Money Creation in a Random Matching Model*

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Abstract

I study money creation in versions of the Trejos-Wright (1995) and Shi (1995) models with indivisible money and individual holdings bounded at two units. I work with the same class of policies as in Deviatov and Wallace (2001), who study money creation in that model. However, I consider an alternative notion of implementability—the ex ante pairwise core. I compute a set of numerical examples to determine whether money creation is beneficial. I find beneficial effects of money creation if individuals are sufficiently risk averse (obtain sufficiently high utility gains from trade) and impatient.

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1 Introduction

The welfare effects of lump-sum money creation differ depending on the model used. In particular, there seems to be a sharp contrast in results de-

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pending on whether or not the model contains heterogeneous agents. Representative-agent models tend to yield results in line with the Friedman rule: the optimal monetary policy is not creation, but destruction financed by taxes. Models with heterogeneous agents do not give a general answer: in some the optimal monetary policy is contractionary, in others it is expansionary.[1] This paper confirms that result in a somewhat new model—actually, the familiar matching model setting of Trejos-Wright (1995) and Shi (1995), but with a notion of implementability that has not been used before to study the effects of money creation. Here a lottery allocation is implementable if it is in the pairwise core (in every meeting).

My work is most closely related to Molico (2006) and to Deviatov and Wallace (2001) who study money creation in versions of the same model. Molico (2006) approximates divisible money and proceeds numerically using a particular bargaining solution—take-it-or leave-it offers by consumers. Hence, his work leaves open whether the results are special to that bargaining solution. Deviatov and Wallace (2001) allow for any outcome which satisfies ex post individual rationality in meetings and work with money holdings in the set $\{0, 1, 2\}$, the smallest set that gives money creation a role in determining the distribution of money holdings. They get an analytical result—money creation is beneficial whenever agents are sufficiently patient—but only because they do not permit those in a meeting to commit to lotteries. Here, I adopt the same set of individual holdings but allow people in a meeting to commit to lotteries, while at the same time requiring that the lottery trades be in the pairwise core for every meeting.

I cannot get analytical results, and, therefore, proceed numerically. For each example studied, I find both the best rate of money creation and the best pairwise-core lottery allocation. In other words, I allow the division of the gains from trade in a meeting to depend both on the money creation rate and on the money holdings of the consumer and producer in the meeting. I find that in general optima do not have take-it-or leave-it offers or any other fixed bargaining rule except in settings where individuals are sufficiently impatient. In that case the optima have binding producer participation constraints in all meetings, which implies take-it-or leave-it offers by consumers.

For many settings I cannot find beneficial money creation. However, when

[1] Examples of models where it is expansionary include Imrohoroglu (1992), Levine (1991) and a generalization by Kehoe, Levine and Woodford (1992), Deviatov and Wallace (2001), Berentsen, Camera, and Waller (2005), Bhattacharya, Haslag, and Martin (2005), and Molico (2006).

people are both sufficiently impatient and risk averse (obtain sufficiently high utility gains from trade), I find that money creation is beneficial. That is quite different from Deviatov and Wallace (2001), whose results apply only if individuals are sufficiently patient. In all examples where money creation is beneficial there is no randomization in meetings.

The rest of the paper is organized as follows. The next section provides the description of the environment; in section 3 I define implementable allocations; section 4 contains a discussion of some general properties of implementable allocations; in section 5 I present numerical examples; section 6 concludes. Description of numerical algorithm is relegated to Appendix.

2 Environment

The background environment is a simple random matching model of money due to Shi (1995) and Trejos and Wright (1995). Time is discrete and the horizon is infinite. There are $N \geq 3$ perishable consumption goods at each date and a $[0,1]$ continuum of each of N types of agents. A type n person consumes only good n and produces good $n+1$ (modulo N). Each person maximizes expected discounted utility with discount parameter $\beta \in (0,1)$. Utility in a period is given by $u(y) - c(x)$, where y denotes consumption and x denotes production of an individual ($x, y \in \mathbb{R}_+$). The function u is strictly concave, strictly increasing and satisfies $u(0) = 0$, while the function c is convex with $c(0) = 0$ and is strictly increasing. Also, there exists $\hat{y} > 0$ such that $u(\hat{y}) = c(\hat{y})$. In addition, u and c are twice continuously differentiable. At each date, each agent meets one other person at random.

There is only one asset in this economy which can be stored across periods: fiat money. Money is indivisible and no individual can have more than two units of money at any given time. Agents cannot commit to future actions (except commitment to outcomes of randomized trades). Finally, each agent's specialization type and individual money holdings are observable within each meeting, but the agent's history, except as revealed by money holdings, is private.

3 Implementable allocations and the optimum problem

The timing in a period is the following. First, there are meetings and trades. Then, the monetary policy is applied. The policy is a probabilistic version of the proverbial helicopter drops of money. Then, the next period begins and the above sequence of actions is repeated.

The pairwise meetings, the inability to commit, the privacy of individual histories, and the perishable nature of the goods imply that any production must be accompanied by a positive probability of receiving money. A trade meeting is a meeting between a potential producer with $i \in \{0,1\}$ units of money and a potential consumer with $j \in \{1,2\}$ units of money.

For each trade meeting, a general lottery trade is represented by a probability measure on $\mathbb{R}_+ \times \{0,1,2\}$ with the interpretation that if (y,k) is randomly drawn from that measure, then y is produced and consumed and k units of money are transferred from the consumer to the producer. In models with indivisible money lotteries help to approximate divisibility of money.[2] That seems important because indivisibility of money may drive some results in those models. Thus, randomization should not be considered a strange ingredient of trade.

As is obvious and spelled out below, only measures that are degenerate on output can be in the pairwise core. Consequently, any trade in the pairwise core can be described by the quantity of goods, y_{ij}, traded in meetings between producers with i units of money and consumers with j units and by a probability distribution $\left(\lambda_{ij}^0, \lambda_{ij}^1, \lambda_{ij}^2\right)$, where λ_{ij}^k is the probability that k units of money are transferred and where $\lambda_{ij}^k = 0$ if $k > \min\{j, 2-i\}$. Finally, let p_i be the fraction of agents in each specialization type who start a date with i units of money and let $p = (p_0, p_1, p_2)$. Then, in terms of p_i and λ_{ij}^k, the transition matrix T for money holdings (implied by the trades) is given by:

$$T = \begin{bmatrix} t_{00} & \frac{1}{N}\left(p_1\lambda_{01}^1 + p_2\lambda_{02}^1\right) & \frac{1}{N}p_2\lambda_{02}^2 \\ \frac{1}{N}\left(p_0\lambda_{01}^1 + p_1\lambda_{11}^1\right) & t_{11} & \frac{1}{N}\left(p_1\lambda_{11}^1 + p_2\lambda_{12}^1\right) \\ \frac{1}{N}p_0\lambda_{02}^2 & \frac{1}{N}\left(p_1\lambda_{12}^1 + p_0\lambda_{02}^1\right) & t_{22} \end{bmatrix}, \quad (1)$$

[2] See, e.g. Berentsen, Molico, and Wright (2002), Berentsen, Camera, and Waller (2004), and Camera (2005) for a discussion.

where t_{mm} denotes a diagonal element of T (the probability that an individual leaves the meeting with the same quantity m of money she brought into that meeting). Because T is a transition matrix, t_{mm} can be recovered from the condition that each row of T sums to unity.

I use the formulation of policy introduced by Deviatov and Wallace (2001). As I said, the policy constitutes a probabilistic version of the proverbial helicopter drops of money at a rate. Under policy (α, δ), at each date each person not at the upper bound receives a unit of money with probability α and then each unit of money disintegrates with probability δ. As now explained, the α part of the policy resembles lump-sum money creation, while the δ part is a stand-in for the normalization that is equivalent to inflation.

With divisible money and no bound on individual holdings, the standard policy is creation at a rate where money is handed out lump-sum to people. In a broad class of settings, this policy is equivalent to such creation followed by a proportional reduction in individual money holdings (see e.g. Lucas and Woodford, 1994). The proportional reduction is nothing but a normalization of individual holdings. Here, because individual holdings are bounded, such a normalization is necessary. Moreover, because money holdings are indivisible, both parts of the policy must be probabilistic. Also, because of the bound, the α part only approximates lump-sum creation because a person at the upper bound cannot receive such a transfer.

The standard policy (regardless of whether it is followed by a proportional reduction in holdings or not) has two effects. It shifts the distribution of real money balances towards the mean and makes money less desirable to acquire or retain. The above (α, δ)-policy also has these effects. In particular, producers are less willing to produce for money (because they may get a transfer without production and may lose any money acquired) and consumers are more willing to part with money (because they may get a transfer and may lose money they retain). Notice that these effects of monetary policy depend on the assumption of equal transfers. Here we take for granted that an equal transfer policy is feasible and that one that depends on money holdings is not. Obviously, more could be achieved by a policy that depends on money holdings.

I also follow Deviatov and Wallace (2001) in their interpretation of sequential individual rationality (which here is a part of the pairwise core notion) as precluding direct taxes. That, among other things, implies that it is not feasible to simply take money from people or to force producers to produce. For that reason I consider only non-negative (α, δ)-policies.

Similar to trades, the creation and destruction parts of the policy yield a pair of transition matrices for money holdings, denoted A and D respectively. According to my description of the policy, they are

$$A = \begin{bmatrix} 1-\alpha & \alpha & 0 \\ 0 & 1-\alpha & \alpha \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$D = \begin{bmatrix} 1 & 0 & 0 \\ \delta & 1-\delta & 0 \\ \delta^2 & 2\delta(1-\delta) & (1-\delta)^2 \end{bmatrix}.$$

Then, our sequence of actions implies that the stationarity requirement is $pTAD = p$.

It is convenient to express individual rationality and pairwise core constraints in terms of discounted expected utilities. Given a meeting of a producer with i and a consumer with j units of money, let

$$\mu_{ij} \equiv \left\{ \left(\lambda_{ij}^0, \lambda_{ij}^1, \lambda_{ij}^2 \right), y_{ij} \right\}.$$

Also, let μ denote the collection of all μ_{ij}. For an allocation (p, μ, α, δ) that is stationary, discounted expected utility of an agent who ends up with i units of money at the end of the period, denoted v_i, is constant. Then, the vector $v \equiv (v_0, v_1, v_2)$ satisfies the following three-equation system of Bellman equations:

$$v' = \beta(q' + TAD\, v'), \tag{2}$$

where q, the vector of (expected) one period returns from trade, is given by:

$$q' = \begin{bmatrix} -\frac{p_1}{N}\lambda_{01}^1 c(y_{01}) - \frac{p_2}{N}\left(\lambda_{02}^1 + \lambda_{02}^2\right) c(y_{02}) \\ -\frac{p_2}{N}\lambda_{12}^1 c(y_{12}) + \frac{p_1}{N}\lambda_{11}^1 \left[u(y_{11}) - c(y_{11})\right] + \frac{p_0}{N}\lambda_{01}^1 u(y_{01}) \\ \frac{p_0}{N}\left(\lambda_{02}^1 + \lambda_{02}^2\right) u(y_{02}) + \frac{p_1}{N}\lambda_{12}^1 u(y_{12}) \end{bmatrix}. \tag{3}$$

Note that an individual with no money can only expect to be a producer, an individual with two units can only be a consumer, and a person with one unit of money can be either a consumer or a producer.

Because T, A, and D are transition matrices and $\beta \in (0,1)$, the mapping $G(x) \equiv \beta(q' + TAD\, x')$ is a contraction. Therefore, (2) has a unique solution which can be expressed as

$$v' = \left(\frac{1}{\beta}I - TAD\right)^{-1} q', \tag{4}$$

where I is the 3×3 identity matrix.

Let

$$\Pi_{ij}^{p} \equiv \sum_{k} \lambda_{ij}^{k} \left(e_{i+k} - e_{i}\right) ADv' - c(y_{ij}) \tag{5}$$

be the expected gain from trade for the producer with i units of money in a meeting with a consumer with j units and let

$$\Pi_{ij}^{c} \equiv \sum_{k} \lambda_{ij}^{k} (e_{j-k} - e_{j}) ADv' + u(y_{ij}) \tag{6}$$

be the gain from the same trade for the consumer (where e_l is the three-component coordinate vector with indices running from 0 to 2).

The ex ante pairwise core notion of implementability gives rise to the following definition:

Definition 1. *An allocation (p, μ, α, δ) is called ex ante pairwise core implementable if (i) $pTAD = p$, (ii) v (given by 4) is non-decreasing, (iii) the participation constraints*

$$\Pi_{ij}^{p} \geq 0 \qquad \textit{and} \qquad \Pi_{ij}^{c} \geq 0 \tag{7}$$

hold for all i and j, and (iv) for every pair (i, j) that corresponds to a trade meeting, μ_{ij} solves

$$\max_{\mu_{ij}} \Pi_{ij}^{c} \tag{8}$$

$$\text{subject to } \Pi_{ij}^{p} \geq \gamma_{ij}$$

for some (meeting-specific) γ_{ij} consistent with the participation constraints (7), where the policy (α, δ) and the value function v are taken as given.

In Definition 1, γ_{ij} can depend on the policy (α, δ). Definition 1 says that an allocation is implementable if it (i) is stationary, (ii) satisfies free disposal of money, (iii) satisfies ex ante individual rationality, and (iv) there is no incentive for defections by pairs in meetings. Ex ante individual rationality in (iii) requires commitment by individuals to outcomes of randomized trades. Lack of such a commitment limits the set of feasible lotteries quite severely. In particular, producers are not willing to produce ex post if the ex post money transfer is zero and consumers are unwilling to transfer their money holdings ex post in some meetings (see, for example, columns 1-4 of Table 1, where the transfers of one unit in $(i, j) = (1, 1)$ meetings are inconsistent with willingness to trade by consumers).[3]

Finally, my optimum problem is to maximize ex ante utility. That is, the optimum problem is to choose (p, μ, α, δ) from among those that are implementable to maximize $pv' \equiv W$.

It is useful to express the objective W in terms of returns. If I multiply (2) by p and use the fact that $pTAD = p$, then I obtain

$$W = pv' = \frac{\beta}{1-\beta} pq'.$$

Then, by writing out the product pq', I get

$$W = \frac{\beta}{1-\beta} \frac{1}{N} \sum_{i=0}^{1} \sum_{j=1}^{2} p_i p_j \left[u(y_{ij}) - c(y_{ij}) \right]. \tag{9}$$

As one would expect, because for every consumer there is a producer, welfare is equal to the net expected discounted utility in all trade meetings.

4 General results

Because Π_{ij}^c and Π_{ij}^p are strictly concave functions of y, randomization over output cannot be a solution to (8).[4] The proof proceeds with replacing any

[3] One way to think of lottery trades is the following. Each pair meets in a place equipped by the production technology and a slot machine. First, consumers deposit their money in the slot machine. Next, producers produce quantity y_{ij} of output. Then, the machine gives some money to the producer and the rest as change to the consumer in accordance with the distribution $\left(\lambda_{ij}^0, \lambda_{ij}^1, \lambda_{ij}^2\right)$. That sequence of events implies ex post willingness to trade by both producers and consumers because production costs are sunk and consumers cannot retain the money, which the machine allocates to the producers.

[4] Berentsen, Molico, and Wright (2002) introduce lotteries in a random matching model of money and give a complete characterization of the ex ante pairwise core for the case of one-unit bound on holdings.

non-degenerate distribution over output by its mean, which increases the objective Π_{ij}^c and relaxes the constraint $\Pi_{ij}^p \geq \gamma$. Such degeneracy implies that my optimum problem is finite dimensional. This allows me to characterize the ex ante pairwise core in terms of the necessary first order conditions. Because of concavity of Π_{ij}^p and Π_{ij}^c these necessary conditions are also sufficient. If an allocation (p, μ, α, δ) has $y_{ij} > 0$ in all trade meetings,[5] then the first order conditions can be conveniently written as

$$\left[(e_{j-k} - e_j) + \frac{u'(y_{ij})}{c'(y_{ij})}(e_{i+k} - e_i)\right] ADv' \begin{array}{l} \geq 0 \text{ if } \lambda_{ij}^k = \overline{\lambda}_{ij}^k \\ = 0 \text{ if } 0 < \lambda_{ij}^k < \overline{\lambda}_{ij}^k \\ \leq 0 \text{ if } \lambda_{ij}^k = 0 \end{array} \tag{10}$$

for all pairs (i, j) corresponding to trade meetings and transfers of positive amounts of money k, where $\overline{\lambda}_{ij}^k \equiv 1 - \sum_{s \geq 1,\, s \neq k} \lambda_{ij}^s$.

The first order conditions (10) yield a set of constraints which an ex ante pairwise core implementable allocation must satisfy in addition to the participation constraints in Definition 1. If the value function ADv' implied by an implementable allocation (p, μ, α, δ) is strictly concave, then (10) has implications for the level of output in some meetings. In particular, if $\lambda_{ij}^k > 0$ and $k \geq j - i$ for some positive k, then $y_{ij} \leq y^*$, the unconstrained maximizer of $u(y) - c(y)$. Because the bound on individual holdings is two units, the only meetings in which output can exceed y^* are those between a producer with zero and a consumer with two units of money.

5 The examples

The numerical algorithm is described in the Appendix. All my examples have $u(y) = y^\kappa$, $c(y) = y$, and $N = 3$. I vary κ and patience, r, where $r \equiv \frac{1}{\beta} - 1$. I compute two sets of examples. The first set contains examples for moderate risk aversion and patience; i.e., all combination of κ and r for $\kappa \in \{0.2, 0.3, 0.4, 0.5, 0.6\}$ and $r \in \{0.01, 0.02, 0.03, ..., 0.25, 0.30, 0.35, 0.40, 0.50, 1.00\}$. In this set, beneficial money creation occurs only for $(\kappa, r) = (0.2, 1.00)$. To further explore the range of parameters for which money creation is beneficial,

[5] A sufficient condition for this is that ADv', where v is the value function implied by an implementable allocation (p, μ, α, δ), is strictly increasing and that $u'(0) = \infty$ and $c'(0) = 0$.

Table 1: Optima when $u(y) = x^{0.6}$. The first best output, $y^* = 0.2789$.

r	0.01	0.05	0.10	0.15	0.20	0.25	0.30	0.40	0.50	1.00
α	0	0	0	0	0	0	0	0	0	0
δ	0	0	0	0	0	0	0	0	0	0
p_0	.2307	.2689	.3241	.3822	.4285	.4629	.4895	.5222	.5454	.6043
p_1	.5593	.4936	.4263	.3894	.3628	.3417	.3246	.3025	.2950	.2727
p_2	.2100	.2374	.2496	.2284	.2087	.1954	.1859	.1753	.1596	.1230
λ_{01}	.2693	.4492	.8033	1	1	1	1	1	1	1
λ_{12}	.3896	.6435	1	1	1	1	1	1	1	1
λ_{11}	.1548	.2620	.4452	.5758	.6796	.7749	.8640	1	1	1
y_{01}	1^*	1^*	1^*	$.8408^*$	$.6203^*$	$.4772^*$	$.3783^*$	$.2506^*$	$.1786^*$	$.0527^*$
y_{12}	1^*	1^*	$.8397^*$	$.5181^*$	$.3489^*$	$.2460^*$	$.1796^*$	$.1022^*$	$.0642^*$	$.0118^*$
y_{11}	$.3973^*$	$.4073^*$	$.3740^*$	$.2983^*$	$.2370^*$	$.1904^*$	$.1549^*$	$.1022^*$	$.0642^*$	$.0118^*$
y_{02}	2.490	2.069	1.245^*	$.8408^*$	$.6203^*$	$.4772^*$	$.3783^*$	$.2506^*$	$.1786^*$	$.0527^*$

I also study examples with $\kappa = 0.2$ and $r \in \{0.80, 0.85, 0.90, 0.95, 1.00, 1.10, 1.20, 1.50, 2.00, 3.00\}$.

Two features are common to every example. There are no binding consumer participation constraints. And, in a meeting of a producer with no money and a consumer with two units, one unit of money changes hands with probability one. (I take advantage of the second feature and do not describe the transfer of money in meetings of producers with nothing and consumers with two units. I also suppress the superscript 1 in the notation for the transfer probabilities, λ^1_{01}, λ^1_{12} and λ^1_{11}.) I attach stars (*) to outputs which correspond to binding producer participation constraints.

In all examples where there is randomization between a transfer of one and a transfer of zero units of money, the pairwise core constraints (10) are binding. If the optima have randomization, then individuals can defect both to a new output y_{ij} and a new (internal) distribution $\left(\lambda^0_{ij}, \lambda^1_{ij}, \lambda^2_{ij}\right)$. Then, bindingness of the first order core constraints is a necessary condition that both consumers and producers cannot benefit from such a defection. The core constraints are slack if there is no randomization (except constraints pertaining to the transfer of one unit in $(0, 2)$ meetings for $r = 0.01$ and

Table 2: Optima when $u(y) = x^{0.4}$. The first best output, $y^* = 0.2172$.

r	0.01	0.05	0.10	0.15	0.20	0.25	0.30	0.40	0.50	1.00
α	0	0	0	0	0	0	0	0	0	0
δ	0	0	0	0	0	0	0	0	0	0
p_0	.2022	.2437	.2715	.2892	.3197	.3448	.3660	.4046	.4327	.4834
p_1	.6240	.5542	.5012	.4647	.4316	.4079	.3898	.3623	.3411	.3134
p_2	.1738	.2021	.2273	.2461	.2487	.2473	.2442	.2331	.2262	.2032
λ_{01}	.1419	.2495	.3793	.5081	.6756	.8439	1	1	1	1
λ_{12}	.2795	.4837	.7280	.9735	1	1	1	1	1	1
λ_{11}	.0903	.1604	.2456	.3294	.4267	.5125	.5884	.7188	.8412	1
y_{01}	1^*	1^*	1^*	1^*	1^*	$.1^*$	$.9862^*$	$.7366^*$	$.5778^*$	$.2436^*$
y_{12}	1^*	1^*	1^*	1^*	$.7431^*$	$.5608^*$	$.4355^*$	$.2836^*$	$.1961^*$	$.0511^*$
y_{11}	$.3232^*$	$.3315^*$	$.3375^*$	$.3384^*$	$.3172^*$	$.2873^*$	$.2560^*$	$.2035^*$	$.1648^*$	$.0511^*$
y_{02}	3.094	2.528	2.181	1.955^*	1.480^*	1.185^*	$.9862^*$	$.7366^*$	$.5778^*$	$.2436^*$

$\kappa \in \{0.2,\ 0.4\}$). Another implication of (10) is that if the transfer of one unit of money results in a switch of money holdings, as it does in a $(0,1)$ meeting and a $(1,2)$ meeting, and if there is randomization, then any pairwise core trade has the first-best level of output y^*. Because production of the first best output occurs in many meetings, I find it helpful to express outputs relative to the first best level in all examples. (Notice that given our choice of $u(y) = y^\kappa$ and $c(y) = y$, y^* is a decreasing function of relative risk aversion, $1 - \kappa$.)

I report some examples from the first set in Tables 1-3 (one table for each value of risk aversion κ). The examples are consistent with the existence of four different regions with respect to the degree of patience r. If r is small enough, then the optima have randomization over the transfers of money in all meetings where the transfer is at most one unit of money. If r belongs to the second region, then the optima have randomization only in meetings where the consumers have one unit. In the next region the optima have randomization over the transfers of money only if both the producer and consumer have one unit. Finally, if r is large enough, one unit of money changes hands with probability one in all trade meetings.

The transfer probabilities λ_{12}, λ_{01}, and λ_{11} increase with impatience. Greater impatience tightens producer participation constraints, implying that larger probabilities of transfers are needed to induce any given level of production. In addition, the optimum quantity of money, $p_1 + 2p_2$, declines with impatience. Less money implies a higher probability of meeting other producers without money in the future and, therefore, helps loosen producer participation constraints.

Examples in the first set are consistent with the optima having at most one nonbinding producer participation constraint, the one in meetings of producers with nothing and consumers with two units of money. In a meeting of a producer with one unit and a consumer with two, lowering the probability of handing over money raises v_2. That is helpful because it loosens producer constraint in the $(1, 1)$ meeting, which, in turn, allows a decrease in λ_{11} and, thus, an increase in p_1 (and, thereby, in the frequency of trade). Likewise, a smaller probability of giving up money in the $(0, 1)$ meeting lowers v_0 which helps to relax the producer constraint in the $(0, 2)$ meeting. This allows a higher y_{02} which, again, pushes up v_2. This accounts for why y_{02} is so high in some examples. The same kind of effect on v_2 could be achieved with a positive λ_{02}^0, but that would reduce the inflow into p_1.

I report examples from the second set in Table 4. There are two features that are common to examples in which money creation is beneficial. One is that all producer participation constraints are binding. That is, the optima have take-it-or-leave-it offers in all meetings — the bargaining rule assumed by Molico (2006). That may be an indication that Molico's results are close to what one would get in his setting by choosing the trades optimally.

The other is the absence of randomization. Such absence may not be accidental; it seems that when randomization is a part of an optimum, it produces beneficial extensive margin effects that dominate those of a policy. To see why this is plausible, consider allocations under no policy. If $\alpha = \delta = 0$, then the stationarity requirement, $pTAD = p$, collapses to a single equation:

$$\lambda_{11}p_1^2 = \lambda_{02}p_0p_2. \tag{11}$$

This along with

$$p_0 + p_1 + p_2 = 1, \quad \text{and} \quad p_i \geq 0, \tag{12}$$

yields the set of all stationary distributions. If the notion of implementability is the ex post notion, then every trade must be accompanied by the transfer of money. In that case every implementable allocation has $\lambda_{11} = \lambda_{02} = 1$. Then,

Table 3: Optima when $u(y) = x^{0.2}$. The first best output, $y^* = 0.1337$.

r	0.01	0.05	0.10	0.15	0.20	0.25	0.30	0.40	0.50	1.00
α	0	0	0	0	0	0	0	0	0	.2498
δ	0	0	0	0	0	0	0	0	0	.1763
p_0	.1494	.1925	.2244	.2448	.2600	.2721	.2824	.3022	.3232	.3686
p_1	.7309	.6563	.5992	.5608	.5314	.5077	.4888	.4572	.4302	.3970
p_2	.1197	.1512	.1764	.1944	.2086	.2202	.2288	.2406	.2466	.2344
λ_{01}	.0453	.0908	.1475	.2021	.2565	.3106	.3611	.4668	.5872	1
λ_{12}	.1525	.2943	.4722	.6435	.8154	.9886	1	1	1	1
λ_{11}	.0335	.0676	.1103	.1513	.1921	.2326	.2704	.3477	.4306	1
y_{01}	1^*	1^*	1^*	1^*	1^*	1^*	1^*	1^*	1^*	$.4353^*$
y_{12}	1^*	1^*	1^*	1^*	1^*	1^*	$.8714^*$	$.6597^*$	$.4929^*$	$.1212^*$
y_{11}	$.2199^*$	$.2296^*$	$.2334^*$	$.2356^*$	$.2356^*$	$.2356^*$	$.2356^*$	$.2296^*$	$.2124^*$	$.1212^*$
y_{02}	4.553	3.921	3.064	2.728	2.497	2.339	2.328	2.143^*	1.703^*	$.4353^*$

equation (11) defines a one-dimensional family of stationary distributions on the simplex (12). If the policy is applied, then it shifts the locus of stationary distributions on that simplex. Deviatov and Wallace (2001) show that for some policies it is feasible to reach distributions consistent with an increased frequency of trades. That external margin (distribution) effect gives rise to higher ex ante utility in their model provided that individuals are patient.

If the notion of implementability is the ex ante notion, then only a fraction of trades must have transfers of money. In that case there are many randomization schemes consistent with any distribution p being a stationary distribution. In particular, any distribution consistent with an increased frequency of trade (i.e. with $p_1^2 > p_0 p_2$) can be reached by setting $\lambda_{02} = 1$ and $\lambda_{11} < 1$. If there is both randomization and an expansionary policy, then reducing the degree of expansion helps to increase the value of money, $v_1 - v_0$ and $v_2 - v_1$. As noted above, that relaxes the producer participation constraint in the $(1, 1)$ meeting and allows for a decrease in λ_{11} and, hence, an increase in p_1. However, a formal argument along these lines seems to be out of reach because the value of money and the constraints are complicated functions of an allocation.

Table 4: Optima when $u(y) = x^{0.2}$ (low patience). $y^* = 0.1337$.

r	0.80	0.85	0.90	0.95	1.00	1.10	1.20	1.50	2.00	3.00
α	0	0	.2483	.2489	.2498	.2522	.2551	.2647	.2795	.2321
δ	0	0	.1741	.1751	.1763	.1789	.1818	.1901	.2018	.1762
p_0	.3689	.3739	.3660	.3674	.3686	.3708	.3727	.3768	.3810	.3920
p_1	.3770	.3705	.3970	.3970	.3970	.3974	.3977	.3993	.4016	.3908
p_2	.2551	.2556	.2370	.2356	.2344	.2318	.2296	.2239	.2174	.2172
λ_{01}	.9538	1	1	1	1	1	1	1	1	1
λ_{12}	1	1	1	1	1	1	1	1	1	1
λ_{11}	.6608	.6963	1	1	1	1	1	1	1	1
y_{01}	1^*	$.9850^*$	$.4787^*$	$.4562^*$	$.4353^*$	$.3972^*$	$.3650^*$	$.2894^*$	$.2109^*$	$.1503^*$
y_{12}	$.2416^*$	$.2177^*$	$.1354^*$	$.1279^*$	$.1212^*$	$.1085^*$	$.0980^*$	$.0748^*$	$.0531^*$	$.0329^*$
y_{11}	$.1593^*$	$.1518^*$	$.1354^*$	$.1279^*$	$.1212^*$	$.1085^*$	$.0980^*$	$.0748^*$	$.0531^*$	$.0329^*$
y_{02}	1.049^*	$.9850^*$	$.4787^*$	$.4562^*$	$.4353^*$	$.3972^*$	$.3650^*$	$.2894^*$	$.2109^*$	$.1503^*$

These findings are very different from Deviatov and Wallace (2001), who demonstrate that there is beneficial money creation whenever there is sufficiently low impatience. The difference is easily explained by their use of ex post individual rationality constraints. Ex post individual rationality implies that the only way to increase p_1 in their model is by means of policy. Here similar distribution effects can be achieved by randomization whenever optima have randomization. But if, because of impatience, it is desirable for consumers in $(1, 1)$ meetings to surrender their money with probability one, then the only way to increase p_1 is again through money creation. With impatient individuals money creation can be beneficial only if individuals derive high utility gains from trade. The choice of $u(y) = y^\kappa$ and $c(y) = y$ in numerical examples implies that if κ is low enough, then the gains from trade are sufficiently high.

6 Concluding remarks

This paper adds to the list of models where money creation is beneficial. Because I work with fully decentralized environment, analytical solutions are not feasible, so I do a series of numerical examples. I compute examples for a case of two-unit bound on individual money holdings. That bound is the lowest for which money creation can have a beneficial role. Given my findings of examples where money creation is beneficial, a natural surmise is that beneficial effects of money creation persist for all higher bounds. However, that seems somewhat difficult to verify because the dimensionality of the optimization problem is proportional to the cube of the bound and even for a case of a three-unit bound numerical analysis is demanding. One way to get around the curse of dimensionality is to work with environments with partially centralized markets, where the distribution of money is "manageable".[6] However, that should be done with caution because the external margin beneficial effect of money creation depends on having heterogeneity in the model.

7 Appendix: the algorithm

Because the beneficial external margin and harmful internal margin effects of money creation are at balance in any optimum, the optima always have some binding participation constraints. If individuals are patient enough, in addition to binding participation constraints the optima have randomization over how much money is transferred in meetings. This implies that some of the constraints in (10) are also binding. Because these constraints are complicated functions of an allocation, closed-form solutions for the optima are out of reach even for the case of a two-unit bound on holdings. For this reason I compute a set of examples.

My optimization problem falls within the class of problems generally referred to as "nonlinear programming problems", for which many standard routines are available. However, as one can see, the constraints in (10) are discontinuous.[7] Another difficulty is that the mapping $F(p) \equiv pTAD - p$

[6] See e.g. Lagos and Wright (2003) and Berentsen, Camera, and Waller (2005).

[7] Each constraint in (10) is equivalent to

$$\left[(e_{j-k} - e_j) + \frac{u'(y_{ij})}{c'(y_{ij})}(e_{i+k} - e_i)\right] ADv' + (\text{sign}(\lambda_{ij}^k) - \text{sign}(\overline{\lambda}_{ij}^k - \lambda_{ij}^k))\vartheta_{ij}^k = 0$$

and

$$\vartheta_{ij}^k \leq 0,$$

where sign(x) is the sign function, and ϑ_{ij}^k is a slack variable.

is ill-behaved at $\alpha = \delta = 0$.[8] This precludes application of routines which require continuous differentiability of the objective and constraints, such as sequential quadratic programming. I overcome this difficulty by designing a hybrid algorithm which combines genetic and conventional smooth optimization techniques.

There are three main steps in this algorithm. First, create an initial population of allocations. Second, amend the population by replacing the worst allocations by better ones. Third, check if the termination criterion is satisfied for the best allocation in the population. If yes, then terminate; if no, then return to the second step.

In the first step, I create a matrix where each row is an allocation. Allocations in the initial population are picked randomly among those which satisfy ex ante individual rationality. The size of the population is a parameter of the algorithm.

To amend the population (the second step), I use several genetic operators. These operators are called selection, crossover, and mutation. I use standard selection and crossover operators, a subset of those described in Houck, Joines and Kay (1996). However, I modify the standard mutation operator. The standard operator alters a single allocation (called "the parent") to produce another allocation (called "the child"). The operator I use is a composition of two independent operators.

The first one is applied only if the parent has at least one of the transfer probabilities λ_{ij}^k at its upper or lower bound or if it has $\alpha = \delta = 0$. The operator pushes a random subset of these variables into the interior. If a better allocation is produced, it replaces the parent in the population. This simple mutation deals with discontinuity of the constraints in (10) and with ill behavior of the mapping $F(p)$ at zero.

The second operator alters only those of the transfer probabilities and policy pairs which are already in the interior. There, because all constraints are twice continuously differentiable, application of smooth methods is possible. This leaves a range of possibilities for what this second operator can be. In particular, one can run a few iterations of a sequential quadratic routine or of the BFGS algorithm[9] (as long as these iterations remain in the interior). The operator I adopt makes use of the gradients in the following way.

First, I compute (reduced) gradients of the objective and of all active

[8]See Deviatov and Wallace (2001), who study the properties of that mapping.

[9]See Judd (1998) for further details.

constraints. Then I compute an orthogonal projection of the gradient of the objective onto the subspace orthogonal to the one spanned by the gradients of the active constraints. After that I randomly pick a search direction in the neighborhood (small cone) of that projection. Going in that search direction is likely to improve the objective and does not violate (at least by much) the active constraints. The child is obtained from the parent by moving along the search direction. However, this procedure often leads to a violation of some constraints even if the parent satisfies all the constraints. In this case the objective implied by the child is reduced by some value which is proportional to the amount by which the constraints are violated. If the penalty parameter is large, even a small violation is costly, and the child dies out of the population quickly. If the parent itself violates the constraints by large amounts, then the search direction is chosen to move the child closer to the feasible region regardless of what happens to the objective. Because the initial population is chosen randomly, this is important in the beginning of search. In other words, the second operator first pushes allocations towards satisfaction of the pairwise core conditions; then it drives the population to the optimum.

The termination criterion in the third step is based on the first order conditions for the Kuhn-Tucker theorem. If the length of the projection of the gradient of the objective onto the subspace orthogonal to that spanned by the gradients of the active constraints is less than the tolerance value, the necessary conditions for the theorem are (approximately) satisfied. Because the probability of selection of parents in the population is an increasing function of the objective, this is sufficient to guarantee that every terminal point is a (local) maximum.

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