To solve the general problem of revenue maximization, we proceed as follows. We first guess one value (or a set of values) for x^*. Then we derive $q_0(x)$ and $q_1(x)$ that maximize the expected revenue (16) as an unconstrained problem. Even though functions A, B, C, D capture implications of local IC constraints (5), there is no guarantee that the solution derived satisfies global IC constraints (5) or the IR constraints (4) of types x other than x^*. We verify if any of such constraints are violated and, if so, we recompute $q_0(x)$ and $q_1(x)$ taking this into account. At the end, we check whether the solution can be improved if we were to start from any different x^*.

The following simple result allows us to search for optimal mechanisms among the symmetric mechanisms.

Proposition 6 *There exists a symmetric optimal mechanism.*

Proof. Consider a solution μ to problem (10). Since the setup is symmetric, mechanism $\mu' = \mu(1-x)$ and a symmetric mechanism $\mu'' = \frac{1}{2}\mu + \frac{1}{2}\mu'$ are also solutions. ■

[14] The functions A, B, C, D can be interpreted as follows. Function $A(x)$ is the virtual valuation or marginal revenue from selling the good located at $s = 0$ to the agent of type x assuming all the agent's types closer to $s = 0$ also purchase the good and $s = 0$ is known. Indeed, for price $P = V - c(x)$ and revenue $R = (V - c(x))x$, the marginal revenue is $MR = V - c(x) - c'(x)x$. Similarly, given $s = 1$, $D(x)$ is the marginal revenue under assumption that all agent's types from x to 1 buy the good. Function $C(x)$ can be interpreted as the lost marginal revenue from not selling the good located at $s = 1$ to types closer to 0 when $s = 1$ is known. Indeed, if the price is $P = V - c(1-x)$, then the types in the interval $[0, x]$ do not buy the good, and the lost revenue is $LR = (V - c(1-x))x$. A similar interpretation can be applied to function $B(x)$. Function $\frac{1}{2}(A(x) + C(x))$ represents the marginal revenue from selling the good to agent x under assumption that all agents from 0 to x also buy it in a setting where the agents do not know the seller's type.

Certainly, it is not a surprise that in a symmetric setup a symmetric optimal mechanism exists. Taking into account implications of symmetry allows us to have simpler proofs.[15] In what follows, given that we look for symmetric solutions and so as to avoid excessive formalization, we specify the mechanisms only on $x < \frac{1}{2}$. Their complete specification for all x can be easily inferred by symmetry.

Lemma 2 *Any symmetric mechanism $\mu \in \mathcal{M}$ satisfies*

$$\forall x \in [0,1], \quad q_0(x) = q_1(1-x), \tag{19}$$

$$\forall x \in \left[0, \frac{1}{2}\right], \quad q_0(x) \geq q_1(x). \tag{20}$$

Proof. Equality (19) is by definition, inequality (20) follows from IC constraints for types x and $1-x$. ■

Before we proceed with deriving optimal mechanisms for linear, convex, and concave cost functions we would like to comment on regularity and introduce some notation.

A complication arises when virtual valuations (marginal costs) are non-monotone.[16] For the sake of simplicity, we are going to make some regularity assumptions that are going to take different forms depending on the cost function.

We define x_A and x_C as solutions to $A(x_A) = 0$ and $C(x_C) = 0$, respectively. For arbitrary V and $c(x)$ these equations may not have a solution or have multiple solutions. We will be using x_A and x_C only when the solutions exist and are unique (on the interval of interest).

To ease the notation and for clarity, in the derivation and description of the optimal mechanism we will omit the specification of allocation and prices for threshold values or types. One can take either the left or the right limit for their values. We also occasionally use $\vec{q}(x)$ to denote an allocation pair, $\vec{q}(x) = (q_0(x), q_1(x))$. In describing the mechanisms, in most of the cases we will specify the payment function $p(x)$ as perceived by the agent instead of $P_0(x)$ and $P_1(x)$, as $p(x)$ is uniquely determined by $Q_0(x)$ and $Q_1(x)$, while $P_0(x)$ and $P_1(x)$ can be defined in multiple ways.

Finally, let

$$r(x) = q_0(x) - q_1(x). \tag{21}$$

Thus, we can express

$$q_0(x)A(x) + q_1(x)C(x) = r(x)A(x) + q_1(x)\left(A(x) + C(x)\right) \tag{22}$$

[15] We do not have to rely on symmetry when deriving the optimal mechanism, the methods are general and applicable in asymmetric settings as well.

[16] For example, when function $A(x)$ crosses 0 multiple times, unconstrained optimization under complete information of $s = 0$ would entail to set $q_0(x) = 1$ whenever $A(x) > 0$. However, such solution violates IC constraints as IC dictates that $q_0(x)$ is non-increasing in x. The conventional solution is to "iron $A(x)$ out," that is maximize $\int_0^1 A(x)q_0(x)\,dx$ while imposing non-increasing $q_0(x)$. Practically, this involves finding all x, at which $A(x)$ crosses 0 from above, and comparing revenues from setting each of these x's as threshold x^*.

and optimize (16) over the choice of $r(x)$ and $q_1(x)$ taking into account that $r(x) + q_1(x) \leq 1$ and $r(x) \geq 0$ (from Lemma 2).

5.1 Linear costs

Suppose that the cost function is $c(x) = cx$. First, we describe three possible mechanisms and then show that one of these three is the optimal one depending on V.

Two mechanisms that we had already seen are: no revelation mechanism μ^{nr},

$$\mu^{nr} = \begin{cases} Q_0(x) = Q_1(x) = 1, \quad p(x) = V - \frac{1}{2}c, \quad \text{for } x \in [0,1]; \end{cases}$$

and full revelation mechanism μ^r,

$$\mu^r(\hat{x}) = \begin{cases} Q_0(x) = 1,\ Q_1(x) = 0,\ p(x) = \frac{1}{2}(V - c\hat{x}), & \text{for } x \in [0, \hat{x}), \\ Q_0(x) = Q_1(x) = 0,\ p(x) = 0, & \text{for } x \in \left(\hat{x}, \frac{1}{2}\right), \end{cases}$$

where $\hat{x} < \frac{1}{2}$ is some threshold type.

A new mechanism, which we refer to as the information discrimination mechanism, is

$$\mu^{id}(\hat{x}) = \begin{cases} Q_0(x) = 1,\ Q_1(x) = 0,\ p(x) = \frac{1}{2}(V - c\hat{x}), & \text{for } x \in [0, \hat{x}), \\ Q_0(x) = Q_1(x) = 1,\ p(x) = V - \frac{1}{2}c, & \text{for } x \in \left(\hat{x}, \frac{1}{2}\right), \end{cases}$$

where $\hat{x} < \frac{1}{2}$ is some threshold type. In this mechanism, each type of principal allocates her good to all but the farthest of the agent's types. When considering the information transmission within the mechanism, this mechanism implies that different information is provided to different types of the agent, i.e. there is information discrimination. The agent's types in the middle of the Hotelling-line do not receive any within the mechanism information to update their prior over the principal's types and purchase the good from both types of the principal, at a price that extracts all their surplus. The agent's types at extremes of the segment receive complete information within the mechanism and purchase only when the good is from the preferred type of the seller.

Proposition 7 *For the case of linear costs the optimal mechanism is*

$$\mu^* = \begin{cases} \mu^{nr}, & \textit{for } V > c, \\ \mu^{id}(x_C), & \textit{for } \frac{c}{2} < V < c, \\ \mu^r(x_A), & \textit{for } V < \frac{c}{2}, \end{cases}$$

where $x_A = \frac{V}{2c}$ *and* $x_C = \frac{c-V}{2c}$.

Proof. Since we are looking for a symmetric mechanism, it suffices to specify $q_0(x)$ and $q_1(x)$ for $x \leq \frac{1}{2}$. Clearly, the agent of type $x < \frac{1}{2}$ obtains at least as much value from bundle $\vec{q}(\frac{1}{2})$ as type $x = \frac{1}{2}$ does. Thus, $U(x) \geq U\left(\frac{1}{2}\right)$ for all x, and so, in the optimal symmetric mechanism IR binds for $x^* = \frac{1}{2}$. One simple, yet logically important argument is that for maximization of (16) with $x^* = \frac{1}{2}$ and symmetry constraints (19) and (20) it is sufficient to maximize $\int_0^{\frac{1}{2}} q_0(x)A(x) + q_1(x)C(x)\ dx$ without regard to what happens on $x > \frac{1}{2}$ (while actually, (19) has to hold). Indeed, using (19) and (18), the second integral of (16) can be expressed as

$$\begin{aligned}\int_{\frac{1}{2}}^{1} q_0(x)B(x) + q_1(x)D(x)\ dx &= \int_{\frac{1}{2}}^{1} q_1(1-x)C(1-x) + q_0(1-x)A(1-x)\ dx \\ &= \int_{0}^{\frac{1}{2}} q_0(z)A(z) + q_1(z)C(z)\ dz,\end{aligned}$$

for the variable change $z = 1-x$. Thus, the second integral is exactly equal to the first one. Altogether, setting $U\left(\frac{1}{2}\right) = 0$, by Lemma 2 and (22), the optimal mechanism solves

$$\max ER = \int_0^{\frac{1}{2}} r(x)A(x) + q_1(x)(A(x) + C(x))\ dx \tag{23}$$

over $r(x)$ and $q_1(x)$ for $x \in \left[0, \frac{1}{2}\right]$, subject to feasibility and IC constraints for the agent.

To solve it, note that $A(x) = V - 2cx$, $C(x) = V + 2cx - c$, and $A(x) + C(x) = 2V - c$. When $V > c$, for all $x \in \left[0, \frac{1}{2}\right]$, $A(x) > 0$, $C(x) > 0$, so it is optimal to set $q_0(x) = q_1(x) = 1$.

When $V < c$, $A(x)$ and $C(x)$ cross 0 on $x \in \left[0, \frac{1}{2}\right]$, $x_A = \frac{V}{2c}$ and $x_C = \frac{c-V}{2c}$. Clearly, if $V < \frac{c}{2}$, then $A(x) + C(x) < 0$, $x_C > x_A$, and is optimal to set $q_1(x) = 0$ for all $x \in \left[0, \frac{1}{2}\right]$, $q_0(x) = 1$ for $x < x_A$, and $q_0(x) = 0$ for $x > x_A$. If $\frac{1}{2}c < V < c$, then $A(x) + C(x) > 0$, $x_A > x_C$, and it is optimal to set $q_1(x) = 1$ and so $\vec{q}(x) = (1,1)$ when $A(x) + C(x) > A(x)$, which happens for $x > x_C$, and set $r(x) = 1$ and so $\vec{q}(x) = (1,0)$ otherwise, for $x < x_C$. These two cases are shown at Figure 2, with $A(x) + C(x)$ represented by dotted lines.

All IC constraints hold. For mechanism $\mu^{id}(x_C)$, the expected payment function is: $p(x) = V - \frac{c}{2}$ for $x \in \left[x_C, \frac{1}{2}\right]$ and $p(x) = \frac{1}{2}(V - cx_C) = \frac{3V}{4} - \frac{c}{4}$ for $x \in [0, x_C)$. The latter is the price at which type x_C is indifferent between $\vec{q} = (1,0)$ and $\vec{q} = (1,1)$ at price $V - \frac{c}{2}$. ■

The gain in revenue from the optimal information discrimination mechanism for $\frac{c}{2} < V < c$ compared to the best alternative among the full revelation and no revelation mechanism can be as high as 50%. Indeed, when V is such that $\pi^{nr} = \pi^r$, then the profit π^{id} from $\mu^{id}(x_C)$ is such $\pi^{id} = \frac{3}{2}\pi^r$.

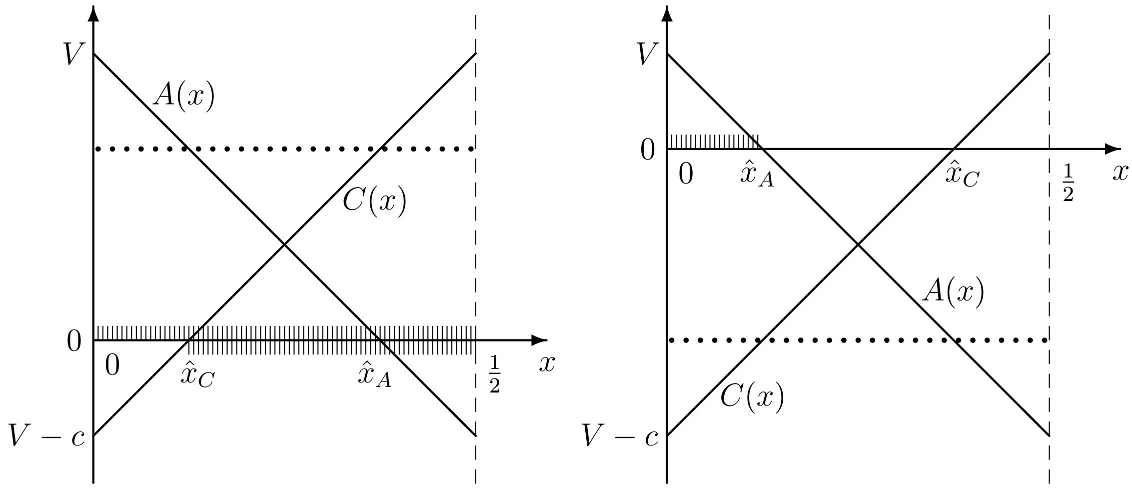


Figure 2: Linear costs optimal mechanism.

5.1.1 Linear costs: implementation of the optimal solution

Mechanisms μ^{nr} and $\mu^{r}(x_A)$ can be trivially implemented: sell the opaque good (without revealing anything) at price $P_{og} = V - \frac{1}{2}c$, and reveal own type and set a price $P = \frac{V}{2c}$, respectively. To implement $\mu^{r}(x_A)$, alternatives to the full information revelation disclosure mechanism exist. For example, the principal may sell information[17] about her type at price $P_i = \frac{1}{4}V - \frac{1}{2}P_g$ and offer to the agent who bought information the possibility to buy the good at an extra expense of P_g, where $0 \leq P_g \leq \frac{1}{2}V$. The choice of P_i and P_g affects where the revenue extracted by the principal comes from. If $P_g = 0$ and $P_i = \frac{1}{4}V$, the agent's types in the two intervals $\left[0, \frac{V}{2c}\right]$ and $\left[1 - \frac{V}{2c}, 1\right]$ pay the same even though, if the principal's type is $s = 0$, the types in $\left[1 - \frac{V}{2c}, 1\right]$ do not buy the good.

Mechanism $\mu^{id}(x_C)$ can be implemented in various forms. One possibility is a two-item menu: (1) the opaque good and (2) "the sale of information" — an option contract that gives a right to purchase the good after learning who is the principal. Another possibility is a different menu: (1) the good without a return option; (2) the good with a return and partial reimbursement option, provided that the agent is communicated the type of the principal before paying the "distance" costs.[18]

The implementing mechanisms can be constructed as follows. For convenience, suppose $c = 1$. Then, $\mu^{id}(x_C)$ is optimal for $\frac{1}{2} < V < 1$ with $x_C = \frac{1-V}{2}$. Considering the first possible implementation, the option entails two payments: the option (infor-

[17] Notice the information that the principal sells is not verifiable, but is credible (because of incentive compatibility).

[18] The exercise of the return and reimbursement right is not conditioned on the ability of the buyer to verifiably prove specific characteristics of the good (e.g. it is of type s and not s'). As such it may be different from other return options used in practice and studied in the literature.

mation) price P_i and the exercise price P_g at which the option holder is entitled to buy the good. The marginal type x_C has to be indifferent between the option and the opaque good at price $P_{og} = V - \frac{1}{2}$,

$$-P_i + \frac{1}{2}(V - x_C - P_g) = V - \frac{1}{2} - P_{og} = 0.$$

The option price P_i and the exercise price P_g are not uniquely defined. Only the overall expected revenue from the sale of the option is determined, $P_i + \frac{1}{2}P_g = \frac{3V-1}{4}$, and so there is a general multiplicity of solutions. In the second possible implementation, the price of the good with a possible return is $P_g + P_i$ with the reimbursement P_g.

External constraints may bring some limitations to which prices are feasible. For example, arbitrage opportunities arising from the possibility of buying information and then, in light of the information acquired, purchase the opaque good may be ruled out. This requires that $P_g \leq P_{og}$. When $P_g = P_{og} = V - \frac{1}{2}$, $P_i = \frac{1}{4}V$.

5.2 Concave costs

Suppose that the cost function is strictly concave, $c' > 0$ and $c'' < 0$ for all $x \in [0, 1]$. Then, $C'(x) > 0$, whereas $A'(x)$ has ambiguous sign. Also,

$$A'(x) + C'(x) = -2c'(x) + 2c'(1 - x) - c''(x)x - xc''(1 - x).$$

Note that for x close to $\frac{1}{2}$, $A(x) + C(x)$ is increasing, while for x close to 0 it is decreasing if $c''(x)$ is bounded. We assume that functions $A(x)$ and $A(x) + C(x)$ are regular, that is, $A(x)$ is strictly decreasing on $\left[0, \frac{1}{2}\right]$, and $A(x) + C(x)$ crosses the x-axis at most twice.

Here we introduce three mechanisms. The first two are modifications of no revelation and information discrimination mechanisms (i.e. μ^{nr} and μ^{id}) with "no sale" region in the middle. These mechanisms are defined as follows:

$$\mu^{nrns}(\hat{x}) = \begin{cases} Q_0(x) = Q_1(x) = 1,\ p(x) = p_{og}, & \text{for } x \in [0, \hat{x}) \\ Q_0(x) = Q_1(x) = 0,\ p(x) = 0, & \text{for } x \in \left(\hat{x}, \frac{1}{2}\right), \end{cases}$$

where $\hat{x} < \frac{1}{2}$ is a threshold, $p_{og} = V - \frac{1}{2}c(\hat{x}) - \frac{1}{2}c(1 - \hat{x})$; and

$$\mu^{idns}(\hat{x}_1, \hat{x}_2) = \begin{cases} Q_0(x) = 1,\ Q_1(x) = 0,\ p(x) = p_g, & \text{for } x \in (0, \hat{x}_1), \\ Q_0(x) = Q_1(x) = 1,\ p(x) = p_{og}, & \text{for } x \in (\hat{x}_1, \hat{x}_2), \\ Q_0(x) = Q_1(x) = 0,\ p(x) = 0, & \text{for } x \in \left(\hat{x}_2, \frac{1}{2}\right), \end{cases}$$

where $\hat{x}_1 < \hat{x}_2 < \frac{1}{2}$ are two thresholds, $p_{og} = V - \frac{1}{2}c(\hat{x}_2) - \frac{1}{2}c(1 - \hat{x}_2)$, and $p_g = p_{og} - \frac{1}{2}(V + c(1 - \hat{x}_1))$.

The last mechanism is a random allocation with information discrimination mechanism: it is a modification of mechanism μ^{id} with two intermediate regions with random allocation. In these regions only the good from the nearest type of the principal is sold and with a type-specific probability. See Figure 3 below for illustration.

$$\mu^{raid}(\hat{x}_1, \hat{x}_2) = \begin{cases} Q_0(x) = 1,\ Q_1(x) = 0,\ p(x) = p_g, & \text{for } x \in (0, \hat{x}_1), \\ Q_0(x) = d(x),\ Q_1(x) = 0,\ p(x) = p^*(x), & \text{for } x \in (\hat{x}_1, \hat{x}_2), \\ Q_0(x) = Q_1(x) = 1,\ p(x) = V - c\left(\frac{1}{2}\right), & \text{for } x \in \left(\hat{x}_2, \frac{1}{2}\right), \end{cases}$$

where $\hat{x}_1 < \hat{x}_2 < \frac{1}{2}$ are thresholds, $d(x) = 1 - \frac{c'(1-x)}{c'(x)}$, $p_g = \frac{1}{2}\left(V + c(1-\hat{x}_1) - 2c\left(\frac{1}{2}\right)\right)$, and $p^*(x) = \frac{1}{2}\left[V - 2c\left(\frac{1}{2}\right) + c(1-x) - \frac{c'(1-x)}{c'(x)}(V - c(x))\right]$.

In terms of within the mechanism information transmission, mechanism μ^{raid} entails a finer information discrimination than μ^{id}. As before, the agent's types at the extremes of the segment receive full information, and the ones in the middle do not receive any information. Now, however, the types located in intermediate areas $(\hat{x}_1, \hat{x}_2)$ and $(1-\hat{x}_2, 1-\hat{x}_1)$ receive a partial amount of within the mechanism information. In particular, they learn only some (not all) information about the principal's type when they do not receive the good.[19]

For any fixed pair (q_0, q_1) with $q_0 \geq q_1 \geq 0$ and price p, the utility of any type $x < \frac{1}{2}$ is higher than the utility of type $x = \frac{1}{2}$. Thus, in a symmetric optimal mechanism type $x^* = \frac{1}{2}$ will have the lowest utility and binding IR constraint. Then, as in the case of linear costs, to find the optimal mechanism under concave costs it suffices to solve (23) subject to global IC and feasibility constraints by comparing $A(x)$ and $A(x)+C(x)$ between themselves and with 0 for $x \leq \frac{1}{2}$. Before we proceed with stating the main proposition, we introduce some notation.

Let $x_{\min} = \arg\min_{x \in \left[0, \frac{1}{2}\right]} A(x) + C(x)$ and V^* be the value at which $A(x_{\min}) + C(x_{\min}) = 0$. Notice that for $V > c(1)$, we have $C(x) > 0$ for all $x < \frac{1}{2}$, and so $A(x) + C(x) > A(x)$; and for $V > V^*$, we have $A(x) + C(x) > 0$. Let $x_= < \frac{1}{2}$ be the type for which $A(x_=) = C(x_=)$ (such type clearly exist for all V, as $A(0) > C(0)$, $A\left(\frac{1}{2}\right) < C\left(\frac{1}{2}\right)$, and is independent of V). Let V^{AC} be the value V at which $A(x_=) = C(x_=) = 0$. Trivially, $V^{AC} < c(1)$ and $V^{AC} \leq V^*$.

Lemma 3 *It is that $c\left(\frac{1}{2}\right) < V^{AC} \leq \min\{V^*, c(1)\}$, when $c(x)$ is strictly concave.*
Proof. *The proofs of this and all subsequent Lemmata are in Appendix A.1.* ■

[19] By (11), we have that $\xi_0(x) = 1$ and $\xi_1(x) = 0$ for $x \in (0, \hat{x}_2)$, and $\xi_0(x) = \xi_1(x) = \frac{1}{2}$ for $x \in \left(\hat{x}_2, \frac{1}{2}\right)$. By (12), we have that $\zeta_0(x) = 0$ and $\zeta_1(x) = 1$ for $x \in (0, \hat{x}_1)$, $\zeta_0(x) = \frac{1}{1+\frac{1}{1-d(x)}}$ and $\zeta_1(x) = \frac{1}{2-d(x)}$ for $x \in (\hat{x}_1, \hat{x}_2)$, and $\zeta_0(x) = \zeta_1(x) = \frac{1}{2}$ for $x \in \left(\hat{x}_2, \frac{1}{2}\right)$. Therefore, in terms of within the mechanism information, the agent's types in the intervals $(0, \hat{x}_1)$ and $(\hat{x}_1, \hat{x}_2)$ differ only in terms of function $\zeta(x)$.

Let

$$W(x) = A(x) + \frac{c'(x)}{c'(1-x)} C(x). \quad (24)$$

Lemma 4 *For $c\left(\frac{1}{2}\right) < V < c(1)$, $W(x)$ crosses 0 at most once.*

Let x_W be the value of x at which $W(x) = 0$, or set $x_W = 0$ if $W(x) > 0$ for all $x < \frac{1}{2}$. Notice that at $V = V^{AC}$, $x_W = x_A = x_C = x_=$. Since $c(x)$ is concave, $\frac{c'(x)}{c'(1-x)} > 1$ for $0 < x < \frac{1}{2}$. Thus, $W(x) > A(x) + C(x)$ for any $x > x_C$, and $W(x) < A(x) + C(x)$ for any $x < x_C$. Therefore, x_W decreases as V increases.

Let

$$R^* = \int_{x_=}^{\frac{1}{2}} A(x) + C(x)dx \quad \text{at } V = V^{AC},$$

For $V < V^{AC}$, let

$$Z^{\#} = \int_{x_A}^{x_W} d(x)A(x)dx + \int_{x_W}^{\frac{1}{2}} A(x) + C(x)dx,$$

and let $V^{\#}$ be the value of V at which $Z^{\#} = 0$.

Lemma 5 *If $R^* > 0$, then $Z^{\#}$ is continuously increasing in V; $V^{\#}$ exists, is unique, and $c\left(\frac{1}{2}\right) < V^{\#} < V^{AC}$.*

If $R^* < 0$, then for $V = V^{AC}$ function $A(x) + C(x)$ crosses 0 twice, once at $x_= = x_A = x_C$ from above and at some $x \in \left(x_=, \frac{1}{2}\right)$ from below. For values $c\left(\frac{1}{2}\right) < V < V^*$, let x_{+1} be the type at which $A(x) + C(x)$ crosses 0 from above if $A(0) + C(0) > 0$, or set $x_{+1} = 0$, and x_{+2} be the type at which $A(x) + C(x)$ crosses 0 from below. Then, let $Z^{\#\#}$ be

$$Z^{\#\#} = \int_{x_{+1}}^{\frac{1}{2}} A(x) + C(x)dx,$$

and let $V^{\#\#}$ be the value for which $Z^{\#\#} = 0$.

Lemma 6 *If $R^* < 0$, then $Z^{\#\#}$ is continuously increasing; $V^{\#\#}$ exists, is unique, $V^{AC} < V^{\#\#} < V^*$.*

Proposition 8 *For the case of concave costs the optimal mechanism is*

$$\mu^* = \begin{cases} \left.\begin{array}{ll} \mu^{nr}, & \text{for } V > c(1), \\ \mu^{id}(x_C), & \text{for } V^{AC} < V < c(1), \\ \mu^{raid}(x_A, x_W), & \text{for } V^{\#} < V < V^{AC}, \\ \mu^{r}(x_A), & \text{for } V < V^{\#}, \end{array}\right| & \text{if } R^* > 0; \\ \left.\begin{array}{ll} \mu^{nr}, & \text{for } V > \max\{c(1), V^{\#\#}\} \\ \mu^{nrns}(x_{+1}), & \text{for } c(1) < V < V^{\#\#} \text{ if } c(1) < V^{\#\#}, \\ \mu^{id}(x_C), & \text{for } V^{\#\#} < V < c(1) \text{ if } V^{\#\#} < c(1), \\ \mu^{idns}(x_C, x_{+1}), & \text{for } V^{AC} < V < \min\{c(1), V^{\#\#}\}, \\ \mu^{r}(x_A), & \text{for } V < V^{AC}, \end{array}\right| & \text{if } R^* < 0. \end{cases}$$

Proof. To find the optimal mechanism under concave costs it suffices to solve (23) subject to global IC and feasibility constraints. As the integrand is linear in q_0 and q_1 (and in r and q_1), the solution that respects only the feasibility constraints is: $\vec{q}(x) = (1,0)$, when $A(x) > 0$ and $C(x) < 0$; $\vec{q}(x) = (1,1)$, when $A(x)+C(x) > 0$ and $C(x) > 0$; and $\vec{q}(x) = (0,0)$, when $A(x) < 0$ and $A(x)+C(x) < 0$. This unconstrained solution does satisfy IC constraints if $A(x)+C(x) > 0$ for all $x \in \left[0, \frac{1}{2}\right]$ (ergo $V > V^*$) or when $A(x)+C(x)$ crosses 0 only once on $x \in \left[0, \frac{1}{2}\right]$ and from above (ergo $V < c\left(\frac{1}{2}\right)$). This implies that the optimal solution is μ^{nr} if $V > V^*$ and $V^* > c(1)$; $\mu^{id}(x_C)$ if $V > V^*$ and $V^* < c(1)$; and $\mu^r(x_A)$ if $V < c\left(\frac{1}{2}\right)$.

For $c\left(\frac{1}{2}\right) < V < V^*$, function $A(x) + C(x)$ crosses 0 twice: first from above and then from below. Then, there may be intermediate types x for which $A(x) < 0$ and $A(x) + C(x) < 0$. For such situations, the unconstrained solution has to be modified to account for binding non-local IC constraints.

We proceed in two steps: (1) determine the implications of non-local IC constraints; and (2) derive the optimal solution.

Step 1. It is convenient to rewrite that incentive constraint (5) in terms of r and q_1. For $x < z \leq \frac{1}{2}$, it becomes

$$r(x) + q_1(x)\delta(x,z) \geq r(z) + q_1(z)\delta(x,z), \tag{25}$$

where

$$\delta(x,z) = 1 - \frac{c(1-x) - c(1-z)}{c(z) - c(x)}.$$

Clearly, $\delta(x,z) \in (0,1)$. Let $\delta(x,x) = 1 - \frac{c'(1-x)}{c'(x)}$ (and so $d(x) = \delta(x,x)$). Then, $\delta(x,z)$ is continuous and decreasing in both x and z for all $x < z \leq \frac{1}{2}$.

As some non-local incentive constraints will be binding in the optimal solution, one has to respect "local" implications of these non-local binding constraints as well. For the case of concave costs, for any incentive compatible allocation we have $U(x) \geq U(z)$ for $x < z \leq \frac{1}{2}$, that is utility increases when x decreases. Suppose that for some y and z, such that $y < z \leq \frac{1}{2}$, $U(y) = U(z|y)$. Global incentive compatibility implies that for all $x \leq \frac{1}{2}$, $U(x) \geq U(z|x)$. As $U(x)$ is continuous and differentiable almost everywhere, for all $x < y$ in the sufficiently small neighborhood of y we must have $U'(x) \leq U'_x(z|x)$, whenever $U'(x)$ exists. That is if we keep decreasing x (starting from y) the utility of truthtelling has to be increasing at least as fast as the utility of pretending to be z. Formally, from (6),

$$\begin{aligned} U'(x) &= -\frac{1}{2}\left[r(x)c'(x) + q_1(x)\left(c'(x) - c'(1-x)\right)\right] \leq \\ U'_x(z|x) &= -\frac{1}{2}\left[r(z)c'(x) + q_1(z)\left(c'(x) - c'(1-x)\right)\right]. \end{aligned}$$

Multiplying both sides by -2 and dividing by $c'(x)$, we obtain

$$r(x) + q_1(x)\delta(x,x) \geq r(z) + q_1(z)\delta(x,x). \tag{26}$$

Note that constraint (26) applies only locally, for x near y for which $U(y) = U(z|y)$. And, for instance, for such y and z, and $x = y$ it differs from (25), as $\delta(x,x) > \delta(x,z)$. In the optimal solution both of these constraints must hold.

Step 2. In deriving the optimal solution for $c\left(\frac{1}{2}\right) < V < V^*$, we consider first the case $c\left(\frac{1}{2}\right) < V < V^{AC}$, and ergo $x_A < x_C$. Clearly, we have $x_{+1} < x_A < x_= < x_C < x_{+2}$. The unconstrained solution is such that IC constraints are violated for $x \in (x_A, x_{+2})$. In order to satisfy these constraints, two are the candidates for the optimal solution: one in which we set $\vec{q}(x) = (1,1)$ for $x > x_{+2}$ and compute $\vec{q}(x)$ for $x \in (x_A, x_{+2})$ taking into account IC constraints (25) and (26); and the other in which we set $\vec{q}(x) = (0,0)$ for $x > x_{+2}$ and also for $x \in (x_A, x_{+2})$.[20]

Suppose $\vec{q}(x) = (1,1)$ or $(r(x), q_1(x)) = (0,1)$ for $x > x_{+2}$. The idea behind computation of $\vec{q}(x)$ for $x \in (x_A, x_{+2})$ can be described as follows. Since both $A(x)$ and $A(x)+C(x)$ are negative on $x \in (x_A, x_{+2})$ we are going to lose expected revenue. Therefore, we would like to sell as little as possible of $(r(x), q_1(x))$ for such x. Incentive constraints (25) and (26) restrict from below how much we "have to" sell. Suppose that the non-local IC binds (26) for such x and z, and ignore all other IC constraints. Then, for any $x \in (x_A, x_{+2})$ we have the following problem

$$\begin{aligned} \max_{r,q_1} \; & rA(x) + q_1\left(A(x) + C(x)\right) \\ \text{s.t.} \quad & r + q_1 d(x) \geq d(x), \\ & r, q_1 \geq 0,\; r + q_1 \leq 1. \end{aligned}$$

Substituting $r = d(x)\left(1 - q_1\right)$ into the objective, we obtain

$$A(x)d(x) + q_1\left(A(x) + C(x) - d(x)A(x)\right) = A(x)d(x) + q_1\frac{c'(1-x)}{c'(x)}W(x).$$

With only one variable to select, q_1, it is optimal to set $q_1(x) = 1$ and thus $r(x) = 0$ for $x \in (x_W, x_{+2})$ as $W(x) > 0$, and set $q_1(x) = 0$ and thus $r(x) = \delta(x,x)$ for $x \in (x_A, x_W)$ as $W(x) < 0$. (Since $x_A < x_C$, we have $x_W \in (x_C, x_{+2})$ as $W(x_C) = A(x_C) + C(x_C)$ and $W(x) > A(x) + C(x)$ for $x > x_C$.) It is straightforward to verify that all the other IC constraints hold for this candidate for optimal solution.

Now we compare this candidate solution with the alternative $(r(x), q_1(x)) = (0,0)$ for $x > x_A$. The difference in the expected revenue between them is

$$ER^*_{x>x_A} = \int_{x_A}^{x_W} d(x)A(x)\,dx + \int_{x_W}^{\frac{1}{2}} A(x) + C(x)\,dx. \tag{27}$$

Thus, if $ER^*_{x>x_A} > 0$, then the first solution is optimal. By Lemma 5, $ER^*_{x>x_A} = Z^\#$ is increasing in V. At $V = V^{AC}$, $R^* = ER^*_{x>x_A}$. If $R^* < 0$, then $ER^*_{x>x_A} < 0$ for all

[20] While it is true that only one of these two candidate solutions is the optimal one, this is not a completely trivial observation. For completeness, the formal argument showing optimality of the obtained solution is presented in Appendix A.2.

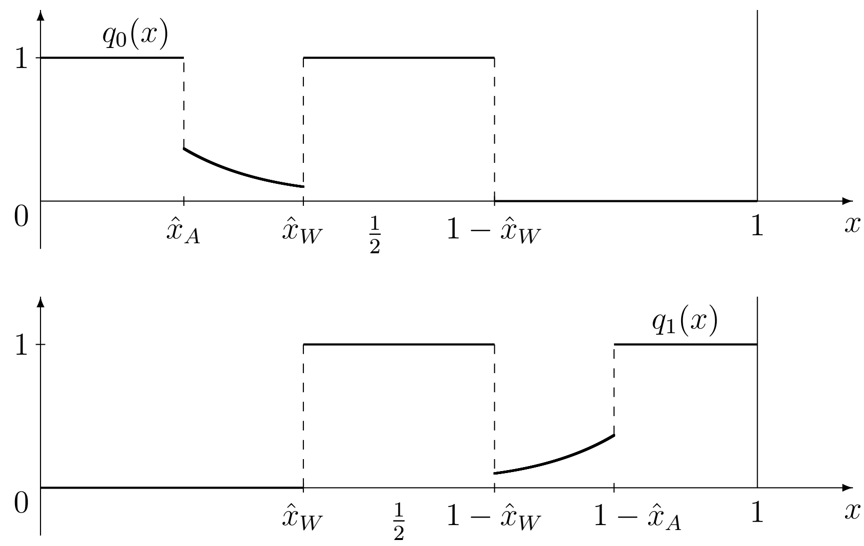


Figure 3: Mechanism μ^{raid}.

$c\left(\frac{1}{2}\right) < V < V^{AC}$, and the optimal mechanism is $\mu^r(x_A)$. If $R^* > 0$, then $\mu^r(x_A)$ is optimal for $c\left(\frac{1}{2}\right) < V < V^{\#}$, and $\mu^{raid}(x_A, x_W)$ is optimal for $V^{\#} < V < V^{AC}$.

The optimal mechanism for the subcase $R^* > 0$ and $V^{\#} < V < V^{AC}$ is presented in Figure 3. The corresponding expected surplus for all the agent's types is depicted in Figure 4.

The remaining subcase, with $V^{AC} < V < V^*$ can be considered in the same way. We have $x_A > x_C$ and $W(x) > 0$ for all $x > x_A$. If $x_{\min} < x_=$, then for all $x_A < x < \frac{1}{2}$, $A(x)+C(x) > 0$. If $V^* > c\,(1)$, the optimal mechanism is μ^{nr} for $c\,(1) < V < V^*$ and $\mu^{id}(x_C)$ for $V^{AC} < V < c(1)$. If $V^* < c\,(1)$, then the optimal mechanism is $\mu^{id}(x_C)$ for $V^{AC} < V < c(1)$. If $x_{\min} > x_=$, we have $x_W < x_A < x_{+1} < x_{+2}$. Then, the optimal solution either involves setting $\vec{q}(x) = (1,1)$ for $x > x_{+2}$, and due to binding global IC constraint optimally (as $W(x) > 0$) set $\vec{q}(x) = (1,1)$ for $x \in (x_{+1}, x_{+2})$, or set $\vec{q}(x) = (0,0)$ for all $x > x_{+2}$. For the rest of x, the solution is the same: $\vec{q}(x) = (1,0)$ for $x < x_C$ and $\vec{q}(x) = (1,1)$ for $x \in (x_C, x_{+1})$. Which option is better depends on $ER^*_{x>x_{+1}} = \int_{x_{+1}}^{\frac{1}{2}} A(x) + C(x)dx$. If $ER^*_{x>x_{+1}} > 0$ it is optimal to set $\vec{q}(x) = (1,1)$ for all $x_{+1} < x < \frac{1}{2}$, and set $\vec{q}(x) = (0,0)$ for these types otherwise. Notice that $ER^*_{x>x_{+1}} = R^*$ at $V = V^{AC}$. By Lemma 6, $ER^*_{x>x_{+1}} = Z^{\#\#}$ increases in V.

Thus, if $R^* > 0$, $ER^*_{x>x_{+1}} > 0$ for all $V > V^{AC}$. If $V^* > c\,(1)$, the optimal mechanism is μ^{nr} for $c(1) < V < V^*$ and $\mu^{id}(x_C)$ for $V^{AC} < V < c(1)$. If $V^* < c\,(1)$, the optimal mechanism is $\mu^{id}(x_C)$ for $V^{AC} < V < V^*$.

If $R^* < 0$, then the relevant comparison is between $V^{\#\#}$, V^* and $c\,(1)$. If $V^{\#\#} > c\,(1)$ (and therefore $V^* > c\,(1)$), then the optimal solution is μ^{nr} for $V^{\#\#} < V < V^*$ and μ^{nrns} for $V^{AC} < V < V^{\#\#}$. If $V^{\#\#} < c\,(1)$ and $V^* < c\,(1)$, then it is $\mu^{id}\,(x_C)$ for $V^{\#\#} < V < V^*$. If $V^{\#\#} < c\,(1)$ and $V^* > c\,(1)$, $\mu^{id}\,(x_C)$ for $V^{\#\#} < V < c\,(1)$

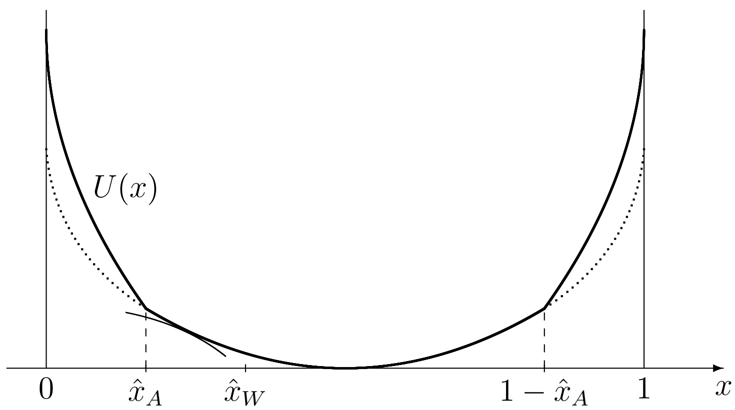


Figure 4: Expected surplus for mechanism μ^{raid}.

and μ^{nr} for $c(1) < V < V^*$. Finally, in both subcases with $V^{\#\#} < c(1)$, it is that $\mu^{idns}(x_C, x_{+1})$ for $V^{AC} < V < V^{\#\#}$.

For all the mechanisms derived, prices are determined using IR constraint $U\left(\frac{1}{2}\right) = 0$, indifference conditions at threshold types, and using (8). ■

Altogether, when V changes the optimal mechanism changes as follows. If we start with V relatively high, we have $\vec{q}(x) = (1, 1)$: only the opaque good is sold. Then, when V decreases the following two things happen in parallel. Once $C(0)$ becomes negative (at $V = c(1)$), full within the mechanism information transmission to types at the edges becomes optimal. Once V crosses V^*, the marginal revenue from selling the good with no within the mechanism information revelation, i.e. the opaque good, becomes negative for some types, and so the gains from selling the opaque good to the types in the middle have to be compared to the losses from selling it to the intermediate types for which $A(x) + C(x) < 0$. For V close to V^*, the gains are clearly larger, but they diminish with V. While V decreases, x_C moves to the right and x_A to the left. If the losses become higher than gains while V is still greater than V^{AC}, it becomes optimal to sell nothing to both intermediate types and the types in the middle, but still sell the opaque good to the remaining intermediate types for whom $A(x) + C(x) > A(x) > 0$. These types, $x \in (x_C, x_{+1})$, will disappear once V reaches V^{AC}. If the losses are still lower than gains when V reaches V^{AC}, then when V decreases further, it becomes optimal to "optimize" the losses. It is interesting to stress that global (instead of local) incentive constraints may be the ones binding. The optimal solution allocates only the most preferred good with probability $\delta(x, x)$ to types $x \in (x_A, x_W)$. This is equivalent to a partial within the mechanism information transmission. When finally losses exceed gains it becomes optimal to have full within the mechanism information disclosure and sell the pure good.

5.3 Convex costs

Suppose that the cost function is strictly convex, that is, $c'(x) > 0$, $c''(x) > 0$ for all x. Then function A is decreasing, function D is increasing, and $C'(x)$ has ambiguous sign. We assume that function $C(x)$ is regular, that is, $C(x)$ is strictly increasing on $\left[0, \frac{1}{2}\right]$.

We define two new mechanisms. The first mechanism is a variation of the random allocation information discrimination mechanism μ^{raid}. In it, in the intermediate regions both goods are sold, but the good from the farthest type of the principal is sold with a type-specific probability, so that all the agent's types in these intermediate regions obtain the same utility in equilibrium. See Figure 5 below for illustration.

$$\mu^{raid2}(\hat{x}_1, \hat{x}_2) = \begin{cases} Q_0(x) = 1,\ Q_1(x) = 0,\ p(x) = \frac{1}{2}\left(V - c\left(\hat{x}_1\right)\right), & \text{for } x \in (0, \hat{x}_1), \\ Q_0(x) = 1,\ Q_1(x) = e(x),\ p(x) = p^*(x), & \text{for } x \in (\hat{x}_1, \hat{x}_2), \\ Q_0(x) = Q_1(x) = 1,\ p(x) = p_{og}, & \text{for } x \in \left(\hat{x}_2, \frac{1}{2}\right), \end{cases}$$

where $\hat{x}_1 < \hat{x}_2 < \frac{1}{2}$ are two thresholds, $e(x) = \frac{c'(x)}{c'(1-x)}$, $p_{og} = V - \frac{1}{2}\left[c\left(\hat{x}_2\right) + c\left(1 - \hat{x}_2\right)\right]$, and $p^*(x) = \frac{1}{2}\left[V - c\left(x\right) + e(x)\left(V - c(1-x)\right)\right]$.

As for μ^{raid}, some types receive partial within the mechanism information revelation. Now, however, partial within the mechanism information transmission occurs when the good is allocated.[21]

An important special case of the random allocation mechanism is $\mu^{raid2}(0, \hat{x}_2)$ with $\hat{x}_1 = 0$, so the types of the agent are divided in two groups: the ones who receive no within the mechanism information transmission (i.e. the opaque good), and the ones who receive partial within the mechanism information revelation (no full revelation to any type of the agent).

The other mechanism is random allocation with "no sale" region in the middle mechanism, μ^{rans}, which is a modification of μ^{raid2} in which the "no sale" region replaces the no within the mechanism information revelation region:

$$\mu^{rans}(\hat{x}_1, \hat{x}_2) = \begin{cases} \mu^{raid2}(\hat{x}_1, \hat{x}_2), & \text{for } x \notin (\hat{x}_2, 1 - \hat{x}_2), \\ Q_0(x) = Q_1(x) = 0,\ p(x) = 0, & \text{for } x \in (\hat{x}_2, 1 - \hat{x}_2). \end{cases}$$

Before we proceed with stating the main proposition, we introduce some notation and establish properties of some functions that will be useful for deriving optimal mechanisms.

[21] By (11), we have that $\xi_0(x) = 1$ and $\xi_1(x) = 0$ for $x \in (0, \hat{x}_1)$, $\xi_0(x) = \frac{1}{1+e(x)}$ and $\xi_1(x) = \frac{e(x)}{1+e(x)}$ for $x \in (\hat{x}_1, \hat{x}_2)$, and $\xi_0(x) = \xi_1(x) = \frac{1}{2}$ for $x \in \left(\hat{x}_2, \frac{1}{2}\right)$. By (12), we have that $\zeta_0(x) = 0$ and $\zeta_1(x) = 1$ for $x \in (0, \hat{x}_1)$, $\zeta_0(x) = 0$ and $\zeta_1(x) = 1$ for $x \in (\hat{x}_1, \hat{x}_2)$, and $\zeta_0(x) = \zeta_1(x) = \frac{1}{2}$ for $x \in \left(\hat{x}_2, \frac{1}{2}\right)$. Therefore, in terms of within the mechanism information, the agent's types in the intervals $(0, \hat{x}_1)$ and $(\hat{x}_1, \hat{x}_2)$ differ only in terms of function $\xi(x)$.

Lemma 7 *Function $C(x)+D(x)$ is strictly increasing on $x<\frac{1}{2}$ and crosses 0 at most once.*

Let x_+ be the solution to $C(x)+D(x)=0$ for $x<\frac{1}{2}$.

Lemma 8 *It is that $V^{AC}<c\left(\frac{1}{2}\right)$, when $c(x)$ is strictly convex.*

Lemma 9 *If $V>c\left(\frac{1}{2}\right)$, then $W(x)>0$ on $0\le x\le\frac{1}{2}$.*

For $V<c\left(\frac{1}{2}\right)$, $W(x)$ is strictly decreasing on $x<\frac{1}{2}$, and $W\left(\frac{1}{2}\right)<0$. Let x_W solve $W(x)=0$ on $x<\frac{1}{2}$ if the solution exists, or set $x_W=0$ if $W(0)<0$.

Proposition 9 *For the case of convex costs the optimal mechanism is*

$$\mu^*=\begin{cases}\mu^{nr}, & \text{for } V>c(1)+\frac{1}{2}c'(1),\\ \mu^{raid2}(0,x_+), & \text{for } c(1)<V<c(1)+\frac{1}{2}c'(1),\\ \mu^{raid2}(x_C,x_+), & \text{for } c\left(\frac{1}{2}\right)<V<c(1),\\ \mu^{rans}(x_C,x_W), & \text{for } V^{AC}<V<c\left(\frac{1}{2}\right),\\ \mu^{r}(x_A), & \text{for } V<V^{AC}.\end{cases}$$

Proof. We maximize the objective function (16) subject to the agent's IR and IC constraints (5), the principal's IC constraint, and $q_0(x)\in[0,1]$, $q_1(x)\in[0,1]$ for all $x\in[0,1]$.

Case 1: Suppose $V>c(1)+\frac{1}{2}c'(1)$. As initial guess, we take $x^*=0$ and, as such, we look at functions B and D to find $\vec{q}(x)$. Since $q_0(x)B(x)+q_1(x)D(x)=q_1(1-x)C(1-x)+q_0(1-x)A(1-x)$, due to symmetry (19) and properties (18) we can express the expected revenue as

$$ER = \frac{1}{2}\int_0^1 q_0(x)B(x)+q_1(x)D(x)\,dx \tag{28}$$

$$= \frac{1}{2}\int_0^{\frac{1}{2}} q_0(x)[A(x)+B(x)]+q_1(x)[C(x)+D(x)]\,dx. \tag{29}$$

Note that for $x<\frac{1}{2}$, $A(x)+B(x)>C(x)+D(x)$, $C'(x)+D'(x)>0$, and $C(0)+D(0)>0$ if $V\ge c(1)+\frac{1}{2}c'(1)$. Thus, the unconstrained optimization of (29) gives $\vec{q}(x)=(1,1)$ for $x\in\left[0,\frac{1}{2}\right]$ and, therefore mechanism μ^{nr}, as solution.

We need to verify if, by choosing a different x^*, the expected revenue can be increased. Suppose a different symmetric solution gives strictly higher expected revenue and in it, the lowest type for which IR binds is $x^*>0$. Then, to find $\vec{q}(x)$, we consider functions A and C for $x < x^*$, and functions B and D for $x > x^*$:

$$\begin{aligned} ER &= \frac{1}{2}\int_0^{x^*} q_0(x)A(x) + q_1(x)C(x)\ dx + \frac{1}{2}\int_{x^*}^{1} q_0(x)B(x) + q_1(x)D(x)\ dx \\ &= \int_0^{x^*} q_0(x)A(x) + q_1(x)C(x)\ dx \qquad (30) \\ &\quad + \frac{1}{2}\int_{x^*}^{\frac{1}{2}} q_0(x)\left[A(x)+B(x)\right] + q_1(x)\left[C(x)+D(x)\right]\ dx. \end{aligned}$$

Clearly, $A\left(\frac{1}{2}\right) > 0$ and $C(0) > 0$. Thus, $A(x) > 0$ and $C(x) > 0$ for all $x < \frac{1}{2}$. So it is optimal to set $\vec{q}(x) = (1,1)$ on $x < x^*$, but doing so will violate IR constraint for types $x < x^*$ since then $U(x) < U(x^*) = 0$. Then, optimization on $x < x^*$ under binding IR constraint necessarily results in IR constraint binding for some (and actually all) $x < x^*$, which contradicts x^* being the smallest x^* for which IR constraint binds.

Case 2. Suppose $c(1) < V < c(1) + \frac{1}{2}c'(1)$, then $C(0) + D(0) < 0$, and $C(0) > 0$.

If we start by setting x^* as any $x \in (0, x_+)$, the unconstrained optimization of (30) produces $\vec{q}(x) = (1,1)$ for $x < x^*$, and $\vec{q}(x) = (1,0)$ for $x \in (x^*, x_+)$. But then, from (6), $U(x) < U(x^*) = 0$ for all $x \in (0, x_+) \setminus \{x^*\}$ Thus, no matter what x^* we start with, optimization of (16) has to be done under tight IR constraint for the whole $[0, x_+]$ interval,

$$U(x) = U(x^*), \text{ for } x \in [0, x_+]. \qquad (31)$$

Combining equations (7) and (15), constraint (31) can be expressed as

$$\frac{1}{2}\int_x^{x^*} q_0(t)c'(t)dt - \frac{1}{2}\int_x^{x^*} q_1(t)c'(1-t)dt = 0.$$

Differentiating with respect to x we get $-c'(x)q_0(x) = -q_1(x)c'(1-x)$, and so

$$q_1(x) = \frac{c'(x)}{c'(1-x)}q_0(x).$$

Since $c'(x)$ is increasing, $q_1(x) < q_0(x) \le 1$. The marginal effect on revenue at any $x \in (0, x_+)$ is $q_0(x)W(x)$. Since $V > c\left(\frac{1}{2}\right)$, $W(x) > 0$, and so it is optimal to select mechanism $\mu^{raid2}(0, x_+)$.

It is trivial to verify that setting x^* as any $x^* > x_+$ is suboptimal. Indeed, if we optimize given such x^*, IR has to bind for $x < x^*$, but then, we can start with any $\hat{x} < x_+$, for which the solution computed above is better.

Case 3. If $c\left(\frac{1}{2}\right) < V < c(1)$, then $C(0) < 0$ and $C\left(\frac{1}{2}\right) + D\left(\frac{1}{2}\right) > 0$. Since $C(x) < D(x)$, we have $x_+ > x_C$; and since $V^{AC} < c\left(\frac{1}{2}\right)$, we also have $x_A > x_C$ on $c\left(\frac{1}{2}\right) < V < c\left(\frac{1}{2}\right) + \frac{1}{2}c'\left(\frac{1}{2}\right)$. Following a similar analysis as in the previous case, the optimization of (16) has to be done under tight IR constraint for the whole (x_C, x_+), and so it is optimal to set $q_0(x) = 1$ and $q_1(x) = \frac{c'(x)}{c'(1-x)}$ for $x \in (x_C, x_+)$. Combined, we have mechanism $\mu^{raid2}(x_C, x_+)$ as the solution of the constrained optimization.

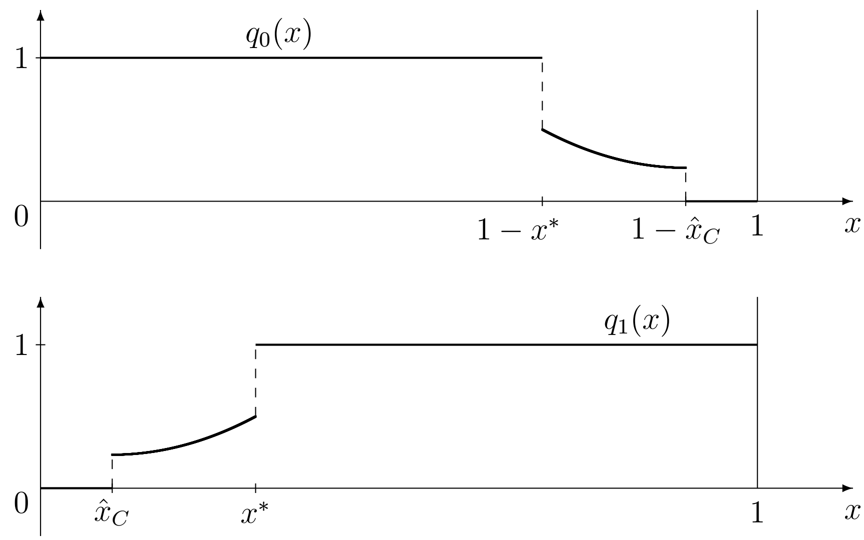


Figure 5: Mechanism μ^{raid2}.

Suppose that we consider a candidate for the optimal mechanism for which IR binds for some $x^* < x_C$. Then IR has to bind for $x \in (x^*, x_+)$, and so it binds for x_C, but then we can reoptimize considering $x^* = x_C$, and obtain higher revenue. The argument for suboptimality of setting $x^* > x_+$ is the same.

The optimal solution for this case is shown in Figure 5. The corresponding expected surplus for the agent is depicted in Figure 6.

Case 4. When $V^{AC} < V < c\left(\frac{1}{2}\right)$, then since $C(x) + D(x) < 0$, it is never optimal to set $\vec{q}(x) = (1,1)$. Still, if we consider $x^* = \frac{1}{2}$, since $x_A > x_C$, and thus $A(x) + C(x) > 0$ on $x \in (x_C, x_A)$, the unconstrained optimization of (30) would require setting $\vec{q}(x) = (1,1)$ for $x \in (x_C, x_A)$, which would violate IR constraint. As we have already seen the expected revenue under binding IR constraint is $q_0(x)W(x)$, so it is optimal to set $q_0(x) = 1$ when $W(x) > 0$ and set $q_0(x) = 0$ otherwise. Clearly, $W(x) > 0$ at x_C, and so $x_W > x_C$. Altogether, mechanism $\mu^{rans}(x_C, x_W)$ is optimal. Note that IR binds for all $x \in \left(x_C, \frac{1}{2}\right]$.

Case 5. If $V < V^{AC}$, and ergo $x_A < x_C$, then $W(x) < 0$ for any $x > x_A$, as $W(x_A) < 0$. This implies that the optimal mechanism is $\mu^r(x_A)$.

■

The optimal mechanism changes when V changes. For V relatively high, we have $\vec{q}(x) = (1,1)$ for all x. Once $C(0) + D(0)$ becomes negative, in addition to the sale of the good with no within the mechanism information transmission (i.e. the opaque good), partial within the mechanism information revelation (i.e. random allocation) becomes part of the optimal mechanism. As V decreases, at some point $C(0)$ become negative. When this happen, in the optimal mechanism the agent's types at the edges learn the principal's type perfectly and buy only if the true type

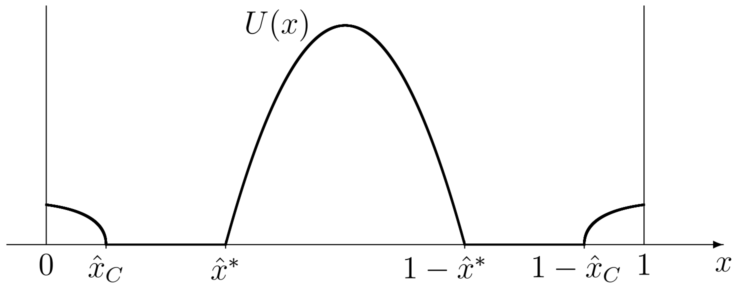


Figure 6: Expected surplus for mechanism μ^{raid2}.

of the principal is the closest one (this mechanism is shown on Figure 5). Once V decreases further, a change in the optimal mechanism occurs at $V = c\left(\frac{1}{2}\right)$, when $W(x)$ becomes negative at $x = \frac{1}{2}$. From that moment, the good with no within the mechanism information transmission is no longer sold and instead agent's types in the middle do not buy anything at all. For even lower values of V, the region with partial within the mechanism information revelation (i.e. random allocation) shrinks, while the "no sale" region widens. Eventually, at $V = V^{AC}$, the random allocation region disappears and the optimal mechanism coincides with the optimal mechanism under complete information disclosure.

5.4 Implementation for concave and convex costs

We have shown that the optimal mechanism takes different forms depending on the value of V. Some of these mechanisms, namely μ^{nr}, μ^{r}, and μ^{id}, are common to the linear case and can be implemented as proposed in Section 5.1.1. For the concave costs case, mechanisms μ^{nrns} and μ^{idns} are slight modifications of mechanisms μ^{nr} and μ^{id}, respectively, and can be implemented similarly. The only difference is that the price of the opaque good is high enough so that some types of the agent prefer not to buy anything. Probabilistic mechanisms μ^{raid} in the concave costs case and μ^{raid2} (and μ^{rans}) in the convex costs case require different implementing solutions.

A way to implement $\mu^{raid}(x_A, x_W)$ is to offer the following menu: a) an opaque good; and b) an infinitum of *options*, each one giving the right to purchase, after being communicated the type of the principal, a lottery ticket that awards the good with probability d. It is convenient to parameterize these options by d. The price of the opaque good is $P_{og} = V - c\left(\frac{1}{2}\right)$. There is an option for each $d(x) = 1 - \frac{c'(1-x)}{c'(x)}$ such that $x \in (x_A, x_W)$; in addition, there is $d = 1$. Each option d has a price for acquiring information P_i^d and a strike price P_{lot}^d. From definition of $\mu^{raid}(x_A, x_W)$, these prices have to satisfy

$$
\begin{aligned}
P_i^{d(x)} + \frac{1}{2} P_{lot}^{d(x)} &= p^*(x), \\
P_i^1 + \frac{1}{2} P_{lot}^1 &= p_g.
\end{aligned}
$$

Notice that, as in the linear costs case, prices P_i^d and P_g^d for each option d are not uniquely determined. Extra concerns, such as preventing arbitrage by buying the information and then buying either the opaque good or a different option after learning the type of the principal may put further constraints at option prices.

Alternatively, $\mu^{raid}(x_A, x_W)$ can be implemented by offering the following menu: a) the opaque good, b) an infinitum of *opaque lotteries* indexed by probability of winning d. By an opaque lottery we mean that, at the moment of the lottery sale, the buyer knows the odds of winning but does not know which prize the lottery offers. Only after the purchase the buyer is communicated the specification of the good that is awarded (i.e. the principal's type), and then he can decide whether to participate in the lottery draft or not. To implement the optimal mechanism, opaque lotteries $d(x)$ for each $x \in (x_A, x_W)$ at price $p^*(x)$ and degenerate lottery 1 at price p_g have to be offered. This second method to implement the optimal mechanism $\mu^{raid}(x_A, x_W)$ is less exposed to arbitrage manipulation by the agent because the agent is always asked to pay everything before learning the type of the principal.

Mechanism $\mu^{raid2}(\hat{x}_1, \hat{x}_2)$ can be implemented by a menu of: a) the opaque good at price $P_{og} = V - \frac{1}{2}\left[c(x_+) + c(1 - x_+)\right]$, b) an infinitum of contracts that give the right to return the product, after being communicated the type of the principal, with probability $1 - e$. Indexed by $1 - e$, contracts $1 - e(x)$ at a price $p^*(x)$ have to be offered for each $x \in (\hat{x}_1, \hat{x}_2)$ and contract 1 has to be offered at a price p_g (if $\hat{x}_1 > 0$), where prices p_g and $p^*(x)$ are specified in the definition of $\mu^{raid2}(\hat{x}_1, \hat{x}_2)$. Alternatively, instead of an infinitum of contracts with a probabilistic option to return, an infinitum of *mixed* or *conditional* lotteries can be offered, indexed by e. A mixed or conditional lottery has different winning odds conditional on the exact good to be allocated, which is determined only after the lottery is purchased. In our case, lottery e, depending on the realized good (principal's) type, either gives the good with probability one or with probability e. The mechanism μ^{rans} is a slight variation of the mechanism μ^{raid2} where the opaque good is not offered.

Generally, as only expected payments from each type of the agent are specified in the optimal mechanism, there is flexibility on how actual prices and options are set and principal's surplus is accumulated over the agent's types in an implementation. For instance, the principal can ask each agent's type to pay upfront, so that each type of the principal earns the same amount for any given realization of the agent's type, or the principal can ask to pay only conditional on the good delivered, or, one can construct implementing mechanisms where some types of the agent pay only to the principal of a certain type, while the other types pay to the other type of the principal. IC for the principal only requires that expected surplus is the same for all the types of the principal.

6 Conclusion

We consider a seller who holds private information about characteristics that makes the good on sale horizontally differentiable. We show that the seller can strategically use her private information to increase her profits. Depending on the shape of the buyers' utility function, the seller's optimal mechanism may entail disclosing, hiding, or selling information, and random allocation. We introduce the notion of information transmission within the mechanism which allows to trace what different types of the buyer learn. We show that in the optimal mechanism different amount of information may be transmitted to different types, which implies that focusing attention only on no, full, or partial public disclosure mechanisms is limiting for the seller.

If the utility from consuming the good is high enough for everyone in the market independently from its specific characteristics, then the seller prefers not to reveal his private information. We show that in many settings the seller maximizes her profits offering simultaneously an opaque good and an option. Purchasing the option, the buyer learns the seller's private information and acquires the right to buy the good at a predetermined exercise price. Selling information allows the seller to cover a wider market, appealing to customers whose willingness to pay for the good is highly sensitive to the content of the seller's private information. Indeed, some customers may be willing to pay for information alone in order to avoid to buy the good if they learn characteristics they dislike.

In different markets we observe practices that resemble the optimal mechanism that we characterized. In the market of financial services, for example, customers are offered both investment products and consulting services by the same provider. Customers may buy consulting services to better learn the characteristics of different investment products and make a final purchase from a more informed standpoint. In the market of education, customers can buy a limited number of introductory classes or enroll directly in a full-length course. In the case of food and beverages, retailers sell both small and big packages of several products. Through the offer of introductory classes and small packages goods, the sellers allow the customers to get a taste of the good before buying it (e.g. wine-tasting). More recently, technology companies have started to offer "buy-back" and "upgrade" programs: paying a fee, customers purchase the right to return their product over time and get a pro-rated refund, in the first case; the right to trade their old devices (e.g. smartphones) for newer models every six or twelve months at a discounted price, in the second case.

One of the common denominators of these business practices is that they all entail the opportunity for the customers of learning their valuation for the good before buying it (or buying only a sample of it, or retaining the right of asking for a refund) when some characteristics of the good are not known to them and better known to the seller. However, customers' valuations may be differently sensitive to this extra information. Some customers may not be affected and prefer to buy directly the final good, other customers may choose to defer such purchase until after they have acquired more information. In that way, if they discover of not liking the good, they can limit their disutility. We offer a rationale for these kinds of mechanisms to appear based on the analysis of the informed principal problem. In horizontally differentiated markets, the optimal mechanism chosen by an informed seller may entail price-discrimination across customers based on their valuation for the seller's private information about the good's characteristics.

References

Anderson, S. P. & Renault, R. (2006), 'Advertising content', *American Economic Review* **96**(1), 93–113.

Balestrieri, F. (2008), 'A modified English auction for an informed buyer', MIT, mimeo.

Balestrieri, F., Izmalkov, S. & Leão, J. (2013), 'The market for surprises: Selling substitute goods through lotteries', NES, mimeo.

Bergemann, D. & Pesendorfer, M. (2007), 'Information structures in optimal auctions', *Journal of Economic Theory* **137**(1), 580–609.

Board, S. (2009), 'Revealing information in auctions: the allocation effect', *Economic Theory* **38**(1), 125–135.

Celik, L. (2012), 'Information unraveling revisited: Disclosure of horizontal attributes', available at SSRN 1719338.

Courty, P. & Li, H. (2000), 'Sequential screening', *Review of Economic Studies* **67**(4), 697–717.

Eső, P. & Szentes, B. (2007), 'Optimal information disclosure in auctions and the handicap auction', *Review of Economic Studies* **74**(3), 705–731.

Ganuza, J.-J. (2004), 'Ignorance promotes competition: An auction model with endogenous private valuations', *The RAND Journal of Economics* **35**(3), pp. 583–598.

Ganuza, J. J. & Penalva, J. (2010), 'Signal orderings based on dispersion and the supply of private information in auctions', *Econometrica* **78**(3), 1007–1030.

Grossman, S. J. (1981), 'The informational role of warranties and private disclosure about product quality', *Journal of law and economics* **24**(3), 461–483.

Grossman, S. J. & Hart, O. D. (1980), 'Disclosure laws and takeover bids', *Journal of Finance* **35**(2), 323–34.

Holmström, B. & Myerson, R. B. (1983), 'Efficient and durable decision rules with incomplete information', *Econometrica* **51**(1), 1799–1819.

Hotelling, H. (1929), 'Stability in competition', *Economic Journal* **39**(153), 41–57.

Jewitt, I. & Li, D. Z. (2012), 'Cheap-talk information disclosure in auctions', University of Oxford, mimeo.

Johnson, J. P. & Myatt, D. P. (2006), 'On the simple economics of advertising, marketing, and product design', *American Economic Review* **96**(3), 756–784.

Kamenica, E. & Gentzkow, M. (2011), 'Bayesian persuasion', *The American Economic Review* **101**(6), pp. 2590–2615.

Koessler, F. & Renault, R. (2012), 'When does a firm disclose product information?', *The RAND Journal of Economics* **43**(4), 630–649.

Krishna, V. (2002), *Auction Theory*, Academic Press, San Diego, California.

Lewis, T. R. & Sappington, D. (1994), 'Supplying information to facilitate price discrimination', *International Economic Review* **35**(2), 309–27.

Maskin, E. & Tirole, J. (1990), 'The principal-agent relationship with an informed principal: The case of private values', *Econometrica* **58**(2), 379–409.

Maskin, E. & Tirole, J. (1992), 'The principal-agent relationship with an informed principal, ii: Common values', *Econometrica* **60**(1), 1–42.

McAfee, R. P. & McMillan, J. (1988), 'Multidimensional incentive compatibility and mechanism design', *Journal of Economic Theory* **46**(2), 335–354.

Milgrom, P. (2008), 'What the seller won't tell you: Persuasion and disclosure in markets', *Journal of Economic Perspectives* **22**(2), 115–131.

Milgrom, P. R. & Weber, R. J. (1982), 'A theory of auctions and competitive bidding', *Econometrica* **50**(5), 1089–1122.

Milgrom, P. & Roberts, J. (1986), 'Relying on the information of interested parties', *The RAND Journal of Economics* **17**(1), 18–32.

Myerson, R. B. (1981), 'Optimal auction design', *Mathematics of Operations Research* **6**(1), 58–73.

Myerson, R. B. (1983), 'Mechanism design by an informed principal', *Econometrica* **51**(6), 1767–97.

Mylovanov, T. & Tröger, T. (2008), 'Optimal auction design and irrelevance of private information', University of Bonn, Bonn Econ Discussion Papers bgse21_2008.

Mylovanov, T. & Tröger, T. (2012), 'Informed-principal problems in environments with generalized private values', *Theoretical Economics* **7**(3), 465–488.

Pavlov, G. (2006), Optimal mechanism for selling substitutes, Department of Economics Working Papers Series WP2006-014, Boston University.

Rayo, L. & Segal, I. (2010), 'Optimal information disclosure', *Journal of Political Economy* **118**(5), 949 – 987.

Riley, J. & Zeckhauser, R. (1983), 'Optimal selling strategies: When to haggle, when to hold firm', *The Quarterly Journal of Economics* **98**(2), 267–289.

Skreta, V. (2011), 'On the informed seller problem: optimal information disclosure', *Review of Economic Design* **15**(1), 1–36.

Sun, M. J. (2011), 'Disclosing multiple product attributes', *Journal of Economics and Management Strategy* **20**(1), 195–224.

Thanassoulis, J. (2004), 'Haggling over substitutes', *Journal of Economic Theory* **117**(2), 217–245.

Yilankaya, O. (1999), 'A note on the seller's optimal mechanism in bilateral trade with two-sided incomplete information', *Journal of Economic Theory* **87**(1), 267–271.

Zhang, J. (2008), 'Auctions with refund policies as optimal selling mechanisms', Queen′s University, mimeo.

A Appendix

A.1 Proofs of Proposition 3 and Lemmata.

Proof of Proposition 3. Clearly, when $V \leqslant y_{\min}$, revealing information is better as otherwise no revenue can be collected. If V is sufficiently high then both virtual valuation functions $\Psi^{nr}(y(x))$ and $\Psi(x)$ are strictly positive for all x, thus every type of the agent will be served in each informational treatment. As $y_{\max} = \max_{x\in[0,1]} \frac{1}{2}\left\{c(x) + \frac{1}{2}c(1-x)\right\} < c(1)$, the profits under no revelation exceed the profits under full revelation disclosure policies: $\pi^{nr} = P^{nr} = V - y_{\max} > V - c(1) = P = \pi^r$.

Now, by continuity, there exists V^*, for which the profits from the optimal mechanisms with full and no revelation coincide. Consider any such V^*. Let x^*, y^*, and $Q^{nr}(y^*)$ be, respectively, the marginal type in mechanism μ^{ck} for $s = 0$, the marginal expected cost type, and the quantity (the probability of the good being sold) in the optimal mechanism under no revelation. By definition,

$$\pi^r(V^*) = (V^* - c(x^*))x^* = \pi^{nr}(V^*) = (V^* - y^*)Q^{nr}(y^*).$$

It has to be that $x^* < \frac{1}{2}$ and, since $\frac{1}{2}c(x^*) + \frac{1}{2}c(1-x^*) > c(x^*)$, then $Q^{nr}(y^*) > x^*$. Indeed, if $x^* \geq \frac{1}{2}$, then by revealing no information and setting $P^{nr} = V - c(x^*)$ the principal can sell to everyone at a profit strictly higher that $\pi^r(V^*)$, contradicting the presumption that $\pi^{nr}(V^*) = \pi^r(V^*)$.

By the envelope theorem, $\frac{\partial \pi^r}{\partial V} = x^* < \frac{\partial \pi^{nr}}{\partial V} = Q^{nr}(y^*)$. Therefore, there can exist only one V^* at which $\pi^r(V^*) = \pi^{nr}(V^*)$, as at any such

V, $\pi^{nr}(V)$ crosses $\pi^r(V)$ from below. ■

Proof of Lemma 3. It is obvious that $V^{AC} < c(1)$ and $V^{AC} \leq V^*$. By definition, $V^{AC} = c(x_=) + c'(x_=)x_= = c(1-x_=) - c'(1-x_=)x_=$. Due to concavity, $c\left(\frac{1}{2}\right) - c(x_=) < c'(x_=)\left(\frac{1}{2} - x_=\right)$ and $c\left(\frac{1}{2}\right) - c(1-x_=) < c'(1-x_=)\left(x_= - \frac{1}{2}\right)$. Then, we can write $c\left(\frac{1}{2}\right) - V^{AC} < c'(x_=)\left(\frac{1}{2} - 2x_=\right)$ and $c\left(\frac{1}{2}\right) - V^{AC} < c'(1-x_=)\left(2x_= - \frac{1}{2}\right)$. Clearly, the right-hand sides of these inequalities are either equal to 0 (if $x_= = \frac{1}{4}$) or are of different signs. Thus, $c\left(\frac{1}{2}\right) - V^{AC} < 0$. ■

Proof of Lemma 4. It is that $W'(x) = w(x)(V - c(1-x))$, where $w(x) = \frac{c''(x)c'(1-x) + c'(x)c''(1-x)}{[c'(1-x)]^2} < 0$. Therefore, the sign of $W'(x)$ depends on $V - c(1-x)$ and it changes exactly once on $x < \frac{1}{2}$ from positive to negative. As $W\left(\frac{1}{2}\right) = A\left(\frac{1}{2}\right) + C\left(\frac{1}{2}\right) > 0$, $W(x)$ crosses 0 at most once. ■

Proof of Lemma 5. At $V = V^{AC}$, $Z^{\#} > 0$, given that $x_A = x_W = x_=$, and $R^* > 0$. At $V = c\left(\frac{1}{2}\right)$, $Z^{\#} < 0$, because $A(x) < 0$ for $x \in \left(x_A, \frac{1}{2}\right)$ and $A(x) + C(x) \leq 0$ for $x \in \left(x_W, \frac{1}{2}\right)$. Monotonicity of $Z^{\#}$ comes from the facts that $A(x)$ and $C(x)$ increase with V, the change in lower bound x_A has a second-order effect as $A(x_A) = 0$, and the change in x_W also has a second-order effect, as at x_W, $d(x)A(x) = A(x) + C(x)$. Therefore, $V^{\#} \in \left(c\left(\frac{1}{2}\right), V^{AC}\right)$. ■

Proof of Lemma 6. Similar to the proof of Lemma 5. ■

Proof of Lemma 7. Indeed $C'(x) + D'(x) = 4c'(1-x) + (1-2x)c''(1-x) > 0$. Given that $C(0) + D(0) = 2V - 2c(1) - c'(1)$, and $C\left(\frac{1}{2}\right) + D\left(\frac{1}{2}\right) = 2V - 2c\left(\frac{1}{2}\right)$, then there exists a unique V, $c\left(\frac{1}{2}\right) < V < c(1) + \frac{1}{2}c'(1)$, such that $C(x) + D(x) = 0$. ■

Proof of Lemma 8. Similar to the proof of Lemma 3. ■

Proof of Lemma 9. It is that $W'(x) = w(x)(V - c(1-x))$, where $w(x) > 0$ when $c(x)$ is convex. Therefore, on $0 < V < c(1)$, $W(x)$ has a minimum at x solving $V = c(1-x)$. At $V = \frac{1}{2}$, the minimum of $W(x)$ is at $x = \frac{1}{2}$, and $W\left(\frac{1}{2}\right) = A\left(\frac{1}{2}\right) + C\left(\frac{1}{2}\right) = 2V - 2c\left(\frac{1}{2}\right) = 0$. As $W(x)$ increases with V for all x, we have the result. ■

A.2 Verification of the optimality of the obtained solution for the concave costs

Now, let us prove that the presented solution in Step 2 of the proof of Proposition 8 is indeed the optimal one. We do so for the subcase $V < V^{AC}$. Suppose solution $(r(x), q_1(x))$ for all $x \in \left[0, \frac{1}{2}\right]$ is optimal. We want to show that such optimal solution has to coincide with the one that is presented in Step 2. First, we must have $(r(x), q_1(x)) = (1, 0)$ on $x < x_A$, as this is optimal on $x < x_A$ and does not violate any constraints. Consider a segment $x \geq x_{+2}$ and let $q_1^* = \max_{x\in\left[0,\frac{1}{2}\right]} q_1(x)$, $x^* = \arg\max_{x\in\left[0,\frac{1}{2}\right]} q_1(x)$.[22] Redefine $\tilde{r}(x), \tilde{q}_1(x)$ on $x \in \left[x_A, \frac{1}{2}\right]$ as follows, $\tilde{r}(x) = q_1^*\delta(x,x)$, $\tilde{q}_1(x) = 0$ on $x < x_W$ and $\tilde{r}(x) = 0$, $\tilde{q}_1(x) = q_1^*$ on $x > x_W$. We claim that the redefined solution is at least as good in expected revenue as the original one (and improves on it, if different on the set of positive measure). To prove this claim, first observe that the redefined solution is incentive compatible (it is actually a fraction q_1^* of the candidate solution computed in step 2). Second, note that the expected surplus of the redefined solution on $x > x_{+2}$ is at least as high, as $\tilde{q}_1(x) \geq q_1(x)$ (and $A+C$ is positive), while $\tilde{r}(x) \leq r(x)$ (and A is negative). For $x < x_{+2}$, the redefined solution optimally allocates $\tilde{r}(x) + \delta(x,x)\tilde{q}_1(x) = \delta(x,x)q_1^*$. The only possibility it generates less expected revenue than the original one is if for some types $x \in (x_A, x_{+2})$

$$r(x)A\left(x\right) + q_1(x)\left(A\left(x\right) + C\left(x\right)\right) > \tilde{r}(x)A\left(x\right) + \tilde{q}_1(x)\left(A\left(x\right) + C\left(x\right)\right).$$

Since $\tilde{r}(x)$ and $\tilde{q}_1(x)$ are those at which $\tilde{r}(x)A\left(x\right) + \tilde{q}_1(x)\left(A\left(x\right) + C\left(x\right)\right)$ is maximized subject to $r + q_1\delta(x,x) \geq q_1^*\delta(x,x)$, it must be that for the same $x \in (x_A, x_{+2})$

$$r(x) + \delta(x,x)q_1(x) < \tilde{r}(x) + \tilde{q}_1(x)\delta(x,x).$$

As $U'(x) = -\frac{1}{2}c'(x)\left[r(x) + \delta(x,x)q_1(x)\right]$, $U'_x(x^*|x) = -\frac{1}{2}c'(x)\left[r(x^*) + \delta(x,x)q_1^*\right]$, $\tilde{U}'(x) = -\frac{1}{2}c'(x)\left[\tilde{r}(x) + \tilde{q}_1(x)\delta(x,x)\right]$, $\tilde{U}'_x(x^*|x) = -\frac{1}{2}c'(x)\delta(x,x)q_1^*$, we have that for these x

$$U'(x) > \tilde{U}'_x(x^*|x) \geq U'_x(x^*|x).$$

Because of the global IC, we then have $U(x) > U(x^*|x)$. But then, looking at the change of equilibrium utility when moving from x^* to x, this implies that the utility had to increase strictly faster than from bundle $(r(x^*), q_1(x^*))$ somewhere along the way. Formally, there must exist an interval (y_1, x^*) and its subinterval (y_1, y_2), with $y > x_A$, for which $U'(y) \leq U'_y(x^*|y)$ for all $y \in (y_1, x^*)$ and $U'(y) < U'_y(x^*|y)$ for all $y \in (y_1, y_2)$.

Then, consider a different solution $(\hat{r}(x), \hat{q}_1(x)) = (r(x), q(x))$ on $x < y_1$ and $(\hat{r}(x), \hat{q}_1(x)) = (\tilde{r}(x), \tilde{q}_1(x))$ on $x > y_1$. Note first that this solution is incentive compatible. In essence, in it, all the types $x > y_1$ obtain a bundle $(r, q) = (0, q_1^*)$; whatever

[22] Strictly speaking, maximum may not exist. If so we can define $q_1^* = \sup_{x\in\left[0,\frac{1}{2}\right]} q_1(x)$, select type x^* with $q_1(x)$ sufficiently close to q_1^* and repeat the argument below, then take the limit.

else they have previously received is no longer available to them, and they would not want to switch to anything that is available for types $x < y_1$, as, now, those bundles will be re-priced accordingly. [By the nature of incentive constraints, lower types (which in this case are those who are closer to the middle) never envy the allocation received by higher types]. As for the types $x < y_1$, they receive their prior allocation and do not want $(0, q_1^*)$ as they did not want $(r(x^*), q_1^*)$. But as $\hat{r}(x) + \delta(x,x)\hat{q}_1(x) = \delta(x,x)q_1^* \leq U'(y) = r(x) + \delta(x,x)q(x)$ for $x \in (y_1, x^*)$ and strictly less for $x \in (y_1, y_2)$, allocation $(\hat{r}(x), \hat{q}_1(x))$ generates strictly more revenue than $(r(x), q(x))$. Thus, we must have $r(x)A(x) + q_1(x)(A(x) + C(x)) \leq \tilde{r}(x)A(x) + \tilde{q}_1(x)(A(x) + C(x))$ for all x, which means that if q_1^* is positive, the optimal solution has to have the form of the candidate optimum computed in Step 2. Finally, optimization over q_1^* results in $q_1^* = 1$ if $ER^*_{x>x_A}$, and $q_1^* = 0$, otherwise.