Conditional and Unconditional Correlatedness and Heteroskedasticity

Stanislav Anatolyev*

Problem

For jointly stationary scalar processes e_t and z_t, define Ω_t to be the σ-field generated by $z_t, z_{t-1}, z_{t-2}, \cdots$, and let e_t have conditional mean zero relative to Ω_t, i.e. $E[e_t|\Omega_t] = 0$. Show that:

(a) It is possible that $E[e_t e_{t-1}|\Omega_t] \neq 0$ almost everywhere, but $E[e_t e_{t-1}] = 0$. That is, an unconditionally uncorrelated process may be conditionally serially correlated.

(b) It is possible that $E[e_t e_{t-1}|\Omega_t]$ is almost surely constant, but $E[e_t^2|\Omega_t]$ is not constant on a set of positive measure, or vice versa. That is, a process may be conditionally homoskedastic in covariance, but conditionally heteroskedastic in variance, or vice versa.

(c) When $E[e_t e_{t-1}|\Omega_t]$ and $E[e_t^2|\Omega_t]$ are measurable with respect to the σ-field generated by some scalar random variable ζ_t in which both are monotinic, it is possible that $E[e_t e_{t-1}|\Omega_t]$ is increasing, but $E[e_t^2|\Omega_t]$ is decreasing in ζ_t, or vice versa. That is, the direction of conditional heteroskedasticity in variance and that in covariance may be opposite.

*New Economic School, Nakhimovsky prospect, 47, room 1721, Moscow, 117418, Russia. E-mail: sanatoly@nes.ru

Suggested Solution

The following example demonstrates all phenomena.

Let z_t be a strict white noise with unit variance. Let a standard bivariate normal white noise $(\epsilon_{vt}, \epsilon_{wt})'$ be independent of the process z_t, and generate $v_t = \epsilon_{vt}\sqrt{1-\alpha+\alpha z_t^2}$ and $w_t = \epsilon_{wt} z_t$, where $0 < \alpha < 1$. Construct e_t as $e_t = v_{t+1} + \theta_v v_t + w_{t+1} + \theta_w w_t$ for $\theta_v, \theta_w \neq 0$, $|\theta_v| < 1$, $|\theta_w| < 1$. Then one can find that $E[e_t|\Omega_t] = 0$ and $E[e_t e_{t-1}|\Omega_t] = \theta_v(1-\alpha) + (\theta_v\alpha + \theta_w)z_t^2$, while $E[e_t^2|\Omega_t] = 2 + \theta_v^2(1-\alpha) + (\theta_v^2\alpha + \theta_w^2)z_t^2$.

If we set $\theta_v + \theta_w = 0$, we have $E[e_t e_{t-1}] = \theta_v(1-\alpha) + (\theta_v\alpha + \theta_w)E[z_t^2] = 0$, while $E[e_t e_{t-1}|\Omega_t] \neq 0$ almost everywhere. This illustrates the phenomenon in (a).

If $\theta_v\alpha + \theta_w = 0$ but $\theta_v^2\alpha + \theta_w^2 \neq 0$, we observe that $E[e_t e_{t-1}|\Omega_t]$ is almost surely constant, while $E[e_t^2|\Omega_t]$ is nonconstant. Conversely, we may have $\theta_v^2\alpha + \theta_w^2 = 0$ but $\theta_v\alpha + \theta_w \neq 0$ (this requires α to be negative, which is possible if z_t has finite support[1]). This illustrates the phenomenon in (b).

Finally, if we set $\zeta_t = z_t^2$, both $E[e_t e_{t-1}|\Omega_t]$ and $E[e_t^2|\Omega_t]$ are $\sigma(\zeta_t)$-measurable. If $\theta_v\alpha + \theta_w$ and $\theta_v^2\alpha + \theta_w^2$ are nonzero and have opposite signs, we observe the phenomenon in (c).

[1]For example, one can take $z_t \sim i.i.d.U[-\sqrt{3}, \sqrt{3}]$, $\alpha = -\frac{1}{2}$, $\theta_v = \frac{1}{\sqrt{2}}$, $\theta_w = \frac{1}{2}$.