On Heterogeneous Size of Stable Jurisdictions

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February 2005 (Preliminary Version)

Abstract

This paper examines a model of multi-jurisdiction formation where individuals' characteristics are uniformly distributed over the finite interval. Every jurisdiction chooses a location of a public good and equally shares the cost of production among its residents. We consider two notions of stability: Nash stability and its refinement — local stability, and examine the existence and characterization of stable partitions. The main feature of our analysis is the *heterogeneity gap* in jurisdiction sizes that may emerge for both stability notions mentioned above. These results may explain the creation of cities of different size even though the individuals' characteristics are represented by the uniform distribution.

Keywords: Stability, Jurisdictions, Public Projects, Heterogeneity Gap.

JEL Classification Numbers: D70, H20, H73.

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1 Introduction

Consider a problem where an urban population has to decide how many horizontally[1] differentiated public projects of Mas-Colell (1983) (say, libraries) to be built in the city, where to locate these libraries, how to assign each individual to a library, and, finally, how to split the burden of financing the libraries among the residents.

The group choice of the locational problem described here consists of three items:

• - *jurisdiction structure*, which is a partition of the set I into subsets of individuals, *jurisdictions*, assigned to the same library;

• - *libraries locations* in each jurisdiction, and

• - *sharing rule*, that is a choice of a contribution scheme in order to cover the total cost of libraries in all jurisdictions.

In this paper we focus on the first part of the problem, namely, a search for a partition of the entire population into several jurisdictions[2]. As far as a library location within each jurisdiction and sharing the libraries cost, we impose two following principles, *efficiency* and *equal share*.

The efficiency requires that each jurisdiction chooses a location of the library in such a way as to minimize the total transportation cost of its residents. We will make an assumption of linear transportation costs, under which the efficiency is equivalent to the majority voting requirement, and each jurisdiction should place the library at the location of its *median* resident. Following Alesina and Spolaore (1997), Casella (2001) and Haimanko et al. (2004a), we also impose the assumption of equal share where all members of the same jurisdiction make equal contributions towards the financing of the public project.[3]

[1]For the analysis of vertically differentiated projects, see Guesnerie (1995), Guesnerie and Oddou (1981,1988), Greenberg and Weber (1986), Jehiel and Scotchmer (2001), Weber and Zamir (1985), Westhoff (1977), Wooders (1978, 1980).

[2]We will use terms *jurisdiction structure* and *partition* interchangeably, throughout the paper.

[3]See Haimanko et al. (2004b) and Le Breton et al. (2004) for alternative approaches to cost sharing mechanisms.

We then introduce two notions of stability. The first is the standard Nash stability requirement which means that a partition is immune against individual moves from jurisdiction to jurisdiction. The alternative notion of stability is *local stability* where small intervals of individuals are allowed to migrate between jurisdictions while taking into account the median location of the public project and the equal share cost mechanism in newly created jurisdictions.

Next, we derive the conditions for the existence of both stability notions, and characterize stable partitions. The special attention will be devoted to the so-called *heterogeneity gap* in sizes of jurisdictions in stable partitions. We show that, unlike in Alesina and Spolaore (1997), our stability notions may yield a stable partition with sharply distinct jurisdiction sizes. Thus, even the uniform distribution of individuals' characteristics in an environment with horizontally differentiated projects, does not guarantee the uniformity of jurisdiction sizes.

2 The Model

We assume that individuals are located over the interval $I = [0, 1]$. The distribution of their locations is assumed to be uniform, in order to keep things as simple as possible, while stressing the stability issues. The analysis of the general case is more involved, while main ideas and notions are highlighted even in the simplest possible case of the uniformly distributed population.

The total mass of I is equal to 1 and we denote by λ the associated (Lebesgue) probability measure. The cost of every library is given by a positive parameter g. The transportation cost incurred by individual[4] t, assigned to a library located at point p, is given by the cost function $d(t, p) = |t - p|$.

For each measurable subset S of I, denote by $M(S)$ the set of *median locations* defined

[4]We will not distinguish between individual t and an individual located at point t.

by:

$$M(S)=\left\{p\in I:\lambda\left(\{t\in S:t\leq p\}\right)=\lambda\left(\{t\in S:t\geq p\}\right)=\frac{1}{2}\lambda(S)\right\}. \quad (1)$$

Since the minimization of the aggregate transportation cost of jurisdiction S:

$$D(S)=\min_{p\in I}\int_S d(t,p)\,dt \quad (2)$$

is attained when $p\in M(S)$, it follows that, whenever jurisdiction S forms, it selects a library location p from the set $M(S)$. Since the set $M(S)$ may contain more than one element, in order to reinforce the median principle we will always choose the mean of the median set $m(S)$, defined by

$$m(S)=M(M(S)). \quad (3)$$

Note that, since $M(S)$ is always an interval, its median is unique and is well-defined. If $M(S)$ itself is a single point, then $m(S)$ obviously coincides with $M(S)$. If the set $M(S)$ is an interval with a nonempty interior, then our selection picks the middle point of that interval.

A sharing rule, $x(t)$ describes the monetary contribution of each individual t towards the cost of the libraries. We assume that each jurisdiction is balances its budget and covers the cost of its own library by assigning required contributions to its members:

$$\int_S x(t)\,dt=g. \quad (4)$$

Whenever jurisdiction S forms, the total cost of individual $t\in S$, given selected library location p and cost allocation x, is

$$|t-p|+x(t). \quad (5)$$

We assume that whenever jurisdiction S forms, it distributes the cost equally among its residents, i.e.,

$$x(t)=\frac{g}{\lambda(S)}. \quad (6)$$

Under efficiency, specified location rule and equal-share, our model turns *hedonic*, where the identity of a jurisdiction is sufficient to determine the total cost, transportation and contribution, of each of its members. Thus, our choice would be to select a jurisdiction structure under various notions of stability.

In order to avoid pathological situations, we assume that the median locations are different for all jurisdictions in the partition. Indeed, it is unreasonable to have two or more jurisdictions with one and the same library location: pooling together would not change the location, hence, transportation costs of members of these jurisdictions, at the same time decreasing contribution (hence, total) costs of everyone involved.

Let us now introduce a concept of n-partition for an arbitrary positive integer n:

Definition 2.1: An n-partition $P = (S_i)_{1 \le i \le n}$ is a jurisdiction structure that consists of n measurable sets of positive measure with pairwise disjoint interior, the union of which being equal to the entire set I.[5] For every n-partition P, every $S \in P$ and every individual $t \in S$ we denote by $v_t(S)$ the payoff of individual $t \in S$ (recall that jurisdiction S chooses $m(S)$ as its library location):

$$v_t(S) = -|m(S) - t| - \frac{g}{\lambda(S)}. \tag{7}$$

An n-partition P is called *stratified* or *consecutive* if every set $S_i \in P$ is an interval.

It is useful to note that a stratified n-partition could be identified with $n-1$ points $0 < x_1 < \ldots < x_{n-1} < 1$ such that $S_i = [x_{i-1}, x_i]$ for $i = 1, \ldots, n$, where, by definition, $x_0 = 0$, and $x_n = 1$. In what follows, we will sometimes identify a stratified n-partition P with a bundle $(x_1, \ldots, x_{n-1})$.

Formula (7) makes sence for an arbitrary $S \subset I$ (not just for $S \in P$) and for all $t \in I$, and expresses the payoff the individual t would receive shall he find himself to be a member of a group S. Obviously, this function is continuous in t (in fact, peace-wise linear).

[5]If one allows for null-set jurisdictions, then its members would incur infinitely high costs; and such a jurisdiction could be merged with any other jurisdiction of positive measure, without affecting the well-being of the members of the latter.

The structure and the size of jurisdictions in the partition will play the major part in our analysis. We introduce the notion of a *heterogeneity gap* of a partition P, which equals to the ratio between the size of the largest and the smallest jurisdiction in the partition, and the *variance* of a partition P, which is the variance of jurisdiction sizes in P:

Definition 2.2: Let an n-partition P of I be given. Let $\underline{s}(P)$ and $\overline{s}(P)$ denote the sizes of the smallest and largest jurisdictions in P, respectively. Then the heterogeneity gap, $H(P)$ is defined by

$$H(P) = \frac{\overline{s}(P)}{\underline{s}(P)}. \tag{8}$$

The only n-partition for which $H(P) = 1$ will be called *homogeneous* as it consists of jurisdictions of the equal size ($\overline{s}(P) = \underline{s}(P) = \frac{1}{n}$). All other partitions, for which $H(P) > 1$, would be called *heterogenous*.

Definition 2.3: The *size variance*, $V(P)$ of an n-partition P is given by:

$$V(P) = \sum_{S \in P} \left(\lambda(S) - \frac{1}{n} \right)^2. \tag{9}$$

For any partition P we have $V(P) \geq 0$, and $V(P) = 0 \Leftrightarrow P$ is homogeneous.

In the next section we turn our attention to Nash stable partitions.

3 Nash Stability

The immediate implication of the Nash stability requirement in our framework amounts to the following:

Definition 3.1: A jurisdiction structure $P = \{S_1, \ldots, S_n\}$ is called *Nash stable* if the following is true:

$$v_t\left(S^t\right) \geq v_t\left(S_i\right) \tag{10}$$

for all $t \in I$ and all $S_i \in P$, where S^t denotes the jurisdiction in P that contains t. For a given project cost g and $n > 0$, the set $\mathcal{N}(g, n)$ will denote the set of all Nash stable n-partitions.

Note that the set of Nash stable jurisdiction structures is the set of pure Nash equilibria of the non-cooperative game where each agent announces the "address" from a certain exogenously given set of addresses, and every group of agents with same address forms a coalition.

Alternatively, we could see a Nash stable jurisdiction structure as a *free individual mobility equilibrium*: no individual has an incentive to move from his current location to another jurisdiction.[6]

Obviously, the set I itself is trivially a Nash stable 1-partition and we continue our examination for the case $n > 1$.

Proposition 3.2: (i) Every Nash stable jurisdiction structure P is stratified.

(ii) Every stratified n-partition $P = (x_1, \ldots, x_{n-1})$ is Nash stable if and only if the cost function of individuals, $u(t) = c_t(S^t)$ is continuous in t (like in Figure 1a, and contrast to a Figure 1b; instead of costs, we could have required continuity of the payoff function, $v_t(S^t)$, which is apparently the same requirement).

Proof of this and other propositions will be given in the Appendix.

[6]We should emphasize here that there is no physical movement of agents' addresses, t (or, equivalently, of their preferences); agents are allowed only to choose their *jurisdictions*.

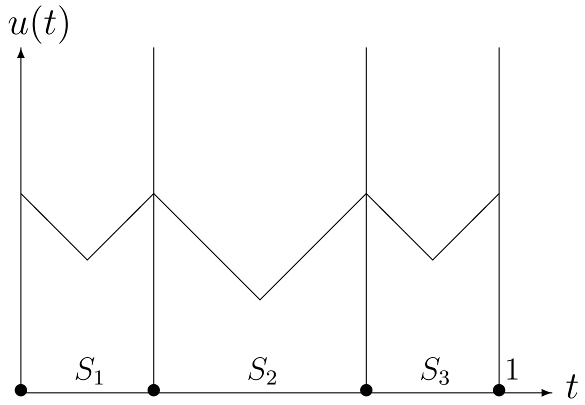

Figure 1a. A Nash stable stratified n-partition.

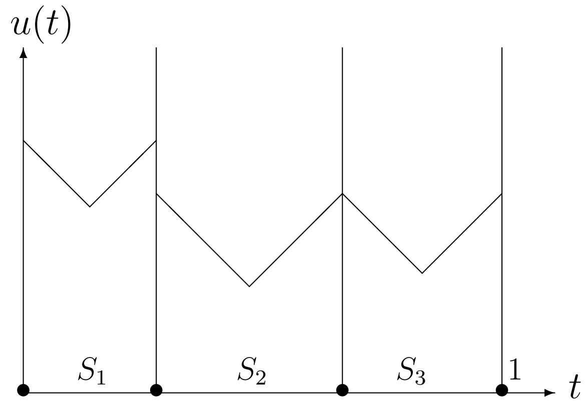

Figure 1b. An unstable stratified n-partition.

Figure 1. A geometric representation of Nash stable and unstable partitions.

The last assertion of Proposition 3.2 implies that every individual located at one of the border points x_i, where $i = 1, \ldots, n-1$, is indifferent between joining the two jurisdictions $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$, or $c_{x_i}(S_i) = c_{x_i}(S_{i+1})$ for every i between 1 and $n-1$. This observation would allow us to formulate the characterization of Nash stable partitions in terms of a simple system of $n-1$ equations over $n-1$ variables.

Since every element S_i of the stratified partition P is an interval $S_i = [x_{i-1}, x_i]$, then, from formula (7), we deduce that

$$x_i - \frac{x_{i-1} + x_i}{2} + \frac{g}{x_i - x_{i-1}} = \frac{x_i + x_{i+1}}{2} - x_i + \frac{g}{x_{i+1} - x_i} \tag{11}$$

holds for all $i = 1, \ldots, n-1$. Let us introduce the function Ψ, where the value $\Psi(s)$, defined by

$$\Psi(s) \equiv \frac{g}{s} + \frac{s}{2}, \tag{12}$$

is the total cost incurred by the peripheral individual who is located at the endpoint of the interval of the size s. (In fact, $\Psi(s) = v_0([0,s]) = v_s([0,s])$.) Then, we can rewrite (11) as

$$\Psi(x_i - x_{i-1}) = \Psi(x_{i+1} - x_i). \tag{13}$$

Thus, in any Nash stable n-partition $P = (x_1, \ldots, x_{n-1})$, we have:

$$\Psi(s_i) = \Psi(s_{i+1}) \quad \text{for all} \quad i = 1, \ldots, n-1, \tag{14}$$

where $s_i = x_i - x_{i-1}$. The function Ψ, whose minimum is attained at $\sqrt{2g}$, is strictly convex, and thus, there could be (maximum) two values of s that satisfy (14). It follows, therefore, that there could exist heterogeneous Nash stable jurisdiction structures. We show that if the project cost g is small enough, then, in addition to a homogenous partition, there indeed exist heterogenous Nash stable n-partitions:

Proposition 3.3: Let integer n be greater than 1. Then

(i) For every n there is a homogeneous n-partition $\left(\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}\right)$.

(ii) Denote

$$g_n = \frac{1}{8\left(n-1\right)}. \tag{15}$$

Then, a Nash stable heterogenous n-jurisdiction structure exists if and only if $g \leq g_n$ for $n \geq 3$; a Nash stable 2-partition exists if and only if $g < g_2$.

(iii) Every heterogenous partition contains jurisdictions of two different sizes, $\underline{s} < \overline{s}$, which satisfy the following equation:

$$\Psi\left(\underline{s}\right) = \Psi\left(\overline{s}\right). \tag{16}$$

In what follows, small size jurisdictions would play a crucial role; thus, their size in what follows will be redenoted simply by s, instead of $\underline{s}$.

Now let us evaluate the heterogeneity gap and the variance across all Nash stable partitions. For every integer $n > 0$ and the cost of the project $g > 0$, denote the heterogeneity gap $H^{\mathcal{N}}\left(n, g\right)$:

$$H^{\mathcal{N}}\left(n, g\right) = \max_{P \in \mathcal{N}(g,n)} H\left(P\right) \tag{17}$$

and the variance

$$V^{\mathcal{N}}\left(n, g\right) = \max_{P \in \mathcal{N}(g,n)} V\left(P\right). \tag{18}$$

If $\mathcal{N}\left(g, n\right) = \emptyset$, then we define $H^{\mathcal{N}}\left(n, g\right) = V^{\mathcal{N}}\left(n, g\right) = -\infty$. We have

Proposition 3.4: (i) For every integer $n > 0$

$$\sup_{g>0} H^{\mathcal{N}}(g, n) = \infty, \tag{19}$$

and

$$\sup_{g>0} V^{\mathcal{N}}(g, n) = 1 - \frac{1}{n}. \tag{20}$$

Notice that $1 - \frac{1}{n}$ is the absolute maximum of variance over the set of all possible n-partitions, not just over Nash stable ones.

In the next section we will analyze jurisdiction structures that satisfy a more tough stability requirement, and estimate the heterogeneity gap and variance for this alternative stability concept.

4 Local stability

Since no single individual has an impact on the library location selected by a jurisdiction, the impact of his migration from one jurisdiction to another on other individuals could be ignored. However, if an individual t finds it beneficial to migrate to another jurisdiction, then other individuals, close to t, would contemplate the same migrational choice. Then the group of individuals willing to migrate has a positive measure and its relocation would affect both the new jurisdiction and that they left behind.

The argument for considering group moves could be made even stronger when examining the situations where no individual could benefit from shifting to another jurisdiction alone, whereas a set of individuals with a positive measure would benefit all its members by making a collective migration decision. The reason lies in the very fact just mentioned: a positive-measured migration affects the location of a group which it enters (as well as the size of the latter). Obviously, to claim stability of a certain partition, one should check whether the latter is prone to such threats.

At the same time, allowing any group of individuals to move seems to overlook difficulties related to a migration decision. Indeed, that decision requires high coordination and communication cost that could be excessive if one considers a large group spread over a long patch of territory. To preserve a spirit of the Nash paradigm, in this paper we consider the requirement that allows only for a migration of a small groups of individuals with almost identical locations and adopt the stability concept of "free mobility equilibrium" of Jehiel and Scotchmer (2001):

Definition 4.1: An n-jurisdiction structure $P = \{S_1, \ldots, S_n\}$ is said to be *locally stable* if there exists $\varepsilon > 0$ for which there is no interval $[x, x+\varepsilon] \subset [0,1]$, together with a jurisdiction $S \in P$, with the property that $\forall t \in [x, x+\varepsilon]$ we have

$$v_t \left(S \cup [x, x+\varepsilon]\right) > v_t \left(S^t\right), \tag{21}$$

where, to recall, S^t denotes the jurisdiction in P that contains t. For a given project cost $g > 0$ and $n > 0$, the set $\mathcal{L}(g,n)$ will denote the set of all locally stable n-partitions.

The discussion in the beginning of this section implies that every locally stable jurisdiction structure is a Nash stable partition (this is proved formally in the Appendix):

Corollary 4.2: For every $n > 1$ and $g > 0$, we have

$$\mathcal{L}(g,n) \subset \mathcal{N}(g,n). \tag{22}$$

Since every locally stable partition is Nash stable, it ought to be stratified. It turns out then, that local stability is implied by a much more simple and tractable condition that no small interval of individuals neighbor to a jurisdiction S would benefit by joining S. Specifically, we have the following

Proposition 4.3: A stratified partition $P = (x_1, \ldots, x_{n-1})$ is locally stable if and only if there is an $\varepsilon > 0$ for which there exists no jurisdiction $S_i \in P$, together with some interval J that belongs to one of the two families of intervals adjacent to jurisdiction S_i,

$$\{[x_i - \mu, x_i]\}_{0<\mu\leq\varepsilon}\,;\ \{[x_{i+1}, x_{i+1}+\mu]\}_{0<\mu\leq\varepsilon} \tag{23}$$

such that

$$v_t\left(S_i \cup J\right) > v_t\left(S^t\right), \tag{24}$$

for all individuals t from J.

In characterizing partitions that are not only Nash stable but also are prone to small group migrations, one is to notice that these partitions could not contain jurisdictions that are "too small". The reason is that in such jurisdictions their small enlargement would lead to a sharp decrease of costs; thus, a small interval adjacent to such a jurisdiction would have a stimulus to migrate to that small jurisdiction, and we would have no local stability. The next proposition makes this reasoning clear and precise:

Proposition 4.4: Let P be a Nash stable n-jurisdiction structure for $n > 1$.

(i) Then P is locally stable if and only if

$$\lambda\left(S\right) \geq \sqrt{\frac{2g}{3}} \tag{25}$$

for every jurisdiction $S \in P$.

(ii) Let $g > 0$ be given. Then a homogeneous n-jurisdiction structure is locally stable if and only if $n \leq \sqrt{\frac{3}{2g}}$.

One can notice that the size of a jurisdiction, s, is the only parameter which value matters when checking whether the jurisdiction is migration-proof or not. At first sight it looks surprising, for one may expect the size of a neighboring jurisdiction to enter the criterion; however, it does not. The explanation is that the size of the neighboring jurisdiction has already been taken into account when assuming that the partition under consideration is Nash stable. This guarantees equal payoffs at the border, and all that matters afterwards is how the payoff in the new jurisdiction is modified after migration. As for the payoff at home jurisdiction, it increases linearly with the slope 1 while approaching the center of the jurisdiction, irrespective of the size of the home jurisdiction.

Now, we turn to a description of heterogeneous locally stable n-partitions. From the previous section we now the conditions under which heterogeneous Nash stable partitions exist; it turns out then, that there are additional requirements implying that not every heterogeneous Nash stable partition is locally stable. The complete list of conditions for existance of heterogeneous locally stable n-partitions is given in the appendix; here we present a summary of results.

Proposition 4.5: (i) Let $n \neq 1, 3, 4, 6$ be given. Then, heterogeneous locally stable n-partition exists for $g \in \left[\underline{g}(n), \overline{g}(n)\right]$;

(ii) Let $n \in \{3, 4, 6\}$. Then, heterogeneous locally stable n-partition exists for $g \in \left[\underline{g}(n), \overline{g}(n)\right] \setminus \{\frac{1}{2n^2}\}$.

Some explanations are needed. As one could have noticed, for homogeneous partitions there is only an upper bound for g, in order to assure stability. At the same time, for heterogeneous partitions a lower bound also emerges. The reason is simple: for very low g, in heterogeneous n-partition the size of big jurisdictions exceeds the most efficient number $\sqrt{2g}$ (where the function Ψ reaches its minimum) by much; hence, the border individuals suffer from being far from the location of their groups. This in turn means that their costs are high, therefore, costs are also high for the individuals in the small jurisdictions (by continuity of the cost function, being assured at every Nash stable partition). The only way, however, for costs to be high is for the size of small groups to be extremely low (because g is very small), even when compared to the efficient size of $\sqrt{2g}$.

As a result, we have that, first, the size of small jurisdictions is less then the cutoff value of $\sqrt{\frac{2g}{3}}$ from Proposition 4.4 (hence, for small g there are no heterogeneous locally stable n-partitions), and, second, that in heterogeneous locally stable partitions, the sizes of small and big groups should not differ by much from each other. Hence, the *heterogeneity gap* is bounded from above, for this type of stability requirement.[7]

[7]For some technical reasons explained in the Appendix, a locally stable n-partition fails to exist when g

To make the last statement precise, let us evaluate, as in the previous section, the heterogeneity gap and the variance across all locally stable n-partitions. For every integer $n > 0$ and the project cost $g > 0$, we denote the heterogeneity gap by $H^{\mathcal{L}}(n, g)$:

$$H^{\mathcal{L}}(n, g) = \max_{P \in \mathcal{L}(g,n)} H(P) \tag{26}$$

and the variance

$$V^{\mathcal{L}}(n, g) = \max_{P \in \mathcal{L}(g,n)} V(P). \tag{27}$$

If $\mathcal{L}(g, n) = \emptyset$, then, again, both $H^{\mathcal{L}}(n, g)$ and $V^{\mathcal{L}}(n, g)$ are by default equal to $-\infty$. We have

Proposition 4.6: (i) For every integer $n > 1$

$$\max_{g>0} H^{\mathcal{L}}(g, n) = 3; \tag{28}$$

and (ii) for every integer $n > 1$

$$\max_{g>0} V^{\mathcal{L}}(g, n) = \frac{1}{3n} \tag{29}$$ [8]

The results obtained so far could be illustrated graphically. First, look at the Figure 2. One can see several graphs of the Ψ-function, for different values of g. All these graphs have the same two asymptotas, the vertical axe and the line $g = \frac{s}{2}$. The lower g, the closer is the graph to this asymptotas.

Further on, there are two more lines on this graph. The first is the line $\left(\sqrt{2g}, \sqrt{2g}\right)$. It crosses all the Ψ-graphs at their minima. The second line, $\left(\sqrt{\frac{2g}{3}}, 2\sqrt{\frac{2g}{3}}\right)$, crosses the graphs in the points corresponding to the minimal sizes of jurisdictions in locally stable partitions. The value $3\sqrt{\frac{2g}{3}}$ then corresponds to the size of big jurisdictions, in the extreme case. Therefore, 3 is the maximum value of heterogeneity gap for locally stable partitions, in accordance with Proposition 4.6.

equals exactly to $\frac{1}{2n^2}$, if $n = 3, 4, 6$. For other values of n, this phenomenon is not observed anymore.

[8] This result is exact for $n = 4m$, with m integer. For other values of n, the maximum variance is very close to this number (see the Appendix for detailed explanation).

Now, look at the Figure 3. It illustrates Proposition 4.4: one can see the curve $\bar{g}_{LS} = \dfrac{1.5}{n^2}$ which gives us the upper bound on g in locally stable homogeneous n-partitions. For $n = 1$, the 1-partition is trivially locally stable for all values of g, which is also expressed in the figure.

Next, let us move to the Figure 4. It presents bounds on g, both upper and lower, on the existence and local stability of heterogeneous Nash stable n-partitions. The lower bound for local stability is given by the curve $\underline{g_{LS}}$; the upper bounds for existence and local stability are, respectively, given by the curves g_{exist} and $\overline{g_{LS}}$. Note, in accordance with Proposition 4.5, that the $\overline{g_{LS}}$-curve starts from $n = 4$: for $n = 3$, existence imply local stability. The case $n = 2$ is treated distinctly, and for this case the upper and lower bounds on existence and local stability, respectively, are given by $g = 0.125$ and $g \equiv 0.094$.

Finally, the bounds on existence and local stability of homogeneous and heterogeneous n-partitions are summarized in the Table 1 below (for values of n between 2 and 7).

Table 1. Critical values for existence and local stability

of homogeneous and heterogeneous n-partitions, for $n = 2, 3, 4, 5, 6, 7$.

Cutoff values of g for $n =$	2	3	4	5	6	7
homogeneous $\overline{g_{LS}}$	0.375	0.167	0.094	0.06	0.041	0.03
heterogeneous g_{exist}	0.125	0.063	0.041	0.032	0.025	0.02
heterogeneous $\overline{g_{LS}}$	NA	NA	NA	0.03	0.023	0.018
heterogeneous $\underline{g_{LS}}$	0.094	0.03	0.015	0.0085	0.006	0.004

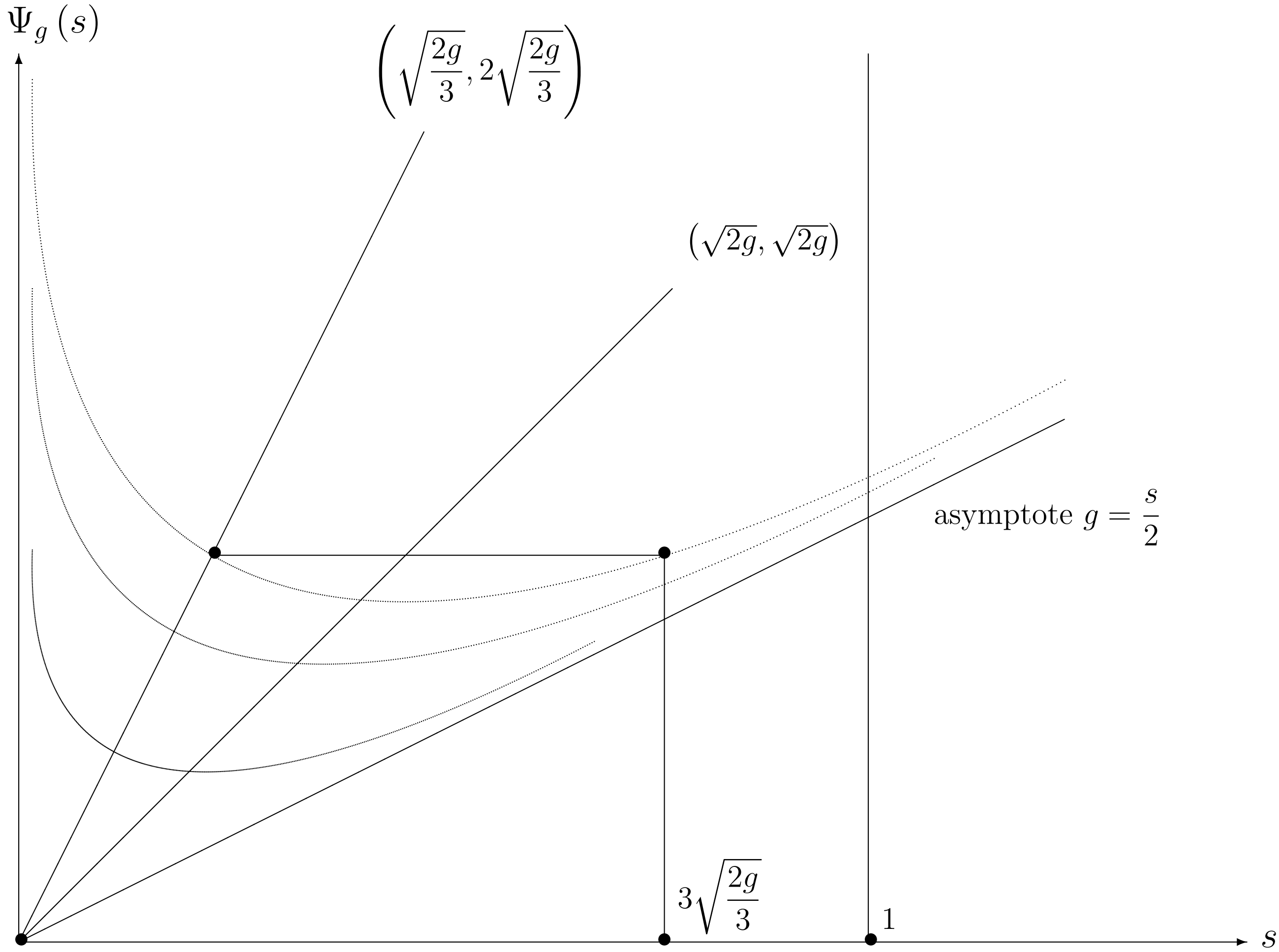

Figure 2. Total costs function $\Psi(s)$ of the border individual in the jurisdiction of the size s, for alternative values of parameter g, and the lower bounds on sizes of small jurisdictions in locally stable partitions.

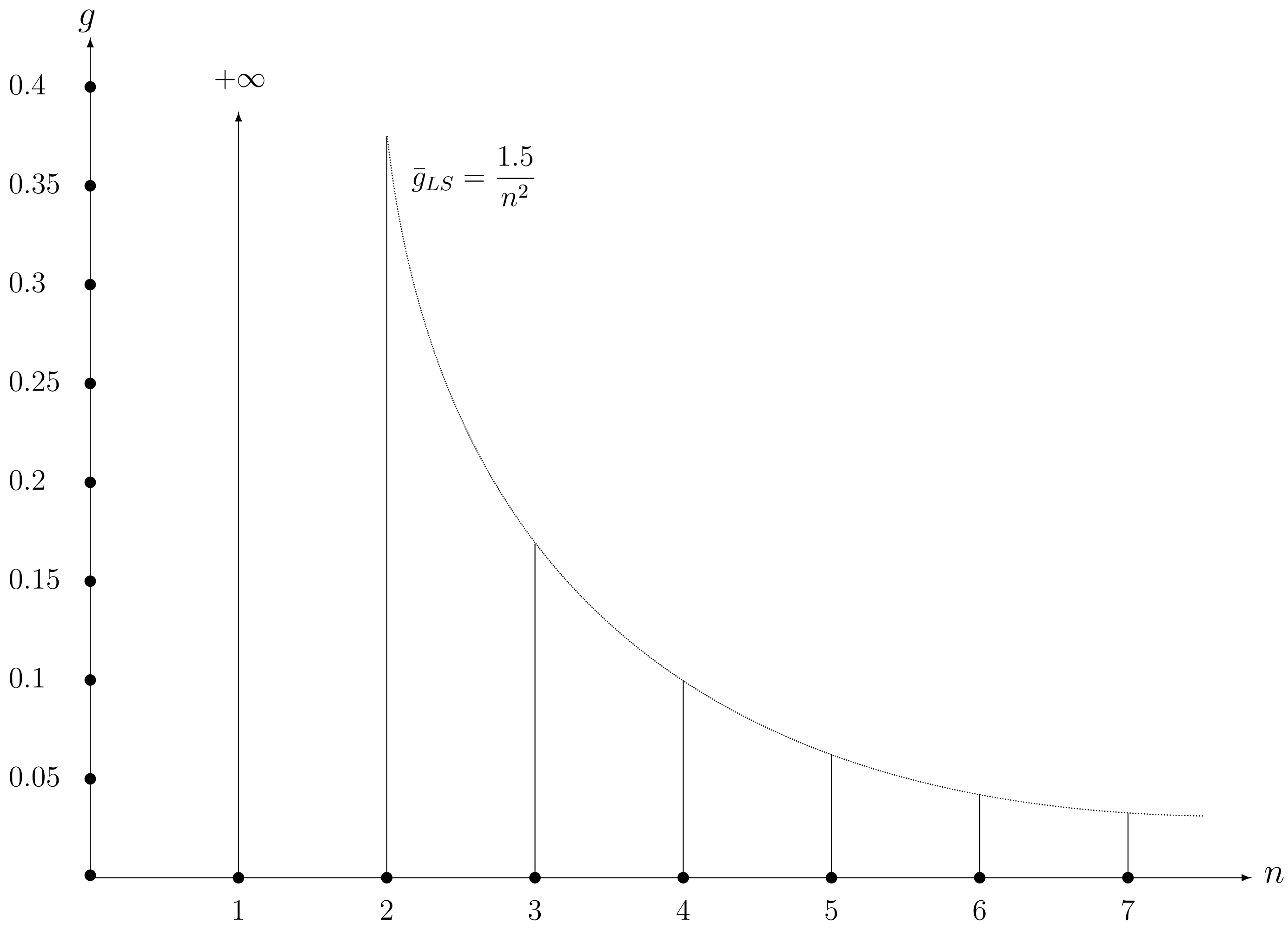

Figure 3. Zones of local stability of homogeneous n-partitions for alternative n

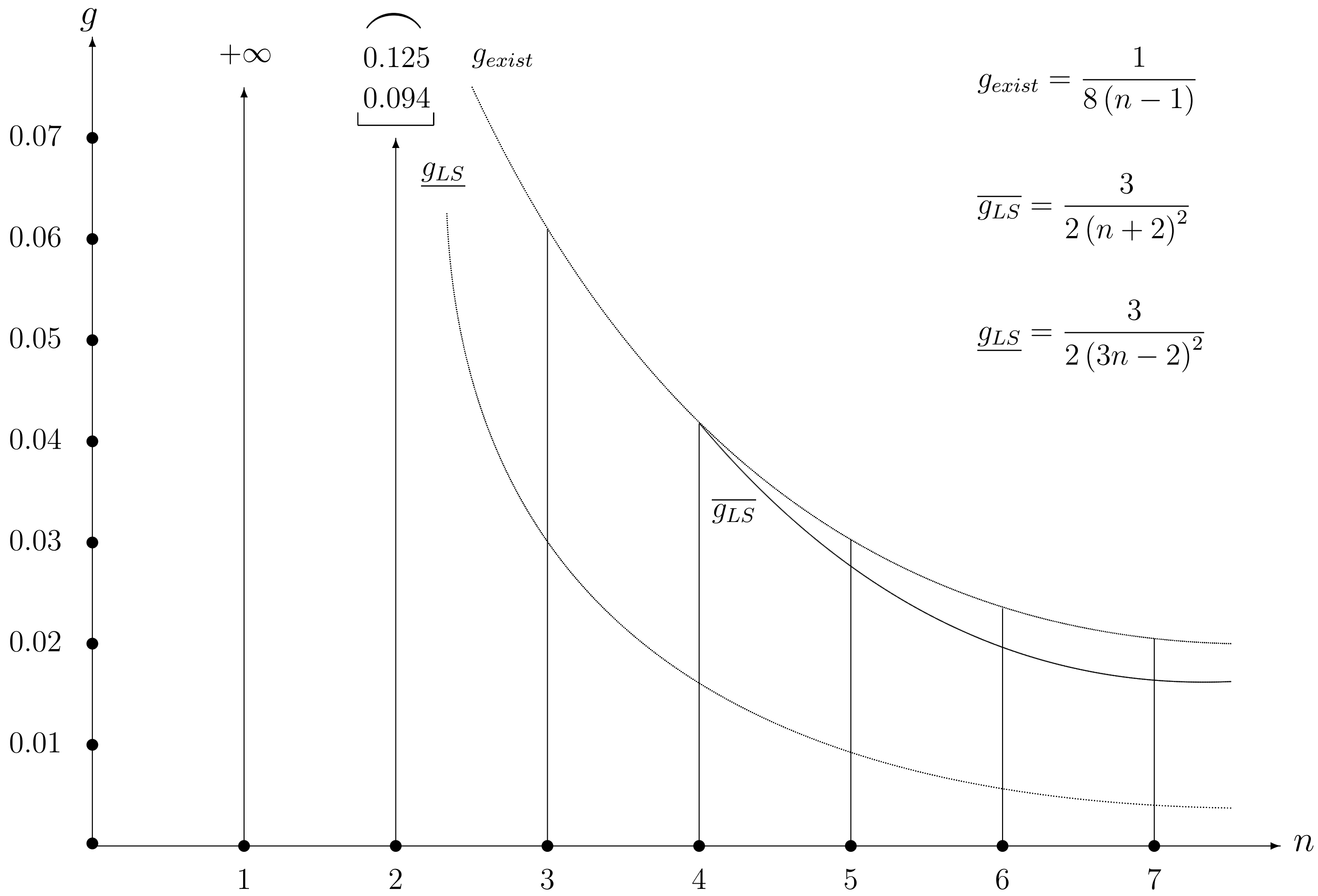

Figure 4. Existence and local stability of heterogeneous n-partitions

Appendix

Proof of Proposition 3.2 (i) By contradiction, suppose that we have a non-stratified Nash stable jurisdiction structure P; hence, there exist $S, T \in P$, and $s_1, s_2 \in S,\ \ t \in T$ such that $s_1 < t < s_2$. Consider $v_t(S)$ and $v_t(T)$ as functions of t. In what follows, it is more convenient to operate with costs functions, rather than with payoffs, so let us denote $c_t(S) := -v_t(S)$. Now, three cases are distinct:

Case 1. $\forall t \in I \quad c_t(T) \leq c_t(S)$. In this case, a partition P is by no means Nash stable, since each agent from S would want to move to T if this inequality is strict, and each agent from S located strictly to the same direction from $m(S)$ as $m(T)$ is, would be better off in the jurisdiction T, if this inequality ever turns to an equation (see Figures 5a,b).

Case 2. $\forall t \in I \quad c_t(S) \le c_t(T)$. This case is just symmetric to the previous one, see Figures 5c,d.

Case 3. Neither of functions $c_t(T), c_t(S)$ dominates the other one (see Figure 5e). WLOG, assume that $m(S) < m(T)$ (in the opposite case, just revert the real line). Then, the difference $\delta(t) = c_t(S) - c_t(T)$ is nondecreasing in t, see figure 5f. Moreover, it satisfies two inequalities: $\delta(m(S)) < 0$ and $\delta(m(T)) > 0$, due to assumptions of case 3, and strictly increases on $[m(S), m(T)]$ (it is linear on this segment with a slope of 2).

Now, recall that there are $s_1, s_2 \in S,\ t \in T$ such that $s_1 < t < s_2$. If we observe $c_t(S) < c_t(T)$, then t will move to S, hence, P is not Nash stable; if, on the contrary, $c_t(S) \ge c_t(T)$, i.e. if $\delta(t) \ge 0$, then by nondecreasing nature of δ we should have $t > m(S)$, hence, δ either is strictly increasing or already constant at t, a constant which is greater than zero; in both cases, we have for $s_2 > t$ that $\delta(s_2) > 0$, hence, $c_{s_2}(S) > c_{s_2}(T)$ and s_2 will move to T, hence, again, P is not Nash stable. □

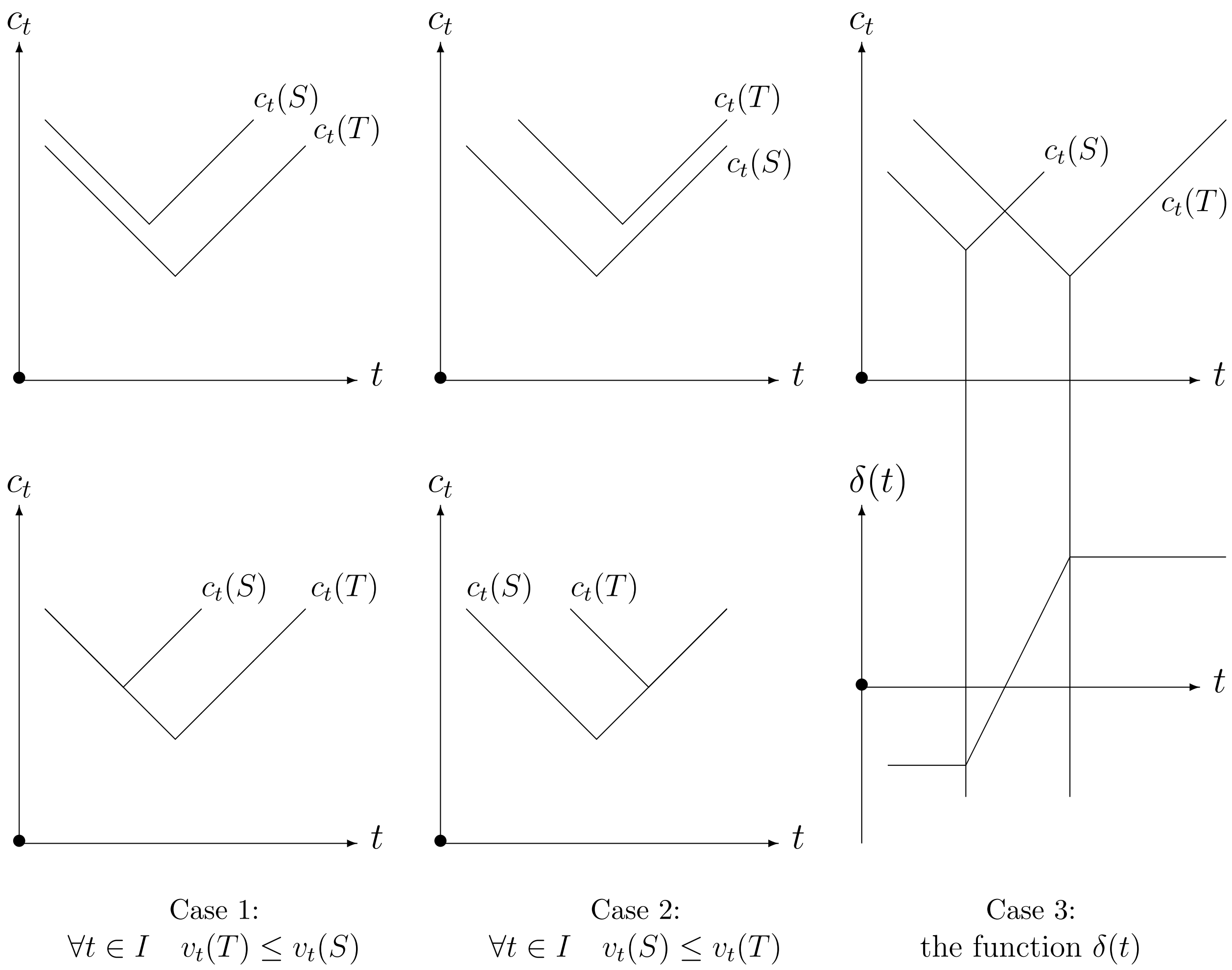

Figure 5. Nash stable partitions are necessary stratified.

(ii) First of all, let us note that the continuity requirement is equivalent to having

$$c_{x_i}(S_i) = c_{x_i}(S_{i+1}) \tag{30}$$

for all $i = 1, \ldots, n-1$, since the only points of possible discontinuity of $c_t(S^t)$ are the very border points $x_i, i = 1, \ldots, n-1$. So, it is suffice to prove the equivalence between Nash stability and the property (30), for an arbitrary stratified partition P.

Let P be Nash stable, and assume that $\exists i: \; c_{x_i}(S_i) \neq c_{x_i}(S_{i+1})$, say, WLOG, that $c_{x_i}(S_i) > c_{x_i}(S_{i+1})$. Due to continuity of functions $c_t(S_i)$ and $c_t(S_{i+1})$, there exists a positive μ such that $\forall t \in [x_i - \mu, x_i]$ we still have that $c_t(S_i) > c_t(S_{i+1})$. Take μ small enough to have for all these t that $t \in S_i$. Then, it means that $c_t(S^t) > c_t(S_{i+1})$ everywhere on the segment

$[x_i - \mu, x_i]$. Take an arbitrary such t and observe that this individual would unanimously want to move to the jurisdiction S_{i+1}, hence, P is not Nash stable, a contradiction.

Conversely, let $\forall i = 1, \ldots, n-1 \quad c_{x_i}(S_i) = c_{x_i}(S_{i+1})$, or equivalently, that $u(t) = c_t(S^t)$ is a continuous function in t. For an arbitrary subset $S \subset I$ (a possible jurisdiction), consider the following function

$$\Delta_S(t) = v_t\left(S^t\right) - v_t(S) = c_t(S) - c_t\left(S^t\right) = c_t(S) - u(t), \tag{31}$$

which gives the difference between a payoff that an individual t receives "at home", and that he would have received would he move to the jurisdiction S. We claim that this function is continuous and (nonstrictly) single-dipped, attaining its minimum at $m(S)$ (in fact, this function is peace-wise linear, look at the Figure 6a).

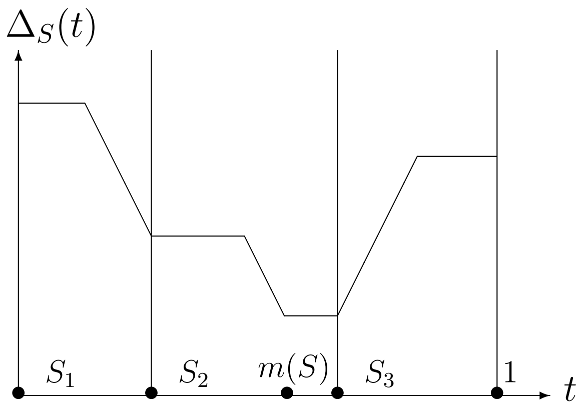

Figure 6a. A graphical representation of a function $\Delta_S(t)$, where $S \notin P$

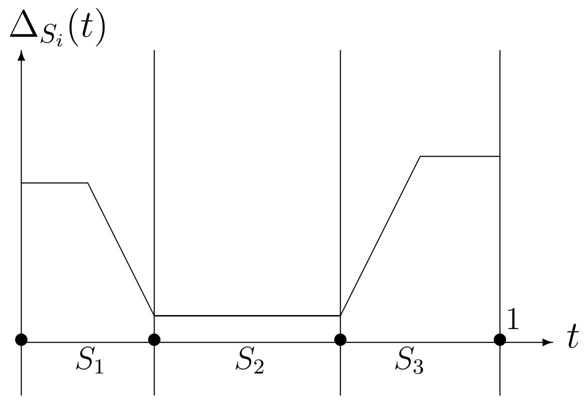

Figure 6b. A graphical representation of a function $\Delta_{S_2}(t)$, where $S_2 \in P$

Indeed, continuity follows directly from our assumption and formula (7); as for single-dippedness, it is suffice to prove that this function is non-increasing for $t \leq m(S)$. The case $t \geq m(S)$ is just mirror-symmetric. Continuity assures that one could separately check that this function is non-increasing within each jurisdiction S_i lying to the left of $m(S)$ (and within one of the two parts of the only jurisdiction that contains $m(S)$, i.e. the part lying to the left of $m(S)$).

But it is easy. Recalling that we are dealing with the case $t \leq m(S)$, write

$$\begin{gathered}\Delta_S(t) = v_t(S^t) - v_t(S) = \\ |t - m(S)| - |t - m(S_i)| + \frac{g}{\lambda[S]} - \frac{g}{\lambda[S_i]} = \\ m(S) - t - |t - m(S_i)| + Const = \\ \begin{cases} Const, & t \leq m(S_i); \\ Const - 2t, & t \geq m(S_i). \end{cases}\end{gathered} \tag{32}$$

Here, $Const$ means the part of the expression that does not depend on t. These constants are different in different parts of this system of equations.

Now, knowing that $\Delta_S(t)$ is single-dipped with a minimum at $m(S)$, just replace S with each of the sets S_j (look at the figure 6b). We get that $\Delta_{S_j}(m(S_j)) = 0$, and we know that it is minimum. Hence, no agent exists with $v_t(S^t) < v_t(S_j)$, which by definition means that P is Nash stable. □

Proof of Proposition 3.3: (i) Follows directly from formula (13): all the equations are satisfied when $s_i = \dfrac{1}{n}$ $\forall i = 1, \ldots, n$. □

(ii) Let P be a Nash stable heterogenous n-jurisdiction structure. Then, as (13) should hold, we conclude that there are exactly two different values in the bundle $(s_1, \ldots, s_n)$ (at least two, due to heterogeneity, and no more than two, since equation $\Psi(s) = w$ has at most two solutions). Let us denote the common value of $\Psi(s_k)$ by w, and calculate these two different jurisdiction sizes. This just means to solve the equation

$$\Psi(s) = \frac{s}{2} + \frac{g}{s} = w, \tag{33}$$

which gives $s = w \pm \sqrt{w^2 - 2g}$ for $w \geq \sqrt{2g}$ and no solution for $w < \sqrt{2g}$. Exact value $w = \sqrt{2g}$ does not satisfy us as well, for in this case there is only one solution $s = w$ and no heterogeneity in the corresponding partition.

So, in every heterogeneous n-partition we should have $w > \sqrt{2g}$, and there are two sizes of jurisdictions, $\underline{s} = w - \sqrt{w^2 - 2g}$ and $\overline{s} = w + \sqrt{w^2 - 2g}$. Let there be k small jurisdictions (of the size s) and l big jurisdictions (of the size $\overline{s}$), where $k + l = n$. Then, we must have $ks + l\overline{s} = 1$, because the size of the overall society is 1. Therefore,

$$l\left(w + \sqrt{w^2 - 2g}\right) + k\left(w - \sqrt{w^2 - 2g}\right) = nw + (l - k)\sqrt{w^2 - 2g} = 1. \tag{34}$$

As $l > 0$ in heterogeneous partitions, we must have $k < n$ and since both k and l are integers, we have that $l - k \geq 2 - n$. This means that

$$1 = nw + (l - k)\sqrt{w^2 - 2g} \geq nw - (n - 2)\sqrt{w^2 - 2g} \geq \min_{w > \sqrt{2g}} \left\{nw - (n - 2)\sqrt{w^2 - 2g}\right\}. \tag{35}$$

But this minimum is easy to calculate. Matters are somehow different for $n = 2$ and $n > 2$, so consider them in turn. Let us start with the case $n = 2$.

For $n = 2$, the expression $nw - (n - 2)\sqrt{w^2 - 2g}$ reduces just to $2w$ and, naturally, is minimized when $w \to \sqrt{2g}$; its minimum is not being reached within the region of heterogeneity (i.e. $\sqrt{2g} < w < +\infty$), hence for heterogeneous 2-partitions we have

$$1 = 2w > 2\sqrt{2g} \Leftrightarrow g < g_2. \tag{36}$$

For $n > 2$, the expression $nw - (n - 2)\sqrt{w^2 - 2g}$ reaches its minimum when

$$w = \frac{n\sqrt{2g}}{\sqrt{n^2 - (n-2)^2}} > \sqrt{2g}; \tag{37}$$

and this minimum is equal to $\sqrt{\left[n^2 - (n-2)^2\right] \cdot 2g}$. Hence, if a heterogeneous Nash stable n-partition exists, we must have

$$1 \geq \sqrt{\left[n^2 - (n-2)^2\right] \cdot 2g} \Leftrightarrow g < g_n, \tag{38}$$

after trivial exercises. This completes the "only if" part of a proof. Before turning to the "if" part, let us make the following observation. It will be useful in what follows.

Proposition A.1: The expression

$$nw - (k - l)\sqrt{w^2 - 2g} \tag{39}$$

increases monotonically from $\sqrt{2g}$ to $+\infty$ for $k \leq l$; and it is single-dipped with the minimum at

$$w = \frac{n}{\sqrt{n^2 - (k-l)^2}}\sqrt{2g} \tag{40}$$

if $k > l$. This expression is unbounded from above when w approaches infinity.

Proof of Proposition A.1 reduces to using elementary calculus techniques: it is suffice to consider the derivative of (39). □

Let us now have either $n = 2$ and $g < g_2$ or $n > 2$ and $g \leq g_n$. We will now demonstrate that there exists a heterogeneous Nash stable n-partition in such cases.

Indeed, when $n = 2$, let $w = \frac{1}{2}$. As $g < g_2 = \frac{1}{8}$, we know that $w^2 = \frac{1}{4} > 2g$, hence, $w^2 - 2g > 0$ and $s = w - \sqrt{w^2 - 2g} \neq w + \sqrt{w^2 - 2g} = \overline{s}$, and a partition $\{s, \overline{s}\}$ is the desired heterogeneous 2-partition, since $s + \overline{s} = 2w = 1$.

When $n > 2$, we should use the continuity argument. From now on, we will refer to an n-partition which contains k small and l big jurisdictions, $k + l = n$, as a (k,l)-*partition*. Now, if $g = g_n$, then that very partition associated with $w = \frac{n\sqrt{2g}}{\sqrt{n^2 - (n-2)^2}} > \sqrt{2g}$ which minimizes the expression $nw - (n-2)\sqrt{w^2 - 2g}$ is the desired heterogeneous (k,l)-partition, if we take $k = n - 1$ and $l = 1$.

If $g < g_n$, the collection of $n - 1$ small jurisdictions and 1 big jurisdiction corresponding to that value of w is not an n-partition anymore, for in this case we have

$$(n-1)\, s + \overline{s} < 1. \tag{41}$$

According to Proposition A.1, the expression (39) is unbounded from above, and by continuity will come across 1 for a certain w. This very value of w will generate a desired heterogeneous $(n-1,1)$-partition. □

Proof of Proposition 3.4: (i) Again, recall Proposition A.1. According to this proposition, for a given small g we can always find a unique heterogeneous Nash stable (k,l)-partition with $w \in \left[\frac{n}{\sqrt{n^2 - (k-l)^2}} \sqrt{2g}, +\infty \right)$. Take, for example, such a partition with $k = n - 1$ and $l = 1$.

As g approaches zero, the size of small jurisdictions, apparently, approaches zero as well (they are smaller than $\sqrt{2g}$); the size of the only large jurisdiction, hence, is close to 1. Therefore,

$$\lim_{g \to 0} \frac{\overline{s}}{s} = +\infty. \tag{42}$$

□

(ii) As for the variance, let us first deduce the general formula for $V(P)$ for an arbitrary stratified partition $P = (x_1, \ldots, x_{n-1})$. In what follows, it will be useful to characterize such a partition P not by a bundle of x_i, but alternatively by a bundle of sizes of jurisdictions: $P = (s_1, \ldots, s_n)$, where $\sum_{i=1}^{n} s_i = 1$ and $s_i = x_i - x_{i-1} > 0$ for $i = 1, \ldots, n$; that is, stratified partitions one to one correspond to points lying in an interior of the standard n-symplex. Now, using formula (9), we get:

$$\begin{gathered} V(P) = \sum_{i=1}^{n} \left(s_i - \frac{1}{n} \right)^2 = \\ \sum_{i=1}^{n} s_i^2 - \left(\frac{2}{n} \right) \sum_{i=1}^{n} s_i + n \left(\frac{1}{n} \right)^2 = \\ \sum_{i=1}^{n} s_i^2 - \frac{1}{n}, \end{gathered} \tag{43}$$

after simple rearrangements. As $\forall i = 1, \ldots, n \;\; s_i \geq 1$, we have $s_i^2 \leq s_i$ and therefore $\sum_{i=1}^{n} s_i^2 \leq \sum_{i=1}^{n} s_i = 1$, hence, for an arbitrary stratified partition $P = (s_1, \ldots, s_n)$ we must have $V(P) \leq 1 - \frac{1}{n}$.

Now, take again a Nash stable $(n-1, 1)$-partition for $g \to 0$. We now that the size s of small jurisdictions approaches zero, whereas the size $\overline{s}$ of the only big jurisdiction approaches 1; hence, in this case $\sum_{i=1}^{n} s_i^2 \to 1$ as $g \to 0$, and as a consequence, we have $V(P) \to 1 - \frac{1}{n}$, its maximum value. □

Proof of Corollary 4.2: Let us prove it carefully, from the negation. Assume P is not Nash stable, hence, $\exists i: \; c_{x_i}(S_i) \neq c_{x_i}(S_{i+1})$, say, WLOG, $c_{x_i}(S_i) > c_{x_i}(S_{i+1})$. Due to continuity of functions $c_t(S_i)$ and $c_t(S_{i+1})$, there exists a positive μ such that $\forall t \in [x_i - \mu, x_i]$ we have still that $c_t(S_i) > c_t(S_{i+1})$. Take μ small enough to have for all these t that $t \in S_i$ and $m([x_i - \mu, x_i] \cup S_{i+1}) \geq x_i$. Then, it means that $c_t(S^t) > c_t(S_{i+1})$ everywhere on the segment $[x_i - \mu, x_i]$.

Now, we claim that $c_t(S_{i+1}) > c_t([x_i - \mu, x_i] \cup S_{i+1})$ for all $t \in [x_i - \mu, x_i]$. Indeed, it follows from formula (7), since we have for all such t that, first, $t < m([x_i - \mu, x_i] \cup S_{i+1}) < m(S_{i+1})$, hence, personalized distant costs decrease after simultaneous migration, compared to individual migrational move of any t, and second, $\lambda[[x_i - \mu, x_i] \cup S_{i+1}] > \lambda[S_{i+1}]$, which means that per capita cost of a project decreases as well.

Hence, knowing that any $t \in [x_i - \mu, x_i]$ wishes to migrate even individually, we infer that organized migration is desired for all such t even more. Therefore, the segment $[x_i - \mu, x_i]$ would unanimously want to move to the jurisdiction S_{i+1}, which means that P is not locally stable, a contradiction.□

Proof of Proposition 4.3: We assume that the property stated in Proposition 4.3 holds, and prove local stability. Let ε be that from the stated property (i.e. no adjacent interval J with $\lambda[J] < \varepsilon$ exists that wish to move), and additionally require that $\varepsilon < \min\limits_{i=1,\ldots,n} \lambda[S_i]$.

Suppose, by negation, that there exists some (not adjacent to some S_i) interval J with $\lambda[J] < \varepsilon$ which members all increase their payoffs in $J \cup S_i$. The case $J \cap S_i \neq \emptyset$ immediately leads to having $J \setminus S_i$ as the adjacent interval with $\lambda[J \setminus S_i] < \varepsilon$ that is prone to a migration, so assume that $J \cap S_i = \emptyset$. Then, at least, $m(S_i \cup J) \notin J$ (recalling that $\varepsilon < \min\limits_{i=1,\ldots,n} \lambda[S_i]$, hence, $\lambda[J] < \lambda[S_i]$).

Now, using the definition of a function Δ given above (see formula (31) for $S = S_i \sqcup J$, rewrite the formula (24) in the form of

$$\forall t \in J \quad \Delta_{S_i \sqcup J}(t) < 0. \tag{44}$$

But in this case, due to properties of the function $\Delta_S(t)$ established in the proof of Proposition 3.2, we must have that $\Delta_{S_i \sqcup J}(t) < 0 \; \forall t$ lying between the point $m(S_i \cup J)$ and the set J, and in particular for all t lying between the jurisdiction S_i and the interval J.

And as a final step, one concludes that the interval J' of the size $\lambda[J]$ which is adjacent to S_i and lies to the same side of S_i as J itself, would want to move to S_i, since $\forall t \in J'$ we have

$$\Delta(t) < 0 \Leftrightarrow v_t\left(S^t\right) < v_t(S_i \cup J) = v_t(S_i \cup J'), \tag{45}$$

because $m(S_i \cup J) = m(S_i \cup J')$ and $\lambda[S_i \cup J] = \lambda[S_i \cup J']$. But this is a contradiction. □

Proof of Proposition 4.4: (i) Consider a jurisdiction S of a size s in the Nash stable partition P. The question is: Under which circumstances, this jurisdiction is stable with respect to possible migrational "from abroad"?

We know that it is enough to analyze stability with respect to migration threats from small adjacent intervals. Let such an interval J of the size μ be given. It is migration-proof if and only if its far-most individual t is migration-proof, that is, if

$$\Delta_{S \sqcup J}(t) = v_t\left(S^t\right) - v_t(S \sqcup J) \geq 0 \tag{46}$$

for this individual t.

We claim that $v_t(S^t) = -\Psi(s) + \mu$. Indeed, as P is Nash stable, the payoff of the individual at the border of S (which is equal to $-\Psi(s)$) is equal to a payoff this border agent would get in S^t; but the latter payoff differs from $v_t(S^t)$ by exactly μ.

Further on, $v_t(S \sqcup J) = -\Psi(s + \mu)$, obviously (when J moves to S, there emerges a new jurisdiction $S \sqcup J$ of the size $s + \mu$, with the individual t at the border). Hence, we have that S is migration-proof if and only if for small μ

$$\begin{gathered} \Delta(t) = v_t(S^t) - v_t(S \cup J) \geq 0 \Leftrightarrow \\ -\Psi(s) + \mu + \Psi(s + \mu) \geq 0 \Leftrightarrow \\ \Psi(s + \mu) - \Psi(s) \geq -\mu. \end{gathered} \tag{47}$$

So, S is migration-proof if and only if there $\exists \varepsilon > 0$ such that

$$\forall \mu \leq \varepsilon \quad \Psi(s+\mu) - \Psi(s) \geq -\mu \Leftrightarrow \frac{d\Psi(s)}{ds} \geq -1, \tag{48}$$

because $\dfrac{d^2\Psi(s)}{ds^2} = \dfrac{2g}{s^3} > 0$.

But $\dfrac{d\Psi(s)}{ds} = \dfrac{1}{2} - \dfrac{g}{s^2}$, hence, necessary and sufficient conditions for a jurisdiction S to be migration-proof is that $\dfrac{1}{2} - \dfrac{g}{s^2} \geq -1 \Leftrightarrow s \geq \sqrt{\dfrac{2g}{3}}$. And a given Nash stable partition P is locally stable if and only if all its jurisdictions are migration-proof, hence, all their sizes satisfy the equation (25). □

(ii) Follows immediately from (i), since the size of jurisdictions in the homogeneous n-partition is equal to $\dfrac{1}{n}$; hence, this partition is locally stable if and only if

$$\frac{1}{n} \geq \sqrt{\frac{2g}{3}} \Leftrightarrow n \leq \sqrt{\frac{3}{2g}}. \tag{49}$$

□

Proof of Proposition 4.5: In fact, Proposition 4.5 is imprecise. Its strict formulation is given below, in Proposition A.2; and here, we will prove this latter proposition, after formally stating it.

Proposition A.2: Let $n > 1$ be given. Then:

(i) For $n = 2$, heterogeneous n-partition is locally stable for $g \in \left[\dfrac{3}{32}, \dfrac{1}{8}\right)$ and is locally unstable for $g < \dfrac{3}{32}$.

(ii) Locally stable n-partition for $n \geq 3$ exists if and only if one of the following statements holds:

- $n = 3$ and $g \in \left[\dfrac{3}{2(3n-2)^2}, \dfrac{1}{2n^2}\right) \sqcup \left(\dfrac{1}{2n^2}, g_n\right]$;
- $n = 4$ or $n = 6$ and $g \in \left[\dfrac{3}{2(3n-2)^2}, \dfrac{1}{2n^2}\right) \sqcup \left(\dfrac{1}{2n^2}, \dfrac{3}{2(n+2)^2}\right]$;

• $n = 5$ or $n \geq 7$ and $g \in \left[\frac{3}{2(3n-2)^2}, \frac{3}{2(n+2)^2}\right]$.

Precisely:

(iii) For $g \in \left[\frac{3}{2(3n-2)^2}, \frac{1}{2n^2}\right)$, the heterogeneous n-partition which contains 1 small and $n-1$ big size jurisdictions, is locally stable;

(iv) For $g > \frac{1}{2n^2}$, on the contrary, it is the heterogeneous n-partition which contains 1 big and $n-1$ small size jurisdictions that is locally stable if and only if $g \leq \frac{3}{2(n+2)^2}$ for $n > 3$ and $g \leq g_3$ for $n = 3$;

(v) For $g = \frac{1}{2n^2}$, the heterogeneous n-partition which contains $\frac{n}{2}+1$ small and $\frac{n}{2}-1$ big size jurisdictions for even n, and $\frac{n+1}{2}$ small and $\frac{n-1}{2}$ big size jurisdictions for odd n is locally stable, but only when $n \neq 3, 4, 6$.

Proof of Proposition A.2: (i) For $n = 2$, as we know, heterogeneous 2-partition exists and is essentially unique for $g < g_2 = \frac{1}{8}$ (up to a permutation of jurisdictions). It is locally stable if and only if the size of the small jurisdiction is at least equal to $\sqrt{\frac{2g}{3}}$.

But for $n = 2$ we know that $w = \frac{1}{2}$ (see the proof of Proposition 3.3), hence, we deduce that heterogeneous 2-partition is locally stable if and only if

$$\frac{1}{2} - \sqrt{\frac{1}{4} - 2g} \geq \sqrt{\frac{2g}{3}} \Leftrightarrow g \geq \frac{3}{32}, \tag{50}$$

after trivial rearrangements. □

(ii) Now, let us consider a case $n \geq 3$. Except for $n = 3$, there emerges a new parameter under control which is the composition of a heterogeneous n-partition. That is, for $k = 1, 2, \ldots, [n/2]$ and $l = n - k$ we have different (k, l)-partitions which are various types of an n-partition.

It is time now to recall Proposition A.1. The expression (39) behaves in a way prescribed in this proposition, and it is useful to analyze this expression as a function not of w but of

s, the size of small jurisdictions (the two variables, w and s are monotone transformations of each other on the interval of heterogeneity, $w \in \left(\sqrt{2g}, +\infty\right)$, or alternatively, on the interval $s \in \left(0, \sqrt{2g}\right)$). So, we introduce the function

$$L_{(k,l)}\left(s\right) \tag{51}$$

which equals to the size of a Nash stable collection of k small jurisdictions of the size s, and l big jurisdictions (these data uniquely determine the overall size). After trivial arrangements, one gets that

$$L_{(k,l)}\left(s\right) = ks + \frac{2gl}{s}. \tag{52}$$

Its behavior as a function of s for alternative n and (k, l) such that $k + l = n$ is graphically demonstrated in the figure 7 below. This function is single-dipped on the interval of heterogeneity, $s \in \left(0, \sqrt{2g}\right)$; it attains its minimum in the point $s_{min}\left(k, l\right) = s = \sqrt{l/k}\sqrt{2g}$ for $k > l$, and is monotonically increasing for $k \leq l$. All these facts are trivially checked from its functional form (52).

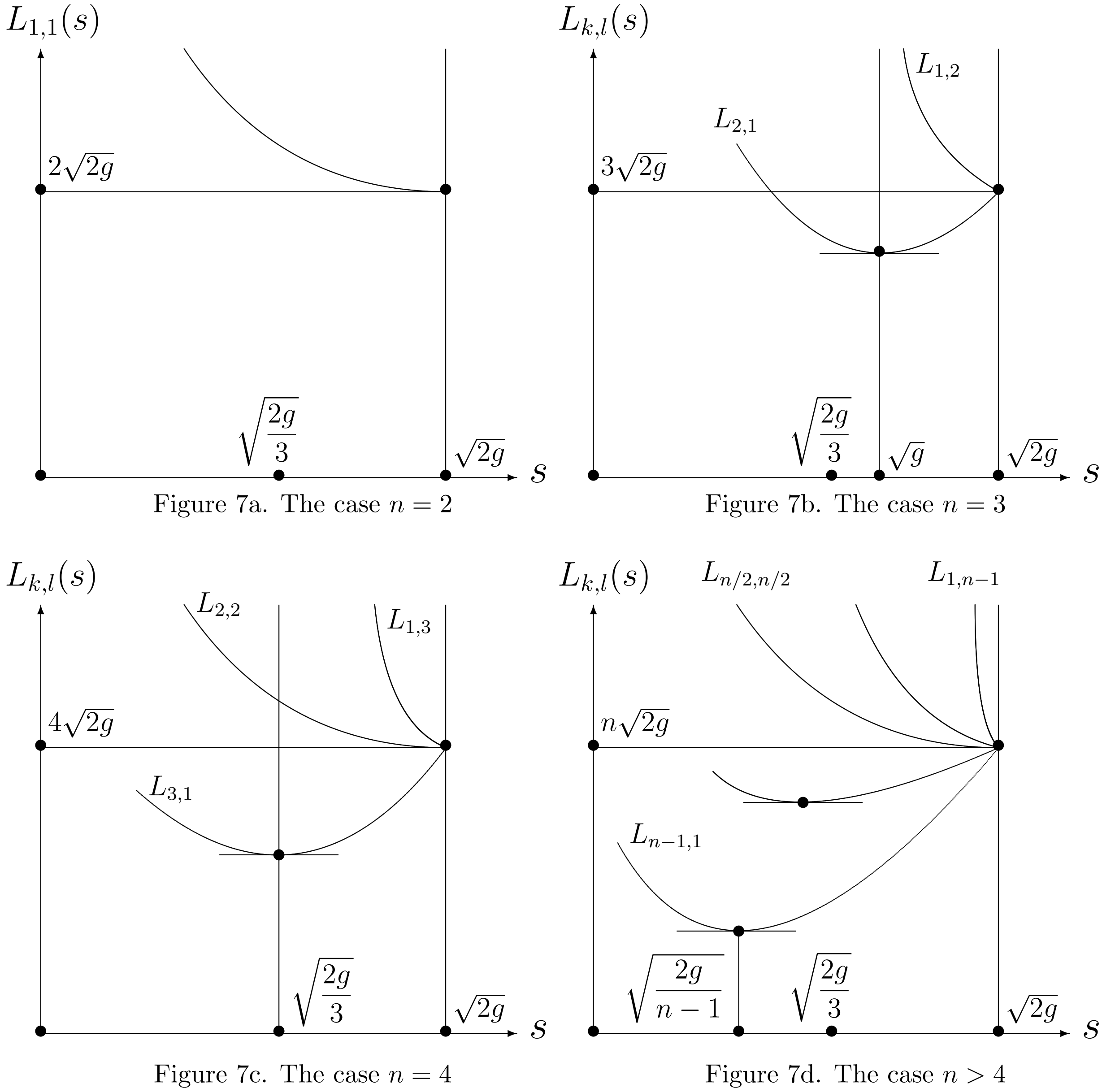

Figure 7a. The case $n = 2$

Figure 7b. The case $n = 3$

Figure 7c. The case $n = 4$

Figure 7d. The case $n > 4$

Figure 7. The overall size of the composition of k small and l big jurisdictions, $L_{k,l}(s)$, as a function of the size of small jurisdictions, s (where $k + l = n$), for $n = 2,\ n = 3,\ n = 4$ and for arbitrary n

Now, we should consider cases from Proposition A.2 in turn, in the following order.

First, consider the case $n = 3$. If $g > g_3$ then there is no heterogeneous Nash stable 3-partition at all, which we know (Proposition 3.3). If $g \in \left(\frac{1}{2n^2}, g_3\right] = \left(\frac{1}{18}, \frac{1}{16}\right)$, then we have that $L_{(2,1)}(s_{min}) < 1$ and $L_{(2,1)}\left(\sqrt{2g}\right) > 1$, hence, for some $s \in \left[s_{min}(2,1), \sqrt{2g}\right)$ we will have

$L_{(2,1)}(s) = 1$ and the corresponding $(2,1)$-partition is locally stable since $s \geq s_{min}(2,1) = \sqrt{g} > \sqrt{\frac{2g}{3}}$, recall Proposition 4.4.

If we now consider the case $n > 3$ and the same interval for g, we will find that $s_{min}(n-1,1) \leq \sqrt{\frac{2g}{3}}$, a critical value of the size of small jurisdictions. Hence, if g is close to g_n (and $s_{min}(n-1,1) < \sqrt{2g}$ strictly, which is equivalent to having $n \geq 5$), then both heterogeneous Nash stable $(2,1)$-partitions — that with $s < s_{min}$, and the one with $s > s_{min}$ — are locally unstable, because for both of them we have $s < \sqrt{2g}$. In fact, we claim the following:

Proposition A.3: If $g \in \left(\frac{3}{2(n+2)^2}, g_n\right]$ then there are no locally stable heterogeneous n-partitions.

Proof of Proposition A.3: For any $s < \sqrt{2g}$ and any k, l with $k + l = n$ it is obvious that $L_{(k,l)}(s) \geq L_{(n-1,1)}(s)$; if there exists a locally stable heterogeneous k,l-partition, then $s \geq \sqrt{2g} \geq s_{min}(n-1,1)$ and hence $1 = L_{(k,l)}(s) \geq L_{(n-1,1)}(s) \geq L_{(n-1,1)}\left(\sqrt{2g}\right) > 1$, where the second inequality is due to single-dippedness of $L_{(n-1,1)}(s)$ and the third (strict) inequality is due to the lower bound on g, which is trivial to check. A contradiction. □

If, on the contrary, $g \in \left(\frac{1}{2n^2}, \frac{3}{2(n+2)^2}\right]$ then as in the case of $n = 3$, the function $L_{(n-1,1)}$, being evaluated at the points $\sqrt{\frac{2g}{3}}$ and $\sqrt{2g}$, produces two values of which one is greater than 1 while the second is lower. Hence, there exists a value of s such that $L_{(n-1,1)}(s) = 1$ within the interval of heterogeneity. This value gives us a desired heterogeneous locally stable n-partition, and completes the proof of part (iv), and of some part of part (ii) of Proposition A.2. □

Let us now turn to the interval $g < \frac{1}{2n^2}$ (leaving the most difficult case of $g = \frac{1}{2n^2}$ for the end). In this case we have for all k, l that $L_{(k,l)}\left(\sqrt{2g}\right) > 1$, hence, in order to get a heterogeneous n-partition, one should turn attention to partitions with $k \leq l$ (look at the Figure 7). More precisely:

Proposition A.4: If $g < \frac{1}{2n^2}$, then the existence of locally stable heterogeneous n-partitions imply that there exists a locally stable heterogeneous $(1, n-1)$-partition.

Proof of Proposition A.4: Indeed, $\forall s < \sqrt{2g} \quad \forall k, l : k + l = n$ we have $L_{(k,l)}(s) \leq L_{(1,n-1)}(s)$. Now, if there exists a locally stable (k, l)-partition with small jurisdictions of the size s, and $k \neq 1$, then we can write down the following inequality:

$$1 = L_{(k,l)}(s) < L_{(1,n-1)}(s). \tag{53}$$

At the same time, $1 < L_{(1,n-1)}\left(\sqrt{2g}\right)$; hence, there exists some $s' \in \left(s, \sqrt{2g}\right)$ such that $L_{(1,n-1)}(s') = 1$, and $s' > s \geq \sqrt{2g}$, since we assumed that s was the size of small jurisdictions in a locally stable (k, l)-partition. But this implies that s' is the size of small jurisdictions in a locally stable $(1, n-1)$-partition. □

Proposition A.5: In the studied case (that of $g < \dfrac{1}{2n^2}$), a locally stable $(1, n-1)$-partition exists if and only if the following inequality holds:

$$L_{(1,n-1)}\left(\sqrt{2g}\right) \geq 1. \tag{54}$$

Proof of Proposition A.5: Recall that under $k \leq l$ (which is the case for $k = 1$) the function $L_{(1,n-1)}(s)$ is monotonically decreasing on the interval of heterogeneity, $s \in \left(0, \sqrt{2g}\right)$. We know that $L_{(1,n-1)}\left(\sqrt{2g}\right) < 1$. The rest is clear from the Figure 7, or from the continuity argument. □

But the condition (54) is equivalent to having $g \geq \dfrac{3}{2(3n-2)^2}$ (it is trivial to check), which completes the proof of part (iii) and some more portion of (ii) of Proposition A.2. □

Finally, let us turn to the "difficult" case of $g = \dfrac{1}{2n^2}$. This case presents new insights since now we observe that $\forall\, k, l : k + l = n \quad L_{(k,l)}\left(\sqrt{2g}\right) = 1$. This does not allow for direct continuity argument: the latter one would lead us to the *homogeneous* partition.

First of all, notice that in this case no heterogeneous partition exists with $k \leq l$: for such values of k and l we have that $L_{(k,l)}(s) > 1$ over the interval of heterogeneity. Hence, one should again turn attention to the case $k > l$. But matters are different now. We have:

Proposition A.6: For $g = \dfrac{1}{2n^2}$, if a locally stable heterogeneous (k,l)-partition exists, then there exists a locally stable $\left(\dfrac{n}{2}+1, \dfrac{n}{2}-1\right)$-partition if n is even, and a locally stable $\left(\dfrac{n+1}{2}, \dfrac{n-1}{2}\right)$-partition if n is odd.

Proof of Proposition A.6: The argument of the proof is nearly the same. Take, say, a case of n even. If a certain (k,l)-partition (not coinsiding with the one mentioned in Proposition A.6) is stable with s being the size of small jurisdictions, then we must have $L_{\left(\frac{n}{2}+1, \frac{n}{2}-1\right)}(s) > 1$; and for s close to $\sqrt{2g}$ we have $L_{\left(\frac{n}{2}+1, \frac{n}{2}-1\right)}(s) < 1$. Therefore, somewhere between there exists s' such that the corresponding $\left(\dfrac{n}{2}+1, \dfrac{n}{2}-1\right)$-partition is locally stable with s' being the size of small jurisdictions (look at the Figure 7). □

The rest is tedious though quite straightforward. Due to single-dippedness of $L_{\left(\frac{n}{2}+1, \frac{n}{2}-1\right)}(s)$, the condition that it could come across 1 is equivalent to having $L_{\left(\frac{n}{2}+1, \frac{n}{2}-1\right)}\left(\sqrt{\dfrac{2g}{3}}\right) \geq 1$ (look at the Figure 7). After some rearrangemets, one gets that $L_{(k,l)}\left(\sqrt{\dfrac{2g}{3}}\right) \geq 1 \Leftrightarrow k \leq l\sqrt{3}$. For even n and $k = \dfrac{n}{2}+1$ it holds from $n = 8$, hence, exceptions are $n = 4$ and $n - 6$; for odd n and $k = \dfrac{n+1}{2}$ it holds from $n = 5$, hence, the only exception is $n = 3$.

This completes the final part of part (ii), proofs part (v) and completes the whole proof of Proposition A.2, henceforth, of Proposition 4.5 as well. □

Proof of Proposition 4.6: (i) From the fact that $s \geq \sqrt{\dfrac{2g}{3}}$ in locally stable heterogeneous partitions we infer that $\overline{s} \leq 3\sqrt{\dfrac{2g}{3}}$, see Figure 2 (or just calculate directly the second root of the equation $\Psi(s) = \Psi\left(\sqrt{\dfrac{2g}{3}}\right)$). Hence, in any locally stable heterogeneous n-partition we must have $H(P) \leq 3$.

To assure that this value is being reached, just take an $(n-1,1)$-partition for the value of $g = \dfrac{3}{2(3n-2)^2}$; the size of its small jurisdictions is exactly equal to $\sqrt{\dfrac{2g}{3}}$, hence, the size of big ones is equal to $3\sqrt{\frac{2g}{3}}$. □

(ii) To evaluate the variance, one has to make some preparations. First, notice that $\forall\, k,l:\ k+l=n$ there exists a value of g such that a (k,l)-partition with the size of small jurisdictions equal exactly to $\sqrt{\frac{2g}{3}}$ is locally stable. Indeed, to find such g, one ought just to solve the following equation:

$$k\sqrt{\frac{2g}{3}}+3l\sqrt{\frac{2g}{3}}=1 \Leftrightarrow g=\frac{3}{2\left(n+2l\right)^{2}}, \tag{55}$$

after simple rearrangements. Next proposition stays that in finding a heterogeneous locally stable n-partition with maximum size variance, $V\left(P\right)$ one can confine himself to these very partitions with the size of small jurisdictions equal exactly to $\sqrt{\frac{2g}{3}}$.

Proposition A.7: The size variance, $V\left(P\right)$ reaches its maximum on the subset of locally stable n-partitions with $H\left(P\right)=3$, i.e. with the least possible size of small jurisdictions, $s=\sqrt{\frac{2g}{3}}$.

Proof of Proposition A.7: Consider an arbitrary locally stable (k,l)-partition, and let $s\neq\sqrt{\frac{2g}{3}}\Leftrightarrow H\left(P\right)<3$. This means two things, which is clear from Figure 2: that $s>\sqrt{\frac{2g}{3}}$ and that $\overline{s}<3\sqrt{\frac{2g}{3}}$. Hence, in the formula (9) for variance every term increases when replacing this partition with the one with $s=\sqrt{\frac{2g}{3}}$ (hence, $\overline{s}=3\sqrt{\frac{2g}{3}}$). □

Now, to find the partition with the maximum variance, one should just compare variances of partitions with $H\left(P\right)=3$ for different k,l (and hence, different values of g). We have the equation $ks+3ls=1$ for (k,l)-partition with $H\left(P\right)=3$, which states the very fact that the size of the overall population amounts to 1. Hence, $s=\frac{1}{k+3l}=\frac{1}{n+2l}$, since $k+l=n$, and $\overline{s}=\frac{3}{n+2l}$. Using formula (43) for variance, we get essentially the following maximization problem to solve (replacing $k = n - l$):

$$\max_{l=1,\dots,n-1} (n-l)\frac{1}{(n+2l)^2} + l\frac{9}{(n+2l)^2} \Leftrightarrow$$
$$\max_{l=1,\dots,n-1} \frac{n+8l}{(n+2l)^2}. \tag{56}$$

Where $l \in \mathbf{R}_+$, we could have used calculus to obtain that $l = \frac{n}{4}$, and one should have $k = \frac{3n}{4}$ small jurisdictions and $l = \frac{n}{4}$ big ones. This would give us that the maximum variance is equal to $\frac{3n}{\left(n+\frac{n}{2}\right)^2} - \frac{1}{n} = \frac{1}{3n}$, as stated in Proposition 4.6. In reality, however, n could not be of the form $4m$, hence, maximum variance is reached on one of the two nearest integer values of l.

The proof of Proposition 4.6 is complete. □

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