Lecture notes for the course **INVESTMENT THEORY**

Lecture 1. Introduction

Practical questions

- How to choose assets to invest?
  - Find under/overpriced relative to the “fair value”
  - Risk-return trade-off
  - Diversification effects
- Why do investors select different optimal portfolios?
  - Risk attitude / horizon
  - Life cycle
  - Taxes
  - Other income or liabilities
- How to manage risks?
  - Avoid
  - Insure
  - Hedge
  - Speculate
- How to evaluate performance of your money manager?
  - Risk adjustment
  - Style analysis

Asset pricing issues

- Relative vs absolute pricing
  - No-arbitrage condition
  - Modelling risk-return preferences
- Rational vs behavioural approaches
  - Different degrees of rationality
  - Limits of arbitrage

Is it sufficient to have one non-satiated investor to set prices to fair values?

Structure of the course

- No-arbitrage pricing
  - Options / forwards
- Choice under uncertainty: risk vs return
  - Mean-variance preferences
- Portfolio theory
- Equilibrium models: absolute pricing
  - CAPM
  - Multi-factor models

Applications

- Asset pricing
  - CF: investment projects
- Portfolio management
  - Tactical vs strategic allocation
- Risk management
  - Hedging vs speculation
- Performance evaluation

What is the practical use of the (abstract) models?

Index models: $R_{i,t} = \alpha_i + \Sigma_k \beta^k_i I^k_t + \varepsilon_{i,t}$,

where $R_{i,t}$ is asset *i*’s return in period *t*,
$E(\varepsilon_{i,t})=0$, $cov(I^k, \varepsilon_i)=0$, and $E(\varepsilon_i\varepsilon_j)=0$ for $i\neq j$

- Performance attribution
- Correlation structure: $cov(R_i, R_j) = \beta_i\beta_j\sigma^2_M$
  - Historical correlations are poor forecasters
- Systematic risk: $var(R_i) = \beta_i^2\sigma^2_M + \sigma^2(\varepsilon)_i$
  - The idiosyncratic risk can be diversified away

Which indices would you use in the stock market?

The single index model with market return: $R_{i,t} = \alpha_i + \beta_i R^M_t + \varepsilon_{i,t}$

The market model: no assumption about $E(\varepsilon_i\varepsilon_j)=0$ for $i\neq j$

- Relation of market beta to
  - Dividend payout (div to earnings): -
  - Asset growth: +
  - Leverage (debt to assets): +
  - Liquidity (current A to current L): -
  - Total assets: -
  - Earning variability: +
  - Earnings beta: +

Multi-index models:
- Industry indices
- Macro factors
  - Oil price, inflation, exchange rates, SR/LR interest rates
- Style indices
  - Small-cap vs large-cap
  - Value vs growth
  - Momentum vs reversal
- Statistical factors
  - Principal components

Salomon Brothers:
-Economic growth
-Default spread
-LR interest rates
-SR interest rates
-Inflation shock
-US dollar
-Orthogonalized market index

General approach to asset pricing: (e.g., *Cochrane, ch. 2*)

Assume time-separable UF: $\max_{\xi} U(C_t) + \delta E_t U(C_{t+1})$

s.t. $C_t = E_t - p_t'\xi$ and $C_{t+1} = E_{t+1} + p_{t+1}'\xi$

where C is consumption, E is endowment, δ is discount factor,
p is the vector of asset prices, ξ is the vector of asset quantities

FOC: $-p_t U'(C_t) + E_t[\delta U'(C_{t+1})\, p_{t+1}] = 0$

assuming non-satiation: $p_t = E_t[\delta U'(C_{t+1})/U'(C_t)\, p_{t+1}] = E_t[M_{t+1}\, p_{t+1}]$

for a single asset: $E_t[M_{t+1} P_{i,t+1}] = P_{i,t}$ or $E_t[M_{t+1} R_{i,t+1}] = 1$ (1)

where $M_{t+1} \equiv \delta U'(C_{t+1})/U'(C_t)$
- IMRS: intertemporal marginal rate of substitution
- SDF: stochastic discount factor
- PK: pricing kernel

(1) is *basic pricing equation*
- Current asset prices are positive and linear functions of their future payoffs
  - Assuming M>0, which is equivalent to the absence of arbitrage
- Small M in a particular state of the world implies that this state is "cheap" in a sense that investors are not willing to pay a high price to receive wealth in it

(1) for a risk-free asset => $E_t[M_{t+1}] = 1/R_F$.

(1) => $E_t[M_{t+1} R_{i,t+1}] = E_t[M_{t+1}]\, E_t[R_{i,t+1}] + \mathrm{cov}(M_{t+1}, R_{i,t+1}) = 1$

Then $E[R_i] = R_F - \mathrm{cov}(M, R)/E(M) = R_F + [\mathrm{cov}(M, R)/\mathrm{var}(M)]\,[-\mathrm{var}(M)/E(M)]$ (2)

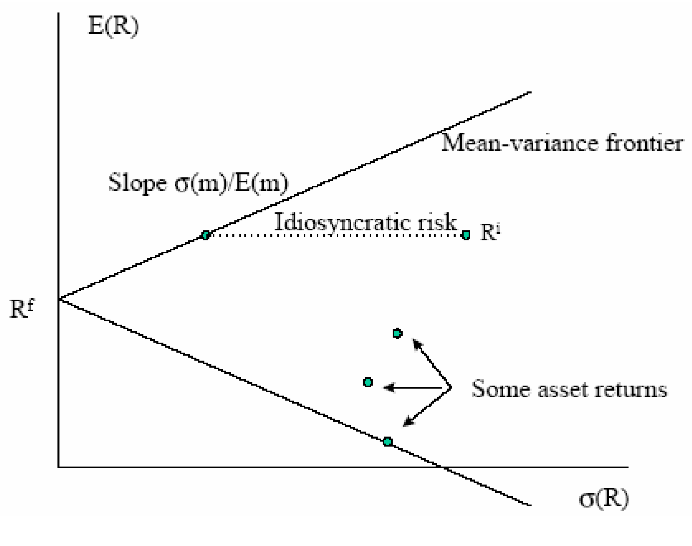
or $E[R_i] = R_F + \beta_M \lambda_M$ *Beta pricing equation*
- Expected return of each asset is equal to the risk-free rate plus a risk adjustment, which depends on the covariance between asset return and consumption.
  - Assets positively correlated with consumption make it more volatile and must offer higher expected return.
- Each AP model implies specific choice of risk factors spanning M

(2) => $E[R_i] - R_F = -[\rho_{M,i}\sigma_M/E_M]\, \sigma_i$

Since $-1 \le \rho \le 1$, $|E[R_i] - R_F| \le [\sigma_M/E_M]\, \sigma_i$
- *Mean-variance frontier*: two lines from R_F with $|\rho_{M,i}|=1$
- All portfolios on the efficient frontier:
  - Have -1 correlation with the pricing kernel
  - Are perfectly correlated with each other
  - Enough to know one efficient ptf to get all

Constraint on the Sharpe ratio:

$|(E[R_i] - R_F)/\sigma_i| \le \sigma_M/E_M$



Lectures 2-3. Arbitrage pricing of derivatives

Forward (e.g., *Hull, ch. 3*)

- Obligation to buy or sell the underlying asset at T at the settlement price K
- The contract is worth zero at t
- Payoff at T: S_T-F

Assumptions:

- Perfect markets
- Lending and borrowing at rate r (cont. compounding)
- No credit risk
- No arbitrage

Then (otherwise arbitrage)

- In the primary market:
    - Unique settlement price
- In the secondary market:
    - If the current forward price is F, then the current value of long forward with the settlement price K is $e^{-rT}(F-K)$

Forward price F:

- For assets with no income (zero-coupon bonds, stocks without dividends): $F = Se^{rT}$
    - Value of the long position: $(F-K)e^{-rT} = S-Ke^{-rT}$
    - Otherwise: borrow S, buy the asset, short forward (or vice versa)
- For assets with known income I (coupon bonds, stocks with known dividends): $F = [S-PV(I)]e^{rT}$
    - Value of the long position: $(F-K)e^{-rT} = S-PV(I)-Ke^{-rT}$
    - One should use risk-free rates for appropriate maturities to compute PV(I)!
- For assets with known dividend yield q (foreign currency, stock index): $F = Se^{(r-q)T}$
    - Value of the long position: $(F-K)e^{-rT} = Se^{-qT} -K^{-rT}$
- For assets with storage costs U (commodities): $F \le (S+ PV(U))e^{rT}$ or $F \le Se^{(r+u)T}$
    - Otherwise: borrow S+ PV(U) for r, buy the asset, pay costs, short forward
    - If short-sale restrictions (e.g. for gold), then there must be a sufficient # people holding the asset purely for investment: sell gold, lend at r, long forward => $F = (S+PV(U))e^{rT}$
    - If there are benefits from holding the asset (e.g., consumption): $F< (S+PV(U))e^{rT}$
        - The convenience yield *y* is s.t. $F = (S+U)e^{(r-y)T}$
- The cost of carry *c*:
    - For an investment asset: $F = S\,e^{cT}$
    - For a consumption asset: $F = S\,e^{(c-y)T}$

The risks of a forward position: $F = E(S_T)e^{(r-k)T}$, where *k* is the discount rate for the basic asset

- Long forward and PV(F) in cash: $-Fe^{-rT}$ at 0, S_T at T
    - $NPV = -Fe^{-rT} + E(S_T)e^{-kT} = 0$ at the competitive market
- Normal backwardation: $F < E(S_T)$
    - Speculators tend to hold long positions and require compensation for their risks
- Contango: $F > E(S_T)$

How to reconcile this with the no-arbitrage forward price formula?

Forwards vs futures

- When the interest rate is deterministic, futures price = forward price
- When the interest rate is stochastic and positively correlated with the underlying asset, futures price > forward price

- The margin proceeds will be re-invested at higher rate
- Liquidity risk for futures due to margin requirements
  - Metallgesellschaft: $1.3 bln loss after closing positions in 5-10 year oil forwards hedged with short-term futures

Options (e.g., *Hull, ch. 7*)

- Right to buy or sell basic asset at T at the exercise price X
- Types: call/put, European/American
- Asymmetric payoff
- Moneyness: in-at-out of the money options
- Intrinsic vs time value
- Factors affecting the option's value:
  - Current stock price, exercise price, maturity, stock volatility, r, dividends

Variable	*c*	*p*	*C*	*P*
S_0	+	−	+	−
K	−	+	−	+
T	?	?	+	+
σ	+	+	+	+
r	+	−	+	−
D	−	+	−	+

Notation:
t=0: current period
T: exercise date
X: exercise price
S: stock price
r: cont. risk-free rate
c, p: prices of European call/put options
C, P: prices of American call/put options

Payoff at T: $c_T=\max(S_T-X,0)$, $p_T=\max(X-S_T,0)$

No-arbitrage relationships: upper and lower bounds

- Lower bounds: option price ≥ 0 (1)
- Upper bounds: $c_t \leq S_t$, $C_t \leq S_t$, $p_t \leq Xe^{-rT}$, $P_t \leq X$ (2)
- European vs American: $c \leq C$, $p \leq P$ (3)
- For $T_2>T_1$, $C(T_2) \geq C(T_1)$, $P(T_2) \geq P(T_1)$ (4)
- Lower bound for call on non-dividend-paying stock: c_t (C_t) > $\max(0, S_t - PV(D) - Xe^{-rT})$ (5)
  - Proof: compare two portfolios, $P_A>P_B$
    - A: call and borrow PV of the strike price, payoff at T = $\max(X, S_T)$
    - B: stock, payoff at T = S_T
- Lower bound for put on non-dividend-paying stock: $p_t > \max(0, Xe^{-rT} + PV(D) - S_t)$ (6)
  - For American put, the early exercise is always possible: $P_t \geq \max(0, X - S_t)$ (7)
- European put-call parity: $c - p = S - PV(D) - Xe^{-rT}$ or $c - p = Se^{-qT} - Xe^{-rT}$
  - Proof: consider the following 2 portfolios, $P_C=P_D$
    - C: European call + PV of the strike price in cash, worth $\max(S_T, X)$ at T
    - D: European put + stock, also worth $\max(S_T, X)$ at T
- American put-call relationship: $S - PV(D) - X \leq C - P \leq S - Xe^{-rT}$
- For dividend-paying stocks: call is better to exercise at the end, put – at the beginning

Theorem: early exercise of American call on non-dividend-paying stock is not optimal

- Proof: (5) => $C_t > S_t - X$
- If the trader wants to exercise early and hold the stock till maturity, he'd better earn interest on X
- If he expects the stock price go down, he'd better short the stock. In any case, he is better off selling the option rather than exercising it!
  - If others also thought that the stock is overpriced, the stock price would go down.
- If the stock pays a dividend, it may be optimal to exercise the option before the ex-dividend date.

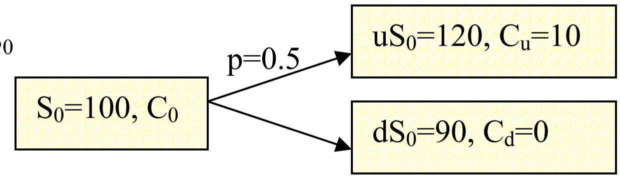
Lecture 4. Binomial model

One-period binomial model (e.g., *Hull, ch. 9*)

Two benchmark assets:

- Bond (money market account): r_F=3%, $B_1=1.03B_0$
- Stock with risky payoff

What is the price of the call option on the stock with the exercise price 110?



"Heuristic" approach: $E[C_1]=5$ and $C_0 = E[C_1] / (1+r_F) = 4.85$,

or maybe, since it's risky: $C_0 = E[C_1] / (1+E[R_S]) = 4.75$?

No-arbitrage approach:

Construct a ptf of *a* stocks and *b* bonds to replicate the option's payoff

$$120a + 1.03b = 10$$
$$90a + 1.03b = 10$$

It follows that $a = 1/3$

$$b = -90a/1.03 = -29.1$$

By the law of single price, $C_0 = aS_0 + B_0 = 4.2$

Conclusions:

- The option's price does not depend on the state probabilities and stock's expected return
    - o It is relative to the stock's price, which already incorporates all risks
- With 2 assets and 2 states (complete market), we can construct replicating ptf for any derivative!

Solving the binomial model for general parameters:

Construct a risk-free ptf of Δ stocks and one short option

$$\Delta uS_0 - C_u = \Delta dS_0 - C_d$$

It follows that $\Delta = [C_u-C_d] / [S_0(u-d)]$.

The price of this ptf in t=1: $P_1 = [d/(u-d)]C_u - [u/(u-d)]C_d$

In t=0 (after dicounting): $P_0 = (1/R_F) ([d/(u-d)]C_u - [u/(u-d)]C_d) = \Delta S_0 - C_0$

Thus, $C_0 = (1/R_F) ([(R_F-d)/(u-d)]C_u + [(u-R_F)/(u-d)]C_d) = (1/R_F)E^Q[C_1]$

Where $q \equiv (R_F-d)/(u-d)$ and 1-q are **"risk-neutral" probabilities**

In the "risk-neutral" world all assets have same (riskless) expected return!

Another approach: via **state prices** (Arrow-Debreu assets)

S_k: price of the security with unit payoff in state k and zero payoff in other states

If we know state prices, we can price any asset as $P_0 = s'P_1$.

In our model, $S_k = q_k/R_F$.

Generalizations of a binomial model:

- Many periods
    - o Two periods: $C_0 = (1/R^2) (q^2C_{uu} + 2q(1-q)C_{ud} + (1-q)^2C_{dd})$
- Many states
    - o For the market to be complete, # assets must be at least the same as # states
    - o Otherwise, some assets will not be priced uniquely

From binomial model to Black-Scholes:

Fine the price of European call with exercise date T and price X

One period: $c = e^{-r\Delta t} (qC_u + (1-q)C_d)$

Two periods: $c = e^{-2r\Delta t} (q^2C_{uu} + 2q(1-q)C_{ud} + (1-q)^2C_{dd})$

n periods: $c_t = e^{-rn\Delta t} \Sigma_{j=0:n} [n! / (j!(n-j)!)q^j(1-q)^{n-j} \max(0, u^jd^{n-j} S_t-X)]$

(def) a is min # of up moves to make the option in-the-money, i.e. $\max(0, u^jd^{n-j} S_t-X)>0$ for $j \geq a$

Thus, $c_t = S_t\,[\Sigma_{j=a:n}\, n! \,/\, (j!(n-j)!)(e^{-r\Delta t}qu)^j(e^{-r\Delta t}(1-q)d)^{n-j}] - X\,[e^{-rn\Delta t}\,\Sigma_{j=a:n}\,[n! \,/\, (j!(n-j)!)q^j(1-q)^{n-j}]$
or $c_t = S_t\,\Phi_1(a,n,q^*) - X\,\Phi_2(a,n,q)$,
where $q^* = e^{-r\Delta t}qu$ (Exercise: prove that $e^{-r\Delta t}(1-q)d = 1-q^*$)
$\Phi_1(a,n,q)$ is cdf of binomial distribution as f(# successes, # trials, prob. of success)

Assume
- No transaction costs
- No SS restrictions
- Continuous-time trade
- Constant R_F and σ^2
- Log-normal distribution of prices
- Parameters matching the properties of the stock price distribution
  - Volatility: $u=e^{\sigma\Delta t}$, $d=1/u$ (Cox-Ross-Rubinstein, 1979)
  - Expected return: $p=(E[R_S]-d)/(u-d)$
  - When we move to risk-neutral distribution $q=(R_F-d)/(u-d)$, the volatility does not change! (*Girsanov's theorem*)

Then price of call: $c_t = S_t e^{-qT}\,N(d_1) - Xe^{-rT}\,N(d_2)$
price of put: $p = Xe^{-rT}\,N(-d_2) - Se^{-qT}\,N(-d_1)$
where $d_1 = [\ln(S/X) + T(r-q+\sigma^2/2)] \,/\, [\sigma\sqrt{T}]$, $d_2 = d_1 - \sigma\sqrt{T}$
q is cont. dividend yield

- In case of discrete dividends, their current value is subtracted from S
- In case of stochastic interest rates, use zero rate for T
- For a given price, σ is *implied volatility*
  - How does it differ for call and put with the same strike and exercise date? Use put-call parity!
- *Volatility smiles*: plot implied volatility as f(strike)
  - U-shape for foreign currency options: non-constant volatility and jumps
  - Decreasing for equity options: the company's leverage, "crashophobia"
- Option's delta: dC/dS, positive for call, negative for put
  - For call: $\Delta = e^{-qT}\,N(d_1)$, for put $\Delta = -e^{-qT}\,N(-d_1)$
- Other greeks (sensitivities to parameters):
  - Rho (risk-free rate), vega (volatility), theta (time)
- $SN(d_1)$ shows how much it costs to hedge the option position
- $N(d_2)$ is the probability that the option be exercised

Binomial model is more flexible:
- Three parameters (u, d, p) instead of two (μ and σ) in BS
- Can embed different possibilities between t and T
  - Time-varying volatility, dividends, etc.
- Can price exotic options (with path-dependent payoffs)

Black-Scholes applications:
- Valuation of embedded options
  - Convertible and callable bonds
- Valuation of real options
  - The firm's equity, investment projects
- Measuring default probability and credit risk

Lectures 5-6. General static model with discrete states

Objective: necessary conditions for the equilibrium under no arbitrage

Key concepts: arbitrage and law of single price
- Two assets with identical payoff must have same price
- Relative pricing

Larry Summers (JF, 85): “ketchup economy” with two groups of economists
- Some study ketchup market as a part of the whole economic system
- Others sit in the ketchup dept, with big salary and research on the ketchup market showing that two small bottles of ketchup should cost the same as one big one (with deviations within the transaction costs) *financial economists*!

General static model with discrete states:
- One period, two dates t=0,1
- K states (k=1,…,K): $\Omega=\{w_k\}$ with $\Pr(w_k)>0$ (same across agents)
- N assets (n=1,…,N) with current prices p_n and future payoffs $d_n = (d_{n1},\dots, d_{nK})'$ (Kx1)
- I agents (i=1,…,I) with utility U_i

The payoff matrix: $D=(d_1,\dots, d_K)$ (KxN)
The vector of current prices: $p=(p_1,\dots, p_N)$ known
The vector of future payoffs: $d=(d_1,\dots, d_N)$ uncertain
Consumption in t=1: $C_i = d'\theta_i + e_i$,
where $\theta_i = (\theta_{i,1},\dots, \theta_{i,N})'$ is agent i’s ptf (# of each asset),
e_i is stochastic endowment in t=1 (stochastic revenue on some fixed investment)

Optimization problem: $\max_{\theta i} E[U_i(C_i)]$ (1)
s.t. $p'\theta_i = 0$ (budget constraint: self-financing ptf)
$C_i = d'\theta_i + e_i$

The equilibrium: $(p^*, \{\theta^*_i\})$, if θ^*_i solves (1) with $p=p^*$ and $\Sigma_i\theta_i = 0$ (Nx1).
- The latter condition implies equilibrium in the asset market
- Automatically, in the goods market: $\Sigma_i c_i = \Sigma_i e_i$

Our objectives:
- Relation between current prices p*
- Optimal θ_i given p*
- Necessary conditions for $(p^*, \{\theta^*_i\})$ to be in the equilibrium

Assuming
- No transaction costs
- No short sales restrictions
- The presence of a non-satiable agent

Finding an equilibrium for a general problem is quite hard, unless impose additional structure (like quadratic UF). Therefore, we concentrate on NC

The role of assets:
- Suppose there are none. Then $c_i = e_i$, no choice!
- Adding assets, we enlarge the choice set. Should we add as many assets as possible?

(def) **arbitrage**: if there exists θ s.t. $\theta' p < 0$ & $D\theta \geq 0$
or $\theta' p = 0$ & $0 \neq D\theta \geq 0$

Equivalently: the system $0 \neq \begin{pmatrix} -p' \\ D \end{pmatrix}\theta \geq 0$ does not have a solution

Asset pricing in t=0: three approaches

Approach 1. **Pricing via replicating ptf**

(def) m is a **redundant asset** if exists α s.t. $d_m = \Sigma_n \alpha_n d_n$

- m's payoff is replicated in all states
- α is a **replicating ptf**

The current price of m must be: $p_m = \Sigma_n \alpha_n p_n$ (otherwise arbitrage)

Example A: $D = \begin{pmatrix} 1\,0\,1 \\ 2\,1\,4 \end{pmatrix}, p = \begin{pmatrix} 1.5 \\ 0.5 \\ ? \end{pmatrix} \cdot$, $p_3 = p_1 + 2p_2$

(def) the market is **complete** if Rank(D)=K

- In the complete market, one can find a replicating ptf for any additional asset
- Necessary condition: N≥K

Approach 2. **Pricing via state prices**

(def) **Arrow-Debreu security** (state-contingent claim): $d=e_k=(0,\ldots,1,\ldots,0)'$

Finding **state prices**:

For any m, $d_m = \Sigma_k d_{mk}e_k$ => $p_m = \Sigma_n d_{mk}S_k$ or P = D'S

In the complete market, state prices are unique and we can divide D into two blocks: $D=(D_1 \mid D_2)$. Then $S = (D_1')^{-1}P_1$

Example from the binomial model: $D = \begin{pmatrix} 120 & 1.03 \\ 90 & 1.03 \end{pmatrix}, p = \begin{pmatrix} 100 \\ 1 \end{pmatrix}, d_c = \begin{pmatrix} 10 \\ 0 \end{pmatrix}, p_c = ?$

Replicating ptf: $\alpha = D^{-1}d_c = \begin{pmatrix} 120 & 1.03 \\ 90 & 1.03 \end{pmatrix}^{-1}\begin{pmatrix} 10 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.33 \\ -29.1 \end{pmatrix}$. The option's price: $p_c = p'\alpha = 4.2$.

State prices: $S = D'^{-1}\, p = \begin{pmatrix} 120 & 90 \\ 1.03 & 1.03 \end{pmatrix}^{-1}\begin{pmatrix} 100 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.42 \\ 0.55 \end{pmatrix}$. The option's price: $p_c = S'd_c = 4.2$.

In the incomplete market, state prices are not unique: P = D'S implies N restrictions on K unknowns

Example A (cont.): add one more state $D = \begin{pmatrix} 1\,0 \\ 2\,1 \\ 1\,2 \end{pmatrix}, p = \begin{pmatrix} 1.5 \\ 0.5 \end{pmatrix}$

The system P = D'S: $1.5 = S_1 + 2S_2 + S_3$

$0.5 = S_2 + 2S_3$

Solution: $S_3 = \lambda$, $S_2 = 0.5 - 2\lambda$, $S_1 = 0.5 + 3\lambda$

Since *S_k>0 in absence of arbitrage*, $0 < \lambda < ¼$

Approach 3. **Pricing via pricing kernel**

Solving optimization problem: $\max_{\theta i} E[U_i(d'\theta_i + e_i)]$ s.t. $p'\theta_i = 0$

Lagrangean: $L = E[U_i(d'\theta_i + e_i)] - \lambda_i(p'\theta_i)$
FOC: $E[U'_i(C_i)]d = \lambda_i p$
Since $\lambda>0$ for a non-satiated agent, we obtain the necessary condition for the equilibrium:

$$p = E[\pi d] \text{ or } 1 = E[\pi R] \qquad (*)$$

where $\pi \equiv U'_i(C_i) / \lambda_i$: is pricing kernel / IMRS / SDF
This is the fundamental pricing equation

- The current price is a positive and linear function of future payoffs (d>0 => p>0)

Rewrite (*) as $p = \Sigma_k \Pr(w_k)\, \pi_k\, d_k = \Sigma_k S_k d_k$
Thus, $S_k = \Pr(w_k)\, \pi_k$

- $S_k > 0 \Leftrightarrow \pi_k > 0$ when there is no arbitrage!
- In the complete market, the PK is unique!

The first fundamental theorem of finance:
There is no arbitrage ⇔ there exists PK $\pi > 0$ s.t. $p = E[\pi d]$

Stiemke's lemma: either the system $0 \neq \begin{pmatrix} -p' \\ D \end{pmatrix} \theta \geq 0$ has a solution,
or the system $D'S = P$ has a solution $S_k>0$, $k=1,\ldots,K$

- Positive state prices imply no arbitrage

Is PK unique?
□ Suppose there are two PKs: π and π^*
(*) implies that for any θ, $p'\theta = E[\pi d'\theta]$
$p'\theta = E[\pi^* d'\theta]$
Thus, $0 = E[(\pi - \pi^*) d'\theta]$
If the market is complete, then for every state k we can choose θ s.t. $d'\theta = e_k$ (Arrow-Debreu security).
Then $\pi_k = \pi^*_k$ for all k w.p. 1and the PK is unique. ■

The second fundamental theorem of finance:
The market is complete ⇔ the PK is unique

Link to the risk-neutral probabilities:
Assume that there is an asset with strictly positive payoff (numeraire): e.g. N, s.t. $d_{Nk}>0$, $k=1,\ldots,K$
Since both π and d_N are positive w.p. 1, $p_N = E[\pi d_N] > 0$.
Then we can rewrite (*) as $p = \Sigma_k \Pr(w_k)\, \pi_k\, d_k = p_N \Sigma_k [(1/p_N) \Pr(w_k)\, \pi_k\, d_{Nk}]\, d_k / d_{Nk}$
Introduce the new prob distribution $Q(w_k) = \Pr(w_k)\, \pi_k\, d_{Nk} / p_N = S_k d_{Nk} / \Sigma_k S_k d_{Nk}$

- $Q(w_k) > 0$
- $\Sigma_k Q(w_k) = 1$
- P and Q are equivalent prob distributions: $\Pr(w_k) > 0 \Leftrightarrow Q(w_k) > 0$

Then (*) becomes $p / p_N = E^Q[d / d_N]$

- After scaling down by numeraire and transformation of prob distribution, the asset prices become expectations of their payoffs!

If there is a risk-free asset: $p = (1/R_F)\, E^Q[d]$ and $Q(w_k) = S_k / \Sigma_k S_k$

- All assets have same exp return, when calculated wrt Q
- Risk-neutral probabilities are just normalized state prices
- Q is called equivalent risk-neutral prob distribution

If there is no risk-free asset: $p = E(\pi)\, E[\pi/E(\pi)\, d] = E(\pi)\, E^Q[d]$ and $Q(w_k) = \Pr(w_k)\, \pi_k / E(\pi_k)$

- Q is called equivalent martingale prob distribution

Lecture 7. Modeling investor preferences

Modeling choice under uncertainty: maximization of the **expected utility function** E[U(W)]

Nonsatiation: U'(W) > 0

Risk attitude: aversion / neutrality / loving
- Concave / linear / convex UF
- U(E[W]) > / = / < E[U(W)]
- For any W, risk premium $\pi > 0$

(def) (insurance) **risk premium** π: $E[U(W)] = U(E[W]-\pi)$
- Difference between the expected result and certainty equivalent

Arrow-Pratt measure of local risk premium (**absolute risk aversion**): ARA = -U''(W) / U'(W)
- Invariant to strictly positive affine transformation of the cardinal UF
- Measures only *local* premium

Motivation: add an actuarily neutral lottery Z to the current wealth W
- Z has zero expectation and variance σ^2

The risk premium compensates for this risk: $E[U(W+Z)] = U(W+E(Z)- \pi)$
From the Taylor's series expansion: $\pi \approx -(\sigma^2/2)\, U''(W) / U'(W)$

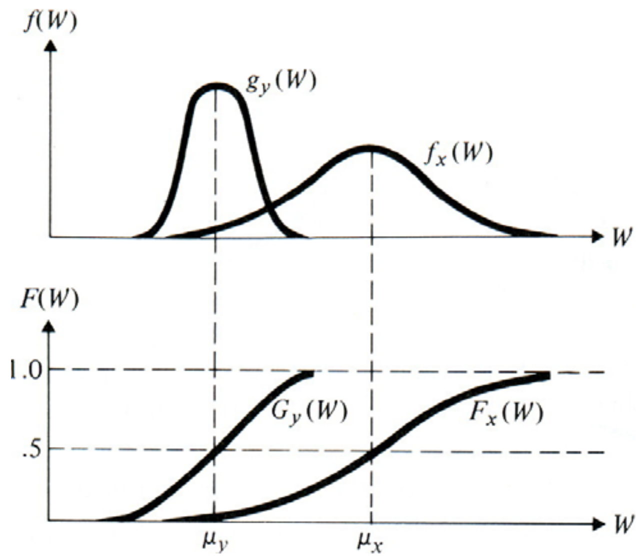
Relative risk aversion: RRA = -W U''(W) / U'(W)

Examples of UFs:

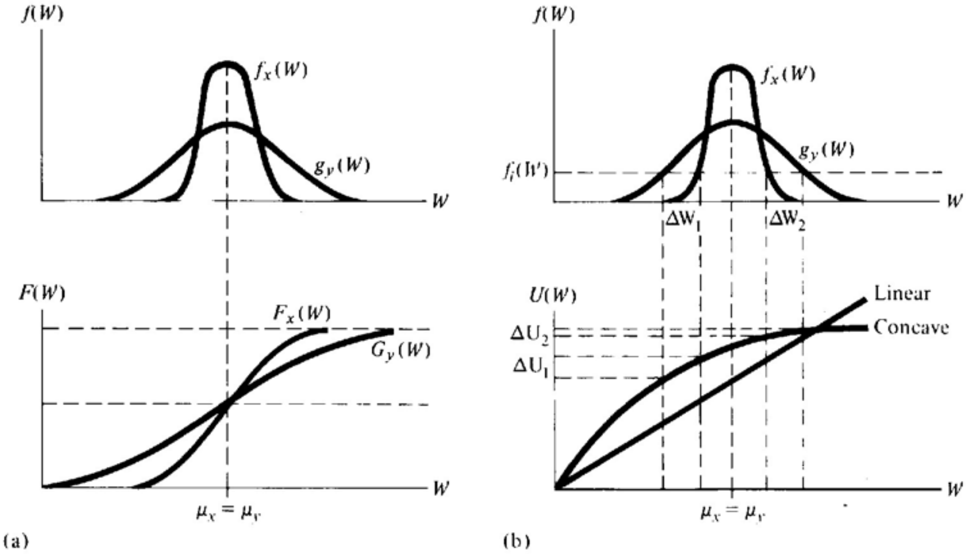
	U	U'	U''	ARA	RRA
Power	$(1/a)W^a$	W^{a-1}	$(a-1)W^{a-2}$	$(1-a)/W$	1-a
Log	ln(W)	W^{-1}	$-W^{-2}$	1/W	1
Exponential	$(1/a)e^{-aW}$	$-e^{-aW}$	ae^{-aW}	a	aW
Quadratic	$W-(a/2)W^2$	1-aW	-a	a/(1-aW)	aW/(1-aW)

- Constant ARA (RRA) implies a constant amount (fraction) invested in the risky asset
  - The result holds only in the case of a singly risky asset (or fixed ptf of risky assets)!
- Empirically: DARA (the risky asset is a normal good), CRRA≈2 or even DRRA
  - Closest: power UF with a=-1
- Quadratic UF: least attractive
  - Satiation: decreasing after the peak
  - Both ARA and RRA increasing in wealth

First-order stochastic dominance:
An individual receives greater wealth from X in every ordered state of nature: $X »_1 Y$ iff
- Any (non-satiable) individual with increasing UF prefers X to Y
- $F_X(W) \leq G_Y(W)$, for any W
  - $\neq$ for strict dominance
- $X =_d Y + \xi$, such that $\xi \geq 0$
  - '$=_d$': equal in distribution



Second-order stochastic dominance:
$X \gg_2 Y$ iff
- Any (risk-averse) individual with concave UF prefers X to Y (non-satiable for strict dominance)
- $\int^W [G_Y(W)-F_X(W)]dW \geq 0$ for any W
  - $\neq$ for strict dominance
- $Y =_d X + \xi + \varepsilon$, such that $\xi \leq 0$ and $E[\varepsilon|X+\xi]=0$
  - It is often assumed that E[X]=E[Y]: $Y =_d X + \varepsilon$, such that $E[\varepsilon|X]=0$ (*mean-preserving spread*)



Advantages of stochastic dominance concepts
- Based on exp. utility theory, for general UF
- Applies to an arbitrary distr. function, uses the whole distribution
- Second-order SD stronger, can be used to exclude dominated assets (portfolios)

Motivation for the mean-variance preferences
Taylor's series expansion for UF:
$U(W) = U(E[W]) + U'(E[W])(W-E[W]) + \frac{1}{2} U''(E[W])(W-E[W])^2 + R_3$
Assuming convergence and putting the sign of expectations under the sum,
$EU(W) = U(E[W]) + \frac{1}{2} U''(E[W])\sigma^2_W + E[R_3]$, where $E[R_3] = \Sigma_{n>2} (1/n!)\, U^{(n)}(E)\, m_n(E[W])$
$E[R_3]=0$ and EU is a function of the first two moments of the distribution if we assume
- *Quadratic utility*: $U^{(n)} = 0$, n>2
- *Normal distribution*: $m_n = f(m_1,m_2)$

Lectures 8-9. Portfolio theory

Return statistics
- Simple returns R vs log returns $R^*=\ln(R)$
  - Time aggregation: holding-period return = sum (log-returns)
  - Cross-aggregation: ptf return = wtd avg (simple returns)
- Time aggregation of the portfolio return: t=1,…,T
  - Holding-period return: $R_{p,[1:T]} = \Sigma_k w_{k,0} R_{k,[1:T]}$
    - Keep the portfolio till the end without rebalancing
  - Fixed-weight return: $R_{p,t} = \Sigma_k w_{k,0} R_{k,t}$
    - Assume regular rebalancing to sustain fixed weights
- High to low frequency assuming iid log returns
  - $\sqrt{T}$ rule for sigma

Definitions
- Feasible set: $\{w: w'l=1\}$
- Minimum variance frontier: $\{w_p$: for any q s.t. $m_p=m_q$, $\sigma^2_p \le \sigma^2_q\}$
- Efficient frontier: $\{w_p$: for any q s.t. $\sigma^2_p=\sigma^2_q$, $m_p \ge m_q\}$
    - Includes optimal portfolios that could be chosen by risk-averse investors

Graphical analysis in the MV space

Identify part of the frontier where short sales are required.

- Risk-free asset and risky asset
    - Straight line: $\sigma^2_p = w^2\ \sigma^2_S$
- Two risky assets with different correlation
    - Hyperbola: $\sigma^2_p = w^2\ \sigma^2_1 + 2w(1-w)\ \rho\sigma_1\sigma_2 + (1-w)^2\ \sigma^2_2$
- N risky assets: $R_p = w'R$, $\sigma^2_p = w'\Sigma w$
- Risk-free asset and N risky assets:
    - Back to the straight line!
    - Unique efficient portfolio of risky assets (Tobin, 1958)

Diversification: lower dispersion is achieved with
- Lower correlation
- Larger # assets

Demonstrate that indifference curves of a risk-averse investor are increasing and convex.

Example: assume same dispersion and covariance and equal weights
- $\sigma^2_p = \sigma^2/N + (N-1)/N\ \sigma_{i,j}$
- As $N\to\infty$, the portfolio variance is determined by covariance (*systematic risk*)

Markowitz model: find efficient portfolio with given utility function
Assumptions:
- Concave UF, $E[U] = U(\mu, \sigma^2)$ with $U_1\ge 0$, $U_2\le 0$
- No taxes, no transaction costs, no short sales restrictions
- Investors are price-takers
- N liquid and perfectly divisible assets, $R \sim N(\mu, \Sigma)$ where Σ is non-singular

One-period optimization problem: $\mathrm{Min}_w\ \tfrac{1}{2}w'\Sigma w$ s.t. $w'\mu=m$ and $w'l=1$
Lagrangean: $\tfrac{1}{2}w'\Sigma w + \lambda(m-w'\mu) + \gamma(1-w'l)$

FOC:	$\Sigma w - \lambda\mu - \gamma l = 0$	(1)	
	$w'\mu=m$	(2)	
	$w'l=1$	(3)	
(1) =>	$w^* = \Sigma^{-1}(\lambda\mu + \gamma l)$	(4)	
l'*(4) =>	$1 = \lambda l'\Sigma^{-1}\mu + \gamma l'\Sigma^{-1}l$	(5)	$1 = \lambda B + \gamma C$
μ'*(4) =>	$m = \lambda\ \mu'\Sigma^{-1}\mu + \gamma\ \mu'\Sigma^{-1}l$	(6)	$m = \lambda A + \gamma B$

$D = AC-B^2>0$, since $(B\mu-Al)'\ \Sigma^{-1}\ (B\mu-Al)' = A(AC-B^2)>0$ (Σ^{-1} is positive definite)

$\lambda = (mC-B)/D$, $\gamma = (A-mB)/D$

Unique optimal portfolio with the given exp return m:

$$w^* = (1/D)\ [(mC-B)\Sigma^{-1}\mu + (A-mB)\Sigma^{-1}l)$$
$$= (1/D)\ [A\Sigma^{-1}l - B\Sigma^{-1}\mu) + (1/D)\ [C\Sigma^{-1}\mu - B\Sigma^{-1}l)m = g + hm = (1-m)g + m(g+h)$$

- g is a ptf with zero exp return, g+h has exp return of unity (h is zero-investment ptf)
- Any MVE ptf is LC of g and g+h

$\sigma^2_p = w'\Sigma w = (1/D)\,[Cm^2 - 2Bm + A]$
- Parabola in (μ, σ^2) space
- Hyperbola in (μ, σ) space
  - Global minimum variance ptf: $m_G=B/C$, $\sigma^2_G=1/C$, $w_G= \Sigma^{-1}l/C$
  - Asymptotes $B/C \pm \text{sqrt}(D/C)\sigma$

Two-fund separation theorem
Let x_A and x_B be two minimum variance portfolios with different mean returns. Then
a. Every min-var ptf x_C is a linear combination of x_A and x_B
b. Every linear combination of x_A and x_B: $\alpha x_A + (1-\alpha)x_B$ is a min-var ptf
c. If x_A and x_B are efficient portfolios, then any convex linear combination of x_A and x_B is efficient portfolio

□a: Choose α s.t. $m_C = \alpha m_A + (1-\alpha)m_B$.
By definition, $x_C = g + h[\alpha m_A + (1-\alpha)m_B] = \alpha[g + hm_A] + (1-\alpha)[g + hm_B]$. ■
□c: Same with $0\le\alpha\le1$ implying $m_A \le m_C \le m_B$. ■

Implications:
- Every investor can achieve the optimal risk-return trade-off with any two MVE mutual funds
- If every investor owns MVE ptf, then market ptf is also efficient
  - Later used to derive CAPM

Markowitz model with a risk-free asset
- R_F: T-bill or short-term deposit rate
- Previous model is applicable if
  - There is no R_F
  - Maximization of *real* return, when there is no perfect hedge for inflation
- Can't use the previous set-up, since Σ will become singular
- w: vector of portfolio weights of risky assets
- $R_p = w'R + (1-w'l)R_f = R_f + w'(R- R_f l)$

Optimization problem: $\text{Min}_w\ ½w'\Sigma w$ s.t. $R_f + w'(\mu - R_f l) = m$
Lagrangean: $½w'\Sigma w + \lambda(m- R_f - w'(\mu - R_f l))$
FOC: $\Sigma w = \lambda(\mu - R_f l)$
$w'(\mu - R_f l) = m - R_f$
It follows that $w^* = \lambda\, \Sigma^{-1}(\mu - R_f l)$
$\lambda = (m - R_f) / H$,
where $H = (\mu - R_f l)'\Sigma^{-1}(\mu - R_f l) = A - 2BR_f + CR_f^2 > 0$

Unique optimal portfolio with the given exp return m:
$w^* = (1/H)\, \Sigma^{-1}(\mu-R_f l)\, (m-R_f)$
$\sigma^2_p = w'\Sigma w = (1/H)\, (m-R_f)^2$
- MV frontier: two straight lines $m = R_f \pm \sigma\sqrt{H}$
  - Slope: Sharpe coefficient $S_i = (E[R_i]-R_f)/\sigma_i$
- Any efficient portfolio is LC of R_f and (unique) tangent portfolio T
  - Standard assumption: $R_f < B/C$ (R_f lower than global min-var ptf)

Characterizing the **tangent portfolio** T:
- At T, invest only in risky assets: $l'w^* = 1$. This implies $(B-R_fC)(m_T-R_f) = A - 2BR_f + CR_f^2$
- $m_T = (A-R_fB)/(B-R_fC)$
- $w_T = \Sigma^{-1}(\mu-R_f l)/(B-R_fC)$

Limitations of the standard Markowitz model

- MV preferences (motivated by quadratic utility or normal distribution)
- Same R_f for borrowing and investing
    - If not, the efficient frontier will consist of three parts
- No frictions
    - Taxes: if no distinction between capital gain and dividend taxes, scale returns
    - SS and other restrictions: solve the model with linear inequality restrictions using Kuhn-Tucker conditions
- Static problem: tactical allocation
    - Intertemporal continuous-time ptf theory by Merton (1971, 1973): strategic allocation
    - Three-fund separation: R_f, tangent ptf, and hedging ptf (max corr with the state variable)

Generalized Markowitz model with fixed liabilities and futures

Examples of fixed liabilities:

- Fixed investments
    - Currency risk for the exporter who receives income, say, in 3 months
    - Illiquid assets (real estate)
- Part of future income depends on risk factors
    - Labor income
    - Pension fund liabilities protected from inflation

Assume

- RF: risk-free rate (different notation than before!)
- R_S: return on N risky assets
- $X_{(Kx1)}$: absolute exposure to K risk factors with return $R_X = (X_1 - X_0) / X_0$
- $q = X / W$: relative exposure (per unit of wealth)
- $\lambda_{(Lx1)}$: (zero-investment) futures positions with pseudo-return $R_F = (F_1 - F_0) / F_0$
- Σ_{SS}, Σ_{SF}, Σ_{SX},…: covariance matrices
- $R_p = RF + w'(R_S - RF l) + \lambda' R_F + q' R_X$
    - $m_p = RF + w'(\mu_S - RF l) + \lambda' \mu_F + q' \mu_X$
    - $\sigma_p^2 = (w', \lambda') \begin{bmatrix} \Sigma_{SS} & \Sigma_{SF} \\ \Sigma_{FS} & \Sigma_{FF} \end{bmatrix} \begin{pmatrix} w \\ \lambda \end{pmatrix} + 2(w', \lambda') \begin{bmatrix} \Sigma_{SX} \\ \Sigma_{FX} \end{bmatrix} q + q' \Sigma_{XX} q$

Optimization problem: $\mathrm{Max}_w\ \gamma m_p - \frac{1}{2} \sigma^2_p$

FOC: $\gamma(\mu_S - RF l) - \Sigma_{SS} w - \Sigma_{SF} \lambda - \Sigma_{SX} q = 0$

$\gamma \mu_F - \Sigma_{FF} \lambda - \Sigma_{FS} w - \Sigma_{FX} q = 0$

$$\begin{pmatrix} w^* \\ \lambda^* \end{pmatrix} = -\begin{bmatrix} \Sigma_{SS} & \Sigma_{SF} \\ \Sigma_{FS} & \Sigma_{FF} \end{bmatrix}^{-1} \begin{bmatrix} \Sigma_{SX} \\ \Sigma_{FX} \end{bmatrix} q + \gamma \begin{bmatrix} \Sigma_{SS} & \Sigma_{SF} \\ \Sigma_{FS} & \Sigma_{FF} \end{bmatrix}^{-1} \begin{bmatrix} \mu_S - RF l \\ \mu_F \end{bmatrix}$$

Special cases:

- K=0, L=0: standard model $w^* = \gamma \Sigma_{SS}^{-1} (\mu_S - R_f l)$
- N=0, use only futures to hedge risks: $\lambda^* = -\Sigma_{FF}^{-1} \Sigma_{FX} q + \gamma \Sigma_{FF}^{-1} \mu_F$
    - Hedging + speculative components
    - Min dispersion: $\gamma = 0$
    - Ideal hedge: $F_1 = X_1$ and $\lambda^* = -q$
- L=0: $w^* = -\Sigma_{SS}^{-1} \Sigma_{SX} q + \gamma \Sigma_{SS}^{-1} (\mu_S - R_f l)$
    - For K=1, increase share of assets negatively correlated with X

Main application problem: estimation of the inputs

- Choice of the length of the period
  - Small T:
    - Imprecise estimation of the mean return
    - For N>>T, possible to construct portfolios of risky assets close to R_f => excessive leverage, unstable solutions and high reallocation costs
  - Large T:
    - Ignoring time variation in (co)variances
- Reduction of the # parameters in the covariance matrix
  - Historical correlations: ½N(N-1) for N assets, too many (~11,000 for N=150)
  - Multi-index model: $R = \alpha + \beta F + \varepsilon$
    - F: factors or industry indices
    - # parameters reduces to N(2+K)+2K
  - Average correlations (say, for companies from the same industry)
  - Mixed models
- Two-step procedure
  - First: choose optimal allocation on the level of asset classes
  - Second: optimize within the classes

Lectures 10-11. CAPM

Another rewrite of the **Markowitz model without a risk-free asset**:

$\text{Max}_w\ \gamma\mu'w - \frac{1}{2}w'\Sigma w - \lambda(w'l - 1)$

FOC: $\gamma\mu - \Sigma w - \lambda l = 0$

$w'l = 1$

It follows that $w^* = \Sigma^{-1}(\gamma\mu - \lambda l) = \gamma\Sigma^{-1}[\mu-(B/C)l] + \Sigma^{-1}l / C$

$1 = \gamma B - \lambda C,\ \lambda = (\gamma B - 1)/C$

Relation between γ and m: $\mu'w=m$ implies that $\gamma = (mC-B)/D = (mC-B)/(AC-B^2)$

Rewrite $w^* = \gamma\Sigma^{-1}[\mu-(B/C)l] + \gamma\,[(AC-B^2)/(mC-B)]\,\Sigma^{-1}l / C = \gamma\Sigma^{-1}\,(\mu - [(A-mB)/(B-mC)]l)$

or $w^* = \gamma\Sigma^{-1}\,(\mu - m_Z l)$,

where $m_Z = (A-mB)/(B-mC)<B/C$ is exp return of the orthogonal portfolio: $cov(R_P, R_Z)=0$

□ $w'_P\Sigma w_Z = \gamma\,(\mu - m_Z l)'\,\Sigma^{-1}\,\Sigma\,w_Z = 0$ ■

Graphically, for MVE ptf P:

- Z is always on the inefficient part
- In (μ, σ) space, m_Z is the intersection point of the tangent line to hyperbola at P and vertical axis
- In (μ, σ^2) space, m_Z is the intersection point of the line coming through P & GMV and vertical axis
- There is no orthogonal ptf for GMV ptf
- If there is R_F, it is an orthogonal ptf for any asset: $w^* = \gamma\Sigma^{-1}\,(\mu - R_F l)$

Deriving beta relation

Let P be a MVE ptf and Q – arbitrary ptf.

$cov_{PQ} = w'_P\Sigma w_Q = \gamma_P\,(\mu - m_Z l)'\,\Sigma^{-1}\,\Sigma\,w_Q = \gamma_P\,(m_Q - m_Z)$ (1)

For Q=P, $var_P = \gamma_P(m_P - m_Z)$ (2)

Define $\beta_{QP} = cov_{PQ} / \sigma^2_p$. Excluding γ_P from (1) and (2), we obtain

$m_Q = m_Z + \beta_{QP}(m_P - m_Z) = (1-\beta_{QP})m_Z + \beta_{QP}m_P$ (3)

Thus, exp return of *any* asset is LC of exp returns of a MVE ptf and its orthogonal ptf !

From Markowitz to CAPM:

- Same assumptions adding homogenous expectations

It follows that

- All investors work with the same MV frontier
- All investors hold a MVE ptf
- Since the market ptf (value-wtd index) is convex LC of MVE ptf's, it is also MVE.

Writing the beta relation for the market ptf, we obtain **zero-beta CAPM** by Black (1972):

$$E[R_i] = E[R_{MZ}] + \beta_{iM} (E[R_M] - E[R_{MZ}])$$

When there is R_F, it serves as an asset with zero beta.
Standard CAPM by Sharpe (1964), Lintner (1965), and Mossin (1965):

$$E[R_i] = R_F + \beta_{iM} (E[R_M] - R_F)$$

Standard (informal) derivation of the CAPM:
□ Every agent i holds optimal ptf $w_i^* = \gamma_i\Sigma^{-1} (\mu - m_Z l)$.
Consider LC of w_i^* with weights proportional to the agents' wealth $W_{0,i}/W_{0,M}$:
$w_M = \gamma_M\Sigma^{-1} (\mu - m_Z l)$ where γ_M is wealth-wtd γ or $\Sigma w_M = \gamma_M (\mu - m_Z l)$
Multiply this by e_i' and by w_M':
$\sigma_{i,M} = e_i'\Sigma w_M^* = \gamma_M (E[R_i] - m_Z)$
$\sigma^2_M = w_M'\Sigma w_M^* = \gamma_M (w'_M \mu - m_Z) = \gamma_M (E[R_M] - m_Z)$
Excluding γ_M, we obtain CAPM equation. ■

Graphically

- Capital market line in (μ, σ) space: $E[R_i] = R_F + \text{Sharpe}_M \sigma_i$
- Security market line in (μ, β) space: $E[R_i] = R_F + \beta_i \lambda_M$, where λ_M is the market premium for risk

Interpretation:

- Riskier assets earn a risk premium, since risk-averse investors are only willing to take on risk if they are compensated with expected return.
- Covariance determines the risk premium. An asset can have high variance but small covariance. However, the individual risk can be diversified away. Market only prices the non-diversifiable part of risk.
- The assumptions are unrealistic. Empirically "beta is dead." However, we use CAPM, since it is a basic and robust model.

Applications of the CAPM:

- Portfolio management:
    - Every investor holds same ptf of risky assets, which is the market ptf. No need in optimization – any optimal ptf is a combination of the market ptf and R_F
    - If the market premium for risk is positive, then every risky asset has a positive share in the market ptf
- Risk management:
    - Implies TS regression $R_{i,t} - R_F = \beta_i (R_{M,t} - R_F) + \varepsilon_{i,t}$
    - Total risk is a sum of the systematic and idiosyncratic risk: $\sigma^2_i = \beta^2_i \sigma^2_M + \sigma^2_\varepsilon$
- Asset pricing / finding the appropriate discount rate or cost of capital:
    - Only systematic risk is priced
    - $P_0 = E[P_1] / (1 + R_F + \beta_i \lambda_M)$
- Performance evaluation:
    - In the TS regression $R_i - R_F = \alpha_i + \beta_i (R_M - R_F) + \varepsilon_i$
    - Jensen's alpha $\alpha_i > 0$ implies stock-picking ability

Questions for understanding:

- Can MV investor include an asset with exp return below R_F with positive weight in his ptf?
    - Example: R_F=5%, two risky assets with exp returns of 0% and 20% and -99% correlation. The ptf of 50-50 in risky assets provides exp return of 10% and almost no risk.
- Is it possible in the CAPM that the equilibrium exp return is below R_F?
    - Yes, in case of negative covariance with the market.
- What is the nonsystematic risk for the assets on the CML?
    - Zero, since $\sigma^2_p = \beta^2_p\, \sigma^2_M + \sigma^2_\varepsilon \geq \beta^2_p\, \sigma^2_M$
    - Increase in the nonsystematic risk implies shift to the left in (μ, σ) space
- What is the correlation between the assets on the CML and market ptf?
    - CAPM equation: $E[R_i] = R_F + (E[R_M] - R_F)\rho_{i,M}\, \sigma_i / \sigma_M = = R_F + Sharpe_M\, \sigma_i\, \rho_{i,M}$
    - The highest slope equal to the Sharpe ratio of market ptf is achieved when ρ=1. Intuitively, all assets on the CML consist of R_M and R_F and therefore are perfectly correlated with market ptf.
    - Graphically: ρ is the slope of the line coming through R_F and given asset, relative to $Sharpe_M$. Example of ρ<0: car insurance.

Extensions of the standard CAPM:

- No R_F: zero-beta CAPM
    - Investors hold different risky ptf's
    - CML is the upper half of hyperbola in (μ, σ) space
- No risk-free borrowing:
    - CML will consist of two parts: straight line [R_F, R_M] and hyperbola above R_M
- Different risk-free borrowing and lending rates:
    - CML will consist of three parts: straight line [R_F, R_{T1}], hyperbola including R_M, and straight line above R_{T2}
    - The market portfolio will be between the two tangent ptf's

Necessary conditions for the efficiency of the market ptf:

- All investors have same inv opportunity set restricted to the securities in the market index
- No short-selling restrictions
- No taxes

Otherwise: market ptf will not belong to the efficient set

What is required for the market ptf to be efficient ?

When the market ptf is inefficient:

- Heterogeneous expectations:
    - Fama (1976): market ptf will efficient for avg-wtd (μ, σ), but inefficient for every individual investor
- Short-sale restrictions:
    - The MVE frontier will shift to the right and consist of multiple segments with changing compositions of the MVE ptf's
    - LC of MVE ptf's will not be MVE!
- Taxes:
    - Each investor will have his own after-tax opportunity set
- Alternative assets: human capital, foreign assets
    - The local part of the global optimal ptf will not necessarily be MVE wrt local opportunity set
- No MV preferences or normal distribution
- Inefficient pricing by the market due to irrational behavior

- o Example: investors may under- and overreact to certain types of news, overvalue companies that are currently highly profitable
- o Then cap-wtf ptf will overweight expensive securities

Conclusion: all investors should hold MVE ptf's for market ptf to be efficient!

Empirical tests of the CAPM: usually for portfolios to reduce noise

- Time series tests:
  - o Standard CAPM: $R_i - R_F = \alpha_i + \beta_i (R_M - R_F) + \varepsilon_i$
    - H_0: $\alpha_i = 0$ for every i=1,…,N; rejected

Gibbons, Ross, and Shanken (1989) test with finite-sample correction:

$$\left(\frac{T-N-1}{N}\right)\frac{\hat{\alpha}'\hat{\Sigma}^{-1}\hat{\alpha}}{1+\hat{S}_m^2} \sim F_{N,T-N-1}$$

- α: the estimated vector of alphas
- Σ: the estimated covariance matrix of residuals
- S_q: the Sharpe ratio of the tangency portfolio estimated ex post (based on N assets and T observations)

  - o Zero-beta CAPM: $R_i = \alpha_i + \beta_i R_M + \varepsilon_i$
    - H_0: exists R_Z s.t. $\alpha_i = (1- \beta_i)R_Z$ for every *i*; rejected
- Fama-McBeth methodology:
  - o First: estimate betas
  - o Second: CS regression $R_i = \gamma_0 + \gamma_1\beta_i + \gamma_2 x_i + \varepsilon_i$
    - H_0: $\gamma_0 = R_F$, $\gamma_1 = E[R_M] - R_F$, $\gamma_2 = 0$
    - Results: $\gamma_0 > R_F$, $\gamma_1 < E[R_M] - R_F$ (not very significant), $\gamma_2 \neq 0$ "*beta is dead*"
- Caveats:
  - o Joint hypothesis problem: testing CAPM and market efficiency at the same time
  - o Roll's critique: we can't observe the true market ptf
  - o Tests assume that means and variances are constant over time
    - CAPM may hold conditionally, but fail unconditionally
  - o Anomalies are concentrated among small, illiquid stocks
  - o Data mining: since late 70s "fishing license" for anomalies

Lectures 12-13. Multi-factor models

APT (arbitrage pricing theory)
Main idea: no arbitrage in equilibrium
(def) impossible to construct self-financed ptf (w'l=0) with $0 \neq R_P \geq 0$

Assumptions:

- Perfect competition (price-takers)
- No taxes, no transaction costs, no short sales restrictions
- Homogeneous expectations about the DGP: K-factor model

$$R_i = E[R_i] + \beta'_i F + \varepsilon_i \qquad (1)$$

or (in vector form) $R = E[R] + B F + \varepsilon$

where errors are white noise,
factors have zero expectation, orthogonal to each other and to the errors

- There is at least one non-satiated agent (to exclude arbitrage opportunities)

Deriving the APT formula for the exp return:

□ RHS of (1) is like return on the portfolio of R_F (with weight $E[R_i]/R_F$) and factors (with weights $\beta_{i,k}$). By the law of single price, both ptf's should have the same price:

$P(R_i) = E[R_i]\, P(1) + \beta'_i\, P(F) + P(\varepsilon_i)$

The **exact APT**: $\varepsilon=0$

Since $P(R_i)=1$ and $P(1)=1/R_F$, $E[R_i] = R_F + \beta'_i\, [-R_F\, P(F)]$ or

$$E[R_i] = R_F + \beta'_i\, \lambda$$

The expected return is a linear function of factor betas!

λ_k: factor *k*'s risk premium, excess return of the factor-mimicking ptf (with unit exposure to factor *k* and zero exposure to other factors) ■

The **approximate APT**: $P(\varepsilon_i)=0$

- For well-diversified portfolios: as $N\to\infty$, $var(\varepsilon)\to 0$
- $R^2\to 1$

Otherwise: $P(\varepsilon_i)$ can get away from zero even for small ε and APT relation is broken

Graphically

- SML: $E[R_i] = R_F + \beta_i`\, \lambda$, in K-dimensional plane
- Proof that SML is a straight line for the case of one factor:
  - If A, B, and C are not on the same line, then arbitrage is possible with exp return of m_1-m_2 (the difference between the intersection points of lines AB and AC with the vertical axis)

Interpretation:

- More general model (multiple factors) based on weaker assumptions
  - If single market factor, back to the CAPM equation
- Key principle: absence of arbitrage
  - In practice, pure arbitrage is hardly possible due to the basis risk and estimation error
- CAPM vs APT:
  - If $\lambda_k = x_k\, (E[R_M] - R_F)$, back to the CAPM with $\beta_{i,M} = \Sigma_k \beta_{i,k}\, x_k$

Pros and cons of APT compared to CAPM:

- No need for the assumption of quadratic utility or normal return distribution
- No need for the market ptf
- Can be estimated with a subset of the assets
- Easy to generalize for multiple periods

- Approximate APT is not testable
- In practice need to estimate DGP
- The factors and their # are unknown

Applications of the APT:

- Asset allocation:
  - (K+1)-fund separation
- Risk management:
  - Decomposition of risk: $var(w'R) = var(w'BF) + var(w'\varepsilon) = w'B\, \Sigma\, B'w + w'diag(\sigma^2_\varepsilon)w$
  - Hedging: $w'R - w'E[R] = w'BF + w'\varepsilon$, choose w to achieve desired factor exposure $w'B=a$
    - Example: $B' = (0.1\ -0.2\ 0.4)$, then $w'=(2/3,\ 1/3,\ 0)$ eliminates systematic risks
- Asset pricing / finding the appropriate discount rate or cost of capital
- Performance evaluation:
  - In the TS regression $R_i - R_F = \alpha_i + \Sigma_k\, \beta_{k,i}\, F_k + \varepsilon_i$

What is more important: Jensen's alpha or Sharpe ratio?

Empirical estimation of the APT factors:

- Statistical:
  - Factor analysis or principal components

- Macroeconomic: business cycles, confidence, etc.
    - Inflation, ind and consumption growth, oil prices, default and term spreads
- Firm-specific: proxies for the costs of fin distress
    - Size (market cap), book-to-market, momentum, liquidity, fin leverage, P/E, D/P
    - Fama-French three-factor model: market (R_M-R_F), HML (BE/ME), SMB (cap)
    - Carhart's four-factor model: add one-year momentum factor

Another extension: **intertemporal (dynamic) model**

- Intertemporal portfolio choice: Merton's continuous-time model
    - Assume that prices follow Ito processes $dP_i/P_i = \alpha_i(x, t)dt + \sigma_i(x, t)dW_t$
        - Simplest case: lognormal distribution when α_i and σ_i are constants
    - The cond mean and variance are functions of the state variable(s) x
    - Assume time-separable but not state-separable utility
    - Under certain conditions: two-fund separation and CAPM
        - Log utility
        - All asset returns are uncorrelated with dx
        - All assets have return distributions independent of x
    - In general: (m+2)-fund separation and multi-factor ICAPM
        - Benchmark portfolios: R_F, tangency ptf, and m hedging portfolios
        - Hedging portfolios have max corr with the state variables that represent shifts in the investment opportunity set and tastes (U depends on x)
- Contributions:
    - Generalizing static mean-variance theory
        - Considering both the consumption and ptf selection over time
        - Dropping the quadratic utility assumption
    - More realistic and analytically tractable
    - Strategic asset allocation:
        - In contrast to tactical (static) asset allocation, which is OK only in case of const inv opportunity set and liabilities proportional to inv opportunities
        - Takes into account shifting exp returns & LR risk properties of each asset class

General approach:
$E_t[M_{t+1} R_{i,t+1}] = 1$ or $E_t[M_{t+1} P_{i,t+1}] = P_{i,t}$
where M_{t+1} is pricing kernel

Beta form:
$E[R_i] = R_F - \text{cov}(M, R)/E(M) = R_F + [\text{cov}(M, R)/\text{var}(M)] [-\text{var}(M)/E(M)] = R_F + \beta_M\lambda_M$

- CAPM: $M_{t+1} = a + bR_{M,t+1}$
- APT and ICAPM: $M_{t+1} = a + b'f_{t+1}$

Mean-variance frontier:
$E[R_i] - R_F = -[\rho_{M,i}\sigma_M/E_M]\ \sigma_i$
Since $-1 \le \rho \le 1$, $|E[R_i] - R_F| \le [\sigma_M/E_M]\ \sigma_i$
Constraint on the Sharpe ratio: $|(E[R_i] - R_F)/\sigma_i| \le \sigma_M/E_M$