Electoral Threshold, Representation, and Parties' Incentives to Form a Bloc.

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Abstract

In most countries with proportional representation systems, there is an election threshold, i.e., minimum share of popular vote (usually between 3 and 7 percent) that a party or an electoral bloc must receive in order to be represented in the parliament. We present a theoretical study of the effects of electoral threshold on ex-post representation in such systems. In a model where professional politicians are office-motivated and strategic voters care both about ideology and the professional skills of candidates, we find that distortion effects of electoral threshold depend crucially on the ability of a politician to commit to party affiliation prior to learning her position in the electoral list. In particular, without such ability, marginal increase in electoral threshold does not affect expected proportionality of representation. We also study the effect of electoral threshold on incentives for small parties to form a bloc. We find that while small parties are jointly more motivated to form a bloc if the threshold is high, an individual small party does not necessarily find it more attractive, since the marginal benefit from increased probability of excluding its potential bloc partner may outweigh marginal costs from the lower probability of being represented itself.

1 Introduction.

The history of modern democracy now counts more than 200 years, and it has become clear that democratic institutions may take very different forms. There are presidential and parliamentary systems, there are highly centralized countries and federations where local authorities

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have more power than central governments. While all democracies have parliaments, the ways they work, and the ways they are elected, differ substantially.

It is traditionally accepted that majoritarian election rules tend to favor local interests, precisely because smaller communities are more likely to have a representative in the parliament under majoritarian than under proportional system, while PR systems tend to favor creation of strong nation-wide parties (Sartori 1976, Lijphart 1994, local interests support in SMD systems is consistent with Ferejohn 1974). The trade-off between local interests support and strong parties is non-trivial: while majoritatian systems tend to lead to higher accountability, strong parties are particularly important in terms of economic performance. For instance, Riker (1964) suggests that decentralization is good for growth only in the presence of strong nation-wide parties; this theory has found recent empirical support by Enikolopov and Zhuravskaya (2007). Because both extreme cases have certain merit, it may well be possible that some kind of mixed system is optimal, and indeed there are quite a lot countries that have a mixed elections system implemented. Cox (1990) provides a major overview of PR and SMD systems, while more recent studies (Massicotte & Blais, 1999, Shugart & Wattenberg, 2001) consider mixed electoral systems. Apparently, recent attention to mixed systems was inspired by several countries switching them from PR (Italy) and purely majoritarian systems (Japan, New Zealand), as well as creation of new democracies with mixed system (Russia).

In the literature, it is usually assumed that parties maximize their share of votes or probability of getting more votes than other parties; the first assumption makes sense if the parliament is formed on proportional basis but the winner does not automatically allowed to form the government, the latter one better suits other cases. An early result in the field was established by Downs (1957), stating, basically, that one would observe policy convergence to the position of the median voter. Since then, a number of studies were devoted to understanding whether or not this is the case in reality, and especially why not. For example, Alesina (1988) considers the case where parties have ideological preferences but are imperfectly informed about the position of the median voter; if they have to commit to the policy announced during elections, they will trade off likelihood of winning for policy closer to their ideal point. Recently, much attention has been devoted to subversion of democratic institutions by special interest groups; Kunicová and Rose-Ackerman (2005) argues that PR electoral systems are more likely to be subverted than SMD.

What appears, however, to lie beyond the scope of theoretical research on parliamentary elections is the electoral threshold in proportional districts. Presumably, this is the case because its impact may seem quite obvious: it prevents very small parties from getting (or at least being sure) to get into the parliament, and thus enables formation of a few parties which may actually be broad coalitions representing large spectra of particular interests. However, this does not appear to necessarily be the case. In this paper, we build a model that explicitly studies the effects of electoral threshold; it turns out that while high threshold tends to help larger parties, it may actually inhibit formation of strong multi-party system, especially if politicians may switch parties at low personal costs.

To keep analysis tractable, we consider the simplest case of two parties participating in elections in one proportional district with electoral threshold; the parties are uncertain about the exact distribution of votes. The larger party is known to pass the threshold for sure, while for the smaller party it is not evident a priori. The voters do not vote randomly; instead, they have ideological preferences but they also value professionalism, or valence, of politicians. Before finalizing their choice they can observe both party lists, and this makes the choice of party affiliation non trivial for politicians. We assume that politicians are assigned positions in the party list they choose randomly and distinguish between two cases: one where politicians can change party affiliation after observing their place in the list and one where they cannot.

We suggest a model to evaluate direct and indirect effect of an increase in electoral threshold. Higher electoral threshold makes a larger party more attractive for professional politicians, and thus disproportionately many politicians will decide to join it in equilibrium, which reinforces voters' inclination to vote for a larger party; higher attractiveness for voters in turn attracts more professional politicians, and so on. Therefore, higher electoral threshold favors large party both directly (by raising expected number of seats it wins for any given share of votes) and indirectly (by attracting professional politicians which tends to increase its share of votes). Chances that the large party will get a disproportionately large share of seats in the parliament are greater if electoral threshold increases. It then follow that, instead of helping establishing several strong parties, electoral threshold may rather favor the largest party while effectively destroying incentives to build new parties because few professional politicians would be interested in building a party with uncertain chances of winning any parliament seats. In other words, while high electoral threshold may help preserve an existing system with a few strong parties (for which passing the threshold is not a problem), it is hardly going to help establish a strong party system from scratch.

The two effects outlined above are both typically present when a politician must commit to a party affiliation prior to learning her position on the party list. However, our main finding is that under an alternative assumption (arguably realistic in countries with limited experience with parliamentary democracy), under which politicians may switch their party affiliation at any time, the equilibrium number of politicians affiliated with each party (and hence that party expected vote share) is unaffected on the margin by an increase in the level of threshold.

Not only is higher electoral threshold unlikely to help build competitive party system, it is also uncertain that it will stimulate already existing small parties to merge or build an electoral bloc. We extend the model to allow for two small parties who must decide whether to coalesce. We show that the total expected number of seats the two party get is never increasing in the level of threshold (regardless of the nature of uncertainty with respect to their number of votes); this finding suggests that the higher the electoral threshold, the more advantageous it is for small parties to form a bloc. However, we further show that the expected number of seats each small party alone gets may or may not be decreasing in the level of threshold. That last finding explains why it is often so difficult for small parties to reach an agreement to form a coalition: first, when it is difficult to agree on transfers, coalescing may not be individually rational for the parties. Second, each of the parties may be excessively optimistic about its electoral prospects, which will make it unwilling to negotiate a transfer.

We have certain anecdotal evidence confirming that such effects may indeed be significant in real elections. In Russia, there have been four elections to State Duma, the lower House of the parliament, under mixed system, with a single proportional district accounting for half of seats. In all these elections, party supported by the Kremlin was expected to win, and in the last two elections this turned out to be the case. However, this prediction was so commonly shared that a very large number of federal and local politicians (especially those without strong ideological preferences) decided to pledge allegiance to the Kremlin-supported party, even if they had supported another party in previous elections. As a result, Russia's political arena has been experiencing ongoing politicians' migration from one party to another because each time politicians wanted to join would-be winner. Finally, in the latest (2003) elections, United Russia party got a huge support both from most politicians and voters and therefore managed to get the majority of seats, i.e., a single large party has been chosen. However, after that, mixed electoral system was replaced with a purely proportional one and electoral threshold was raised from five to seven per cent, despite the fact that even five-percent threshold prevented at least two significant liberal parties from getting to the parliament in the last elections. These changes were justified by desire to create a stable multi-party (or at least two-party) system. However, as the model in our paper suggests, given that one very large party has already been established, these reforms are likely to reinforce its positions and inhibit formation of another strong party.[1] Moreover, it remains a question whether a single large party may be called strong one. Indeed, since most politicians associate with a large party just because other parties are not guaranteed to get into parliament rather than for ideological reasons, it is very possible that such party will lose most its supporters once its dominance is challenged.

2 The Model.

There is a continuum of voters (its measure is unimportant) and a continuum of measure β of identical "recognizable" professional politicians. There is a continuum[2] parliament of measure one, which is filled by deputies elected in pure PR way. All politicians are purely office motivated, utility of each is equal to one if she becomes a member of parliament and zero otherwise. Everybody is risk neutral.

There are two parties, S and L, representing different ideologies. Ideology of party S is less popular, it is on average supported by share $\alpha < 1/2$ of the voters; party L is on average supported by share $1 - \alpha > 1/2$ of the voters. There is, however, an exogenous shock e to these shares. Specifically, ideology of party S is supported by share $\alpha + e$ of the voters and that of party L is supported by share $1 - \alpha - e$ of the voters, where $e \sim U[-\varepsilon, \varepsilon]$ for some $\varepsilon \in [0, \alpha]$. Nobody observes realization of e until after the elections.

[1]This seems to be recognized by the ruling elite, which has recently created a new party Just Russia – allegedly competitive with the existing United Russia, also backed by Kremlin.

[2]We use continuum parliament to abstract away from indivisibility of a single seat, but this assumption is not crucial for basic insights.

There is an electoral threshold $\tau \geq 0$. If a party gets vote share lower than τ it is not represented in the parliament and all seats are divided proportionally among parties who have vote shares higher than τ. In the case of two parties this means that if one of the parties does not pass the threshold all the seats go to the other party. To keep things interesting we assume that $\alpha - \varepsilon < \tau < \alpha + \varepsilon$ and that $1 - \alpha - \varepsilon > \tau$. These inequalities imply that, as soon as all voters vote according to their preferred ideology, the Party L is certain to pass the electoral threshold while Party S may or may not pass it. In addition, we assume that $\beta \in [1-\alpha-\varepsilon, 1+\alpha+\varepsilon]$, so that each professional politician has a positive chance can receive a seat but some of them may end up not gaining seats. Assume that at an interior equilibrium $\beta_L > 0$ politicians join the list of party L and $\beta_S > 0$ join the list of the party S; we then have $\beta_L + \beta_S = \beta$.

Each party maximizes its representation in the parliament and does not care about who personally represents it. Voters care about ideology but also about professionalism of parliament members; specifically, assume that utility of each voter is $\lambda \times prof + (1 - \lambda) \times par$, where $prof$ is the number of politicians in the parliament and par is the number of seats voter's preferred party gets. Parameter λ measures how much a voter cares about ideology of his party compared to the professionalism of the elected candidate. Given preferences of voters, each party will place all the professional politicians it has on the top of its list, and fill the rest of the list with generic – i.e., not recognizable – names.

We distinguish two alternative regimes. Under the first regime politicians may change their party affiliation at any time (even after the lists are put together) while under the second regime they must commit to a party affiliation before they know where in its list they find themselves.

3 No commitment.

In this section we assume that a politician may leave either party list at any time. He can then offer his service to the other party, which will be happy to have him on the list, above all non-politicians. Therefore, at any equilibrium the probability that the lowest politician on the list will get a seat is the same for both parties (provided there is at least one politician in each list).

The main result of this section is

Theorem 1 *At any interior equilibrium in which there are at least τ politicians in the list of party S and that $\beta_L < 1-\tau$ and $\beta_S > \tau$. Then, as long as chances for party S to obtain at least τ votes are strictly between 0 and 1, at equilibrium the equilibrium values of β_L and β_S – and hence the expected number of votes each party gets – is unaffected by a marginal increase in τ.*

The intuition behind Theorem 1 is as follows. At any equilibrium politicians lowest on party lists have equal chances of passing to the parliament – otherwise some politicians would prefer to switch party affiliation (assumptions $\beta_L < 1-\tau$ and $\beta_S > \tau$ rule out corner solutions). Because of this, a marginal increase in electoral threshold does not induce a voter to switch vote from one party to the other and hence does not affect probabilities for politicians lowest on their lists to obtain seats.

Proof. At any equilibrium either (1) some supporters of party S's ideology vote for party L or (2) some supporters of party L's ideology vote for party S or (3) all voters vote for party whose ideology they support. We consider cases (1) only; case (2) is similar and case (3) is limiting for both (1) and (2).

Assume that all supporters of party S ideology with $\lambda \geq \lambda_S$ vote for party L. Consider a supporter of party S with weight parameter λ. He knows that if he casts his (infinitesimal) vote dv for party L, rather than S, then the latter will receive $\lambda_S(\alpha+e) - dv$ votes. With probability

$$P\{\lambda_S(\alpha+e) - dv < \tau\} = \frac{\frac{\tau+dv}{\lambda_S} - \alpha + \varepsilon}{2\varepsilon}$$

party S will not be represented in the parliament and the voter will receive utility $\lambda\beta_L$. With the remaining probability $1-p_S$ party S will obtain vote share distributed uniformly over $[\tau, \lambda_S(\alpha+\varepsilon) - dv]$; total expected utility of the voter with parameter λ equals

$$\left[\frac{\frac{\tau+dv}{\lambda_S} - \alpha + \varepsilon}{2\varepsilon}\right]\lambda\beta_L + \frac{1}{2\lambda_S\varepsilon}\left[I_S + \int\limits_{1-\beta_L}^{\lambda_S(\alpha+\varepsilon)-dv} \lambda(\beta_S + 1 - w) + (1-\lambda)w dw\right], \qquad (1)$$

where

$$I_S = \int_{\tau}^{\beta_S} \lambda(\beta_L + w) + (1-\lambda)w dw + \int_{\beta_S}^{1-\beta_L} \lambda(\beta_L + \beta_S) + (1-\lambda)w dw$$

is independent on dv. The voter will cast his vote for party L if (1) has positive coefficient at

dv, i.e., if

$$\lambda\beta_L - (1-2\lambda)\lambda_S(\alpha+\varepsilon) - \lambda(1+\beta_S) > 0$$

or

$$\lambda[\beta_L - 1 + \beta_S + 2\lambda_S(\alpha+\varepsilon)] > \lambda_S(\alpha+\varepsilon); \tag{2}$$

at equilibrium, (2) must be equivalent to $\lambda > \lambda_S$, which gives equilibrium condition

$$\lambda_S = \frac{\alpha+\varepsilon+1-\beta_L+\beta_S}{2(\alpha+\varepsilon)} \tag{3}$$

Condition (3) represents voters' reaction function to politicians' choice of β_S and $\beta_L = \beta - \beta_S$. The higher β_L and the lower β_S, the lower λ_S, i.e., the more party S supporters will vote for party L

At any equilibrium with $\beta_L < 1-\tau$ and $\beta_S > \tau$ the probability p_S that the politician occupying position β_S on party S list will obtain a seat is

$$p_S = P\{\lambda_S(\alpha+e) > \beta_S\} = 1 - \frac{\frac{\beta_S}{\lambda_S} - \alpha + \varepsilon}{2\varepsilon};$$

Similarly, the probability that the politician occupying position β_L on party L list will obtain a seat is

$$p_L = P\{\lambda_S(\alpha+e) < 1-\beta_L\} = \frac{\frac{1-\beta_L}{\lambda_S} - \alpha + \varepsilon}{2\varepsilon};$$

At equilibrium $p_S = p_L$ which gives

$$1 - \beta_L + \beta_S = 2\alpha\lambda_S. \tag{4}$$

Equation (4) can be thought of politicians' reaction function. The higher λ_S, the less supporters of party S vote for party L, the less attractive it is to be on party L list, the lower β_L and higher β_S.

Therefore, at equilibrium λ_S, β_L and $\beta_S = \beta - \beta_L$ are defined by conditions (3) and (4). As neither of them involve τ, nor do expressions for equilibrium λ_S, β_L and β_S, which completes the proof for case (1).

In case (2) the argument is similar; it is straightforward to verify that voters' reaction function is

$$\lambda_L = \frac{1}{2 - (\alpha + \varepsilon) - \beta_L + \beta_S}, \tag{5}$$

where all party L's supporters with $\lambda > \lambda_L$ vote for party S. Politicians' reaction function is

$$1 - \beta_L + \beta_S = 2\alpha\lambda_L + 2(1 - \lambda_L); \tag{6}$$

again, as neither of expressions (5) and (6) involve τ, nor do expressions for equilibrium λ_L, β_L and β_S, which completes the proof for case (2). □

Theorem 1 has an important limitation that it only applies to interior equilibria. One can show, however, that all equilibria with $\beta_L > 1 - \tau$ or $\beta_S < \tau$ or both are unstable with respect to small changes in β_S and β_L. The only remaining options for stable corner solutions involve either no politicians in party S list or party S passing to parliament with certainty. Note that in both these cases a small change in τ also has no effect on either allocation of politicians in each party list or voters' decision.

4 Incentives to form a bloc for small parties.

In this section we extend the analysis above to allow for two small parties rather than one. We address the question whether two small parties will benefit from merging into an electoral bloc and if so, how their incentives to merge respond to an increase in τ.

Two forces are at work, pushing in opposite directions. On one hand, facing a higher threshold a party is less likely to pass to the parliament alone, which pushes it towards forming a bloc. On the other hand if a party passes the threshold it gets higher number of seats on average, since the other party is more likely to fail and there will on average be more seats per party who passes the threshold.

When there are two small parties and each gets a random number of votes, parties incentives to merge may be sensitive to whether the random outcomes of the vote are correlated. If we believe that the two small parties each compete with the larger party on its own policy dimension, it is reasonable to assume that there is little correlation between the number of votes they get. Alternatively, one may assume that small parties compete with each other for the same electoral base (if two parties are considering forming an electoral bloc, they are likely to appeal to the same electoral base), in which case their election results are likely to have negative correlation. Finally, one may think that the source of uncertainty lies in public general perception of current events or policy, so that, for example, leftist ideas altogether may at times become more or less popular; in this case it is natural to assume that votes for small parties with similar ideologies are positively correlated.

We consider two small parties, Party 1 and Party 2, with results $\varphi + e_1$ and $\varphi + e_2$ with the same expected vote φ and random shocks e_1 and e_2 distributed uniformly over intervals $[-\varepsilon_1, \varepsilon_1]$ and $[-\varepsilon_2, \varepsilon_2]$ for some $\varepsilon_i \leq \varphi$, $i = 1, 2$. We denote $q = \varepsilon_2/\varepsilon_1$ and without loss of generality assume $q \leq 1$, i.e., that Party 1 has at least as volatile vote as Party 2. All the other votes go to large Party 3.

We consider the following three alternative assumptions:

Assumption I. Random shocks e_1 and e_2 are independent.

Assumption P. Random shocks e_1 and e_2 are (perfectly) positively correlated: $e_2 = qe_1$.

Assumption N. Random shocks e_1 and e_2 are (perfectly) negatively correlated: $e_2 = -qe_1$.

As it turns out, implications of all three assumptions I, P, N on parties' incentives to merge are identical, as the following theorem demonstrates.

Theorem 2 *Consider two small parties of the same average vote $\varphi \leq 1/4$ and uniform shocks e_1 and e_2. Assume that threshold τ is such that $2\varphi - \varepsilon_1 - \varepsilon_2 > \tau$ and $1 - 2\varphi - \varepsilon_1 - \varepsilon_2 > \tau$, i.e., that both the large party and the bloc of two small parties are certain to pass the threshold. Then, under either of assumptions I, P or N*

(i) The total expected number of seats for both small parties is nonincreasing in τ; in particular, it never exceeds 2φ.

(ii) Depending on parameters τ, φ and q each of the small parties' expected number of seats may or may not be decreasing in τ.

(iii) If $q = 1$ (i.e., parties are symmetric) the expected number of seats for each small party is nonincreasing in τ; in particular, it never exceeds φ.

Proof. See Appendix.

Theorem 2 shows that, despite higher incentives to form a bloc at higher level of electoral threshold τ, any one of the parties may in fact be discouraged from doing so (unless the parties are close to each other in terms of volatility of vote). It is straightforward as far as the less volatile Party 2 is concerned: consider $\tau \in [\varphi - \varepsilon_1, \varphi - q\varepsilon_1]$. Then Party 2 is certain to pass the threshold and is interested in excluding Party 1, so higher τ within the above range is beneficial for Party 2. It is less apparent that more volatile Party 1 may also benefit from higher threshold: if q is very small and $\tau \in [\varphi - q\varepsilon_1, \varphi + q\varepsilon_1]$ than expected loss of mandate to Party 1 generated by an increase in τ can be offset by a much higher increase in probability that Party 2 will not pass the threshold. Details for this case are given in the Appendix.

In conclusion, although an increase in τ usually pushes small parties towards forming a coalition, this conclusion is not universal. A party with unstable electorate may instead be discouraged from forming a bloc. Therefore, if forming a bloc requires mutual consent and parties are not flexible enough in redistributing positions in the list of the bloc, chances of actually coming to an agreement may in fact decrease as the threshold increases.[3]

5 Conclusion.

In this paper, we aim at filling the apparent gap in theoretical research on electoral systems. We build a model that considers potential effects of electoral threshold and find that these effects may differ significantly from the conventional viewpoint that a high threshold encourages formation of strong political parties. We identify two effects that make higher electoral threshold advantageous for large parties. The first (direct) effect is that higher electoral threshold makes smaller parties less likely to pass it, which results in higher expected representation of large parties. The second (indirect) effect is that with higher electoral threshold, it is more attractive for professional office motivated politicians to be on a larger party list; voters, who care not only about representation of their favorite ideologies, but also about professionalism of politicians in the parliament, are more likely to vote for larger parties if there are more politicians on their lists, which in turn results in higher representation of larger parties. We find, somewhat surprisingly, that the indirect effect does not play as long as candidates are free to change their party affiliations at any time; we show, in this case, that a marginal increase in the level of threshold does not affect allocation of politicians between party lists. In contrast, when a politician must commit to party affiliation prior to learning his or her position on the list, the indirect effect of higher threshold level is likely to further favor larger parties.

[3] If each party is overestimating volatility of its electoral base compared to that of the other party, they *both* may be discouraged from forming a bloc by a higher threshold.

When the initial distribution of parties' strengths is uneven, electoral threshold tends to benefit the largest party the most, both in terms of share of votes it receives and the share of politicians that opt to support it. Both effects are likely to inhibit formation of alternative strong parties, which may be well aligned with the single largest party interests, but is definitely bad for society as a whole. In principle, such results may be driven not only by electoral thresholds, but also by other forms of discrimination of small parties.

One possible conclusion that may be drawn from our paper is that a strong multi-party system can not be built by creating parties one at a time, because the first strong party may prevent formation of strong rivals.

There are a number of issues left behind in this paper. One important limitation of the model is that politicians are assumed to be purely office motivated. It would be interesting to see how (if at all) the conclusions change if politicians care not only about office, but also about the well being of a particular constituency; parties will then compete for having a popular politicians in the list by offering some benefits to the corresponding constituency. In a purely proportional system a politician has no chance to gain a seat but to associate herself with a particular party; in contrast, in a mixed system a candidate may choose to run independent. It would be instructive to see how politicians' and parties' incentives change from one regime to the other and how the effect of raised threshold interacts with that of the shift from a mixed to a purely proportional electoral system, which are the two components of the legislature change in Russia effective 2007.

Another natural and intriguing extension of the model is the optimal level of electoral threshold. As this paper shows, high electoral threshold means high dispropotionality in representation; on the other hand, low threshold would typically result in lower accountability of parliament because of its higher fragmentation and hence high likelihood of broad coalitions. Theoretical foundations of optimal balance, and hence optimal electoral threshold, remain to be developed.

There are definitely more questions than answers, and we hope that this paper will inspire further research on this subject.

6 Appendix.

Proof of Theorem 2 As discussed in the text, for $q < 1$ Party 2 is interested in higher threshold level as long as $\tau < \varphi - \varepsilon_2$. Here we focus on the less obvious case when none of the small parties is certain to pass the threshold: $\tau \in [\varphi - \varepsilon_2 < \tau < \varphi + \varepsilon_2]$.

We treat each assumption I, P, N in turn.

Consider Assumption I (i.e., independent shocks) first. The total number of seats for Party 1 is

$$\begin{aligned} M_1^I(\tau) &= \frac{1}{4\varepsilon_1\varepsilon_2}\left[\int_{\tau-\varphi}^{\varepsilon_1} de_1 \int_{-\varepsilon_2}^{\tau-\varphi} \frac{\varphi + e_1}{1-\varphi-e_2} de_2 + \int_{\tau-\varphi}^{\varepsilon_1} de_1 \int_{\tau-\varphi}^{\varepsilon_2} (\varphi + e_1) de_2\right] \\ &= \frac{1}{2\varepsilon_1}\left[\int_{\tau-\varphi}^{\varepsilon_1} (\varphi + e_1) de_1\right] \frac{1}{2\varepsilon_2}\left[\int_{-\varepsilon_2}^{\tau-\varphi} \frac{de_2}{1-\varphi-e_2} + \int_{\tau-\varphi}^{\varepsilon_2} de_2\right] \\ &= \frac{1}{2\varepsilon_1}\left[\frac{(\varepsilon_1+\varphi)^2}{2} - \frac{\tau^2}{2}\right] \cdot \frac{1}{2\varepsilon_2}\left[\ln\frac{1+\varepsilon_2-\varphi}{1-\tau} + \varepsilon_2 - \tau + \varphi\right]. \end{aligned}$$

Its derivative with respect to τ is

$$m_1^I(\tau) = \frac{\tau}{4\varepsilon_1\varepsilon_2}\left\{\left[\frac{1}{1-\tau} - 1\right]\left[\frac{(\varepsilon_1+\varphi)^2}{2} - \frac{\tau^2}{2}\right] - \tau\left[\ln\frac{1+\varepsilon_2-\varphi}{1-\tau} + \varepsilon_2 - \tau + \varphi\right]\right\}. \quad (7)$$

It is straightforward to verify that $m_1^I(\tau) > 0$ for $\varepsilon_1 = \varphi$, $\tau = \varphi + \triangle\varphi$ and $\varepsilon_2 = \triangle\varphi$ for small enough $\triangle\varphi$. This shows claim (ii) of the theorem.

To show part (ii), calculate the total mandate that two parties get as

$$M_1^I(\tau) + M_2^I(\tau) = \frac{1}{2\varepsilon_1}\left[\frac{(\varepsilon_1+\varphi)^2}{2} - \frac{\tau^2}{2}\right] \cdot \frac{1}{2\varepsilon_2}\left[\ln\frac{1+\varepsilon_2-\varphi}{1-\tau} + \varepsilon_2 - \tau + \varphi\right] +$$
$$\frac{1}{2\varepsilon_2}\left[\frac{(\varepsilon_2+\varphi)^2}{2} - \frac{\tau^2}{2}\right] \cdot \frac{1}{2\varepsilon_1}\left[\ln\frac{1+\varepsilon_1-\varphi}{1-\tau} + \varepsilon_1 - \tau + \varphi\right].$$

Its derivative with respect to τ is

$$m_1^I(\tau) + m_2^I(\tau) = \frac{\tau}{4\varepsilon_1\varepsilon_2}\left\{\left[\frac{1}{1-\tau} - 1\right]\left[\frac{(\varepsilon_1+\varphi)^2}{2} - \frac{\tau^2}{2}\right] - \tau\left[\ln\frac{1+\varepsilon_1-\varphi}{1-\tau} + \varepsilon_1 - \tau + \varphi\right] +\right.$$
$$\left.\left[\frac{1}{1-\tau} - 1\right]\left[\frac{(\varepsilon_2+\varphi)^2}{2} - \frac{\tau^2}{2}\right] - \tau\left[\ln\frac{1+\varepsilon_2-\varphi}{1-\tau} + \varepsilon_2 - \tau + \varphi\right]\right\}.$$

To show that the above expression is negative for any $\varepsilon_1, \varepsilon_2 \in [0, \varphi]$ it suffices to show that

$$\left[\frac{1}{1-\tau} - 1\right]\left[\frac{(\varepsilon+\varphi)^2}{2} - \frac{\tau^2}{2}\right] - \tau\left[\ln\frac{1+\varepsilon-\varphi}{1-\tau} + \varepsilon - \tau + \varphi\right] \quad (8)$$

is negative for any $\varepsilon \in [0, \varphi]$. The derivative of (8) with respect to ε is

$$\left[\frac{1}{1-\tau} - 1\right](\varphi + \varepsilon) - \frac{\tau}{1+\varepsilon-\varphi} - \tau \leq 2\varphi\left[\frac{1}{1-\tau} - 1\right] - 2\tau < 0,$$

since $\tau < \varphi + \varepsilon < 2\varphi < 1/2$. This shows claim (i).

To show claim (iii), plug constraint $\varepsilon_1 = \varepsilon_2$ into expression (7). It then becomes identical to (8) up to a positive multiplier, and (8) is shown above to be negative. This shows claim (iii) and competes the proof for the case of independent shocks (Assumption I).

We now turn to the case of positive correlation of vote (Assumption P). Consider $\tau > \varphi$. The expected number of seats Party 1 gets is

$$M_1^P(\tau) = \frac{1}{2\varepsilon_1}\left[\int_{\tau-\varphi}^{\frac{\tau-\varphi}{q}} \frac{\varphi + e_1}{1-\varphi-qe_1}de_1 + \int_{\frac{\tau-\varphi}{q}}^{\varepsilon_1}(\varphi+e_1)de_1\right].$$

Its derivative with respect to τ equals

$$m_1^P(\tau) = \frac{1}{2\varepsilon_1}\left[\frac{1}{q}\frac{\varphi+\frac{\tau-\varphi}{q}}{1-\tau} - \frac{\tau}{1-\varphi-q(\tau-\varphi)} - \frac{1}{q}\left(\varphi+\frac{\tau-\varphi}{q}\right)\right].$$

It is straightforward to verify that $m_1^P(\tau) > 0$ for $\varepsilon_1 = \varphi$, $\tau = \varphi + \triangle\varphi$ and $q = \triangle\varphi/\varphi$ for small enough $\triangle\varphi > 0$. This shows claim (ii) of the theorem.

For $\tau \in [\varphi, \varphi + q\varepsilon_1]$ (conclusion for other values of τ is similar) the total expected number of seats the two small parties get is

$$M_1^P(\tau) + M_2^P(\tau) = \frac{1}{2\varepsilon_1}\left[\int_{\tau-\varphi}^{\frac{\tau-\varphi}{q}} \frac{\varphi + e_1}{1-\varphi-qe_1}de_1 + \int_{\frac{\tau-\varphi}{q}}^{\varepsilon_1}(\varphi+e_1)de_1 + \int_{\frac{\tau-\varphi}{q}}^{\varepsilon_1}(\varphi+qe_1)de_1\right].$$

Its derivative with respect to τ is

$$m_1^P(\tau) + m_2^P(\tau) = \frac{1}{2\varepsilon}\left[\frac{1}{q}\frac{\varphi+\frac{\tau-\varphi}{q}}{1-\tau} - \frac{\tau}{1-\varphi-q(\tau-\varphi)} - \frac{1}{q}\left(\varphi+\frac{\tau-\varphi}{q}\right) - \frac{\tau}{q}\right] \leq$$

$$\frac{1}{2\varepsilon}\left[\frac{1}{q}\frac{\varphi+\frac{\tau-\varphi}{q}}{1-\tau} - \frac{1}{q}\left(\varphi+\frac{\tau-\varphi}{q}\right) - \frac{\tau}{q}\right] = \frac{1}{2\varepsilon}\frac{\tau}{q}\left[\frac{\varphi+\frac{\tau-\varphi}{q}}{1-\tau} - 1\right] \leq \frac{\tau}{q}\left[\frac{2\varphi}{1-\tau} - 1\right] < 0$$

for $\varphi \leq \frac{1}{4}$. This shows claim (i) of the theorem.

Claim (iii) is trivial under assumption P: if $q = 1$ none of the parties can pass the threshold unless the other party also passes it, so none can possibly be interested in a higher threshold.

Finally, we prove the theorem under Assumption N. For $\tau \in [\varphi - q\varepsilon_1, \varphi]$ the expected number of seats that Party 1 gets is

$$M_1^N(\tau) = \frac{1}{2\varepsilon_1}\left[\int_{\tau-\varphi}^{\frac{\varphi-\tau}{q}}(\varphi+e_1)de_1 + \int_{\frac{\varphi-\tau}{q}}^{\varepsilon_1}\frac{\varphi+e_1}{1-\varphi+qe_1}de_1\right].$$

Its derivative with respect to τ equals

$$m_1^N(\tau) = \frac{1}{2\varepsilon_1}\left[-\frac{1}{q}\left(\varphi+\frac{\varphi-\tau}{q}\right) - \tau + \frac{1}{q}\frac{\varphi+\frac{\tau-\varphi}{q}}{1-\tau}\right].$$

It is straightforward to verify that $m_1^P(\tau) > 0$ for $\varepsilon_1 = \varphi$, $\tau = \varphi - \triangle\varphi$ and $q = \triangle\varphi/\varphi$ for small enough $\triangle\varphi > 0$. This shows claim (ii) of the theorem.

For $\tau \in [\varphi - q\varepsilon_1, \varphi]$ (analysis for other values of τ is similar) the total expected number of seats the two small parties get is

$$M_1^N(\tau)+M_2^N(\tau) = \frac{1}{2\varepsilon_1}\left[\int_{\tau-\varphi}^{\frac{\varphi-\tau}{q}}(\varphi+e_1)de_1 + \int_{\frac{\varphi-\tau}{q}}^{\varepsilon_1}\frac{\varphi+e_1}{1-\varphi+qe_1}de_1 + \int_{\tau-\varphi}^{\frac{\varphi-\tau}{q}}(\varphi-qe_1)de_1 + \int_{-\varepsilon_1}^{\tau-\varphi}\frac{\varphi-qe_1}{1-\varphi-e_1}de_1\right].$$

Its derivative with respect to τ is

$$m_1^N(\tau)+m_2^N(\tau) = \frac{1}{2\varepsilon}\left[-\frac{1}{q}\left(\varphi+\frac{\varphi-\tau}{q}\right)-\tau+\frac{1}{q}\frac{\varphi+\frac{\varphi-\tau}{q}}{1-\tau}-\frac{\tau}{q}-[\varphi-q(\tau-\varphi)]+\frac{\varphi-q(\tau-\varphi)}{1-\tau}\right] =$$

$$\frac{1}{2\varepsilon}\left[\frac{\tau}{q}\left(\frac{\varphi+\frac{\varphi-\tau}{q}}{1-\tau}-1\right)+\tau\left(\frac{\varphi-q(\tau-\varphi)}{1-\tau}-1\right)\right] < 0$$

for $\varphi \leq \frac{1}{4}$. This shows claim (i) of the theorem.

Finally we show claim (iii) by setting $\varepsilon_2 = \varepsilon_1 = \varepsilon$. The expected number of seats that each of Party 1 and Party 2 get is

$$M^N(\tau) = \frac{\tau-\varphi+\varepsilon}{2\varepsilon}\cdot 0 + \int_{\tau-\varphi}^{\varphi-\tau}\frac{\varphi+e}{2\varepsilon}de + \int_{\varphi-\tau}^{\varepsilon}\frac{1}{2\varepsilon}\frac{\varphi+e}{1-\varphi+e}de.$$

Its derivative with respect to τ is given by

$$m^N(\tau) = F'(\tau) = \frac{1}{2\varepsilon}\left[\frac{2\varphi-\tau}{1-\tau}-2\varphi\right] < 0.\blacksquare$$

7 References

Alesina, A., (1988), Credibility and Policy Convergence in a Two-Party System with Rational Voters, American Economic Review 78, 796-805.

Cox, G.. (1990). “Centripetal and Centrifugal Incentives in Electoral Systems.” American Journal of Political Science 34: 903–35.

Downs, A. (1957). An Economic Theory of Democracy. New York: Harper and Row.

Enikolopov, R., and E. Zhuravskaya (2007), Decentralization and Political Institutions, forthcoming in Journal of Public Economics.

Ferejohn, J. A. (1974). Pork Barrel Politics: Rivers and Harbors Legislation, 1947–1968. Stanford: Stanford University Press.

Kunicová, J., and S. Rose-Ackerman (2005) Electoral Rules and Constitutional Structures as Constraints on Corruption. British Journal of Political Science, 35: 573-606.

Lijphart, A. (1994). Electoral systems and party systems: A study of 27 democracies, 1945-1990. New York: Oxford Up.

Massicotte, L., & Blais, A. (1999). Mixed electoral systems: a conceptual and empirical survey. Electoral Studies, 18: 341-366.

Riker, W. (1964) "Federalism: Origins, Operation, Significance," Little, Brown and Co, Boston, MA.

Sartori, G. (1976). Parties and party systems: A framework for analysis. New York: Cambridge University Press.

Shugart, M. S. and M. P Wattenberg (2001). Mixed-member electoral systems: A definition and typology. In: M. S. Shugart&M. P.Wattenberg (Eds.), Mixed-member electoral systems: the best of both worlds? (pp. 9-24). New York: Oxford University Press.