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Strategic Asset Allocation in Money Management

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This article analyzes the dynamic portfolio choice implications of strategic interaction among money managers. The strategic interaction emerges as the managers compete for money flows displaying empirically documented convexities. A manager gets money flows increasing with performance, and hence displays relative performance concerns, if her relative return is above a threshold; otherwise she receives no (or constant) flows and has no relative concerns. We provide a tractable formulation of such strategic interaction between two risk averse managers in a continuous-time setting, and solve for their equilibrium policies in closed-form. When the managers' risk aversions are considerably different, we do not obtain a Nash equilibrium as the managers cannot agree on who loses (getting no flows) in some states. We obtain equilibria, but multiple, when the managers are similar since they now care only about the total number of losing states. We recover a unique equilibrium, however, when a sufficiently high threshold makes the competition for money flows less intense. The managers' unique equilibrium policies are driven by chasing and contrarian behaviors when either manager substantially outperforms the opponent, and by gambling behavior when their performances are close to the threshold. Depending on the stock correlation, the direction of gambling for a given manager may differ across stocks, however the two managers always gamble strategically in the opposite direction from each other in each individual stock.

Key words: Money Managers, Strategic Interaction, Portfolio Choice, Relative Performance, Incentives, Risk Shifting, Fund Flows, Tournaments.

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Данная работа исследует динамический выбор оптимального портфеля при наличии стратегического взаимодействия между управляющими портфелями. Стратегическое взаимодействие возникает в результате того, что управляющие конкурируют за приток активов под управление. Активы достаются управляющему если его относительная доходность превышает некое пороговое значение, в противном случае управляющий не получает активов. Мы аналитически исследуем стратегическое взаимодействие двух не расположенных к риску портфельных управляющих в непрерывном времени. Нами получены следующие основные результаты. Если коэффициенты неприятия риска значительно отличаются, то равновесия Нэша не существует так как ни один управляющий не желает быть проигравшим (не получить активов) в некоторых состояниях. С другой стороны, при близких значениях коэффициентов неприятия риска существует множество равновесий Нэша, так как в данном случае управляющим важно лишь общее количество проигрышных состояний. Единственное равновесие Нэша существует, однако, если вышеуказанное пороговое значение достаточно высоко, что ведет к менее активной конкуренции за приток новых активов. Оптимальный портфель в данном случае определяется механизмами преследования и уклонения если один из управляющих значительно опережает своего оппонента, и механизмом гэмблинга если доходности управляющих близки. В зависимости от корреляции между рисковыми активами, стратегия гэмблинга может вести к разнонаправленному изменению инвестиций управляющих в рисковые активы для данного управляющего. Тем не менее, для данного актива второй управляющий всегда стратегические изменяет свои инвестиции в противоположном направлении относительно оппонента.

Ключевые слова: управляющие активами, стратегическое взаимодействие, оптимальный портфель, относительная доходность, стимулы, приток активов, турниры.
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1. Introduction

This paper analyzes the dynamic portfolio strategies of money managers in the presence of strategic interactions, arising from each manager's desire to perform well relative to the other managers. There are several reasons why managers may care about relative performance. First, given the prevalent finding in money management that the money flows to relative performance relationship is increasing and convex (Chevalier and Ellison (1997), Sirri and Tufano (1998)) for individual mutual funds, Gallaher, Kaniel, and Starks (2006) for mutual fund families, Agarwal, Daniel, and Naik (2004), Ding, Getmansky, Liang and Wermers (2007) for hedge funds), a fund manager has incentives to outperform the peers so as to increase her assets under management, and hence, her compensation.[1] Relative concerns may also arise within fund families as funds with high relative performance are likely to be advertised more (Jain and Wu (2000)) and also to be cross-subsidized at the expense of other family members (Gaspar, Massa, and Matos (2006)). Moreover, money managers may care about their relative standing due to psychological aspects of human behavior, such as envy or crave for higher social status.

When discussing the interaction of fund managers in the presence of relative performance concerns, Brown, Harlow, and Starks (1996) appeal to the notion of a *tournament* in which managers are the competitors and money flows are the prizes awarded based on relative ranking. Since then, a rapidly growing empirical literature (Busse (2001), Qiu (2003), Goriaev, Nijman, and Werker (2005), Reed and Wu (2005)) address the tournament hypothesis by looking at how risk taking behavior responds to relative performance. While several recent theoretical studies attempt to analyze mutual fund tournaments, their results are obtained under fairly specialized economic settings (as discussed below). To our best knowledge, ours is the first comprehensive analysis of the portfolio choice effects of strategic interactions within a workhorse dynamic asset allocation framework, allowing us to derive a rich set of implications. The effects of strategic considerations are likely to be the strongest when there is a small number of funds competing against each other. One natural real-life case is when several top-performing funds compete for the leadership. Another example is the strategic interaction of money managers within a fund family comprised of a small number of funds, as documented by Kempf and Ruenzi (2008).

We consider two risk averse money managers, interpreted as either mutual fund managers, hedge fund managers, or simply traders.[2] We adopt a familiar Merton (1969)-type

[1] A standard explanation for a positive relation between money flows and relative performance is that investors respond to widely published fund rankings (*MorningStars*, *Business Week*, *Forbes*, *Institutional Investor*) when choosing which fund to invest into.

[2] In addition to the money management industry, our analysis can also be applied to study the behavior of traders working in the same investment bank. Indeed, while it may not be explicitly written in a contract, it is common knowledge that promotion of traders much depends on their relative (to colleagues) success.

continuous-time economy for investment opportunities with multiple correlated risky stocks, and assume constant relative risk aversion (CRRA) preferences for a normal manager with no relative performance concerns. We model relative performance concerns by postulating that a manager's objective function depends (positively) on the ratio of her horizon investment return over the other manager's return, in addition to her own horizon wealth. Our interpretation for the presence of these relative performance concerns is based on money flows, which capture the desire of a money manager to attract new money by outperforming the other manager. We formally justify this in our analysis by considering an option-like flow-performance specification that exhibits a convexity, consistent with the above-cited empirical evidence on fund flows. If the manager's relative performance is above a certain threshold, she receives money flows increasing with performance, and hence displays relative performance concerns. Otherwise if she performs relatively poorly, she receives no (or constant) flows and her objectives are as for a normal manager with no relative concerns. The presence of the threshold captures inertia whereby investors respond to relative performance and award a fund with money flows only after this fund outperforms the opponent by a certain margin (Huang, Wei, and Yan (2007)). In characterizing managers' behavior, we appeal to the pure-strategy Nash equilibrium concept, in which each manager strategically accounts for the dynamic investment policies of the other manager, and the equilibrium policies of the two managers are mutually consistent.

Solving for the managers' best responses reveals that a manager only chooses two outcomes at the horizon: "winning" by outperforming the other manager and attracting flows, or "losing" by underperforming and getting no flows, and never opting for a "draw." This is due to the local convexity around the performance threshold, inducing the manager to gamble so as to end up either a winner or a loser, thus avoiding a draw. Moreover, we show that an important feature of a manager when outperforming, and hence displaying relative performance concerns, is that she either "chases" the opponent, increasing her investment policy in response to the opponent's increasing hers, or acts as a "contrarian," decreasing her investments in response to the opponents's increasing hers. In our formulation, a manager is a chaser if her risk aversion coefficient is greater than unity (more empirically plausible), and is a contrarian otherwise.

Moving to equilibrium, we demonstrate that the gambling behavior caused by local convexities leads to the potential non-existence of a pure-strategy Nash equilibrium, since when both managers' performances are close to their thresholds (in the convex region), they cannot agree on who the winner is. Such a situation occurs when the managers' attitude towards risk are considerably different, where one manager may want to outperform the other by

Hence, a trader concerned about her career is likely to have relative performance considerations as one of her objectives.

just a little to become a winner, while the other manager wants to underperform by a lot to be content with being a loser. If on the other hand, the managers' risk aversions are sufficiently similar, an equilibrium obtains, with one manager emerging as a winner and the other as a loser. However, since the managers are very similar, the opposite outcome may also occur, where the earlier winner and loser positions are switched, giving rise to multiplicity. We show that both of these non-existence and multiplicity issues are resolved when the performance threshold is sufficiently high. This is because with a high threshold, the two managers cannot both be close to their thresholds and gamble at the same time, which is the main cause for the non-existence and multiplicity. This unique equilibrium is more likely to exist for higher values of risk aversions or lower money flow elasticities as this dampens the gambling incentives.

It is recognized in other economic settings that strategic interactions may lead to outcomes such as multiplicity or non-existence of equilibrium. However, the economic mechanisms and factors determining which of the outcomes occur seem to be specific to our framework.[3] In particular, we believe that our analysis is the first to identify an important economic role played by the performance threshold and risk aversion heterogeneity in determining whether unique or multiple equilibrium exists. It is also worth noting that in related non-strategic works where the gambling behavior is present the optimal portfolios are unique (Carpenter (2000), Basak, Pavlova, and Shapiro (2007)), and hence the possibility of multiple equilibria or no equilibrium at all is not anticipated in the existing finance literature.

When equilibrium is unique, we provide a full characterization of the equilibrium investment policies. The investments of the two managers at any interim point in time depend on their performance relative to each other. If a manager significantly outperforms the opponent, both managers are far from the convex region of their objectives and so the chasing and contrarian behaviors dominate the gambling incentives. In particular, when outperforming a chaser moves her investment policy *towards* the opponent's normal policy, while a contrarian tilts her policy *away* from the opponent's normal policy. For the underperforming manager, the relative concerns are weak since the likelihood of her attracting money flows at year-end is low, and hence her equilibrium policy is close to the normal policy. When both managers' performance is close to the threshold, the gambling incentives dominate the chasing and contrarian behaviors, inducing the two managers to gamble in the opposite direction from each other in each individual stock. The exact direction of a manager's gambling in each

[3]In other words, the important question of *when* multiplicity or non-existence occurs cannot be addressed by relying on the intuition or formal conditions developed in other strategic settings where these features are also present. For example, given the complexity of our framework it does not appear possible to check whether the conditions for existence (Baye, Tian, and Zhou (1993)) or multiplicity (Cooper and John (1988)) of pure-strategy equilibrium are satisfied in our setting.

stock is determined by the relation between the managers' risk aversions and the Sharpe ratios of the risky stocks.

Considering a setting with two risky stocks and looking at the effect of stock correlation, we show that the managers' policies are driven by the interaction of a diversification effect, the extent of benefits to diversification, and a substitution effect, the extent of the stocks acting as a substitute to each other. A higher stock correlation reduces the diversification benefits and so the managers tend to decrease their investments in both stocks. As the correlation increases further, the substitution effect kicks in leading to a decrease in the managers' investments in the less favorable stock (with a relatively low Sharpe ratio) and an increase in the more favorable stock (with a relatively high Sharpe ratio). As a result, the managers' investments in the more favorable stock are non-monotonic in correlation. Moreover, as the substitution effect grows stronger for high correlation values, each manager strategically changes the direction of gambling in the less favorable stock in order to amplify the gambling in the opposite direction in the more favorable stock.

Above equilibrium investments determine whether a manager ends up as a winner or as a loser at the investment horizon depending on economic conditions. In good states, the more risk averse manager performs worse (consistent with normal behavior) and hence is a loser, while the less risk averse manager with the better performance is a winner, getting the money flows. In bad states, the opposite holds, with the more risk averse manager emerging as the winner, and the less risk averse as the loser. In intermediate states, around their performance thresholds, both managers are losers, not getting any flows. Multiple equilibria obtains when the performance threshold is low, which rules out the unique equilibrium outcome of both managers being losers in intermediate states. Each manager now wants to differentiate herself from the opponent, and so both outcomes, with either manager being a winner and the other a loser, constitute a (multiple) equilibrium. The reason why a manager may be content with being a loser in a certain state is that choosing a relatively low wealth and losing in that state allows the manager to choose a relatively high wealth and win in another state. In good and bad states the multiple equilibrium outcomes are as in the unique equilibrium.

Our paper is related to several strands of literature. First is the literature on the effects of strategic considerations on portfolio managers' choices. Most of these works adopt single- or two-period settings and often assume risk neutrality (Goriaev, Palomino and Prat (2003), Taylor (2003), Palomino (2005), Li and Tiwari (2005), Chiang (1999), Loranth and Sciubba (2006)). Our goal is to characterize optimal portfolios in a standard dynamic asset allocation setting with risk averse managers. We find risk aversion to be the critical driving factor in much of our analysis, including chasing/contrarian behavior, risk shifting, existence of equilibrium. In a dynamic setting like ours, Browne (2000) investigates a portfolio game

between two managers. He primarily focuses on the case when the managers face different financial investment opportunities and have practical objectives (maximizing the probability of beating the other manager, minimizing the expected time of beating the other manager). None of these objective functions display local convexities, which we find to have important implications for equilibrium investment policies as well as for the number of equilibria.

If a peer group consists of a large number of competing funds, strategic interactions are likely to be less pronounced. In this case, the behavior of each fund manager is better described by assuming that she seeks to perform well relative to an exogenous benchmark. The manager's behavior in this case has been recently investigated in Basak, Shapiro, and Tepla (2006), van Binsbergen, Brandt, and Koijen (2007), Cuoco and Kaniel (2007). Several works, including Carpenter (2000), Basak, Pavlova, and Shapiro (2007), have demonstrated that convexities in managers' objective functions have significant implications for the optimal portfolios, leading to risk shifting behavior. We contribute to this literature by investigating how the risk shifting motives interact with strategic considerations and recover novel economic implications. First, strategic interactions can lead to non-existence or multiplicity of equilibrium. Second, the economic mechanism behind the chasing and contrarian behavior, which drives the equilibrium policies across the whole range of relative performance apart from the gambling region, is not present in the above non-strategic studies. Third, our two managers always gamble in the opposite direction from each other in each individual stock, and it is the manager's risk aversion *relative* to that of the opponent that determines the direction of gambling. This is notably different from the related works which find that the direction of gambling is either determined by the *absolute* value of the manager's risk aversion or is always the same. Fourth, we reveal that changing the correlation between risky stock returns leads to rich patterns in the behavior of strategic managers, which are not present in the existing non-strategic works.

Finally, our paper is related to the literature that examines the role of relative wealth concerns in finance. DeMarzo, Kaniel, and Kremer (2007, 2008) show that relative wealth concerns may play a role in explaining financial bubbles and excessive real investments. These papers are close in spirit to our work since they also demonstrate how relative wealth concerns may arise endogenously. However, their mechanism for the emergence of relative concerns is a general equilibrium one, and so is notably different from ours. In DeMarzo et al., there is a scarce consumption good whose price increases with the cohort's wealth, implying that an investor's relative wealth determines the quantity of the good she can afford. Abel (1990), Gomez, Priestley, and Zapatero (2008), among many others, demonstrate that models with the "catching-up-with-the-Joneses" feature can explain various empirically observed asset pricing phenomena. Goel and Thakor (2005) investigate how envy leads to corporate investment distortions.

Remainder of the paper is organized as follows. Section 2 presents the economic set-up and provides the money flows justification for relative performance concerns. Section 3 describes the managers' objective functions and characterizes their best responses. Section 4 analyzes the issues of non-existence, uniqueness, and multiplicity of equilibrium. Section 5 characterizes the unique and multiple equilibrium and investigates the properties of the equilibrium investment policies. Section 6 concludes. Proofs are in the Appendix.

2. Economy with Strategic Asset Allocation

2.1. Economic Set-Up

We adopt a familiar dynamic asset allocation framework along the lines of Merton (1969). We consider a continuous-time, finite horizon $[0, T]$ economy, in which the uncertainty is driven by an N-dimensional standard Brownian motion $\omega = (\omega_1, \ldots, \omega_N)^\top$. Financial investment opportunities are given by a riskless bond and N correlated risky stocks. The bond provides a constant interest rate r. Each stock price, S_j, follows a geometric Brownian motion

$$dS_{jt} = S_{jt}\mu_j dt + S_{jt}\sum_{k=1}^{N}\sigma_{jk}d\omega_{kt}, \qquad j = 1, \ldots, N,$$

where the stock mean returns $\mu \equiv (\mu_1, \ldots, \mu_N)^\top$ and the nondegenerate volatility matrix $\sigma \equiv \{\sigma_{jk},\ j, k = 1, \ldots N\}$ are constant, and the (instantaneous) correlation between stock j and ℓ returns is given by $\rho_{j\ell} = \sum_{k=1}^{N}\sigma_{jk}\sigma_{\ell k}/\sqrt{\sum_k \sigma_{jk}^2 \sum_k \sigma_{\ell k}^2}$.

Each money manager i in this economy dynamically chooses an investment policy ϕ_i, where $\phi_{it} \equiv (\phi_{i1}, \ldots, \phi_{iN})^\top$ denotes the vector of fractions of fund assets invested in each stock, or the *risk exposure*, given initial assets of W_{i0}. The investment wealth process of manager i, W_i, follows

$$dW_{it} = W_{it}[r + \phi_{it}^\top(\mu - r\bar{1})]dt + W_{it}\phi_{it}^\top\sigma d\omega_t, \tag{1}$$

where $\bar{1} = (1, \ldots, 1)^\top$. Dynamic market completeness (under no-arbitrage) implies the existence of a unique state price density process, ξ, with dynamics $d\xi_t = -\xi_t r dt - \xi_t \kappa^\top d\omega_t$, where $\kappa \equiv \sigma^{-1}(\mu - r\bar{1})$ is the constant N-dimensional market price of risk in the economy. The state-price density serves as the driving economic state variable in a manager's dynamic investment problem absent any market imperfections. The quantity $\xi_t(\omega)$ is interpreted as the Arrow-Debreu price per unit probability $\mathcal{P}$ of one unit of wealth in state $\omega \in \Omega$ at time t. In particular, each manager's dynamic budget constraint (**??**) can be restated as (e.g.,

Karatzas and Shreve (1998))

$$E[\xi_T W_{iT}] = W_{i0}. \tag{2}$$

This allows us to equivalently define the set of possible investment policies of managers as being the managers' horizon wealth, W_{iT}, subject to the static budget constraint (**??**).

2.2. Modeling Strategic Interaction

We envision a money manager, interpreted either as a mutual fund manager, a hedge fund manager or a trader, who seeks to increase the terminal value of her portfolio. This is consistent with maximizing her own compensation given the widespread use of the linear fee structure in the money management industry. The key feature of our setting is that the manager experiences money flows which depend on her relative performance within a peer group – we formalize this idea below. In this paper, we look at the scenario when all fund managers within this peer group have relative performance concerns driven by money flows, leading to the strategic interaction among the managers. We expect the effects of such interaction to be most pronounced when the peer group is comprised of a small number of funds. In this case, each manager is a major player in the competition for money flows implying that her decisions may significantly affect the other managers' actions. One natural real-life example is when the few very top funds attempt to become year-end top-performers. Another example is the competition of fund managers within fund families. Kempf and Ruenzi (2008) document the presence of strategic interactions in families that are comprised of a small number of funds.

In the presence of relative concerns, the objective function of manager i has the general form

$$v_i(W_{iT}, R_{iT}), \tag{3}$$

where v_i is increasing in horizon wealth, W_{iT}, and horizon relative return, R_{iT}. We consider a framework with two fund managers, indexed by $i = 1, 2$. The relative returns of managers 1 and 2, R_{1T} and R_{2T}, capture the relative performance concerns and are defined as the ratio of the two managers' time-T investment returns:

$$R_{1T} = \frac{W_{1T}/W_{10}}{W_{2T}/W_{20}}, \quad R_{2T} = \frac{W_{2T}/W_{20}}{W_{1T}/W_{10}}. \tag{4}$$

We normalize both managers' initial assets to be equal, $W_{10} = W_{20}$, without loss of generality (see Remark 3).

We now provide a rational justification for the objective function (**??**) by showing that it arises naturally in a setting where managers care directly only about their own wealth and

that its specific form used in this paper corresponds to the empirically observed fund-flows to relative performance relation. Towards this, the economic setting is extended as follows. The manager continues to invest beyond date T up until an investment horizon T'. Motivated by the empirical evidence for both mutual funds and hedge funds (Chevalier and Ellison (1997), Sirri and Tufano (1998), Agarwal, Daniel, and Naik (2004), Ding, Getmansky, Liang and Wermers (2007)), we assume that the manager experiences money flows at a rate f_T. The flow-performance relationship f_T is (weakly) increasing and convex in the manager's relative performance over the period [0,T], with $f_T > 1$ denoting an inflow and $f_T < 1$ an outflow. The manager's investment horizon T' (e.g., expected tenure, compensation date), of course, need not coincide with the money flows date T (e.g., quarter- or year-end). So, the objective function (**??**) is interpreted as the manager's indirect utility of post-flows horizon wealth.[4]

Manager i, $i = 1, 2$, is assumed to have CRRA preferences defined over the overall value of assets under management at time $T' > T$:

$$u_i\left(W_{T'}\right) = \frac{\left(W_{T'}\right)^{1-\bar{\gamma}_i}}{1-\bar{\gamma}_i}, \quad \bar{\gamma}_i > 0, \bar{\gamma}_i \neq 1. \tag{5}$$

Chevalier and Ellison (1997) document that for top-performing mutual funds the flow-performance relation is roughly flat until a certain threshold and then increases sharply. Similarly, Ding, Getmansky, Liang, and Wermers (2007) find that for a certain group of hedge funds, comprised to a large extent of top-performers, the flows sensitivity shoots up in the region of high past performance.[5] Based on this, we consider a flow-performance relation that resembles the convex payoff profile of a call option – it is flat until a certain *performance threshold* η and then increases with relative performance – given by $f_T = k\mathbb{1}_{\{R_{iT}<\eta\}} + k(R_{iT}/\eta)^{\alpha}\mathbb{1}_{\{R_{iT}\geq\eta\}}$. In this specification, α denotes the *flow elasticity*, the elasticity of money flows with respect to manager's relative performance when it is above the performance threshold η. The parameter k reflects the idea that a manager, being a top-performer, is likely to experience money inflows even when her relative performance is below the threshold η. This feature can be captured by setting $k > 1$. The performance threshold $\eta \geq 1$ captures the potential inertia of individual investors who do not respond to relative performance unless a manager outperforms the opponent by a certain margin. The case of $\eta = 1$ corresponds to the absence of inertia, where investors immediately reward manager i with money flows once her performance R_{iT} exceeds that of the opponent. The stronger

[4]Alternatively, the objective specification (**??**) could be interpreted as capturing the well-known psychological feature that people care about their relative standing in the society or in their profession.

[5]While Ding et al. find a linear relationship between flows and past performance for the universe of hedge funds, they document a convex relationship once they look at the so-called defunct database. One of the main reasons why a hedge fund becomes defunct is when it had an exceptionally good past performance and, as a result, no longer needs to advertise itself to potential investors by reporting the information to data vendors. Given that we focus on top-performing hedge funds, this evidence supports our assumption that the flow-performance relation is convex.

the inertia the higher the performance threshold η. For completeness, we comment on the case of $\eta < 1$ in Remark 1. It should be noted that the presence of the inertia does not imply any sort of investors' irrationality. On the contrary, Huang, Wei, and Yan (2007) formally show that performance thresholds endogenously arise when rational investors face information acquisition costs or transaction costs.[6]

The original optimization problem of maximizing the expected value of (**??**) is equivalent to maximizing the time-T indirect utility function v_{iT}, defined as

$$v_{iT} \equiv \max_{\phi_i} E_T \left[u_i \left(W_{iT'} \right) \right]$$

subject to the dynamic budget constraint (**??**) for $t \in [T, T']$, given the time-T assets value augmented by money flows, $W_{iT} f_T$. Lemma 1 presents the time-T indirect utility function for our flow-performance specification.

Lemma 1. *For the flow-performance function* $f_T = k\mathbb{1}_{\{R_{iT}<\eta\}} + k(R_{iT}/\eta)^{\alpha}\mathbb{1}_{\{R_{iT}\geq\eta\}}$, $\alpha > 0$, *the time-T indirect utility function is*

$$v_{iT} = \begin{cases} \frac{k^{1-\bar{\gamma}_i}}{1-\gamma_i} W_{iT}^{1-\bar{\gamma}_i} & R_{iT} < \eta \\ \frac{k^{1-\bar{\gamma}_i}\eta^{-\theta(1-\gamma_i)}}{1-\gamma_i} \left(W_{iT}^{1-\theta} R_{iT}^{\theta} \right)^{1-\gamma_i} & R_{iT} \geq \eta, \end{cases} \tag{6}$$

where

$$\theta = \frac{\alpha}{1+\alpha}, \tag{7}$$

$$\gamma_i = \bar{\gamma}_i + \alpha(\bar{\gamma}_i - 1), \tag{8}$$

with the properties that $\theta \in [0, 1)$, $\gamma_i > \bar{\gamma}_i$ *if and only if* $\bar{\gamma}_i > 1$.

Lemma 1 provides a rational justification for managers' having relative performance concerns, and quantifies the link between the shape of the flow-performance relation and the parameters of the manager's objective function. When the manager performs relatively poorly, she gets performance-insensitive money flows, and so inherits the objectives of a *normal manager* with no relative performance concerns. When the manager's relative return is above the performance threshold however, she gets money flows increasing with performance, and hence displays relative concerns, with the parameter θ capturing the manager's *relative*

[6] A real-life example of the use of thresholds is provided by the following description of the investment strategy of DAL Investment Company with 2 billion under management: "We rank them [mutual funds] based on the average performance ... Then we review the rankings to select the funds that are in the top 10% of their risk category for the portfolio. We have specific sell thresholds, and when they reach the threshold, we replace them with the current leaders."

performance bias, the extent to which she biases her objectives towards relative performance concerns. The special case of $\theta = 0$ (or equivalently $\alpha = 0$) corresponds to a normal manager. From (**??**), we also observe that the manager's attitude towards risk changes in the presence of money flows. Indeed, while $\bar{\gamma}_i$ represents the manager's *intrinsic risk aversion* (over terminal wealth $W_{iT'}$), the parameter γ_i captures her *effective risk aversion* (over the composite $W_{iT}^{1-\theta} R_{iT}^{\theta}$) in the presence of relative performance concerns.[7] Moreover, the manager's attitude towards risk is increased ($\gamma_i > \bar{\gamma}_i$) by the presence of money flows ($\alpha > 0$) for intrinsic risk aversions greater than unity ($\bar{\gamma}_i > 1$), and is decreased for intrinsic risk aversions less than unity.

We note that for our subsequent maximization problems to be well-defined, we will assume that the indirect utility function (**??**) is concave in the region $R_{iT} \geq \eta$. This is true if and only if $\bar{\gamma}_i > 1 - 1/(1+\alpha)$, which always holds for a relatively risk averse manager $\bar{\gamma}_i > 1$. We also note that a local convexity in our flow-performance specification leads to a local convexity in the resulting objective function (**??**), which will be shown to have important implications for many of our results. In addition to the fact the local convexities arise endogenously for a realistic flow-performance specification, Koijen (2008) provides an alternative supporting evidence for the presence of convexities in managers' objective functions. He derives the optimal portfolios of mutual fund managers for several different specifications of the objective function and then uses the portfolio data to back out the managers' risk aversions implied by each specification. Koijen uncovers that the only specification leading to plausible estimates of risk aversion is the one featuring local convexities.

2.3. Nash Equilibrium Policies

In this paper, we appeal to the Nash equilibrium notion to characterize managers' behavior in their strategic interaction via relative performance concerns.[8] In what follows, we assume that managers have common knowledge of the flow-performance relation and each other's objectives, in particular attitude towards risk $\bar{\gamma}_i$. In reality, an average fund manager may not know precisely her peers' risk aversions. However, this seems to be less of a concern in our setting given that we focus on the very top managers. Indeed, their portfolio strategies are often in the spotlight attracting a lot of attention from the analysts and financial media. Moreover, the managers' risk aversions can be estimated from the past performance data,

[7] The difference between the two risk aversion parameters stems from the fact that manager i assesses a gamble $(W_{iT}+\epsilon, W_{iT}-\epsilon)$ differently in cases with and without money inflows. In the latter case, ϵ represents an actual change in wealth . In the former case, changing wealth by ϵ leads to money in- or outflows and so effectively the manager faces a different gamble.

[8] As we employ the concept of Nash equilibrium, we assume that the managers act *non-cooperatively*. In principle, it is possible that other types of behavior, such as herding (Scharfstein and Stein (1990)), can take place. However, the empirical literature largely fails to find evidence of herding among fund managers (Grinblatt, Titman, and Wermers (1995), Dass, Massa, and Patgiri (2008)).

as demonstrated by Koijen (2008). In order to define a Nash equilibrium, we first introduce the best response policies. Throughout the paper, a symbol with a hat $\hat{}$ denotes an optimal best response quantity, while one with an asterisk * denotes an equilibrium quantity.

Definition 1. *For a given manager 2's dynamic policy ϕ_2, manager 1's best response $\hat{\phi}_1$ is the solution to the following maximization problem:*

$$\begin{aligned} &\max_{\phi_1} \quad E\left[v_1(W_{1T}, R_{1T})\right] \\ \textit{subject to} \quad &dW_{1t} = W_{1t}[r + \phi_{1t}^\top(\mu - r\bar{1})]dt + W_{1t}\phi_{1t}^\top \sigma d\omega_t. \end{aligned} \tag{9}$$

Similarly, for a given manager 1's dynamic policy ϕ_1, manager 2's best response $\hat{\phi}_2$ is the solution to the problem:

$$\begin{aligned} &\max_{\phi_2} \quad E\left[v_2(W_{2T}, R_{2T})\right] \\ \textit{subject to} \quad &dW_{2t} = W_{2t}[r + \phi_{2t}^\top(\mu - r\bar{1})]dt + W_{2t}\phi_{2t}^\top \sigma d\omega_t \end{aligned} \tag{10}$$

Definition 2. *A pure-strategy Nash equilibrium is a pair of investment policies (ϕ_{1t}^*, ϕ_{2t}^*, $t \in [0, T]$) such that:*
(i) ϕ_1^* is manager 1's best response to ϕ_2^*,
(ii) ϕ_2^* is manager 2's best response to ϕ_1^*.

In a Nash equilibrium, each manager strategically accounts for the actions of the other manager, and the pure-strategy equilibrium policies of the two managers are mutually consistent. As discussed previously, in our set-up, for a given horizon wealth W_{iT} satisfying the budget constraint (**??**) there exists a unique portfolio policy ϕ_{it}, $t \in [0, T]$, replicating it. Hence, for an equilibrium outcome in investment policies (ϕ_{1t}^*, ϕ_{2t}^*, $t \in [0, T]$), there is always an equivalent outcome in terms of horizon wealth policies (W_{1T}^*,W_{2T}^*). We make use of this duality by solving for the equilibrium horizon wealth, and then deriving the corresponding equilibrium investment policies. While we need to impose a certain restriction on the class of admissible dynamic policies in order to use this technique, it is straightforward to prove (proof available from the authors) that the resulting dynamic policies also constitute an equilibrium within the class of strategies without this restriction.[9]

[9]Specifically, when solving for manager 1's best-response horizon wealth, we assume that manager 2's horizon wealth is a fixed function of the state variable ξ_T. This corresponds to the restriction that manager 2 chooses her dynamic strategy at time 0 and does not subsequently change it when observing the dynamics of manager 1's wealth. It can be easily shown that if we start from the equilibrium strategies with this restriction and allow each manager to react to the trades of the other, then neither manager would optimally deviate from her time-0 strategy. A potentially interesting issue is whether there are other equilibria within the class of strategies without the restriction. Addressing this question turns out to be complex, and is beyond the scope of the current paper.

3. Managers' Objectives and Best Responses

In our analysis, we investigate a setting in which manager 1 is guided by the objective function

$$v_1(W_{1T}, W_{2T}) = \begin{cases} \frac{k^{1-\bar{\gamma}_1}}{1-\gamma_1} W_{1T}^{1-\bar{\gamma}_1} & W_{1T} < \eta W_{2T} \\ \frac{k^{1-\bar{\gamma}_1}}{1-\gamma_1} \left(W_{1T}^{1-\theta_1} (\frac{W_{1T}}{\eta W_{2T}})^{\theta_1} \right)^{1-\gamma_1} & W_{1T} \geq \eta W_{2T}, \end{cases} \tag{11}$$

as given by Lemma 1, where $\eta \geq 1$ is the performance threshold, and manager 2's objective function is as in (**??**) with subscripts 1 and 2 switched. Here, the convexity of flow-performance relation leads to an asymmetric perception of outperformance and underperformance by the manager, whereby only the former affects her normal objectives.[10] For ease of discussion, we henceforth refer to the manager with a below-threshold performance as not getting any flows (instead of getting constant flows), since all our subsequent results are not affected by the magnitude of constant flows k (Propositions 1–4).

Before proceeding with the formal analysis, we provide some basic intuition regarding how the optimal horizon wealth of manager 1 may be affected by manager 2's choice of horizon wealth in the presence of relative performance concerns when outperforming ($W_{1T} \geq \eta W_{2T}$). Suppose that manager 2's horizon wealth increases. This has the following two effects on manager 1. First, higher manager 2's wealth implies lower money flows for manager 1. As a result, manager 1 wants to increase her wealth so as to restore the previous level of flows. Second, higher W_{2T} reduces the incremental effect of a unit change in manager 1's wealth, making it costlier for manager 1 to increase her wealth. In the pivotal logarithmic case ($\gamma_1 = 1$), the two effects exactly offset each other. For a relatively risk averse manager, $\gamma_1 > 1$, the first effect dominates and manager 1 increases her wealth W_{1T}. For a relatively risk tolerant manager, $\gamma_1 < 1$, the second effect dominates and manager 1 decreases her wealth. Note that for $\gamma_1 > 1$, manager 1's response essentially means that manager 1 is *chasing* manager 2. On the other hand, for $\gamma_1 < 1$ manager 1 is a *contrarian* to manager 2. Chetty (2006), in different economic settings, and Koijen (2008), specifically for fund managers, document a substantial heterogeneity in the estimates of relative risk aversion, which suggests that both types of behavior – chasing and contrarian – are likely to be present in our money management context. The latter study also finds the average risk aversion of fund managers to be above unity, implying that one can expect the average real-life manager to be a chaser.

To highlight further features of the objective function (**??**), we plot in Figure **??** its typical shape. From Figure **??**, there are three distinct regions of the objective function,

[10] An alternative psychological interpretation is that that such an asymmetry can be due to the well-documented fact that people tend to attribute their success to skill and failure to bad luck (Zuckerman (1979)).

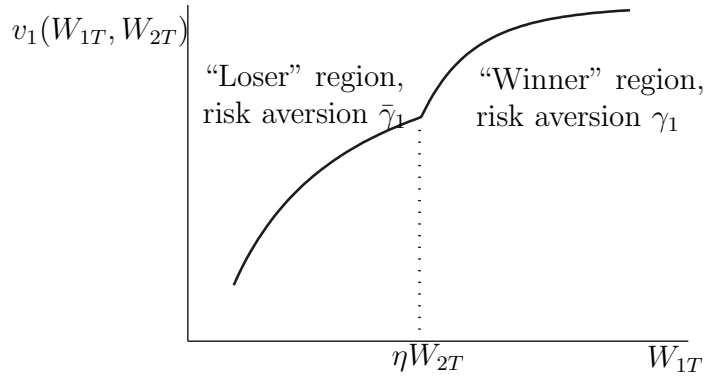


Figure 1: Manager 1's objective function.

depending on manager 1's relative performance at the horizon T, or equivalently on the relation between manager 1's wealth and the threshold level ηW_{2T}. When her wealth is above the threshold, the manager gets money flows, and hence we label her as the *winner*. In this case, she is in the region of objectives augmented by relative performance concerns, driven by her effective risk aversion γ_1. When manager 1's wealth is below the threshold, she does not get money flows, and so is labelled as the *loser*. In this case, the manager finds herself in the region of normal objectives with no relative performance concerns, driven by her intrinsic risk aversion $\bar{\gamma}_1$. Finally, when the performance is around the threshold level, the manager is in the region of local convexity. Consequently, there are two main differences from the conventional CRRA objective function. The first is that now both intrinsic and effective risk aversions directly enter the objective specification, and thus have distinct effects on the optimal policies. The second, and a major, difference is the presence of the local convexity in (**??**) which, as established in the existing literature (Basak, Pavlova, and Shapiro (2007) and Carpenter (2000)), leads to risk shifting behavior. We show that such convexities coupled with strategic interactions can result in multiple equilibria or no pure-strategy equilibrium. Given the novelty of these phenomena, we provide a detailed analysis of the economic conditions under which they may occur in the next Section 4. In Section 5, we fully characterize the equilibrium policies when an equilibrium (unique or multiple) exists and also investigate the effects of various parameters on the equilibrium policies.

We now determine the managers' best responses. The managers' optimization problems are non-standard since their objective functions are not globally concave. Nevertheless, interior solutions turn out to exist since the managers' risk aversions limit the sizes of their gambles over the locally convex regions. In Proposition **??** we report the managers' best responses explicitly in closed form.

Proposition 1. *The best responses of managers 1 and 2 are given by*

$$\hat{W}_{1T} = \begin{cases} (y_1\xi_T)^{-1/\bar{\gamma}_1} & y_1\xi_T > b_1(\eta W_{2T}) \ \ (loser) \quad (12) \\ (1+\alpha)^{1/\gamma_1}(y_1\xi_T)^{-1/\gamma_1}(\eta W_{2T})^{\theta(\gamma_1-1)/\gamma_1} & y_1\xi_T \leq b_1(\eta W_{2T}) \ \ (winner), \quad (13) \end{cases}$$

$$\hat{W}_{2T} = \begin{cases} (y_2\xi_T)^{-1/\bar{\gamma}_2} & y_2\xi_T > b_2(\eta W_{1T}) \ \ (loser) \quad (14) \\ (1+\alpha)^{1/\gamma_2}(y_2\xi_T)^{-1/\gamma_2}(\eta W_{1T})^{\theta(\gamma_2-1)/\gamma_2} & y_2\xi_T \leq b_2(\eta W_{1T}) \ \ (winner), \quad (15) \end{cases}$$

where the boundary functions $b_i(\cdot)$ are given by

$$b_i(W) = (1+\alpha)^{\bar{\gamma}_i/\theta}\left(\bar{\gamma}_i/\gamma_i\right)^{\bar{\gamma}_i\gamma_i/(\gamma_i-\bar{\gamma}_i)} W^{-\bar{\gamma}_i} \tag{16}$$

and $y_i > 0$ solves $E[\xi_T\hat{W}_{iT}] = W_{i0}, i = 1, 2$. Moreover, when manager i is a winner, her associated relative performance $\hat{R}_{iT}$ is bounded from below by the minimum outperformance margin, $\bar{\eta}_i$, given by $\bar{\eta}_i = (1+\alpha)^{-1/\alpha}(\bar{\gamma}_i/\gamma_i)^{-\bar{\gamma}_i/(\gamma_i-\bar{\gamma}_i)}\eta > \eta$. When manager i is a loser, her relative performance $\hat{R}_{iT}$ is bounded from above by the maximum underperformance margin, $\underline{\eta}_i$, given by $\underline{\eta}_i = (1+\alpha)^{-(1+\alpha)/\alpha}(\bar{\gamma}_i/\gamma_i)^{-\gamma_i/(\gamma_i-\bar{\gamma}_i)}\eta < \eta$.

Focusing on manager 1, she chooses whether to be a winner or a loser depending on the level of the threshold wealth, ηW_{2T}, relative to the cost of wealth in that state, ξ_T, where the threshold wealth affects manager 1 through the decreasing boundary function $b_1(\cdot)$. For a relatively low threshold wealth (low manager 2's wealth or performance threshold), manager 1 optimally becomes a winner, outperforming the threshold ηW_{2T}, in which case her best response (**??**) is given by by a normal policy, $(y_1\xi_T)^{-1/\gamma_1}$, augmented by the component $W_{2T}^{\theta_1(\gamma_1-1)/\gamma_1}$, accounting for relative performance concerns. The additional component formalizes the basic intuition offered previously. When a chaser ($\gamma_1 > 1$), manager 1 increases her optimal wealth in response to manager 2's increasing hers. Vice versa, if a contrarian ($\gamma_1 < 1$), manager 1 decreases her wealth in response to manager 2's increasing hers. Otherwise, for a relatively high threshold wealth, manager 1 opts to be a loser, in which case her best response (**??**) follows the normal policy.

An important feature here is that a manager only considers two outcomes: winning or losing, never opting for a "draw" by choosing her relative performance $\hat{R}_{iT}$ to be equal or close to the threshold η. This is due to the convexity of her the objective function around the threshold, inducing her to gamble so as to end up either a winner or a loser, thus avoiding a draw. As presented in Proposition **??**, formally, there exists a manager-specific *minimum outperformance margin* $\bar{\eta}_i$, greater than η, so that a winner's relative performance can never be below this margin. Similarly, there is a *maximum underperformance margin* $\underline{\eta}_i$, less than η, so that a loser's relative performance can never be above this margin.

4. Existence and Uniqueness of Pure-Strategy Equilibrium

As we demonstrate below, the managers' risk-shifting motives, together with their strategic interaction, may prevent an equilibrium to occur. In this Section, we investigate the underlying economic mechanisms and establish the conditions for the existence and uniqueness of a Nash equilibrium.

To find a Nash equilibrium, we look for mutually consistent best responses of the two managers. Since for a given state of nature, represented by a realization of ξ_T, a manager is either a winner or a loser, there are four possibilities for an equilibrium with two managers. However, both managers cannot be winners as the performance threshold η is greater or equal to 1, and so at most one manager can get money flows. Hence, for each ξ_T there are three possible Nash equilibrium outcomes, denoted by (manager 1 outcome, manager 2 outcome): (winner, loser), (loser, winner), or (loser, loser). Note that the condition for the (loser, loser) outcome is $1/\eta < W_{1T}/W_{2T} < \eta$, which is only possible if $\eta > 1$. A Nash equilibrium exists if the three possible outcomes fully cover the interval $(0, +\infty)$, which represents all possible states of the world ξ_T. There are multiple equilibria whenever any two intervals of ξ_T over which two outcomes occur overlap, meaning that for some ξ_T the outcome is not uniquely defined.

The existence of an equilibrium, however, is not always possible in this setting. To see how non-existence may arise, assume for ease of exposition that the performance threshold η equals one, ruling out the potential (loser, loser) outcome. Consider a situation when manager 1's minimum outperformance margin $\bar{\eta}_1$ is close to one (her relative performance has to exceed $\bar{\eta}_1$ for winning (Proposition **??**)), while manager 2's maximum underperformance margin $\underline{\eta}_2$ is much lower than one (her relative performance has to be below $\underline{\eta}_2$ for losing). In this situation, in some states manager 1 may want to outperform manager 2 by just a little, while manager 2, whenever she decides to be a loser, wants to underperform by a lot. Hence, the managers cannot agree on the winning/losing margin, resulting in the non-existence of equilibrium.

This non-existence issue would be resolved if the two managers had (sufficiently) similar risk aversions, and hence similar performance margins. Indeed, an equilibrium would now exist because the winner's outperformance margin $\bar{\eta}_1$ is consistent with the loser's underperformance margin $\underline{\eta}_2$. However, if in a certain state of the world the outcome with manager 1 a winner and manager 2 a loser constitutes an equilibrium, the opposite outcome is also likely to be an equilibrium as the managers are (almost) symmetric – in other words, the multiplicity of equilibrium arises. Intuitively, since each manager is aware of her own and

the other's budget constraint, a manager is indifferent between being a winner and a loser in a particular state since choosing a relatively low wealth (being a loser) in one state will allow her to have a relatively high wealth (be a winner) in another state.

Proposition **??** formalizes the discussion above by providing conditions on the model's parameters for the existence and uniqueness of an equilibrium.

Proposition 2.
(i) A unique pure-strategy Nash equilibrium occurs when

$$\eta > \max\left[(B/A)^{1/(2\bar{\gamma}_1\bar{\gamma}_2)}, (C/A)^{1/(2\gamma_1\bar{\gamma}_2)}, (B/D)^{1/(2\bar{\gamma}_1\gamma_2)}\right]; \tag{17}$$

(ii) multiple pure-strategy Nash equilibria occur when

$$\max\left[A\eta^{\bar{\gamma}_1\bar{\gamma}_2}, C\eta^{-\bar{\gamma}_2(\gamma_1+\theta(\gamma_1-1))}\right] \leq \min\left[B\eta^{-\bar{\gamma}_1\bar{\gamma}_2}, D\eta^{\bar{\gamma}_1(\gamma_2+\theta(\gamma_2-1))}\right]; \tag{18}$$

*(iii) if neither (**??**) nor (**??**) is satisfied, there is no pure-strategy Nash equilibrium.*
In the above, the constants A, B, C, D are given by

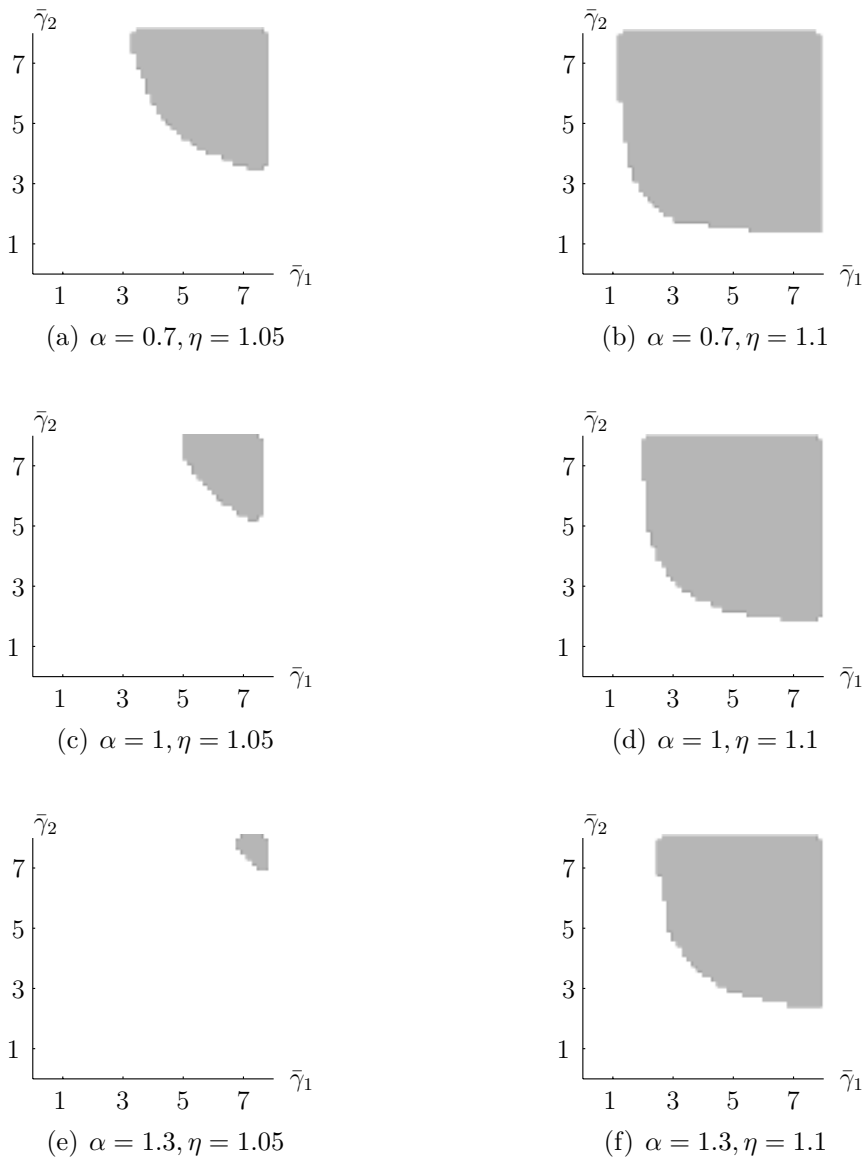
$$A = (1+\alpha)^{-\bar{\gamma}_1\bar{\gamma}_2/\theta}\left(\bar{\gamma}_1/\gamma_1\right)^{-\bar{\gamma}_1\gamma_1\bar{\gamma}_2/(\gamma_1-\bar{\gamma}_1)}, B = (1+\alpha)^{\bar{\gamma}_1\bar{\gamma}_2/\theta}\left(\bar{\gamma}_2/\gamma_2\right)^{\bar{\gamma}_1\gamma_2\bar{\gamma}_2/(\gamma_2-\bar{\gamma}_2)}, \tag{19}$$

$$C = (1+\alpha)^{\gamma_1\bar{\gamma}_2/\theta-\bar{\gamma}_2}\left(\bar{\gamma}_2/\gamma_2\right)^{\gamma_1\gamma_2\bar{\gamma}_2/(\gamma_2-\bar{\gamma}_2)}, D = (1+\alpha)^{\bar{\gamma}_1-\gamma_2\bar{\gamma}_1/\theta}\left(\bar{\gamma}_1/\gamma_1\right)^{-\bar{\gamma}_1\gamma_1\gamma_2/(\gamma_1-\bar{\gamma}_1)}. \tag{20}$$

Proposition **??** reveals that a unique equilibrium occurs when the performance threshold η is above a certain critical value.[11,12] As discussed, the non-existence arises when a potential equilibrium is destroyed by a loser who, after observing that the winner's performance is only slightly higher, increases her performance in an attempt to become the winner. When η increases, it is less likely that the loser would try to become the winner, since it would require a larger increase in performance. As for the multiplicity, it occurs when in some states the managers may switch their ranking, with the winner becoming the loser, and vice versa. When the performance threshold is high, both managers optimally choose to be losers

[11] To see that the unique- and multiple-equilibria conditions, (**??**) and (**??**), cannot hold simultaneously, note that from (**??**) it follows that $\eta > (B/A)^{1/(2\bar{\gamma}_1\bar{\gamma}_2)}$, or equivalently, $A\eta^{\bar{\gamma}_1\bar{\gamma}_2} > B\eta^{-\bar{\gamma}_1\bar{\gamma}_2}$, while (**??**) implies that $A\eta^{\bar{\gamma}_1\bar{\gamma}_2} < B\eta^{-\bar{\gamma}_1\bar{\gamma}_2}$, contradicting (**??**).

[12] When the performance threshold η is equal (or close) to 1, the outcome (loser, loser) cannot happen in equilibrium. Hence, for an equilibrium to exist, the conditions for the two remaining outcomes have to cover the interval $(0, +\infty)$. As shown in the Appendix, the outcome (winner, loser) is an equilibrium when ξ_T is higher than the left-hand side in the multiple-equilibria condition (**??**), while (loser, winner) is an equilibrium when ξ_T is lower than the right-hand side in (**??**). Hence, if (**??**) is satisfied, for every $\xi_T \in (0, +\infty)$ at least one of the two outcomes is an equilibrium. In the region where ξ_T is between the left- and right-hand side of (**??**), any of the two outcomes can occur in equilibrium, giving rise to multiple equilibria. The condition for uniqueness where the performance threshold η is above a certain critical value (**??**) admits the outcome (loser, loser) for some states ξ_T. It turns out that in this case the regions corresponding to the three equilibrium outcomes cover all possible states of the world and do not overlap. Hence, a unique equilibrium occurs.



(a) $\alpha = 0.7, \eta = 1.05$ (b) $\alpha = 0.7, \eta = 1.1$

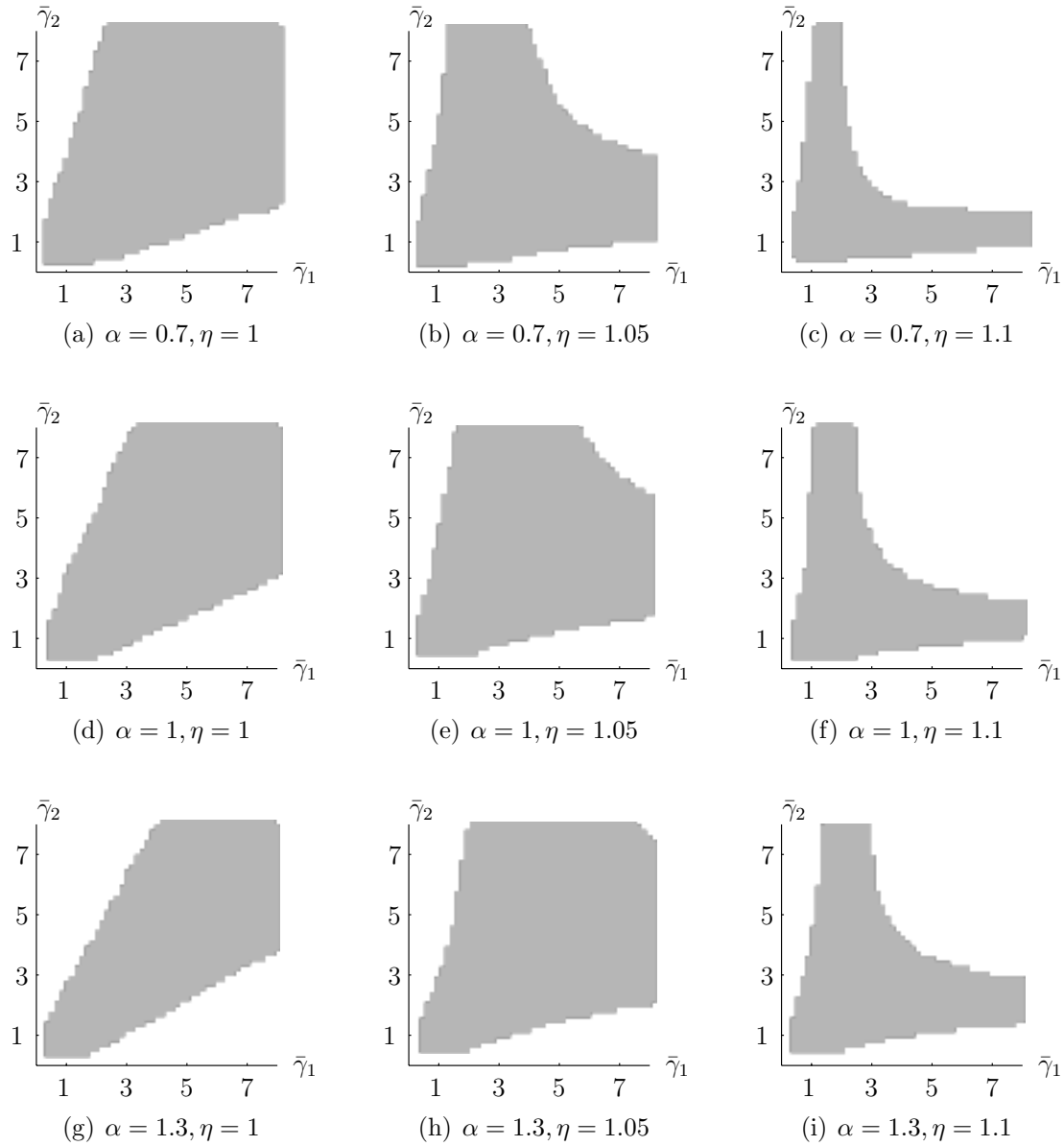
(c) $\alpha = 1, \eta = 1.05$ (d) $\alpha = 1, \eta = 1.1$

(e) $\alpha = 1.3, \eta = 1.05$ (f) $\alpha = 1.3, \eta = 1.1$

Figure 2: Unique Nash equilibrium. The filled area corresponds to the pairs of managers' intrinsic risk aversions $(\bar{\gamma}_1, \bar{\gamma}_2)$ for which the unique pure-strategy Nash equilibrium occurs. The flow elasticity α increases as we go down the plots. The relative performance threshold η increases as we go from the left to the right.

in some states of the world. In this case, the mechanism behind multiplicity – switching rankings – does not work since the rankings are the same, and so the unique equilibrium obtains. Given the convoluted nature of conditions for uniqueness and multiplicity, (**??**) and (**??**), we investigate when they are likely to be satisfied numerically.

Figure **??** depicts the values of the managers' intrinsic risk aversions, $\bar{\gamma}_1$ and $\bar{\gamma}_2$, for which the unique equilibrium occurs, i.e. when (**??**) is satisfied. For a given flow elasticity α and performance threshold η, the unique equilibrium is likely to exist for higher values of the risk aversions. The reason is that in states of the world when *both* managers are losers, high risk



(a) $\alpha = 0.7, \eta = 1$ (b) $\alpha = 0.7, \eta = 1.05$ (c) $\alpha = 0.7, \eta = 1.1$

(d) $\alpha = 1, \eta = 1$ (e) $\alpha = 1, \eta = 1.05$ (f) $\alpha = 1, \eta = 1.1$

(g) $\alpha = 1.3, \eta = 1$ (h) $\alpha = 1.3, \eta = 1.05$ (i) $\alpha = 1.3, \eta = 1.1$

Figure 3: Multiple Nash Equilibria. The filled area corresponds to the pairs of managers' intrinsic risk aversions $(\bar{\gamma}_1, \bar{\gamma}_2)$ for which multiple Nash equilibria obtain. The flow elasticity α increases as we go down the plots. The performance threshold η increases as we go from the left to the right.

aversions prevent each manager from gambling that would destroy the equilibrium. As we have just discussed, increasing the performance threshold η works against both non-existence and multiplicity, hence the region $(\bar{\gamma}_1, \bar{\gamma}_2)$ where the unique equilibrium exists expands when η increases, as seen by comparing the right panels to the left ones in Figure **??**. Finally, increasing the flow elasticity α (moving from the top to bottom plots in Figure **??**) amplifies the incentives to gamble for both managers, leading to a smaller region of $(\bar{\gamma}_1, \bar{\gamma}_2)$ for which the equilibrium exists.

Figure **??** plots the values of the managers' intrinsic risk aversions for which multiple equilibria occur, i.e., when (**??**) is satisfied. When the performance threshold is low – plots (a), (d), (g) on the left – multiple equilibria obtain provided that the risk aversions are not very different, so that the winner's outperformance margin is consistent with the loser's underperformance margin. As the flow elasticity increases, moving from plot (a) down to (g), the gambling behavior becomes more pronounced, and so the set of risk aversions for which equilibria obtain shrinks. As the performance threshold η increases, we get the unique equilibrium for high risk aversions, as depicted in Figure **??**. Hence, we no longer have multiple equilibria in the region of high risk aversion, as evident in Figure **??**, plots (c), (f), and (i) on the right. Finally, we note that increasing the flow elasticity α has a larger effect when the performance threshold is low (plots (a), (d), and (g)) than when the threshold is high (plots (c), (f), and (i)). This is explained by recalling that the non-existence arises when *both* managers are in the convex region and cannot agree on who the winner is. If the threshold η is high, only one manager can be in her risk-shifting region in a given state. For example, manager 1 is in her risk-shifting region when her relative performance R_{1T} is close to η, meaning that the relative performance of manager 2, $R_{2T} = 1/R_{1T}$, is close to $1/\eta$, which is far from her gambling level η when η is high. As a result, increasing the flow elasticity α has little impact on the existence of equilibrium when η is high.

Given the novelty of our results pertaining to multiplicity and non-existence within a standard workhorse asset allocation framework, it is natural to address their robustness. In parallel work, we investigate a strategic game among money managers in a similar setting but when local convexities are absent and show that a pure-strategy equilibrium always obtains and is unique. Hence, the key assumption that leads to multiplicity and non-existence is the convexity of the flow-performance relationship, and consequently the local convexities in the managers' objective functions. As discussed earlier, there is extensive empirical literature supporting this assumption.

Remark 1. *Low performance threshold,* $\eta < 1$. While in our setting the assumption that the performance threshold η is higher than or equal to unity seems justified (see discussion of investor inertia in Section 2.2), it is possible to envision settings in which the threshold is lower than one. For example, Murphy (1999) documents that the prevalent executive compensation contract in the U.S. is the so called 80/120 plan, whereby the manager receives a fixed base salary plus a bonus if her performance exceeds 80% of a pre-specified performance standard. If we take an industry with a few large companies, it may well be that the CEOs of these companies behave strategically, and our framework could be applied once the performance threshold is set at 0.8. It turns out the condition for multiplicity (**??**) equally applies to this case. However, unlike the case of $\eta > 1$, unique equilibrium is not possible. When η drops below a certain value, (**??**) is no longer satisfied and the equilibrium does

not exist. To understand why low performance threshold leads to non-existence, we note that there emerges an additional outcome when both managers are winners. In this (winner, winner) outcome, both managers' objective functions are augmented by relative concerns, and so each manager's actions affect the other's marginal utility. This imposes an additional restriction on the managers' behavior, as compared to the case $\eta > 1$, where we always have at least one loser whose marginal utility is not affected by the winner's actions. This restriction leads to the non-existence of equilibrium.

5. Equilibrium Investment Policies and Properties

In this Section, we describe the (possibly multiple) equilibrium policies and investigate their properties.

5.1. Characterization of Unique Equilibrium

We start with the case of the unique equilibrium that admits a more comprehensive analysis. Proposition **??** fully characterizes the unique investment policies and horizon wealth profiles.

Proposition 3. *Assume the condition for the existence and uniqueness of a Nash equilibrium (**??**) is satisfied. The equilibrium investment policy of manager 1, ϕ_1^*, is given by*

$$\begin{aligned}\phi_{1t}^* &= (\sigma^\top)^{-1}\kappa\Big\{(1-N(d(\tilde\gamma,\beta)))\tilde y_1 Z(\tilde\gamma,t)\xi_t^{-1/\tilde\gamma}/\tilde\gamma \\ &- n(d(\tilde\gamma,\beta))\tilde y_1 Z(\tilde\gamma,t)\xi_t^{-1/\tilde\gamma}/(\|\kappa\|\sqrt{T-t}) + N(d(\bar\gamma_1,\beta))y_1^{-1/\bar\gamma_1}Z(\bar\gamma_1,t)\xi_t^{-1/\bar\gamma_1}/\bar\gamma_1 \\ &+ n(d(\bar\gamma_1,\beta))y_1^{-1/\bar\gamma_1}Z(\bar\gamma_1,t)\xi_t^{-1/\bar\gamma_1}/(\|\kappa\|\sqrt{T-t})\Big\}/W_{1t}^*, \qquad (21)\end{aligned}$$

where $N(\cdot)$ is the standard-normal cumulative distribution function, $n(\cdot)$ is the corresponding probability density function, $\|\cdot\|$ denotes the norm and

$$\begin{aligned}\tilde y_1 &= y_1^{-1/\gamma_1}y_2^{-\alpha(\bar\gamma_1-1)/(\gamma_1\bar\gamma_2)}((1+\alpha)^{-1}\eta^{-\alpha(\bar\gamma_1-1)})^{-1/\gamma_1},\ \tilde\gamma = \frac{\gamma_1\bar\gamma_2}{\bar\gamma_2+\alpha(\bar\gamma_1-1)}, \\ Z(z,t) &= e^{((1-z)/z)\left(r+\|\kappa\|^2/(2z)\right)(T-t)},\ d(z,x) = \frac{\ln(x/\xi_t)+(r+(2-z)\|\kappa\|^2/(2z))\,(T-t)}{\|\kappa\|\sqrt{T-t}}, \\ \beta &= \left[\eta^{\bar\gamma_1\bar\gamma_2}y_2^{-\bar\gamma_1}y_1^{\bar\gamma_2}(1+\alpha)^{-\bar\gamma_1\bar\gamma_2/\theta}\left(\bar\gamma_1/\gamma_1\right)^{-\bar\gamma_1\gamma_1\bar\gamma_2/(\gamma_1-\bar\gamma_1)}\right]^{1/(\bar\gamma_1-\bar\gamma_2)},\end{aligned}$$

*$y_i > 0$ solves $E[\xi_T W_{iT}^*] = W_{i0}$, and W_{1t}^* is given in the Appendix. The equilibrium portfolio policy of manager 2, ϕ_{2t}^*, is as in (**??**) with subscripts 1 and 2 switched.*

The associated equilibrium outcomes and wealth profiles (W_{1T}^*, W_{2T}^*) at the horizon are as follows.

When $\xi_T^{\bar{\gamma}_1-\bar{\gamma}_2} \geq yA\eta^{\bar{\gamma}_1\bar{\gamma}_2}$*, the managers are in (winner, loser) and*

$$W_{1T}^* = \tilde{y}_1 \xi_T^{-(\bar{\gamma}_2+\theta(\gamma_1-1))/(\gamma_1\bar{\gamma}_2)}, \quad W_{2T}^* = (y_2\xi_T)^{-1/\bar{\gamma}_2}.$$

When $yB\eta^{-\bar{\gamma}_1\bar{\gamma}_2} \leq \xi_T^{\bar{\gamma}_1-\bar{\gamma}_2} < yA\eta^{\bar{\gamma}_1\bar{\gamma}_2}$*, the managers are in (loser, loser) and*

$$W_{1T}^* = (y_1\xi_T)^{-1/\bar{\gamma}_1}, \; W_{2T}^* = (y_2\xi_T)^{-1/\bar{\gamma}_2}.$$

When $\xi_T^{\bar{\gamma}_1-\bar{\gamma}_2} < yB\eta^{-\bar{\gamma}_1\bar{\gamma}_2}$*, the managers are in (loser, winner) and*

$$W_{1T}^* = (y_1\xi_T)^{-1/\bar{\gamma}_1}, \; W_{2T}^* = \tilde{y}_2 \xi_T^{-(\bar{\gamma}_1+\theta(\gamma_2-1))/(\gamma_2\bar{\gamma}_1)}.$$

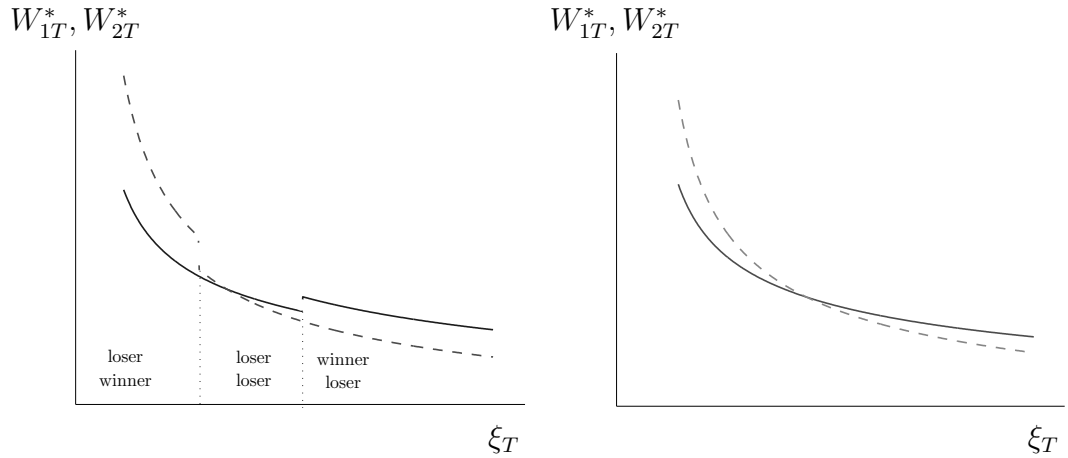
where A *and* B *are as given in (**??**) of Proposition **??**,* $y = y_2^{-\bar{\gamma}_1} y_1^{\bar{\gamma}_2}$*, and*

$$\tilde{y}_2 = y_2^{-1/\gamma_2} y_1^{-\alpha(\bar{\gamma}_2-1)/(\gamma_2\bar{\gamma}_1)} ((1+\alpha)^{-1}\eta^{-\alpha(\bar{\gamma}_2-1)})^{-1/\gamma_2}.$$

We first look at the managers' equilibrium horizon wealth. Figure **??** plots the equilibrium (latter part of Proposition **??**), as well as the normal wealth profiles, as a function of the state price density ξ_T. From Figure **??**(a), in good states (low ξ_T), the less risk averse manager 2 has a higher equilibrium wealth than manager 1, in line with the normal wealth profiles as depicted in Figure **??**(b). In these states, manager 1 is a loser and manager 2 is a winner, getting the money flows. As we move into intermediate states (middle-ξ_T region), manager 2's relative performance decreases, and after hitting her minimum outperformance margin $\bar{\eta}_2$, jumps down as manager 2 optimally becomes a loser, no longer getting any flows. Finally, as economic conditions deteriorate (high ξ_T), manager 1's relative performance increases, and after reaching the maximum underperformance margin $\underline{\eta}_1$, it jumps upwards as manager 1 becomes a winner and receives money flows. From the viewpoint of potential fund investors, Figure **??** illustrates the importance of accounting for the managers' relative performance concerns. The effect is most pronounced in good and bad states, where the presence of strategic interactions strongly amplifies the difference between the returns on the managers' portfolios.

Turning to the equilibrium investment policies, we first investigate the managers' behavior in the deep outperformance and underperformance regions at an interim point in time $t < T$. Corollary 1 characterizes the managers' limiting behavior in the interim bad (high ξ_t) and good (low ξ_t) states, henceforth referred to as the interim (winner, loser) and (loser, winner) regions, respectively.

Corollary 1. *In the interim (winner, loser) region, the limiting equilibrium investment poli-*



(a) Equilibrium horizon wealth profiles (b) Normal horizon wealth profiles

Figure 4: Managers' unique equilibrium horizon wealth. Equilibrium and normal horizon wealth profiles for the more risk averse manager 1 (solid plots) and the more risk tolerant manager 2 (dashed plots). The plots are typical.

cies of managers 1 and 2, $\phi_1^*(\infty) = \lim_{\xi_t \to \infty} \phi_1^*(\xi_t)$ *and* $\phi_2^*(\infty) = \lim_{\xi_t \to \infty} \phi_2^*(\xi_t)$*, are*

$$\phi_1^*(\infty) = \frac{1}{\gamma_1}(\sigma^\top)^{-1}\kappa + \frac{\theta(\gamma_1 - 1)}{\gamma_1 \bar{\gamma}_2}(\sigma^\top)^{-1}\kappa, \quad \phi_2^*(\infty) = \frac{1}{\bar{\gamma}_2}(\sigma^\top)^{-1}\kappa.$$

In the interim (loser, winner) region, the limiting equilibrium investment policies of managers 1 and 2, $\phi_1^*(0) = \lim_{\xi_t \to 0} \phi_1^*(\xi_t)$ *and* $\phi_2^*(0) = \lim_{\xi_t \to 0} \phi_2^*(\xi_t)$*, are*

$$\phi_1^*(0) = \frac{1}{\bar{\gamma}_1}(\sigma^\top)^{-1}\kappa, \quad \phi_2^*(0) = \frac{1}{\gamma_2}(\sigma^\top)^{-1}\kappa + \frac{\theta(\gamma_2 - 1)}{\gamma_2 \bar{\gamma}_1}(\sigma^\top)^{-1}\kappa.$$

Consequently, the directions in which the managers' limiting equilibrium policies deviate from their normal policies are given by

$$\begin{aligned} \operatorname{sgn}(\phi_1^*(\infty) - \phi_1^N) &= \operatorname{sgn}(\bar{\gamma}_1 - 1)\operatorname{sgn}(\phi_2^N - \phi_1^N), && (22) \\ \operatorname{sgn}(\phi_2^*(0) - \phi_2^N) &= \operatorname{sgn}(\bar{\gamma}_2 - 1)\operatorname{sgn}(\phi_1^N - \phi_2^N), && (23) \end{aligned}$$

where manager i's normal policy is $\phi_i^N = (\sigma^\top)^{-1}\kappa/\bar{\gamma}_i$*.*

Equations (**??**)–(**??**) of Corollary 1 reveal that when outperforming, the direction in which manager i adjusts her policy from the normal due to strategic interaction is determined by whether she is a chaser ($\bar{\gamma}_i > 1$) or a contrarian ($\bar{\gamma}_i < 1$), and whether her normal policy exceeds the opponent's normal policy or not. In particular, if manager i is a chaser, she moves her investment policy from the normal *towards* the normal policy of the opponent. On the other hand, if manager i is a contrarian, she tilts her investment policy *away* from the opponent's normal. These implications can be understood by recalling our earlier discussion

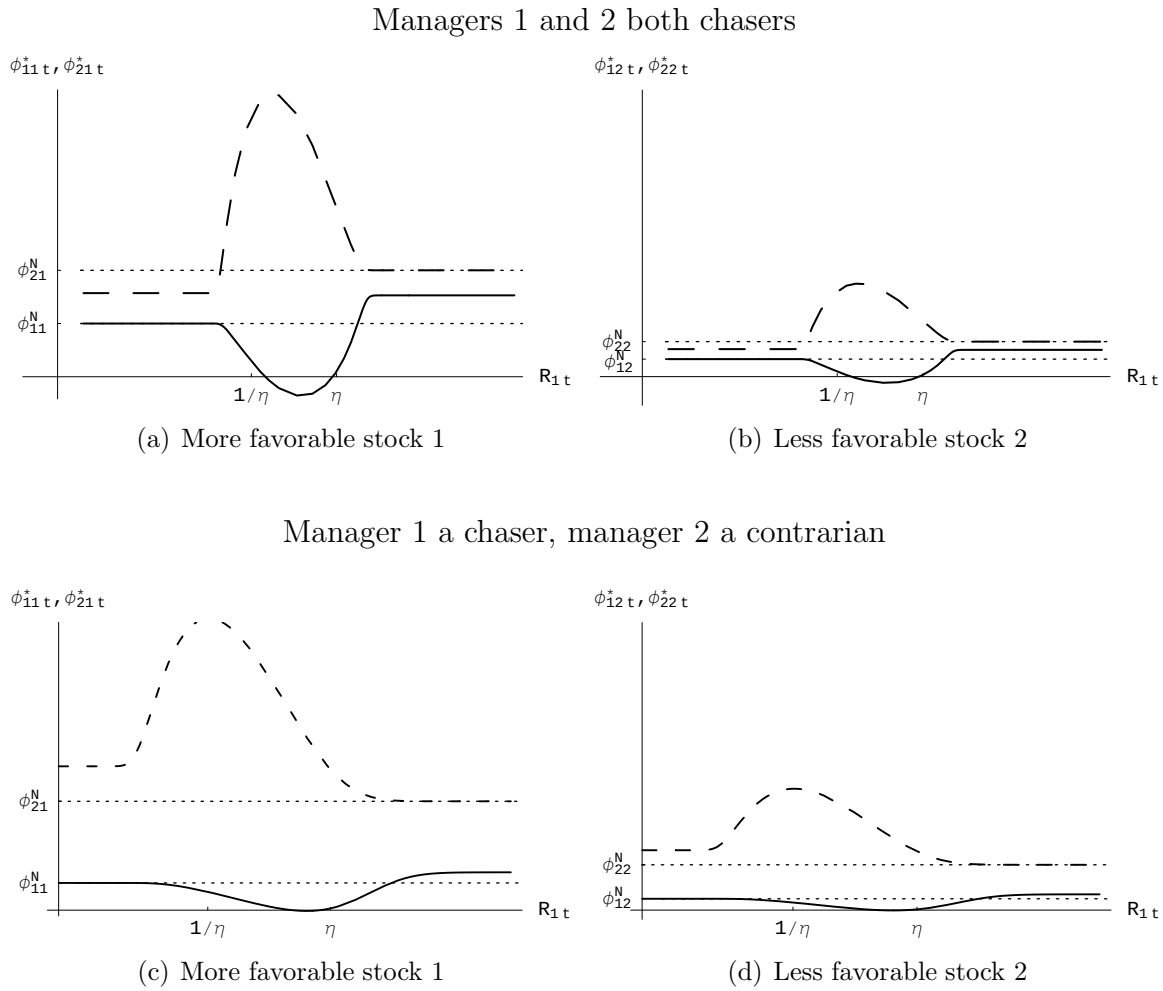
of the chasing and contrarian behavior in the presence of relative performance concerns. For example, consider the case where manager 1 is a chaser ($\bar{\gamma}_1 > 1$) while manager 2 is a contrarian ($\bar{\gamma}_2 < 1$). Here, the normal policy of the more risk averse manager 1 is less risky than that of the more risk tolerant manager 2, $\phi_1^N < \phi_2^N$. Being a chaser, manager 1 moves her policy towards the normal policy of manager 2, leading to a higher than normal risk exposure, as predicted by (**??**). Manager 2 is a contrarian and so moves her policy away from the normal policy of manager 1, leading to a higher than normal risk exposure, as implied by (**??**).

To best highlight the managers' overall asset allocation behavior, Figure **??** plots managers' equilibrium investments (equation (**??**)) as a function of manager 1's relative performance at time t, $R_{1t} = W_{1t}/W_{2t}$, with a relatively high R_{1t} ($> \eta$) corresponding to the interim (winner, loser) region and a relatively low R_{1t} ($< 1/\eta$) to the interim (loser, winner) region. For expositional clarity, we specialize the economy to feature two risky stocks, 1 and 2, and without loss of generality let the stock return volatility matrix to be

$$\sigma = \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{pmatrix}$$

where ρ is the correlation between stock 1 and 2 returns. Here, the market prices of risk are $\kappa_1 = (\mu_1 - r)/\sigma_1$ and $\kappa_2 = [(\mu_2 - r)/\sigma_2 - \rho(\mu_1 - r)/\sigma_1]/\sqrt{1-\rho^2}$. For the parameter values in Figure **??**, stock 1 is the more favorable of the two stocks in the sense that the Sharpe ratio of the stock 1 ($(\mu_1 - r)/\sigma_1$) is relatively high as compared to the Sharpe ratio of stock 2 ($(\mu_2 - r)/\sigma_2$). As a result, both managers tend to invest a larger share of wealth in stock 1 than in stock 2. Apart from this, the profiles of the equilibrium investments in stocks 1 and 2 are similar, as seen by comparing the left panels of Figure **??**, (a) and (c), with the right panels, (b) and (d). Hence, in our discussion we focus only on the equilibrium investments in stock 1 (Figure **??**(a), (c)).

Figure **??**(a) corresponds to the case when both managers are chasers. In the interim (winner, loser) region manager 1 chases manager 2, which from Corollary 1 implies that manager 1 increases her risk exposure as compared to normal and moves her policy towards manager 2's normal policy. For manager 2, the relative performance concerns are weak in the interim (winner, loser) region since the likelihood of attracting money flows at year-end is low. As a result, the equilibrium policy of manager 2 is close to her normal policy. In the interim (loser, winner) region, manager 2, a chaser, decreases her risk exposure relative to normal as she tilts her investment policy towards manager 1's normal policy. The equilibrium policy of manager 1 is close to normal since the effect of relative performance concerns is small. In Figure **??**(c), manager 1 is a chaser but manager 2 is a contrarian. In the interim (winner, loser) region, the outcome of the strategic interaction is as in Figure **??**(a): manager



(a) More favorable stock 1 (b) Less favorable stock 2

(c) More favorable stock 1 (d) Less favorable stock 2

Figure 5: The managers' unique equilibrium investment policies. The time-t investment policies of the more risk averse manager 1 (solid lines) and the more risk tolerant manager 2 (dashed lines) in stock 1 and stock 2. In each panel, the lower dotted line is the normal policy of manager 1 $\phi_1^N = (\sigma^\top)^{-1}\kappa/\bar{\gamma}_1$, the upper dotted line is the normal policy of manager 2 $\phi_2^N = (\sigma^\top)^{-1}\kappa/\bar{\gamma}_2$. Stock 1 is more favorable than stock 2 as it has a relatively higher Sharpe ratio. In panels (a) and (b), the two managers are chasers, and the parameter values are: $r = 0.05, \mu_1 = 0.1, \mu_2 = 0.12, \sigma_1 = 0.15, \sigma_2 = 0.3, \rho = 0.3, \bar{\gamma}_1 = 4, \bar{\gamma}_2 = 2, \alpha = 1.5, \eta = 1.2, t = 0.8, T = 1$, and hence $\theta = 0.6, \gamma_1 = 8.5, \gamma_2 = 3.5$. In panels (c) and (d), manager 1 is a chaser and manager 2 is a contrarian, and the parameter values are: $\bar{\gamma}_1 = 2, \bar{\gamma}_2 = 0.5, \alpha = 0.3, \eta = 1.35, t = 0.95$, and hence $\theta = 0.23, \gamma_1 = 2.3, \gamma_2 = 0.35$, and the remaining parameters are as in panels (a) and (b).

1 increases her risk exposure while manager 2 chooses a policy close to her normal. In the interim (loser, winner) region, Corollary 1 implies that manager 2, a contrarian, increases her risk exposure relative to the normal as she moves her policy away from the manager 1's normal policy. For manager 1, the effect of relative considerations is weak and so she chooses a policy close to the normal.

In the interim (loser, loser) region, the economic mechanism underlying the chasing and contrarian behavior is dominated by the gambling incentives as each manager is in the

convex region of her (conditional) objective function. Hence, the behavior of managers in this region is similar in Figures **??**(a) and (c). Each manager gambles in order to avoid a situation where her relative performance is close to the threshold level, which is achieved by following a policy that is sufficiently different from that of the opponent. As a result, the managers always gamble in the opposite directions from each other in each individual stock. The more risk averse manager 1 optimally decreases her stock holding when gambling while the more risk tolerant manager 2 gambles by increasing hers, as seen in Figures **??**(a) and (c).[13]

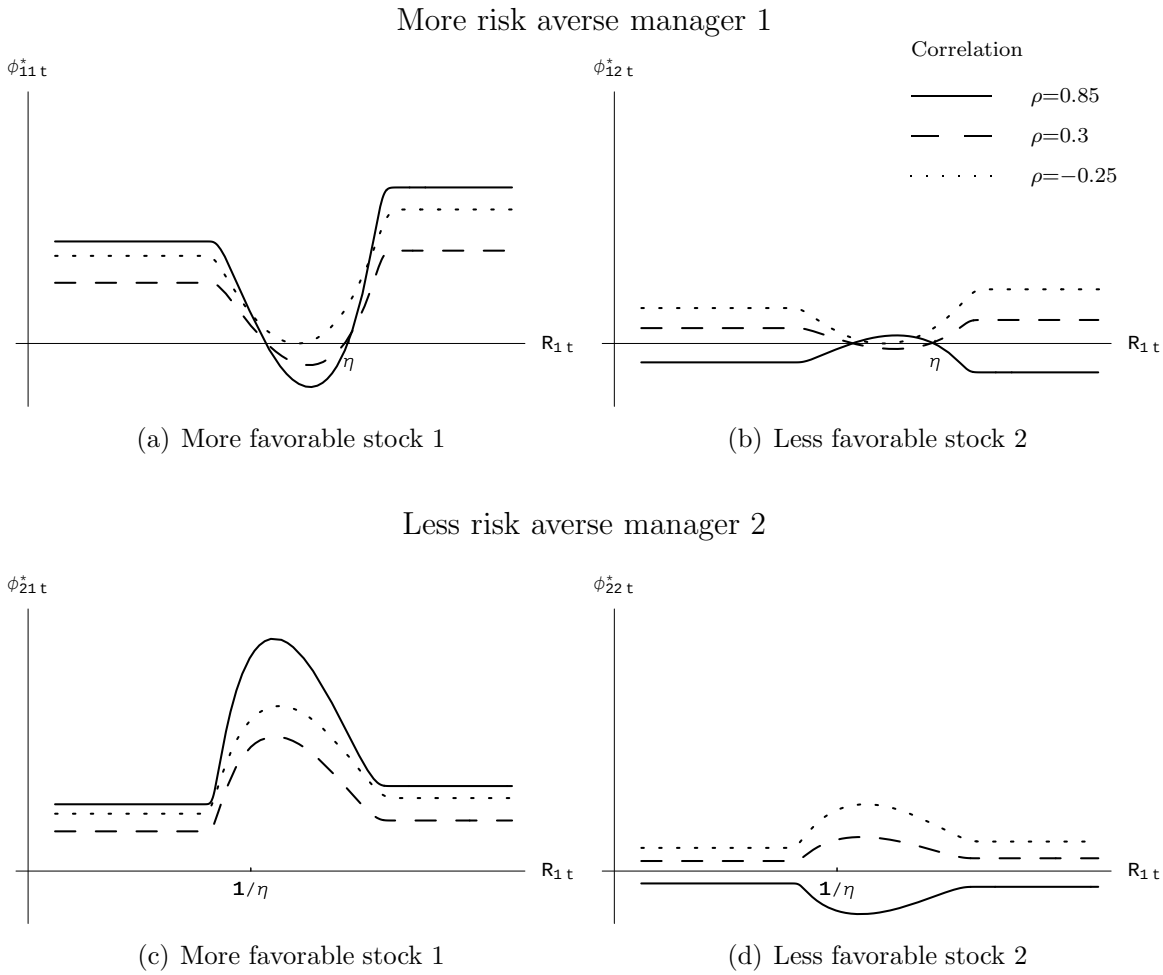
5.2. Effects of Stock Correlation on Unique Equilibrium

Figure **??** illustrates the impact of the correlation between the two stocks, ρ, on the managers' equilibrium investment policies. Several noteworthy features arise. First, changing the correlation can affect the equilibrium policies in a non-monotonic way, as in Figures **??**(a)-(c), as well as monotonically, as in Figure **??**(d). Second, Figures **??**(a) and (b) reveal that the equilibrium investment profiles may cross for different values of correlation. Third, as correlation increases the direction of humps in the interim (loser, loser) region can invert, as depicted in Figures **??**(b) and (d).

Above rich patterns arise due to the interaction of a *diversification* effect, the extent of benefits to diversification, and a *substitution* effect, the extent of the two stocks acting as a substitute to each other. The diversification effect dominates for low correlation values, while the substitution effect dominates for high values of correlation. When the correlation is low, both sources of stock return uncertainty matter considerably and so the two stocks compliment each other by providing a hedge against a specific source of risk. Increasing the correlation reduces the diversification benefits from holding a portfolio of the two stocks, leading the managers to reduce their investments in both stocks. When the correlation is high, the two stocks become close substitutes, in which case the more favorable stock 1 (with a relatively high Sharpe ratio) becomes the primary security through which the managers achieve their desired risk exposures. As the correlation increases further, the managers substitute away from the less favorable stock 2 (with a relatively low Sharpe ratio) into the more favorable stock 1.

Going back to Figure **??**, when the correlation increases from -0.25 to 0.3 (moving from the dotted to dashed lines in all panels of Figure **??**), the diversification effect dominates

[13]We note that mutual fund managers are often not allowed to go short, while according to Figure **??** it may be optimal for a manager to short risky assets in a certain range of interim relative performance. For tractability, we do not explicitly introduce a no-short-sale constraint. We believe that incorporating such a constraint is not going to significantly change our main insights, because if the constraint were present, it would likely lead to less pronounced gambling by the more risk averse manager in the relevant range of the interim relative performance.



(a) More favorable stock 1

(b) Less favorable stock 2

(c) More favorable stock 1

(d) Less favorable stock 2

Figure 6: Effect of stock correlation on the managers' unique equilibrium investment policies. The managers' time-t equilibrium policies for varying levels of the correlation, ρ, between stock 1 and 2 returns. In all panels, dotted line corresponds to $\rho = -0.25$, dashed line to $\rho = 0.3$, solid line to $\rho = 0.85$. The other parameter values are as in Figure **??**(a) and (b).

and both managers reduce their investments in each stock across all three regions of interim relative performance.[14] As the correlation rises further from $\rho = 0.3$ to $\rho = 0.85$ (moving from dashed to solid lines in Figure **??**), in the interim (winner, loser) and (loser, winner) regions the substitution effect induces the managers to increase their investments in the more favorable stock 1 and to finance this by decreasing their holdings in the less favorable stock 2. Hence, the substitution effect works in the opposite direction from the diversification effect for stock 1, leading the equilibrium policies to being non-monotonic in the correlation (Figures **??**(a) and (c)). In the interim (loser, loser) region, the more risk averse manager

[14]While not explicitly highlighted in Figure **??**, we note that the intensity of the chasing behavior, as measured by the magnitude of the deviation of the equilibrium policy from normal, is also reduced with the increase in correlation. The reason is that the managers' normal policies are scaled down proportionally as the correlation increases due to the diversification effect, and so the absolute difference between them, which affects the intensity of chasing, decreases.

1 gambles by decreasing her risk exposure. Since the more favorable stock 1 is the primary means to changing her risk exposure, the downward hump becomes more pronounced, as seen in Figure **??**(a). The substitution effect implies that this decrease in stock 1 holdings is mirrored by an increase in the less favorable stock 2 holdings. This leads to the inverted shape of the equilibrium policy for stock 2, as depicted in Figure **??**(b). The more risk tolerant manager 2 follows the opposite strategy to that of manager 1 and gambles by increasing her risk exposure. Consequently, the size of the upward hump in the primary stock 1 holdings increases (Figure **??**(c)). This larger position in stock 1 is financed by a decrease in the less favorable stock 2 holdings, leading to the inverted hump in Figure **??**(d).[15]

Having fully described the unique equilibrium investment policies of the managers in a strategic setting, we may now compare our results to those obtained in frameworks where a manager competes against an exogenous benchmark (Basak, Pavlova, and Shapiro (2007), Carpenter (2000)). The main similarity between our work and non-strategic studies is the presence of the gambling behavior in one of the three interim regions of our unique equilibrium and their optimal investment policies. On the other hand, there are several important differences. First, strategic interactions can lead to non-existence or multiplicity of equilibrium, as we have elaborated on in Section 4. Second, the economic mechanism behind the chasing and contrarian behavior, which drives the equilibrium policies in the interim (winner, loser) and (loser, winner) regions, is not present in Basak et al. or Carpenter. Third, in the interim (loser, loser) region, where the gambling behavior prevails, the two managers always gamble in the opposite direction from each other in each individual stock, and it is the manager's risk aversion *relative* to that of the opponent that determines the direction of gambling. This is notably different from Basak et al., where for given market parameter values the direction of gambling is determined by the *absolute* value of the manager's risk aversion. In Carpenter, the direction of gambling is always the same. Fourth, by considering a setting with multiple correlated risky stocks we uncover novel patterns of the managers' behavior for varying levels of correlation.

5.3. Characterization of Multiple Equilibria

We now turn to the case of multiple equilibria, which obtains under the multiplicity condition (**??**). Proposition **??** provides the full characterization of all horizon wealth profiles that can

[15]Given that the managers care about the overall risk of their portfolios and that they are heterogeneous in their risk preferences, one could have expected that the mechanism that leads to the shape inversion does not necessarily kick in simultaneously for the two managers. However as discussed earlier, the managers always gamble strategically against each other in the opposite direction in each individual stock. So, manager 1 optimally inverts her equilibrium policy in the less favorable stock 2 if and only if manager 2 inverts hers, reflecting the strategic nature of the managers' interaction.

occur in equilibrium.[16]

Proposition 4. *Assume the condition for the multiplicity of Nash equilibrium (**??**) is satisfied. The equilibrium outcomes and horizon wealth profiles (W^*_{1T}, W^*_{1T}) are as follows. When $\xi_T^{\bar{\gamma}_1-\bar{\gamma}_2} > y \min\left[B\eta^{-\bar{\gamma}_1\bar{\gamma}_2}, D\eta^{\bar{\gamma}_1(\gamma_2+\theta(\gamma_2-1)}\right]$, the managers are in (winner, loser) and*

$$W^*_{1T} = \tilde{y}_1 \xi_T^{-(\bar{\gamma}_2+\theta(\gamma_1-1))/(\gamma_1\bar{\gamma}_2)}, \quad W^*_{2T} = (y_2\xi_T)^{-1/\bar{\gamma}_2}.$$

When $\xi_T^{\bar{\gamma}_1-\bar{\gamma}_2} < y \max\left[A\eta^{\bar{\gamma}_1\bar{\gamma}_2}, C\eta^{-\bar{\gamma}_2(\gamma_1+\theta(\gamma_1-1)}\right]$, the managers are in (loser, winner) and

$$W^*_{1T} = (y_1\xi_T)^{-1/\bar{\gamma}_1}, \; W^*_{2T} = \tilde{y}_2 \xi_T^{-(\bar{\gamma}_1+\theta(\gamma_2-1))/(\gamma_2\bar{\gamma}_1)}.$$

*When $y \max\left[A\eta^{\bar{\gamma}_1\bar{\gamma}_2}, C\eta^{-\bar{\gamma}_2(\gamma_1+\theta(\gamma_1-1))}\right] < \xi_T^{\bar{\gamma}_1-\bar{\gamma}_2} < y \min\left[B\eta^{-\bar{\gamma}_1\bar{\gamma}_2}, D\eta^{\bar{\gamma}_1(\gamma_2+\theta(\gamma_2-1))}\right]$, both outcomes, (winner, loser) and (loser, winner), can occur in equilibrium. Here, the constants A, B, C, D are as in Proposition **??**, and $y, \tilde{y}_1, \tilde{y}_2$ as in Proposition **??**.*

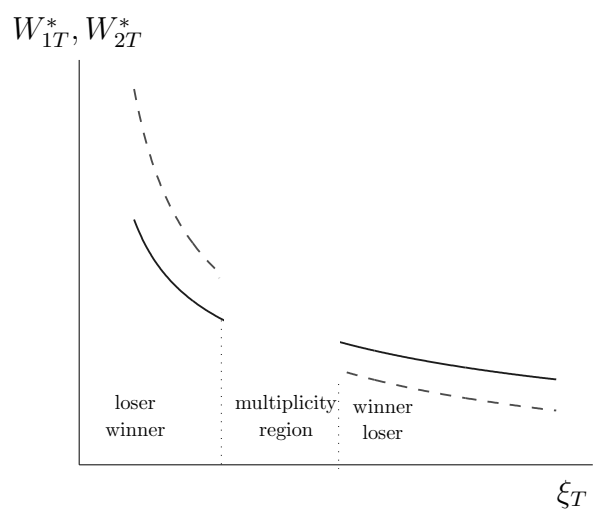


Figure 7: Managers' multiple equilibrium horizon wealth. Multiple equilibrium horizon wealth profiles of the more risk averse manager 1 (solid plot) and the more risk tolerant manager 2 (dashed plot). The plots are typical.

Figure **??** plots the multiple equilibrium wealth profiles. Looking at Figures **??** and **??**, we observe that the multiple equilibrium wealth profiles and the unique ones differ only in the middle-ξ_T region. When the equilibrium is unique, both managers are losers in the middle region, choosing similar horizon wealth profiles (Figure **??**). Now, since the performance threshold is relatively low, such wealth profiles are no longer optimal as both managers have incentives to gamble. So, one of the managers must be a winner, implying that her wealth will

[16]Due to the nature of multiplicity, describing the corresponding investment policies is not straightforward. Loosely, the number of terms in the expressions for equilibrium policies depend on the number of discontinuities in the horizon wealth profiles, which can be infinite.

be considerably higher than the other manager's wealth, preventing the latter from gambling. As discussed in Section 4, in some states the managers are indifferent to switching their rankings. Since now the rankings are different in all states, with neither (winner, winner) nor (loser, loser) outcomes being possible, switching rankings leads to multiple equilibria. Unlike the middle region, when ξ_T is either relatively low or high, the difference between the managers' wealth levels is substantial in the unique equilibrium (Figure **??**). So, even with a low performance threshold η neither of the two managers has incentives to gamble in these regions, leading to the multiple equilibria wealth profiles being the same as the unique-equilibrium ones, as seen by comparing Figures **??** and **??**.

A natural question is whether we can narrow down the set of multiple equilibria and rule out some that are not likely to occur in the real world. While an in-depth investigation of this issue is beyond the scope of this paper, we may put forward the following simple selection criterion. Since real-life money managers incur trading costs, which we have abstracted away from for tractability, the managers would favor the equilibrium with the minimal swings in investments so as to minimize the transaction costs. Since the most pronounced swings are associated with discontinuities in horizon wealth profiles, a natural selection criterion is to pick an equilibrium with the minimum number of the discontinuities, which in our setting amounts to the equilibrium with just one discontinuity. Indeed, inspecting Figure **??** we observe that such an equilibrium is obtained by dividing the middle-ξ_T multiplicity region into two parts, with (loser, winner) outcome occurring in the left part and (winner, loser) outcome occurring in the right part. One can easily see that the structure of the resulting equilibrium wealth profiles is similar to that obtained in the unique equilibrium (Proposition **??**), and so the investment policies underlying this selected equilibrium have the same profile as the unique equilibrium investment policies depicted in Figure **??**.

Remark 2. *Interim Performance.* Starting from the study by Brown, Harlow, and Starks (1996), it has become common in both theoretical and empirical works to consider the interim performance as a pertinent factor that affects risk taking incentives in tournament settings. In our analysis, we can easily accommodate this factor in by introducing an interim performance parameter, $R_{10} > 1$, which gives manager 1 an initial advantage. The time-t performance of manager 1 is now given by $R_{10}W_{1t}/W_{10}$. As a result, manager 1 and 2's relative performances become $R_{10}R_{1t}$ and R_{2t}/R_{10}, respectively, where as before $R_{1t} = W_{1t}/W_{2t}$ and $R_{2t} = W_{2t}/W_{1t}$. In Figures **??**–**??**, we simply need to put manager 1's new relative performance in place of R_{1t}. Alternatively, if one wants to keep the old ones, the graphs should be appropriately scaled along the x-axis by a factor of R_{10}. All our results pertaining to equilibrium existence and multiplicity remain unchanged, and so Figures **??**–**??** are not affected. This is in contrast to a static risk-neutral setting where the difference in the managers' interim performance is found to be critical for the existence of a pure-strategy

Nash equilibrium (Taylor (2003)). The only additional implication of the introduction of interim performance is the possibility of a new equilibrium, in which manager 1 is a winner at the horizon T with certainty, across all states. We do not present this analysis since such an equilibrium can only occur in a knife-edge case when both managers have identical risk aversions.

Remark 3. *Managers with different initial wealth.* If the managers' initial wealth are different, the expressions for best responses in Proposition **??** are augmented by the ratio of the initial wealths. However, all the subsequent results pertaining to equilibrium are virtually unchanged. In particular, the conditions for existence and uniqueness of equilibrium are unaffected (Proposition **??**), and the structures of the unique and multiple equilibria and the shapes of the unique equilibrium investment policies are preserved (Propositions **??**–**??**). The reason is that the fund flows are driven by relative returns, and changing initial wealth does not give a manager any kind of advantage or disadvantage in the competition for fund flows.

6. Conclusion

In this paper, we analyze the equilibrium portfolios of money managers in presence of strategic interactions driven by relative performance concerns with local convexities. We discover the possibility of three distinct results: multiple equilibria, unique equilibrium, or no equilibrium at all. When the equilibrium is unique, we analyze the properties of the equilibrium investment policies. In the other two cases, we elaborate on the underlying economic mechanisms that lead to non-existence or multiplicity, most of which are driven by the risk shifting incentives combined with the strategic interaction of the managers.

Given the novelty of our analysis, we believe there are various promising directions for future research. While we assume that money managers have a perfect knowledge of each other's attitude towards risk, it would be valuable to investigate a perfect Bayesian equilibrium in a more realistic framework where the managers do not have such knowledge but can learn about each other's traits by observing the investment policies. It would also be of interest to extend our framework to investigate the possible strategic interactions among CEOs, whose contracts often include a bonus part for high relative performance (Murphy (1999)). Considering a setting where managers compete against each other and at the same time against an exogenous peer-group benchmark could be an interesting generalization. Another natural extension of our framework would be to incorporate flow-performance relations where money flows depend on discrete rankings, leading to discontinuities of the managers' objective functions. Finally, analyzing the case when the investment opportunities of the

managers are not perfectly correlated would be worthwhile.

Appendix: Proofs

Proof of Lemma 1. Employing martingale methods, given the CRRA preferences (**??**), manager i's optimal time-T' wealth profile $\hat{W}_{iT'}$ is given by the first order condition

$$\hat{W}_{iT'} = (y_{iT}\xi_{T'})^{-1/\bar{\gamma}_i}, \tag{A1}$$

where $y_{iT} > 0$ is the Lagrange multiplier attached to her time-T static budget constraint $E_T[\xi_{T'}\hat{W}_{iT'}] = \xi_T W_{iT} f_T$. The Lagrange multiplier is found by substituting (**??**) into the budget constraint, which yields

$$y_{iT}^{-1/\bar{\gamma}_i} = \frac{\xi_T W_{iT} f_T}{E_T[\xi_{T'}^{1-1/\bar{\gamma}_i}]}. \tag{A2}$$

Plugging (**??**) into (**??**), we find manager i's optimal time-T' wealth profile:

$$\hat{W}_{iT'} = \frac{\xi_T W_{iT} f_T}{E_T[\xi_{T'}^{1-1/\bar{\gamma}_i}]}\xi_{T'}^{-1/\bar{\gamma}_i} \tag{A3}$$

Combining (**??**) and (**??**) yields the time-T indirect utility function

$$v_{iT} \equiv E_T[u_i(\hat{W}_{iT'})] = \frac{1}{1-\bar{\gamma}_i}(\xi_T W_{iT} f_T)^{1-\bar{\gamma}_i}(E_T[\xi_{T'}^{1-1/\bar{\gamma}_i}])^{\bar{\gamma}_i}. \tag{A4}$$

Since ξ_t follows a geometric Brownian motion with constant drift and volatility, we have that $E_T[\xi_{T'}^{1-1/\bar{\gamma}_i}] = a\xi_T^{1-1/\bar{\gamma}_i}$ where a is some constant depending on r, κ, and $T'-T$, and we drop it without loss of generality since it does not affect the optimal behavior. Finally, substituting the expectation into (**??**), we get

$$v_{iT} = \frac{1}{1-\bar{\gamma}_i}(W_{iT} f_T)^{1-\bar{\gamma}_i}. \tag{A5}$$

The indirect utility function (**??**) follows by plugging the flow-performance function into (**??**) after some manipulation. The stated properties of θ and γ_i are immediate.

Q.E.D.

Proof of Proposition ??. We consider only manager 1; for manager 2 the analysis is analogous. Fixing the horizon wealth profile of manager 2, $W_{2T}(\xi_T)$, we look for the manager 1's optimal horizon wealth profile $\hat{W}_{1T}$. Although manager 1's objective function has a region of local convexity, we can still use standard optimization techniques once we concavify the objective function (see Basak, Pavlova, and Shapiro (2007) for a more formal proof in a similar setting). Concavification involves finding the range $[\underline{W}, \overline{W}]$ and the coefficients a and b_1 such that replacing $v_1(\cdot)$ within the range $[\underline{W}, \overline{W}]$ with a chord $a + b_1 W_{1T}$ will result

in a globally concave objective function. Noting that the chord must be tangent to $v_1(\cdot)$ at $\underline{\mathrm{W}}$ and $\overline{W}$, we have the following system of equations to solve for:

$$
\begin{aligned}
a + b_1\underline{\mathrm{W}} &= \frac{\underline{\mathrm{W}}^{1-\bar{\gamma}_1}}{1-\bar{\gamma}_1} && \text{(A6)}\\
a + b_1\overline{W} &= \frac{1}{1-\bar{\gamma}_1}\left(\overline{W}^{1-\theta}\left(\frac{\overline{W}}{\eta W_2}\right)^{\theta}\right)^{1-\gamma_1} && \text{(A7)}\\
b_1 = \underline{\mathrm{W}}^{-\bar{\gamma}_1} &= (1+\alpha)\overline{W}^{-\gamma_1}W_2^{\theta(\gamma_1-1)}, && \text{(A8)}
\end{aligned}
$$

where we have dropped $k^{1-\bar{\gamma}_1}$ from the objective function since it does not affect the optimization problem. Subtracting (**??**) from (**??**) yields

$$
b_1(\overline{W} - \underline{\mathrm{W}}) = \frac{1}{1-\bar{\gamma}_1}\left(\overline{W}^{1-\gamma_1}(\eta W_2)^{\theta(\gamma_1-1)} - \underline{\mathrm{W}}^{1-\bar{\gamma}_1}\right). \tag{A9}
$$

Expressing $\underline{\mathrm{W}}$ and $\overline{W}$ in terms of b_1 and W_2 (from (**??**)) and plugging into (**??**) gives

$$
\begin{aligned}
& b_1(b_1^{-1/\gamma_1}(1+\alpha)^{1/\gamma_1}W_2^{\theta(\gamma_1-1)/\gamma_1} - b_1^{-1/\bar{\gamma}_1})\\
= \; & \frac{1}{1-\bar{\gamma}_1}\left(b_1^{(\gamma_1-1)/\gamma_1}(1+\alpha)^{(\gamma_1-1)/\gamma_1}W_2^{-\theta(\gamma_1-1)^2/\gamma_1}(\eta W_2)^{\theta(\gamma_1-1)} - b_1^{(\bar{\gamma}_1-1)/\bar{\gamma}_1}\right),
\end{aligned}
$$

which after some algebra yields the boundary function (**??**). If $y_1\xi_T$ is higher that the slope of the concavification line b_1, the optimal wealth is to the left from $\underline{\mathrm{W}}$, i.e., manager 1 chooses to be a loser and her normal-type policy (**??**) obtains. Otherwise, she becomes a winner and (**??**) obtains, accounting for relative concerns.

Manager 1's relative performance is closest to the threshold ηW_{2T} when she is indifferent between being a winner and a loser, i.e., when

$$
y_1\xi_T = b_1(\eta W_{2T}) = (1+\alpha)^{\bar{\gamma}_1/\theta}\left(\bar{\gamma}_1/\gamma_1\right)^{\bar{\gamma}_1\gamma_1/(\gamma_1-\bar{\gamma}_1)}(\eta W_{2T})^{-\bar{\gamma}_1}. \tag{A10}
$$

If manager 1 chooses to be a winner, her "minimum outperformance margin" $\bar{\eta}_1$ is obtained by plugging (**??**) into (**??**) and dividing the resulting wealth by W_{2T}. This yields

$$
\begin{aligned}
\bar{\eta}_1 &= (1+\alpha)^{1/\gamma_1}\left((1+\alpha)^{\bar{\gamma}_1/\theta}\left(\bar{\gamma}_1/\gamma_1\right)^{\bar{\gamma}_1\gamma_1/(\gamma_1-\bar{\gamma}_1)}(\eta W_{2T})^{-\bar{\gamma}_1}\right)^{-1/\gamma_1}(\eta W_{2T})^{\theta(\gamma_1-1)/\gamma_1}/W_{2T}\\
&= (1+\alpha)^{(1-\bar{\gamma}_1/\theta)/\gamma_1}\left(\bar{\gamma}_1/\gamma_1\right)^{-\bar{\gamma}_1/(\gamma_1-\bar{\gamma}_1)}(\eta W_{2T})^{(\bar{\gamma}_1+\theta(\gamma_1-1))/\gamma_1}/W_{2T}\\
&= (1+\alpha)^{-1/\alpha}\left(\bar{\gamma}_1/\gamma_1\right)^{-\bar{\gamma}_1/(\gamma_1-\bar{\gamma}_1)}\eta,
\end{aligned}
$$

where in the last equality we use (**??**)–(**??**) to simplify the expressions. If manager 1 chooses to be a loser, her "maximum underperformance margin" $\underline{\eta}_1$ is obtained by substituting (**??**)

into (**??**) and dividing the result by W_{2T}, which yields

$$\underline{\eta}_1 = \left((1+\alpha)^{\bar{\gamma}_1/\theta}\left(\bar{\gamma}_1/\gamma_1\right)^{\bar{\gamma}_1\gamma_1/(\gamma_1-\bar{\gamma}_1)}\left(\eta W_{2T}\right)^{-\bar{\gamma}_1}\right)^{-1/\bar{\gamma}_1}/W_{2T} = (1+\alpha)^{-1/\theta}\left(\bar{\gamma}_1/\gamma_1\right)^{-\bar{\gamma}_1/(\gamma_1-\bar{\gamma}_1)}\eta.$$

Q.E.D.

Proof of Proposition ??. For a given realization of ξ_T, we can have one of the three outcomes: (winner, loser), (loser, winner), or (loser, loser). From the best responses (**??**)–(**??**), we can determine the regions of ξ_T for which these outcomes can occur.

(winner, loser). From (**??**), manager 1 chooses to be a winner if $y_1\xi_T \leq b_1(\eta W_{2T})$. Plugging manager 2's wealth, given by (**??**) as she is a loser, and using the definition of $b_1(\cdot)$ (**??**), yields

$$y_1\xi_T \leq (1+\alpha)^{\bar{\gamma}_1/\theta}\left(\bar{\gamma}_1/\gamma_1\right)^{\bar{\gamma}_1\gamma_1/(\gamma_1-\bar{\gamma}_1)}\left(\xi_T y_2\right)^{\bar{\gamma}_1/\bar{\gamma}_2}\eta^{-\bar{\gamma}_1}.$$

Rearranging, we get

$$\xi_T^{\bar{\gamma}_1-\bar{\gamma}_2} \geq y_1^{\bar{\gamma}_2}y_2^{-\bar{\gamma}_1}\left((1+\alpha)^{\bar{\gamma}_1/\theta}\left(\bar{\gamma}_1/\gamma_1\right)^{\bar{\gamma}_1\gamma_1/(\gamma_1-\bar{\gamma}_1)}\right)^{-\bar{\gamma}_2}\eta^{\bar{\gamma}_1\bar{\gamma}_2} \equiv yA\eta^{\bar{\gamma}_1\bar{\gamma}_2}, \tag{A11}$$

where $y \equiv y_1^{\bar{\gamma}_2}y_2^{-\bar{\gamma}_1}$ and A is as given in (**??**).

From (**??**), manager 2 chooses to be a loser if $y_2\xi_T > b_2(\eta W_{1T})$. Plugging manager 1's wealth W_{1T}, given in (**??**), and expanding b_2 yields

$$y_2\xi_T > (1+\alpha)^{\bar{\gamma}_2/\theta}\left(\bar{\gamma}_2/\gamma_2\right)^{\bar{\gamma}_2\gamma_2/(\gamma_2-\bar{\gamma}_2)}\left(\eta(1+\alpha)^{1/\gamma_1}(y_1\xi_T)^{-1/\gamma_1}(\eta W_{2T})^{\theta(\gamma_1-1)/\gamma_1}\right)^{-\bar{\gamma}_2}.$$

After some algebra, we get

$$\xi_T^{\bar{\gamma}_1-\bar{\gamma}_2} > y_1^{\bar{\gamma}_2}y_2^{-\bar{\gamma}_1}(1+\alpha)^{\gamma_1\bar{\gamma}_2/\theta-\bar{\gamma}_2}\left(\bar{\gamma}_2/\gamma_2\right)^{\gamma_1\gamma_2\bar{\gamma}_2/(\gamma_2-\bar{\gamma}_2)}\eta^{-\bar{\gamma}_2(\gamma_1+\theta(\gamma_1-1))} \equiv yC\eta^{-\bar{\gamma}_2(\gamma_1+\theta(\gamma_1-1))}, \tag{A12}$$

where C is as given in (**??**). The outcome (winner, loser) can occur provided that both (**??**) and (**??**) are satisfied, which means ξ_T satisfies:

$$\xi_T^{\bar{\gamma}_1-\bar{\gamma}_2} > y\max\left[A\eta^{\bar{\gamma}_1\bar{\gamma}_2}, C\eta^{-\bar{\gamma}_2(\gamma_1+\theta(\gamma_1-1))}\right]. \tag{A13}$$

(loser, winner). The expressions are obtained from (**??**) and (**??**) by switching subscripts 1 and 2, leading to the conditions on $\xi_T^{\bar{\gamma}_2-\bar{\gamma}_1}$. For ease of comparison with (winner, loser) case, we then invert the obtained inequalities to get the conditions on $\xi_T^{\bar{\gamma}_1-\bar{\gamma}_2}$:

$$\begin{aligned}
\xi_T^{\bar{\gamma}_1-\bar{\gamma}_2} &\leq y_1^{\bar{\gamma}_2}y_2^{-\bar{\gamma}_1}\left((1+\alpha)^{\bar{\gamma}_2/\theta}\left(\bar{\gamma}_2/\gamma_2\right)^{\bar{\gamma}_2\gamma_2/(\gamma_2-\bar{\gamma}_2)}\right)^{\bar{\gamma}_1}\eta^{-\bar{\gamma}_1\bar{\gamma}_2} \equiv yB\eta^{-\bar{\gamma}_1\bar{\gamma}_2}, && \text{(A14)}\\
\xi_T^{\bar{\gamma}_1-\bar{\gamma}_2} &< y_1^{\bar{\gamma}_2}y_2^{-\bar{\gamma}_1}(1+\alpha)^{\bar{\gamma}_1-\gamma_2\bar{\gamma}_1/\theta}\left(\bar{\gamma}_1/\gamma_1\right)^{-\gamma_2\gamma_1\bar{\gamma}_1/(\gamma_1-\bar{\gamma}_1)}\eta^{\bar{\gamma}_1(\gamma_2+\theta(\gamma_2-1))} \equiv yD\eta^{\bar{\gamma}_1(\gamma_2+\theta(\gamma_2-1))},
\end{aligned}$$

where B and D are given by (**??**) and (**??**), respectively. Combining the two conditions, (loser, winner) can occur for ξ_T satisfying

$$\xi_T^{\bar{\gamma}_1-\bar{\gamma}_2} < y \min\left[B\eta^{-\bar{\gamma}_1\bar{\gamma}_2}, D\eta^{\bar{\gamma}_2(\gamma_2+\theta(\gamma_2-1))}\right]. \tag{A15}$$

(loser, loser). The conditions for this outcome follow from the observation that manager i wants to be a loser in those states in which she does not want to be a winner. Hence, manager 1 wants to be a loser for ξ_T such that (**??**) is not satisfied. Similarly, manager 2 chooses to be a loser when (**??**) does not hold. So, (loser, loser) can occur for ξ_T given by

$$yB\eta^{-\bar{\gamma}_1\bar{\gamma}_2} < \xi_T^{\bar{\gamma}_1-\bar{\gamma}_2} < yA\eta^{\bar{\gamma}_1\bar{\gamma}_2}. \tag{A16}$$

Inspection of (**??**), (**??**), and (**??**) reveals that if (loser, loser) region is not empty, i.e., if

$$B\eta^{-\bar{\gamma}_1\bar{\gamma}_2} < A\eta^{\bar{\gamma}_1\bar{\gamma}_2}, \tag{A17}$$

the three regions can never overlap, meaning that multiple equilibria are not possible. For the unique equilibrium to exist, i.e., for the three regions to fully cover the interval $(0, +\infty)$, it must be the case that

$$A\eta^{\bar{\gamma}_1\bar{\gamma}_2} \geq C\eta^{-\bar{\gamma}_2(\gamma_1+\theta(\gamma_1-1))}, B\eta^{-\bar{\gamma}_1\bar{\gamma}_2} \leq D\eta^{\bar{\gamma}_2(\gamma_2+\theta(\gamma_2-1))}, \tag{A18}$$

in which case the unique equilibrium has the following structure. (winner, loser) occurs for $\xi_T^{\bar{\gamma}_1-\bar{\gamma}_2} > yA\eta^{\bar{\gamma}_1\bar{\gamma}_2}$, (loser, loser) for $yB\eta^{-\bar{\gamma}_1\bar{\gamma}_2} \leq \xi_T^{\bar{\gamma}_1-\bar{\gamma}_2} \leq yA\eta^{\bar{\gamma}_1\bar{\gamma}_2}$, and (loser, winner) for $\xi_T^{\bar{\gamma}_1-\bar{\gamma}_2} < yB\eta^{-\bar{\gamma}_1\bar{\gamma}_2}$. Combining (**??**) and (**??**) yields the condition for the existence and uniqueness of equilibrium (**??**).

If (loser, loser) region is empty, i.e., if (**??**) is not satisfied, then for an equilibrium to exist it must be the case that the remaining two outcomes fully cover $(0, +\infty)$. Hence, from (**??**) and (**??**), we get the multiple equilibria condition (**??**). In a knife-edge case when (**??**) holds as an equality, the equilibrium is unique. In this case, (winner, loser) occurs for ξ_T satisfying (**??**), (loser, winner) occurs for the other ξ_T. In all other cases, when (**??**) holds as a strict inequality, multiple equilibria obtains. The structure of the equilibria is as follows. (winner, loser) occurs for $\xi_T^{\bar{\gamma}_1-\bar{\gamma}_2} > y \min\left[B\eta^{-\bar{\gamma}_1\bar{\gamma}_2}, D\eta^{\bar{\gamma}_1(\gamma_2+\theta(\gamma_2-1)}\right]$, (loser, winner) occurs for $\xi_T^{\bar{\gamma}_1-\bar{\gamma}_2} < y \max\left[A\eta^{\bar{\gamma}_1\bar{\gamma}_2}, C\eta^{-\bar{\gamma}_2(\gamma_1+\theta(\gamma_1-1)}\right]$. The region

$$y \max\left[A\eta^{\bar{\gamma}_1\bar{\gamma}_2}, C\eta^{-\bar{\gamma}_2(\gamma_1+\theta(\gamma_1-1))}\right] < \xi_T^{\bar{\gamma}_1-\bar{\gamma}_2} < y \min\left[B\eta^{-\bar{\gamma}_1\bar{\gamma}_2}, D\eta^{\bar{\gamma}_1(\gamma_2+\theta(\gamma_2-1))}\right]$$

is consistent with both (winner, loser) and (loser, winner) outcomes, hence for such ξ_T we

get the multiplicity of equilibria with these two outcomes. Other structures of equilibrium are not possible, hence there is no pure-strategy Nash equilibrium if neither (**??**) nor (**??**) is satisfied.

Q.E.D.

Proof of Proposition ??. From Proposition **??**, an equilibrium exists and is unique if the uniqueness condition (**??**) is satisfied. In the proof of Proposition **??**, we have established the structure of the managers' equilibrium wealth profiles – for each realization of ξ_T we identified whether manager i, $i = 1, 2$, is a winner or a loser. We now determine the associated equilibrium horizon wealth profiles corresponding to these two outcomes. Focusing on manager 1, from (**??**) her optimal wealth is $(y_1\xi_T)^{-1/\bar{\gamma}_1}$ when she is a loser, i.e., when $\xi_T^{\bar{\gamma}_1-\bar{\gamma}_2} \leq yA\eta^{\bar{\gamma}_1\bar{\gamma}_2}$. Otherwise, when $\xi_T^{\bar{\gamma}_1-\bar{\gamma}_2} > yA\eta^{\bar{\gamma}_1\bar{\gamma}_2}$, manager 1 is a winner, and her best response wealth is given in (**??**). As the performance threshold η is greater than one, manager 2 is a loser whenever manager 1 is a winner, and so in equilibrium chooses $W_{2T}^* = (y_2\xi_T)^{-1/\bar{\gamma}_2}$. Plugging this into (**??**) yields the equilibrium manager 1's wealth when she is a winner:

$$W_{1T}^* = y_1^{-1/\gamma_1} y_2^{-\theta(\gamma_1-1)/(\gamma_1\bar{\gamma}_2)} (1+\alpha)^{1/\gamma_1} \eta^{\theta(\gamma_1-1)/\gamma_1} \xi_T^{-(\bar{\gamma}_2+\theta(\gamma_1-1))/(\gamma_1\bar{\gamma}_2)}.$$

To derive the associated equilibrium investment policy of manager 1, ϕ_{1t}^*, we first determine the time-t value of the equilibrium wealth W_{1t}^*. Since $\xi_t W_{1t}^*$ is a martingale, we have

$$\xi_t W_{1t}^* = E_t[\xi_T W_{1T}^*] = y_1^{-1/\bar{\gamma}_1} E_t[\xi_T^{(\bar{\gamma}_1-1)/\bar{\gamma}_1} \mathbb{1}_{\{\xi_T<\beta\}}] + \tilde{y}_1 E_t[\xi_T^{(\tilde{\gamma}-1)/\tilde{\gamma}} \mathbb{1}_{\{\xi_T\geq\beta\}}], \tag{A19}$$

where β and $\tilde{y}_1$ are as defined in Proposition **??**. To evaluate the two conditional expectations, we use the following property of the truncated log-normal distribution. If x is a log-normally distributed random variable such that $\ln x$ is normal with mean m and variance v^2, then

$$E[x^n \mathbb{1}_{x<a}] = e^{nm+n^2v^2/2}\, N\left((\ln a - m - nv^2)/v\right). \tag{A20}$$

Given that as of time t $\ln \xi_T$ has mean $\ln \xi_t - (r + \|\kappa\|^2/2)(T-t)$ and variance $\|\kappa\|^2(T-t)$, applying (**??**) to (**??**) yields

$$W_{1t}^* = N(d(\bar{\gamma}_1, \beta)) y_1^{-1/\bar{\gamma}_1} Z(\bar{\gamma}_1, t) \xi_t^{-1/\bar{\gamma}_1} + (1 - N(d(\tilde{\gamma}, \beta))) \tilde{y}_1 Z(\tilde{\gamma}, t) \xi_t^{-1/\tilde{\gamma}}, \tag{A21}$$

where $N, d, Z, \tilde{\gamma}, \tilde{y}_1$ are as defined in Proposition **??**. From Itô's Lemma, the coefficient of the Brownian motion ω in the dynamic process for W_{1t}^* is equal to

$$-\kappa_j \xi_t \frac{\partial W_{1t}^*}{\partial \xi_t},$$

where κ_j is the j-th component of the market price of risk vector κ. Equating each of these coefficients to the corresponding diffusion term in (**??**), we obtain the following system of linear equations:

$$\xi_t \frac{\partial W_{1t}^*}{\partial \xi_t} \kappa = W_{1t}^* \sigma^\top \phi_{1t}^*.$$

Solving the system and substituting the derivative of (**??**) with respect to ξ_t into the solution yields manager 1's equilibrium investment policy. For manager 2, the analysis is analogous.

Q.E.D.

Proof of Corollary 1. The limiting equilibrium policies of manager 1 is straightforwardly obtained by letting ξ_t tend to 0 or ∞ in (**??**), and similarly for manager 2.

Substituting the equilibrium limiting policies into (**??**) and multiplying both parts by the inverse of $(\sigma^\top)^{-1}\kappa$, we obtain that (**??**) is equivalent to

$$\operatorname{sgn}\left(\frac{1}{\gamma_1} + \frac{\theta(\gamma_1 - 1)}{\gamma_1 \bar{\gamma}_2} - \frac{1}{\bar{\gamma}_1}\right) = \operatorname{sgn}(\bar{\gamma}_1 - 1) \operatorname{sgn}\left(\frac{1}{\bar{\gamma}_2} - \frac{1}{\bar{\gamma}_1}\right). \tag{A22}$$

Rearranging the argument of $sgn(\cdot)$ on the left-hand side of (**??**) yields

$$\begin{aligned} \frac{\bar{\gamma}_1\bar{\gamma}_2 + \bar{\gamma}_1\alpha(\bar{\gamma}_1 - 1) - \bar{\gamma}_2(\bar{\gamma}_1 + \alpha(\bar{\gamma}_1 - 1))}{\gamma_1\bar{\gamma}_1\bar{\gamma}_2} &= \frac{\bar{\gamma}_1\alpha(\bar{\gamma}_1 - 1) - \bar{\gamma}_2\alpha(\bar{\gamma}_1 - 1)}{\gamma_1\bar{\gamma}_1\bar{\gamma}_2} \\ &= \frac{\alpha}{\gamma_1}(\bar{\gamma}_1 - 1)\left(\frac{1}{\bar{\gamma}_2} - \frac{1}{\bar{\gamma}_1}\right). \end{aligned}$$

Since α and γ_1 are positive, (**??**) obtains. Switching subscripts 1 and 2 in (**??**) gives (**??**).

Q.E.D.

Proof of Proposition ??. In Proposition **??** we show that multiple equilibria occur if the multiplicity condition (**??**) is satisfied. In the proof of Proposition **??**, we describe the structure of the multiple equilibria, i.e., the states in which outcome (winner, loser) or outcome (loser, winner) occurs in equilibrium, and the states in which either of the two outcomes is possible. In Proposition **??**, we describe what horizon wealth manager i, $i = 1, 2$, chooses in equilibrium when she is a winner and when she is a loser. Combining the results of these two Propositions, we obtain the horizon wealth profiles that can occur in the case of multiple equilibria.

Q.E.D.

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