Another Example in which Lump-Sum Money Creation is Beneficial

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Abstract

A probabilistic version of lump-sum money creation is studied in a random matching model with indivisible money and individual holdings bounded at 2 units. Sufficient conditions are obtained for an ex ante optimum from among implementable steady states to involve lump-sum creation of money. The role of that creation is to change the distribution of money holdings to permit more trade to occur. Beneficial money creation is impossible in a version with a 1 unit upper bound on individual holdings, but can almost certainly happen for all higher bounds.

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1 Introduction

A standard exercise to perform on monetary models is to subject them to money creation at a rate, where the creation is accomplished through lump-sum transfers, transfers that do not depend on behavior. Representative-agent models with money in utility or production functions or with cash-in-advance constraints generally give results roughly in line with what has

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come to be called the Friedman rule: the optimum involves not creation, but destruction financed by lump-sum taxes. Models in which money is convincingly essential can give a different answer. We know of two models in which money is convincingly essential and in which lump-sum transfers of money are studied: one is Levine [4] and the generalization of it studied by Kehoe, Levine and Woodford [3]; the other is Molico [8]. Both produce examples in which expansionary policy is beneficial. Here we present another example. We do that because the existing examples are special in ways that may raise doubts about the robustness of the results on beneficial effects.

Levine [4] and Kehoe, Levine and Woodford [3] use a one-good-per-date, pure-exchange model with preference shocks and divisible money. To get money to be essential, they assume that people are anonymous so that only *quid pro quo* spot trades are possible. There are two possible preference realizations at each date and they analyze only equilibria in which at the end of each period all money is held in equal amounts by those who last realized the low preference-for-consumption realization. As the authors make clear, in a model with preference shocks and no risk-sharing arrangements, such degenerate distributions are equilibria only for parameters for which those with high preference-for-consumption realizations want to carry zero wealth from one date to the next. Thus, their analysis leaves open whether beneficial effects of lump-sum money creation could also arise in the more general situation in which precautionary motives for holding money give rise to non-degenerate monetary distributions.

Molico [8] uses a random matching model with divisible money and unbounded individual holdings. As a consequence, he is able to analyze the model only numerically for particular examples. More importantly, he uses a particular bargaining rule: take-it-or-leave-it offers by potential consumers. From the viewpoint of his ex ante welfare criterion, that rule may be a non-optimal way to divide the gains from trade in some meetings. Therefore, part of the role of money creation in his examples may be to counteract a sub-optimal way of dividing the gains from trade in meetings.

We use the same background environment as Molico, but we assume indivisible money and individual holdings bounded at 2 units. That allows us to proceed analytically. Also, because we divide the gains from trade optimally, we are able to isolate the beneficial role of lump-sum money creation. In other words, we can be sure that we are not getting beneficial effects of money creation because we have imposed a sub-optimal trading rule. There is, though, a small price to pay for working with bounded and indivisible money; we must study a probabilistic version of the standard lump-sum money creation policy. We study a model in which holdings are at most 2 units because that is the smallest bound that permits money creation to affect the distribution of holdings in a way that facilitates trade. A plausible conjecture is that the same can happen for all higher bounds.

2 Environment

The background environment is a simple random matching model of money due to Shi [9] and Trejos and Wright [10]. Time is discrete and the horizon is infinite. There are $N \geqslant 3$ perishable consumption goods at each date and a $[0, 1]$ continuum of each of N types of agents. A type n person consumes only good n and produces good $n + 1$ (modulo N). Each person maximizes expected discounted utility with discount parameter $\beta \in (0, 1)$. As regards utility in a period, an agent who produces $y \in R_+$ units of the agent's production good at a date experiences the utility $-y$, while an agent who consumes y units of the agent's consumption good at a date receives the utility $u(y)$. We assume that the function u is strictly concave and increasing, satisfies $u(0) = 0$ and $u'(0) = \infty$, and that there exists $\hat{y} > 0$ such that $u(\hat{y}) = \hat{y}$. At each date, each agent meets one other person at random.

There is only one asset in this economy which can be stored across periods: fiat money. This money is indivisible and no individual can have more than 2 units of money at a time. We assume that an agent's specialization type and individual money holdings are observable. We also assume that agents cannot commit to future actions—that there is sequential individual rationality—and that the agent's history, except as revealed by money holdings, is private.

The pairwise meetings, the inability to commit, the privacy of individual histories, and the perishable nature of the goods imply that any production must be accompanied by a transfer of money. Moreover, the random meetings imply that with positive ex ante probability, there are single-coincidence meetings in which the producer has experienced a long run of being a producer and the consumer has experienced a long run of being a consumer. In such meetings, no matter whether money is bounded or unbounded or indivisible or divisible, the potential consumer will, in general, be unable to offer the producer enough money to induce much production. That opens the way for a potentially beneficial role for redistribution produced through lump-sum creation of money. The redistribution tends to compress the distribution of money holdings and, thereby, lowers the probability of meetings in which producers have a great deal of money and consumers have very little money. In our model with indivisible money and holdings bounded at 2 units, the role of the redistribution is to shift the distribution of money holdings away from the end-points of the support. Of course, as explained below, that potentially beneficial effect of lump-sum money creation may be offset by its undesirable incentive effects.

When first formulated, the randomness of meetings in settings like that described above was adopted because of its simplicity. Here, because the randomness of meetings plays an important role, it ought to be defended on other grounds. The randomness amounts to assuming that people probabilistically encounter consumption opportunities and earnings opportunities. This is a complete-economy version of the kind of uncertainty regarding expenditures and receipts that has long been part of well-known partial equilibrium models of money demand (see, for example, [2] and [7]). Moreover, money aside, some such uncertainty has almost always been assumed in inventory theory and in models of precautionary saving. Therefore, it should not be regarded as a strange ingredient of a model of trade. Finally, our result depends on the presence of uncertainty which produces distributions with unbounded support of runs of being a potential consumer and runs of being a potential producer. In single-coincidence meetings between people who have experienced long runs and in the absence of intervention, the potential consumer will not have enough money to induce the potential producer to produce much. While the existence of such runs is important, their source does not seem important; it could be random meetings or something else–for example, preference shocks as in [3].

3 Policies

We adopt the following timing of events and specification of policies. First there are meetings. After meetings, each person receives one unit of money with probability α. (Those who are at the upper bound and receive a unit must discard it.) Then each unit of money disintegrates with probability δ. Then the next date begins and the sequence is repeated.[1]

[1] Policies much like ours have been studied, but only for the case in which the upper bound on individual holdings is unity (see, in particular, Li [5]). As we will see, if the bound is unity and if the gains from trade are split optimally, then the policy has no scope

This kind of policy is a random version of the standard lump-sum money creation policy. In a model with divisible money, the standard policy is creation of money at a rate with the injections of money handed out lump-sum to people. As is well-known, that policy is equivalent to the following policy: the same injections followed by a reduction in each person's holdings that is proportional to the person's holdings. The proportional reduction is nothing but a normalization (see, for example, [6])[2]. Our policy resembles the second, normalized, policy in two respects. First, the creation part of our policy, the α part, is done on a per person basis, while the disintegration part, the δ part, is proportional to holdings. Second, in a model with divisible money and a nondegenerate distribution of money holdings, the standard policy has two effects: it tends to redistribute real money holdings from those with high nominal holdings to those with low nominal holdings and it has incentive effects by making money less valuable to acquire. Our policy also has these two effects. In particular, as regards incentives, the policy makes producers less willing to acquire money because (a) they may be given money without working for it (the lump-sum transfer part of the policy) and (b) they may lose money for which they have worked (the disintegration part). And, for the same reasons, consumers are more willing to part with money.[3]

Given that the potential beneficial effects of our policy come from redistribution, why not study policies that redistribute directly? The answer is related to the sequential individual rationality that we impose. We interpret that assumption, which in this model is important for the essentiality of monetary exchange, as precluding direct taxes. In particular, it is not feasible to simply take money from people or to force producers to produce. For that reason, we study only non-negative (α, δ) pairs and view any such pair as being accomplished as follows. The creation part is not a problem because it involves giving people something; we view it as accomplished by way of a randomized version of the proverbial helicopter drops of money. The

for beneficial effects.

[2]However, as Edward J. Green points out, the equivalence could fail in a model which posits costs of changing prices.

[3]In some models, lump-sum transfers of money are equivalent to open-market operations. They are not equivalent here. The equivalence requires Ricardian equivalence and, hence, perfect credit markets. Essentiality of money requires imperfect knowledge of individual histories and, hence, imperfect credit markets (see the discussion in [11]). Here, and in [4], [3], and [8], credit markets are excluded completely by way of the assumptions about privacy of individual histories.

random proportional decline in holdings is accomplished by society's choice of the durability of the monetary object. In a model with divisible money, the proportional reduction could be achieved by using as money an object which physically depreciates at the appropriate rate. Here, because of the indivisibility, we assume that the physical depreciation occurs probabilistically (in a "one-hoss-shay" fashion).[4]

4 Implementable allocations and the optimum problem

Given our assumptions, we can restrict attention to what we call *trade meetings*. A trade meeting is a single-coincidence meeting in which the producer does not start with upper-bound money holdings and the consumer starts with positive money holdings. An allocation describes what happens in all such meetings. We restrict attention to allocations that are symmetric across specialization types and are stationary in the following sense: what happens in a trade meeting depends only on the money holdings of the producer and consumer and, in addition, it and the policy, a pair (α, δ), are consistent with a constant and identical distribution of money holdings for each specialization type—a steady state. In a sense to be made precise, we say that such an allocation is implementable if it is also consistent with ex post individual rationality. The optimum problem is to choose an implementable allocation, a policy, and a consistent steady-state initial distribution of money that maximizes ex ante expected utility, utility prior to initial assignments of money. Given the symmetry and the ex ante nature of the criterion, the criterion is a representative-agent criterion.[5]

Although we impose ex post individual rationality, we formulate allocations to permit randomness—to permit different trades in the same kind of meeting. We do this mainly because, with indivisible money, such random-

[4]Some countries have conducted lotteries in which prizes are awarded to those with currency with serial numbers that match some drawn at random. An alternative way to accomplish the δ part of our policy is through the same kind of lottery except that currency with the matching serial numbers is treated by everyone as being worthless.

[5]In principle, our policies could be analyzed taking as given an arbitrary initial distribution of money holdings. However, then, we would have to study non-stationary policies and allocations and would have no reason to use a representative-agent welfare criterion rather than the Pareto criterion.

ness allows for a much richer set of steady state distributions than would be the case if we required that the same trade be made in all meetings of the same type. In a single-coincidence meeting between a producer with i units of money and a consumer with j units, the set of possible transfers of money is $\mathcal{K}_{ij} = \{0, 1, ... \min(j, 2-i)\}$. For trade meetings in which the producer has i units of money and consumer has j units, we let μ_{ij} on $R_+ \times \mathcal{K}_{ij}$ denote a measure with the interpretation that if (y, k) is randomly drawn from $R_+ \times \mathcal{K}_{ij}$ in accordance with measure μ_{ij}, then (y, k) is the suggested trade in that meeting in the sense that it is suggested that y be produced in exchange for k units of money. We let μ be the collection of μ_{ij}'s for $(i, j) \in \{0, 1\} \times \{1, 2\}$.

For each μ_{ij}, it is convenient to define the collection of *k-supports*[6]:

$$\Omega_{ij}^k = (R_+ \times \{k\}) \cap \operatorname{supp} \mu_{ij}, \ k \in \mathcal{K}_{ij}.$$

These represent "k-sections" of the original support of μ_{ij}. The k-supports are disjoint and $\cup_{k \in \mathcal{K}_{ij}} \Omega_{ij}^k = \operatorname{supp} \mu_{ij}$. It is also convenient to let $\lambda_{ij}^k \equiv \mu_{ij}(\Omega_{ij}^k)$, where λ_{ij}^k is the probability that k units of money are transferred in a trade meeting in which the producers starts with i units of money and the consumer with j units. Then we can express the transition matrix for money holdings implied by trades, denoted T, in terms of the λ_{ij}^k as $T = \frac{1}{N} S$, where

$$S = \begin{bmatrix} N - s_{12} - s_{13} & p_1 \lambda_{01}^1 + p_2 \lambda_{02}^1 & p_2 \lambda_{02}^2 \\ p_0 \lambda_{01}^1 + p_1 \lambda_{11}^1 & N - s_{21} - s_{23} & p_1 \lambda_{11}^1 + p_2 \lambda_{12}^1 \\ p_0 \lambda_{02}^2 & p_1 \lambda_{12}^1 + p_0 \lambda_{02}^1 & N - s_{31} - s_{32} \end{bmatrix}. \tag{1}$$

Here p_i denotes the fraction of each specialization type who start the date with i units of money and the entry in the kth row and lth column, s_{kl}, is N times the probability of a trade that results in transiting from having $k-1$ units of money to having $l-1$ units of money.

According to our sequence of actions, trade is followed first by probabilistic lump-sum creation and then by probabilistic proportional destruction. The transition matrix for the creation part is denoted A and that for the

[6]Recall that if μ is a probability measure, the support of μ, denoted $\operatorname{supp} \mu$, is the smallest closed set A such that $\mu(A) = 1$.

destruction part is denoted D. They are given by

$$A = \begin{bmatrix} 1-\alpha & \alpha & 0 \\ 0 & 1-\alpha & \alpha \\ 0 & 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ \delta & 1-\delta & 0 \\ \delta^2 & 2\delta(1-\delta) & (1-\delta)^2 \end{bmatrix}. \tag{2}$$

Notice that an individual can be given at most one unit of money, but can lose two units.

We can now express the requirement that (μ, α, δ) is consistent with a constant distribution of money holdings. A symmetric distribution of money holdings $p \equiv (p_0, p_1, p_2)$ is called stationary with respect to (μ, α, δ) if it satisfies $pTAD = p$.

It is convenient to express the ex post individual rationality restrictions in terms of discounted expected utilities. For (p, μ, α, δ) that is stationary, the discounted expected utility of an agent who ends up with i units money after the destruction stage, denoted v_i, is constant. We let $v \equiv (v_0, v_1, v_2)$. Then v satisfies the following 3-equation system of Bellman equations:

$$v' = \beta(q' + TAD\, v') \tag{3}$$

where q, the vector of (expected) one period returns from trade, is given by

$$q' = \begin{bmatrix} -\frac{p_1}{N}\int_{\Omega_{01}^1} y d\mu_{01} - \frac{p_2}{N}[\int_{\Omega_{02}^1} y d\mu_{02} + \int_{\Omega_{02}^2} y d\mu_{02}] \\ \\ \frac{p_0}{N}\int_{\Omega_{01}^1} u(y) d\mu_{01} + \frac{p_1}{N}\int_{\Omega_{11}^1} [u(y) - y] d\mu_{11} - \frac{p_2}{N}\int_{\Omega_{12}^1} y d\mu_{12} \\ \\ \frac{p_0}{N}[\int_{\Omega_{02}^1} u(y) d\mu_{02} + \int_{\Omega_{02}^2} u(y) d\mu_{02}] + \frac{p_1}{N}\int_{\Omega_{12}^1} u(y) d\mu_{12} \end{bmatrix} \tag{4}$$

Because T, A, and D are transition matrices and $\beta \in (0, 1)$, the mapping $G(x) \equiv \beta(q' + TADx')$ is a contraction. Therefore, (3) has a unique solution which can be expressed as

$$v' = (\frac{1}{\beta}I - TAD)^{-1} q', \tag{5}$$

where I is the 3×3 identity matrix.

We permit each individual to walk away from any realization of μ. In other words, we assume that people in a meeting cannot commit to the outcome of randomization. Therefore, our individual rationality constraints or participation constraints take the following form. If (y_{ij}, k) is in the support of μ_{ij}, then

$$(e_{i+k} - e_i)ADv' - y_{ij} \geqslant 0 \tag{6}$$

and

$$(e_{j-k} - e_j)ADv' + u(y_{ij}) \geqslant 0, \tag{7}$$

where e_l is the 3-component coordinate vector with indices running from 0 to 2. The first inequality pertains to the producer and the second to the consumer. We can now summarize the requirements for implementability.[7]

Definition 1 *(p, μ, α, δ) is called implementable if (i) $pTAD = p$ and (ii) (6) and (7) hold for all (y_{ij}, k) in the support of μ_{ij}.*

Our optimum problem is to maximize ex ante utility. That is, the optimum problem is to choose (p, μ, α, δ) from among those that are implementable to maximize $pv' \equiv W$.[8]

It is useful in what follows to express the objective W in terms of returns. If we multiply (3) by p and use the fact that $pTAD = p$, then we have

$$W = pv' = \frac{\beta}{1-\beta} pq'.$$

[7] We are claiming that the conditions in definition 1 are necessary and sufficient for weak implementability. For sufficiency, given an allocation that satisfies definition 1, we need to provide a game which has that allocation as an outcome. The game can be a very simple coordination game. The strategy set for each agent in a meeting is {yes, no}. If both say yes to a realization from μ, then they carry it out. If either says no, then there is autarky in that meeting. Obviously, if the participation constraints are satisfied, then saying yes is a subgame perfect Nash equilibrium. Necessity, of course, can only hold in the class of stationary and symmetric allocations we are considering. Then, given the privacy of individual histories and ex post individual rationality, our participation constraints must hold.

[8] Because a maximum may not exist, we should really say that for any $\varepsilon > 0$ we seek an implementable allocation that achieves at least $\sup W - \varepsilon$, where, of course, $\sup W$ is defined over the set of implementable allocations. Our arguments below do not depend on whether a maximum exists.

Then, by writing out the product pq', we have

$$w \equiv \frac{(1-\beta)NW}{\beta} = p_0 p_1 \int_{\Omega^1_{01}} z(y) d\mu_{01} + p_0 p_2 \int_{\Omega^1_{02}} z(y) d\mu_{02} + \tag{8}$$
$$p_0 p_2 \int_{\Omega^2_{02}} z(y) d\mu_{02} + p_1^2 \int_{\Omega^1_{11}} z(y) d\mu_{11} + p_1 p_2 \int_{\Omega^1_{12}} z(y) d\mu_{12},$$

where $z(y) \equiv u(y) - y$. As one would expect, because there is a producer for each consumer, from an ex ante view utility is a discounted expected value of the function z.

5 The result

As noted above, expansionary policy gives rise to two effects. First, it tends to tighten participation constraints for producers and to loosen those for consumers. Second, expansionary policy can change the distribution of money holdings p to increase the probability of trade meetings. We doubt that anything can be said generally about which effect dominates. We show that the optimum has expansionary policy provided the parameters are such that the participation constraints are not binding at the optimum subject to $\alpha = \delta = 0$. Roughly speaking, we do this in two steps. We describe the optimum for such parameters and for $\alpha = \delta = 0$. Then we show that there are implementable (p, μ, α, δ) with $\alpha > 0$ that do better. That, of course, implies that for such parameters the optimum is not $\alpha = \delta = 0$. We begin by describing an unconstrained optimum for $\alpha = \delta = 0$.

Lemma 1. *If $\alpha = \delta = 0$, then the optimum subject only to condition (i) in definition 1 and condition (ii) for $k = 0$ is a degenerate μ, denoted μ^*, with support $(y^*, 1)$, where $u'(y^*) = 1$. Moreover, the associated optimal p is $p^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.*

The proof of lemma 1 and the other proofs are in the appendix. Lemma 1 says that if we ignore participation constraints and impose $\alpha = \delta = 0$, then the optimum is a trade of the first-best level of production, that which maximizes $z(y)$, for one unit of money in every trade meeting. Moreover, the best steady-state distribution is uniform, which implies an amount of money per specialization type equal to unity.

The next lemma shows that there is a region of the parameter space, which we describe in terms of the discount factor, for which $(p, \mu, \alpha, \delta) = (p^*, \mu^*, 0, 0)$ is implementable, and, therefore, by lemma 1, is optimal subject to $\alpha = \delta = 0$.

Lemma 2. *Let $(p, \mu, \alpha, \delta) = (p^*, \mu^*, 0, 0)$. There exists a value of the discount factor, β^*, given by*

$$\beta^* = \frac{3Ny^*}{3Ny^* + \sqrt{(3y^*)^2 + 4z(y^*)y^*} - 3y^*},$$

such that if $\beta > \beta^*$, then (6) and (7) are slack. If $\beta = \beta^*$, then (6) and (7) are slack except for (6) for $i = 1$ (when the producer has 1 unit of money) which holds at equality.

Notice that $z(y^*) > 0$ implies that $\beta^* < 1$. Also, β^* is decreasing in $z(y^*)$.

For $\beta \geq \beta^*$, lemmas 1 and 2 completely describe the best (p, μ) subject to $\alpha = \delta = 0$. The final step is to show that for $\beta \geq \beta^*$, there exist (p, μ, α, δ) with $\alpha > 0$ and which are implementable and which imply a higher value of w than the best that can be achieved with $\alpha = 0$. This is done by showing that a relevant derivative of w with respect to α is positive at $(p, \mu, \alpha, \delta) = (p^*, \mu^*, 0, 0)$. For $\beta > \beta^*$, we can compute this derivative while keeping μ constant at μ^*. Because all the participation constraints are slack at $\beta > \beta^*$, implementability is maintained at $\mu = \mu^*$ as we vary α, and, consequently, p. For $\beta = \beta^*$, in order to maintain implementability as we vary α, we permit output when the producer starts with one unit of money to adjust, but, as in μ^*, one unit of money is transferred in every trade meeting.

Proposition 1. *If $\beta \geq \beta^*$, then the optimum is not $\alpha = \delta = 0$.*

The proof shows that the distribution p can be varied from p^* to one which has more trade meetings, a distribution in which $p_1 > \frac{1}{3}$. The measure of trade meetings is increasing in p_1 because people with one unit of money can be either producers or consumers.

Although our discussion of the resemblance between our policy and the standard policy is meant to convince readers that a positive (α, δ) corresponds to an inflationary policy, we can say a little more about this. A policy that resembles the standard lump-sum creation policy ought to lower the benefits of acquiring money. Those benefits are the differences, $v_1 - v_0$ and $v_2 - v_1$.

It is easily shown that they are decreasing in α at $(p, \mu, \alpha, \delta) = (p^*, \mu^*, 0, 0)$. In this sense, our policy is lowering those benefits. One may also wonder what is happening to real balances as we increase α. If $\beta > \beta^*$, then, for sufficiently small α, the price level does not change because in every meeting y^* is exchanged for one unit of money. It turns out that the nominal amount of money, $p_1 + 2p_2$, may either be increasing or decreasing; in particular, $\frac{d(p_1+2p_2)}{d\alpha} = \frac{10-N}{21}$. Thus, real balances may be increasing or decreasing in α. When $\beta = \beta^*$, the price level is increasing in α because the producer with one unit of money produces less as α increases.

Those, of course, are all local statements and do not describe an optimum relative to the optimum constrained by $\alpha = \delta = 0$. It is obvious that for any $\beta \geq \beta^*$, the optimum is such that the participation constraint for producers with one unit of money is binding. That does two things. It tends to make the price level higher. In addition, binding constraints tend to make it better to have less money in the system because less money tends to loosen producer participation constraints. Thus, in the range $\beta \geq \beta^*$, we strongly suspect that real balances at the optimum are lower than real balances at the optimum constrained by $\alpha = \delta = 0$.

6 Concluding remarks

An obvious question is what happens if more general individual money holdings are allowed. We have asserted that the main distinction occurs between an upper bound of unity and anything higher. That money creation cannot help when the bound is unity is, in effect, part of the proof of lemma 1. With an upper bound of unity, all possible distributions are implied by varying the constant amount of money per type. With any higher bound, there is scope for affecting the distribution by a money creation scheme. We are confident that we could produce a version of proposition 1 for any finite bound on individual holdings, but we are also confident that the region of the parameter space that is consistent with production of the first-best level of output and trades of one unit of money in all meetings shrinks as the bound gets large (see Camera and Corbae [1] for a closely related result). That, of course, is not to say that the region of the parameter space where expansionary policy helps shrinks as the bound gets large. It says only that our proof technique becomes less applicable.

In this regard, our proof technique seems completely inapplicable if money holdings are unbounded or if money is divisible with or without a bound, because, in these cases, it would seem vacuous to assume that the optimum with a fixed stock of money has no binding constraints. Therefore, for all parameters, we would then be in the general situation of trading off more favorable money distributions against the tightening of producer constraints for meetings with given money holdings. That is, all situations would be like the two-unit bound case when $\beta < \beta^*$. That, in turn, suggests that results for unbounded money holdings or divisible money will be achieved only by way of numerical examples. And, because the optimization is over large spaces in such cases, the numerical analysis will be demanding.

Those remarks are pertinent to a comparison between what we do and what Molico [8] does. As noted above, he works with the same environment, but with divisible and unbounded money holdings. However, rather than dividing the gains from trade optimally in each kind of meeting, he gives all the gains to the consumer so that the producer's participation constraint is always binding. Almost certainly, that way of dividing the gains from trade is not optimal. Therefore, his findings represent some unknown combination of beneficial effects coming from redistribution and other effects which may be offsetting the inefficient division of the gains from trade. Short of carrying out the optimization problem described above, we do not see how to disentangle those effects.[9]

In addition to studying more general individual money holdings, there are variants of our model that could be studied. These include permitting people to hide money, allowing people to commit in a meeting to the outcome of randomization, and permitting cooperative defection by the pair in a meeting. Although the details will differ, we surmise that the possibility of beneficial money creation exists in all these variants[10].

[9]In the case of indivisible money and a unit upper bound, if the parameters are such that the producer's participation constraint is slack at the first-best level of production in the absence of expansionary policy, then whether there exists a beneficial expansionary policy depends on the bargaining rule. Under the bargaining rule used by Molico [8], there does exist an expansionary policy which would reduce output to the first-best level and not affect the probability of trade.

[10]In fact, our results are unaffected by allowing people to hide money because the allocations used in our arguments are such that the trades offered people with i units of money are at least as good as those offered people with $i-1$ units.

7 Appendix

Proof of lemma 1.

The steady-state condition, which becomes $pT = p$ under $\alpha = \delta = 0$, does not involve the outputs. Therefore, for given λ_{ij}^k, a necessary condition for maximizing W is $y = y^*$ for all y in the support of μ. Then W which satisfies that necessary condition can be written as

$$w = z(y^*)[p_0p_1\lambda_{01}^1 + p_0p_2\left(\lambda_{02}^1 + \lambda_{02}^2\right) + p_1^2\lambda_{11}^1 + p_1p_2\lambda_{12}^1]. \tag{9}$$

Whenever $k = j - i$, the constraint $pT = p$ does not depend on λ_{ij}^k because money holdings are being exchanged. And since the λ_{ij}^{j-i} appear in (9) with non-negative coefficients, we set them at their maxima; namely, $\lambda_{01}^1 = \lambda_{12}^1 = 1$ and $\lambda_{02}^2 = 1 - \lambda_{02}^1$. It follows that w which satisfies necessary conditions for a maximum can be written

$$w = z(y^*)[p_0p_1 + p_0p_2 + p_1^2\lambda_{11}^1 + p_1p_2].$$

Thus, the problem is to maximize $F(p, \lambda_{11}^1, \lambda_{02}^1) = p_0p_1 + p_0p_2 + p_1^2\lambda_{11}^1 + p_1p_2$ subject to $pT = p$.

Now p either has full support or not. If not, then either $p_1 = 0$ or $\lambda_{11}^1 = 0$. (If $p_1 > 0$ and $\lambda_{11}^1 > 0$, then p has full support because there is an inflow into holdings of both 0 and 2 as a result of trade between producers and consumers with one unit.) If $p_1 = 0$ or $\lambda_{11}^1 = 0$ and p does not have full support, then the objective F has the form p_ip_j for $i \neq j$, the maximum of which is $\frac{1}{4}$.

We now find the maximum over full support p's. Consider the Lagrangian

$$L = p_0p_1 + p_0p_2 + p_1^2\lambda_{11}^1 + p_1p_2 - \psi(\lambda_{11}^1p_1^2 - \lambda_{02}^1p_0p_2) - \nu(\sum p_i - 1), \tag{10}$$

where ψ and ν are non-negative multipliers and where we have inserted the explicit form of the constraint, $pT = p$. This constraint reduces to the single equation, $\lambda_{11}^1p_1^2 = \lambda_{02}^1p_0p_2$, which says that the outflow from holdings of 1 unit is equal to inflow. Because p has full support, the first order conditions with respect to the p_i hold at equality. They are

$$p_0 : p_1 + p_2 + \lambda_{02}^1\psi p_2 - \nu = 0, \tag{11}$$

$$p_1 : p_0 + p_2 + 2\lambda_{11}^1(1-\psi)p_1 - \nu = 0, \tag{12}$$

$$p_2 : p_0 + p_1 + \lambda_{02}^1 \psi p_0 - \nu = 0. \tag{13}$$

Again because p has full support, either $\lambda_{11}^1 = \lambda_{02}^1 = 0$ or $\lambda_{11}^1 > 0$ and $\lambda_{02}^1 > 0$. In the first case, it follows from (11)-(13) that the maximum of $F(p, \lambda_{11}^1, \lambda_{02}^1)$ is attained at $p^* = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. Inserting this and $\lambda_{11}^1 = 0$ into F implies that the value of F is $\frac{1}{3}$.

For the second case ($\lambda_{11}^1 > 0$ and $\lambda_{02}^1 > 0$), we substitute from $\lambda_{11}^1 p_1^2 = \lambda_{02}^1 p_0 p_2$ directly into the objective. Then, because the remaining constraint does not involve λ_{02}^1 and because the resulting objective is increasing in λ_{02}^1, we conclude that the optimum in this case has $\lambda_{02}^1 = 1$. Then the sum of (11) and (13) minus twice (12) gives

$$2p_1 - (p_0 + p_2 + 4\lambda_{11}^1 p_1) + \psi(p_0 + p_2 + 4\lambda_{11}^1 p_1) = 0,$$

which can be written as

$$1 - \psi = \frac{2p_1}{p_0 + p_2 + 4\lambda_{11}^1 p_1} > 0. \tag{14}$$

From (10), we have

$$\frac{\partial L}{\partial \lambda_{11}^1} = p_1^2(1-\psi) > 0.$$

where the inequality follows from (14). Therefore, $\lambda_{11}^1 = 1$. Also, if we subtract (11) from (13), we get $p_2 = p_0$. This and $p_1^2 = p_2 p_0$, the explicit form of $pT = p$ with $\lambda_{11}^1 = \lambda_{02}^1 = 1$, imply $p = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. Therefore, the maximum of F in this case is attained at p^* and is equal to $\frac{4}{9}$.

Direct comparison of the three values of maximized objective completes the proof. ■

Proof of lemma 2.

The proof proceeds by explicit computation of the v_i at $(p, \mu, \alpha, \delta) = (p^*, \mu^*, 0, 0)$. In particular, we have

$$v_1 - v_0 = \frac{2\beta[3N(1-\beta)u(y^*) + 4\beta u(y^*) + 2\beta y^*]}{3[N(1-\beta) + 2\beta][3N(1-\beta) + 2\beta]} \equiv h_1(\beta)$$

and

$$v_2 - v_1 = \frac{2\beta[3N(1-\beta)y^* + 4\beta y^* + 2\beta u(y^*)]}{3[N(1-\beta)+2\beta][3N(1-\beta)+2\beta]} \equiv h_2(\beta).$$

It follows that $h_1(\beta)$ and $h_2(\beta)$ are defined and continuous on $[0,1]$, are strictly increasing, satisfy $h_1(\beta) > h_2(\beta)$, $h_1(0) = h_2(0) = 0$,

$$h_1(1) = \frac{2u(y^*) + y^*}{3} \in (y^*, u(y^*))$$

and

$$h_2(1) = \frac{2y^* + u(y^*)}{3} \in (y^*, u(y^*)),$$

where all the inequalities follow from $y^* < u(y^*)$. It follows that consumer participation constraints are slack at all $\beta \in (0,1)$. It also follows that there exists a unique $\beta \in (0,1)$ such that $h_2(\beta) = y^*$. Denote this β^*. Then, aside from the explicit claim about the expression for β^*, all the remaining claims follow from the assertions about $h_1(\beta)$ and $h_2(\beta)$. The explicit expression for β^* is obtained by solving the equation, $h_2(\beta^*) = y^*$.■

Proof of Proposition 1.

We compute a derivative of W with respect to α and evaluate it at $(p, \mu, \alpha, \delta) = (p^*, \mu^*, 0, 0)$. The only requirement is that implementability is maintained as we vary α. In computing the derivative, we keep all money transfers in meetings as they are under μ^*. That is, one unit of money is transferred in every trade meeting in. It follows that the trade matrix T has the form

$$T^* = \begin{bmatrix} 1 - \frac{p_1+p_2}{N} & \frac{p_1+p_2}{N} & 0 \\ \frac{p_0+p_1}{N} & 1 - \frac{p_0+2p_1+p_2}{N} & \frac{p_1+p_2}{N} \\ 0 & \frac{p_0+p_1}{N} & 1 - \frac{p_0+p_1}{N} \end{bmatrix}.$$

The mapping from (α, δ) to p that satisfies $pT^*AD = p$ is not well-behaved at $\alpha = \delta = 0$. At $\alpha = \delta = 0$, there is a one dimensional set of p's that are distributions and that satisfy $pT^*AD = p$. They can be thought as being generated by the set of alternative amounts of money per type, the interval $[0, 2]$. For $(\alpha, \delta) > 0$ and in a neighborhood of $\alpha = \delta = 0$, we can show that there is a unique p that satisfies $pT^*AD = p$ so that the mapping from (α, δ)

to p is a function in that neighborhood, and, moreover, is a differentiable function. After doing that, we will find the unique direction in the (α, δ) plane along which that unique solution approaches p^* as $(\alpha, \delta) \to 0$. Finally, we will compute the derivative of p along that direction and evaluate it at p^*. That, in turn, will allow us to show that welfare is increasing along that direction.

The conditions $\sum p_i = 1$ and $pT^*AD = p$ can be written as the following system of three equations in three unknowns:

$$p_1 + p_2 + p_0 = 1, \tag{15}$$

$$-\xi_1 p_1 + \xi_2 p_2 - \xi_3 p_0 = 0, \tag{16}$$

$$\frac{1}{N}(1-2\alpha)(1-\delta)^2[p_1^2 - p_0 p_2] + \alpha(1-\delta)^2 p_1 - \delta(2-\delta)p_2 = 0, \tag{17}$$

where

$$\begin{aligned}
\xi_1 &\equiv (1-\delta)[(\alpha-\delta)(1-\alpha) - \alpha^2\delta] = \alpha - \delta + o(\alpha,\delta) \\
\xi_2 &\equiv \delta[\alpha\delta + 2(1-\alpha)(1-\delta)] = 2\delta + o(\alpha,\delta) \\
\xi_3 &\equiv \alpha(1-2\alpha)(1-\delta)^2 = \alpha + o(\alpha,\delta)
\end{aligned}$$

and where $o(\alpha, \delta)$ denotes terms of order higher than (α, δ).

Because (15) and (16) are linear, they can be solved uniquely for p_1 and p_2 (in terms of p_0) if

$$\det\begin{pmatrix} 1 & 1 \\ -\xi_1 & \xi_2 \end{pmatrix} = \xi_2 + \xi_1 \neq 0.$$

From the expressions for the ξ_i, it follows that $\xi_2 + \xi_1 = \delta + \alpha + o(\alpha, \delta) > 0$. Therefore, we have,

$$p_1 = \frac{\xi_2 - (\xi_3 + \xi_2)p_0}{\xi_2 + \xi_1} \text{ and } p_2 = \frac{\xi_1 + (\xi_3 - \xi_1)p_0}{\xi_2 + \xi_1}.$$

If we substitute these into (17), the result is a quadratic equation in p_0 or, more simply, x, which we write as $f(x) = ax^2 + bx + c = 0$, where

$$a = -\frac{1}{N}(1-2\alpha)(1-\delta)^2[(\xi_1 + \xi_2)(\xi_3 - \xi_1) - (\xi_2 + \xi_3)^2]$$

$$b = -\frac{1}{N}(1-2\alpha)(1-\delta)^2[\xi_1(\xi_1+\xi_2)+2\xi_2(\xi_2+\xi_3)] - \\ \alpha(1-\delta)^2(\xi_1+\xi_2)(\xi_2+\xi_3) - \delta(2-\delta)(\xi_1+\xi_2)(\xi_3-\xi_1)$$

$$c = \frac{1}{N}(1-2\alpha)(1-\delta)^2\,(\xi_2)^2 + \alpha(1-\delta)^2\xi_2(\xi_1+\xi_2) - \\ \delta(2-\delta)\xi_1(\xi_1+\xi_2)$$

We can rewrite these coefficients as:

$$\begin{aligned} a &= \frac{1}{N}(\alpha^2+3\alpha\delta+3\delta^2)+o(\alpha^2,\alpha\delta,\delta^2) \\ b &= -\frac{1}{N}(\alpha^2+4\alpha\delta+7\delta^2)+o(\alpha^2,\alpha\delta,\delta^2) \\ c &= \frac{1}{N}4\delta^2+o(\alpha^2,\alpha\delta,\delta^2) \end{aligned}$$

Then $f(0) = \frac{1}{N}4\delta^2 + o(\alpha^2,\alpha\delta,\delta^2) > 0$ and $f(1) = -\frac{1}{N}\alpha\delta + o(\alpha^2,\alpha\delta,\delta^2) < 0$. Therefore, there exists a unique solution for p_0 consistent with p being a distribution. That is, for $(\alpha,\delta) > 0$ and in a neighborhood of 0, there exists a unique solution for p. Moreover that solution is differentiable because the coefficients of f are differentiable functions of the parameters.

Now that we have established properties of the mapping from (α,δ) to p in the neighborhood of $(\alpha,\delta)=0$, we can proceed by differentiating $pT^*AD = p$ and evaluating the result at $\alpha=\delta=0$ and $p^* = \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$. This gives the following system of equations:

$$\frac{1}{N}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{bmatrix}\begin{pmatrix} dp_0 \\ dp_1 \\ dp_2 \end{pmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \\ -1 & 2 \end{bmatrix}\begin{pmatrix} d\alpha \\ d\delta \end{pmatrix} \tag{18}$$

Because the first and third components of the left-hand side vector are identical, this system has solutions if and only if $d\delta = \frac{2}{3}d\alpha$. In other words, the direction $\delta = \frac{2}{3}\alpha$ is the unique direction such that $p \to p^*$ when $(\alpha,\delta)\to 0$. (Existence of this path can be confirmed from the quadratic equation $f(x)=0$. In particular, if we set $\delta = \frac{2}{3}\alpha$ and let $\alpha \to 0$, then $f(x) \to 39x^2 - 61x + 16 = 0$, whose roots are $\frac{16}{13}$ and $\frac{1}{3}$.) Using $\sum dp_i = 0$, it follows from (18) that

$$dp_1 = \frac{N}{9}d\alpha \tag{19}$$

along the direction $\delta = \frac{2}{3}\alpha$. As we now show, this is enough to conclude that it is possible to raise welfare with some $(\alpha, \delta) > 0$. The argument is slightly different for $\beta > \beta^*$ and $\beta = \beta^*$.

$\beta > \beta^*$. Here, the participation constraints are slack at $(p, \mu, \alpha, \delta) = (p^*, \mu^*, 0, 0)$ (see lemma 2). Therefore, we can vary (α, δ) from 0 with $\delta = \frac{2}{3}\alpha$ while fixing $\mu = \mu^*$ without violating those constraints. It follows that the derivative of w with $\mu = \mu^*$along the $\delta = \frac{2}{3}\alpha$ path and evaluated at $(p, \mu, \alpha, \delta) = (p^*, \mu^*, 0, 0)$, is given by

$$\begin{aligned} \frac{dw}{d\alpha} &= z(y^*)[\frac{dp_0}{d\alpha}(p_1 + p_2) + p_0(\frac{dp_1}{d\alpha} + \frac{dp_2}{d\alpha}) + \\ &\quad \frac{dp_1}{d\alpha}(p_1 + p_2) + p_1(\frac{dp_1}{d\alpha} + \frac{dp_2}{d\alpha})] \end{aligned}$$

At $p = p^*$, this becomes

$$\begin{aligned} \frac{dw}{d\alpha} &= \frac{z(y^*)}{3}[2\frac{dp_0}{d\alpha} + \frac{dp_1}{d\alpha} + \frac{dp_2}{d\alpha} + 2\frac{dp_1}{d\alpha} + \frac{dp_1}{d\alpha} + \frac{dp_2}{d\alpha}] \\ &= \frac{2z(y^*)}{3}\frac{dp_1}{d\alpha} \end{aligned}$$

where the last equality uses $\sum \frac{dp_i}{d\alpha} = 0$. This and (19) give the result.

$\beta = \beta^*$. Here to maintain implementability as we vary (α, δ), we adjust the supports of the μ_{11} and μ_{12} components of μ, while keeping all other components at their μ^* values. We let the support of μ_{11} and μ_{12} be degenerate at $(y_1, 1)$, where y_1 is determined by the binding producer participation constraint

$$(e_2 - e_1)AD(\frac{1}{\beta^*}I - T^*AD)^{-1}q' - y_1 = 0 \tag{20}$$

with

$$q' = \frac{1}{N}\begin{bmatrix} -(p_1 + p_2)\,y^* \\ p_0 u(y^*) + p_1\,[u(y_1) - y_1] - p_2 y_1 \\ p_0 u(y^*) + p_1 u(y_1) \end{bmatrix}.$$

With $(p_0, p_1, p_2) = p(\alpha, \frac{2}{3}\alpha)$ given by the unique differentiable solution established above, we can write (20) as $g(\alpha, y_1) = 0$, where $g(0, y^*) = 0$ and where

$$\partial g(0,y^*)/\partial y_1 = \frac{2\beta^*\left[\beta^* + (1-\beta^*)N\right]}{4\beta^*\left[\beta^* + 2(1-\beta^*)N\right] + 3(1-\beta^*)^2N^2} - 1 \in (-1, -\frac{1}{2}).$$

It follows from the implicit function theorem that for α in a neighborhood of 0, the y_1 that satisfies (20) is a differentiable function of α.

Since the μ we are now using continues to have degenerate supports, the objective function (8) can be written as:

$$w = p_0(p_1 + p_2)z(y^*) + p_1(p_1 + p_2)z(y_1)$$

Then the derivative of welfare with respect to α evaluated at $\alpha = \delta = 0$ and $p = p^*$ is

$$\frac{dw}{d\alpha} = \frac{2z(y^*)}{3}\frac{dp_1}{d\alpha} + \frac{4}{9}z'(y^*)\frac{dy_1}{d\alpha}.$$

This differs from the corresponding expression for $\beta > \beta^*$ by the presence of an additional term. However, because the derivative $\frac{dy_1}{d\alpha}$ exists and because $z'(y^*) = 0$, this additional term is zero. Therefore, the result again follows from (19).■

References

[1] Camera G, Corbae D. "Money and price dispersion". *International Economic Review* 40 (1999) 985–1008.

[2] Goldman S. "Flexibility and the demand for money", *Journal of Economic Theory*, October, (1974)

[3] Kehoe T., Levine D., Woodford M. "The optimum quantity of money revisited". In *Economic Analysis of Markets and Games: Essays in Honor of Frank Hahn*. Dasgupta P. et al -eds. MIT Press. Cambridge and London, 1992, 501-526.

[4] Levine D. "Asset trading mechanisms and expansionary policy". *Journal of Economic Theory,* 54 (1991), 148-164.

[5] Li V. “The optimal taxation of fiat money in search equilibrium”. *International Economic Review,* 36 (1995), 927-942.

[6] Lucas R E Jr., Woodford M. “Real effects of monetary shocks in an economy with sequential purchases”. University of Chicago Working Paper, 1994.

[7] Miller M, Orr D. “A model of the demand for money for money by firms.” *Quarterly Journal of Economics*, 80 (August 1966) 413-35.

[8] Molico M. ”The distribution of money and prices in search equilibrium”. Unpublished Ph.D. dissertation. The University of Pennsylvania, 1997.

[9] Shi S. “Money and prices: a model of search and bargaining”. *Journal of Economic Theory,* 67 (1995), 467-496.

[10] Trejos A., Wright R. “Search, bargaining, money and prices”. *Journal of Political Economy,* 103 (1995), 118-141.

[11] Wallace, N. “Introduction to modeling money and studying monetary policy,” *Journal of Economic Theory,* 81 (August 1998), 223-231.