Sequential auctions with entry deterrence. *

Andrei Bremzen[†]

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Abstract

The paper studies sequential auctions with potential entry between rounds. In a simple model with two rounds, two initial bidders and one potential entrant, it is shown that every symmetric equilibrium first round bidding function must feature some degree of pooling. In one such equilibrium, the symmetric bidding function is a step function, reflecting the desire of present bidders to hide information from the potential entrant in order to deter entry. Extensions of the simple model which accommodate uncertain entry prospects, multiple incumbents and multiple entrants are discussed.

1 Introduction.

This paper studies sequential auctions with potential entry between rounds. Although sequential auctions have been thoroughly studied, the common assumption in the existing literature is that the set of bidders does not change from one round to the next. It is

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†New Economic School, Nakhimovsky pr. 47 #1721 Moscow 117 418 Russia. E-mail: abremzen@nes.ru.

natural to extend the analysis to the case where new bidders may enter the auction in future rounds, if they find it attractive, possibly at positive entry costs. The purpose of the present paper is to show how potential entry threat changes standard equilibrium predictions.

I consider a repeated English auction with two rounds and three bidders with independent private valuations. Two identical items are for sale, one in each round. Each bidder wants at most one item, so the winner of the first round gets the item and leaves the auction. There are two bidders in the first round. The third bidder can not participate in the first round but can decide to enter in the second round, at a positive entry cost. The third bidder's entry decision is based not only on his own valuation, but also on the observed outcome of the first round (i.e., the final price). This opens the door for strategic behavior in the first round: the decision of a bidder to drop out in the first round not only means that she is not getting the item now but also sends a signal about her valuation to the potential entrant, which may affect his entry decision. Since the entrant's decision is relevant for the payoff to the losing first-round bidder (because it affects competition in the second round), first round bidders will take the entrant's reaction into consideration when deciding at what price to drop out. In particular, each first round bidder has an incentive to pretend to have higher valuation than she actually does, in order to deter entry; the costs of doing so are, however, that the first round bidder may end up winning in the first round, at a price higher than she would be willing to pay. The main focus of the analysis is on how these strategic considerations affect bidders' behavior.

Sequential auctions with a constant number of bidders were first studied by Ortega-Reichert ([18]). He developed a two-person two-stage model of competitive bidding and recognized that the strategies of the bidders may respond not only to the fact that there will be more rounds in the future, but also to the information about their rivals's valuation that has been revealed in previous rounds (such as their bids). He considered a common value auction, where in stage two each bidder updates her estimate of her own valuation of the good based on the first-period bid of her rival (this setup is further analyzed in [13]). Milgrom and Weber ([17]) develop a general model of multistage auctions and compare different procedures for sequential auctions, as well as various information structures, i.e., what information about the bids in past auctions is revealed to the remaining bidders; in their setup, too, all the bidders participate from the first round.[1]

One other area of research of relevance to the present study explores strategic motives of bidding; the general idea is that not only does a bid represent a claim to win the object, but it also conveys some information that the bidder possesses, which may be relevant for other bidders. This idea builds on an auction design that allows for sequential moves; traditional one-stage sealed bid auctions provide no opportunity for signaling. Avery ([1]) studies strategic bidding in English auctions when bidders' valuations are correlated (and thus bids reveal payoff relevant information to competitors). Daniel and Hirshleifer ([6]) build a two-bidder model of an ascending price auction, in which each bidder incurs positive costs every time she submits a bid; jump bidding serves to signal a high valuation to the opponent in order to force him to quit early.[2] In none of this papers strategic motives for bidding include entry deterrence, since all of them assume exogenous set of bidders.[3]

Another relevant study is von der Fehr ([9]). He also considers a two-round sequential auction of identical items with independent valuations and positive participation costs, but with a fundamentally different information structure: in his model, only bidders who participate in the first round observe its outcome, which implies informational disadvantage for entrants and effectively rules out entry in the second round.[4] He finds a perfect Bayesian

[1] Other theoretical studies of sequential auctions with constant set of bidders include Gale and Stageman [12], Kittsteiner et al [15].

[2] See also Fishman ([10]).

[3] There also are a few studies of bidding environments in which submitting or revising a bid is costly and entry decisions are endogenous; this is not an unreasonable assumption, for instance in procurement or construction auctions. These works too are relevant to the current study setup; however, in all of them only single round auctions are considered. See, for example, Gal, Landsberger, Nemirovski ([11]) and Landsberger, Tsirelson ([16]).

[4] Therefore, Fehr's setup is relevant in settings where participation costs reflect costs of paying attention and opportunity costs of time, whereas I view costs of entry as costs of preparing a qualified bid; in particular, I do not assume that a bidder who loses in the first round has to incur participation costs in the second round again.

equilibrium in which first round bids fully reveal valuations of participants; in contrast, I show in my setting such equilibrium can not exist.[5]

A paper that is perhaps the closest to mine is Jeitschko ([14]). He considers the same independent value setup with three bidders and two identical items for sequential sale. In his paper there are also intertemporal signaling motivations that affect bidders' behavior in the first round of the auction. The mechanism though is different: in his model, there in no entry in the second round, so deterrence motives are absent; however, the first price sealed bid format calls for strategic bidding in the second round, and that is where information revealed in the first round is taken into account. In contrast, with english auction format that I am working with there is no strategic bidding in the second round, but the entry decision is nontrivial and driven by the information revealed in the first round, and that is what calls for strategic behavior in the first round.

In this paper I find that the presence (even uncertain) of potential entrant induces the first round bidders to bid strategically, and their equilibrium bidding functions are radically different from bidding functions in an auction without strategic considerations. I show first that, as long as entry costs are strictly positive, in no equilibrium are first round bidding functions monotone in valuation.[6] Next, I show that for any level of entry costs there exists a subgame perfect equilibrium which involves step functions as first round bidding functions, with the number of steps decreasing in entry costs. Therefore, strategic concealment of information about their valuation is an important issue for first round bidders.

The rest of the paper is organized as follows. Section 2 describes the model and shows that every symmetric first-round equilibrium bidding function must feature some degree of pooling. Section 3 studies the case of high entry costs and introduces step functions as equilibrium bidding functions. Section 4 provides full characterization of a symmetric equilibrium in which the (first-round) bidding functions are step functions. Section 5 looks

[5] Bernhardt, Scoones [2], Engelbrecht-Wiggans [7] and Engelbrecht-Wiggans, Menezes [8] study sequential auctions with non-identical objects.

[6] In particular, this implies that the two-stage auction is generally inefficient.

at an extension with exogenous restrictions on entry, Section 6 describes the equilibrium with multiple first round bidders and Section 7 discusses the case of multiple potential entrants. Section 8 concludes.

2 The Model.

Two identical units of a good are offered for sale by means of an ascending price clock auction; the two rounds of auction are conducted sequentially and the outcome of the first round (the final price and the identity of the winner) is publicly observable before the second round starts. In each round, in case of a draw (last active bidders dropping out at the same price), the good is assigned to each of them with equal probability.

There are three risk-neutral bidders, each demanding at most one unit of the good. In the first round only bidders 1 and 2 participate; the winner obtains the good and leaves the auction, the loser passes on to the second round. Upon observing the outcome of the first round, bidder 3 decides whether to enter in the second round. If he enters, he incurs entry cost $c > 0$ and then bids against the remaining first round bidder. If he does not enter, the remaining first round bidder obtains the item for free.[7] The bidders' valuations of the good are their private information; they are independently drawn from the same distribution on $[0, 1]$ with twice continuously differentiable cdf $F(x)$ and pdf $f(x)$. Assume also that $f(x) > 0$ for $x > 0$. Finally, assume strictly increasing hazard rate $h(x) = \frac{f(x)}{1-F(x)}$, in the following strong sense: there exists a constant $m > 0$ such that $h'(x) > m$ for any $x \in [0, 1]$.

I am looking for a subgame perfect Bayesian symmetric equilibrium in weakly undominated strategies of this game. Such an equilibrium consists of

- Drop out strategy $b(v)$ that bidders 1 and 2 follow in the first round;

[7] According to this formulation, bidders are inherently asymmetric in that bidder 3 faces entry costs while those for his rivals are assumed to be sunk. An alternative formulation in which all three bidders have to incur costs to entry results in a similar solution.

- Belief of bidder 3 about the type of his rival as a function of the outcome of the first round;

- Entry decision of bidder 3 as a function of the outcome of the first round;

- Symmetric bidding strategies of bidder 3 and his rival in the second round.

As usual, belief of bidder 3 must be consistent with the prior distribution and strategies played by bidders 1 and 2 in the first round, and his entry decision must be optimal given his belief.

It is straightforward to conclude that if bidder 3 decides to enter, the only pair of weakly undominated strategies for the bidders in the second round is to drop out at prices that are equal to their valuations. Therefore the nontrivial part of the analysis is to characterize first round bidding strategies and the third bidder's entry decision.

2.1 Costless entry.

Assume first that the entry is costless for the third bidder, i.e., $c = 0$. I want to calculate $b(v)$ – the equilibrium strategy of each of the bidders in the first round as a function of her valuation.

In this case the only weakly undominated strategy for bidder 3 is to enter and then stay in until the price reaches his valuation; therefore, his strategy is trivial and the game reduces to a two player game, which, as it turns out, has a solution in weakly dominant strategies.

A bidder with valuation v will choose to drop out in the first round at price $b(v)$, at which she is indifferent between winning in the first round, thus receiving $v - b(v)$, and dropping out, with expected gain from winning the item in the second round equal to $\int_0^v [v - x] f(x) dx = \int_0^v F(x) dx$. Therefore, each bidder's dominant strategy in the first round is to drop out at $b(v) = v - \int_0^v F(x) dx$.

This function is strictly monotone in the valuation, implying that the entrant will be able to deduce the valuation of his opponent. If that valuation is higher than his own, he will be indifferent between entering and not, since he has no chance of winning anyway; to stay out is a weakly dominated strategy and hence is excluded. However, facing any positive entry cost he will strictly prefer not to enter if he knows that he will lose, so the analysis above does not extend to the case of $c > 0$.

2.2 Costly entry: an impossibility result.

In this section we show that in no equilibrium can the first round bidding function be monotone in valuation.

Lemma 1 *For any positive level of entry costs $c > 0$ there exists no subgame perfect symmetric equilibrium bidding function $b(v)$ which is strictly monotone in valuation.*

Proof: Assume the converse, and let $b(v)$ be strictly monotone strategy of bidders 1 and 2 in the first round. Then the potential entrant will be able to correctly deduce valuation v of his opponent from the price that he observes; he will, therefore, enter if and only if his own valuation is at least $v + c$.

Consider the optimization decision of bidder 1 in the first round; suppose that she has high valuation $v > 1 - c$. One particular deviation from $b(v)$ that bidder 1 may consider is playing $b(\tilde{v})$ for some $\tilde{v}$, i.e., to pretend that she is of type $\tilde{v}$; consider small deviations, such that $\tilde{v} > 1 - c$. Such a deviation will give her the payoff of

$$W(v, \tilde{v}) = \int_0^{\tilde{v}} (v - b(x)) f(x) dx + v(1 - F(\tilde{v})).$$

The first term is bidder 1's expected payoff in the first round, given that bidder 2 adheres to $b(v)$. bidder 1 loses in the first round if bidder 2 has valuation higher than $\tilde{v}$, i.e., with probability $1 - F(\tilde{v})$; in this case bidder 3 believes that bidder 1 has valuation $\tilde{v}$ and does not enter, since his own valuation is less than $\tilde{v} + c > 1$; bidder 1 wins the item for free in the second round. Note that, since $b(v)$ is assumed strictly monotone, $W(v, \tilde{v})$ is absolutely continuous in $\tilde{v}$ and has left and right derivatives with respect to $\tilde{v}$ everywhere.

If $b(v)$ is an equilibrium bidding function, then for any v function $W(v,\tilde{v})$ must attain its maximum in $\tilde{v}$ at $\tilde{v}=v$. A first order condition for that its left derivative is nonnegative at $\tilde{v}=v$. We have

$$0 = \frac{d}{d\tilde{v}} W(v,\tilde{v})\bigg|_{\tilde{v}=v-0} = (v - b(v) - v)f(v) = -b(v)f(v). \tag{1}$$

Clearly this can not be the case, since $f(v) > 0$ and $b(v)$ is assumed monotone and hence can not be identically zero. QED.

Remark 1. An analogue of Lemma 1 holds even if there is more then two incumbents (or more than two potential entrants). There exists no separating equilibrium, that is, no equilibrium drop out schedule such that the entrant (in case he observes the complete outcome of the first round, i.e., who dropped out at what price) can deduce valuations of all the bidders from the outcome of the first round. Indeed, if such an equilibrium existed with multiple incumbents, then when all but two first-round bidders have zero valuations, the other two would be in a situation covered by Lemma 1 and the same logic would rule out the possibility of a separating equilibrium.

Remark 2. In the proof of lemma 1 I used the boundedness of the distribution; however, it is straightforward to show that this impossibility result holds for unbounded distributions too under very general regularity restrictions (for example, finite expectation and $f(v) \searrow 0$ as $v \to \infty$ are sufficient).

3 Step function as an equilibrium bidding function.

Now I turn to constructing a symmetric equilibrium strategies $b(v)$ for bidders in the first round.

Consider first the case of very high entry cost, $c \geq 1 - Ev$. Suppose both bidder 1 and bidder 2 drop out immediately at price 0, irrespective of their valuations: $b^{(0)}(v) \equiv 0$. bidder 3 always observes price zero but can make no further inference about his opponent's valuation; he sticks to his prior $v \sim F(v)$ on $[0,1]$. Even if bidder 3 has the highest possible valuation of 1, his expected payoff if he enters, gross of entry costs, will equal $1 - Ev$, not

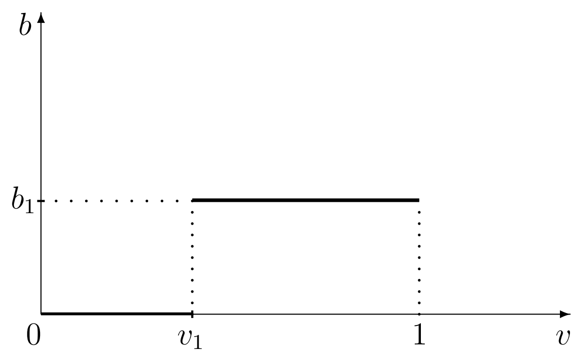

Figure 1: Equilibrium step bidding function $b^(1)(v)$.

enough to cover entry costs, so he will not enter. Therefore, for $c \geq 1 - Ev$ both first round bidders bidding $b^{(0)}(v)$ and bidder 3 never entering is an equilibrium.

If entry costs are not so high ($c < 1 - Ev$), then playing $b^{(0)}(v) \equiv 0$ is no longer an equilibrium: each of the first two bidders expects bidder 3 to enter with positive probability, so each of them finds winning the item in the first period more attractive and will prefer to outbid their rival in the first round.

Instead of $b^{(0)}(v)$, consider the following $b^{(1)}(v)$ for some $v_1 < c$:

$$b^{(1)}(v) = \begin{cases} 0, & 0 \leq v \leq v_1, \\ b_1, & v_1 < v \leq 1. \end{cases}$$

To complete the description of the suggested equilibrium strategies, I must specify entry decision of bidder 3 based on the price p he observes. The entry strategy for bidder 3 is:

- if $p \geq b_1$, do not enter ;
- if $p = 0$, enter if $v > \bar{v} = c + v_1 - \int_0^{v_1} \frac{F(x)}{F(v_1)} dx$;
- if $0 < p < b_1$, enter if $v \geq c$.

The last provision implies specification of bidder 3's beliefs off the equilibrium path: he believes that if a bidder who bids above 0 but below b_1 is of lowest possible type, zero.[8]

[8]This specification of beliefs is consistent with Cho-Kreps intuitive criterion as in [4].

Proposition 1 *The suggested entry strategy is a best reply of the entrant to strategy* $b_1(v)$, *as long as* $v_1 \leq c$ *and* $E(v|v \geq v_1) \geq 1 - c$.

Proof: If the entrant believes that both first-round bidders adhere to $b^{(1)}(v)$, he will conclude, upon observing $p = 0$, that his rival has valuation less than or equal to v_1. This gives conditional density $f(x|x < v_1) = \frac{f(x)}{F(v_1)}$ on $[0, v_1]$ and zero elsewhere. It will only make sense for bidder 3 to enter if his own valuation v is at least c, which immediately means that he will win with probability one in the second round since $v_1 \leq c$. Suppose now that bidder 3 enters and his valuation is $v \geq v_1$. His expected payoff is then

$$\int_0^{v_1} [v - x] \frac{f(x)}{F(v_1)} dx = v - v_1 + \int_0^{v_1} \frac{F(x)}{F(v_1)} dx.$$

He will choose to enter only if his expected payoff is at least c, which is the case when $v \geq \bar{v}$.

Assume now that the entrant observes $p = b_1$. He concludes that his rival is of type higher than v_1, with conditional density $f(x) = \frac{1}{1-F(v_1)}$ on $[v_1, 1]$ and zero elsewhere. Even if his own valuation is 1, his expected payoff is only

$$\int_{v_1}^{1} [1 - x] \frac{f(x)}{1 - F(v_1)} dx = 1 - E(v|v \geq v_1),$$

which by assumption is not enough to cover his entry costs c, so he will not enter, QED.

Now pick b_1 to make an incumbent with valuation v_1 indifferent between bidding zero and b_1. Assume that bidder 1 adheres to $b^{(1)}(v)$ and consider bidder 2 who values the object at some $v \bar{>} v$. When bidder 2 bids zero, with probability $F(v_1)$ bidder 1 will bid zero, in which case with probability $\frac{1}{2}$ bidder 2 gets the object for free and with probability $\frac{1}{2}$ (or for sure if bidder 1 has valuation $v > v_1$) bidder 2 passes on to the second round in which bidder 3 of type $\bar{v}$ or below does not enter and bidder 2 gets the object for free; bidder 3 of type above $\bar{v}$ enters and wins since by assumption $v \leq \bar{v}$. Payoff from bidding 0 to bidder 2 with valuation above $\bar{v}$ is also readily calculated. bidder 2's total expected payoff from bidding zero is equal to

$$\pi_0(v) = \begin{cases} \frac{F(v_1)}{2}v + \left(1 - \frac{F(v_1)}{2}\right)\bar{v}v, & v \leq \bar{v}, \\ \frac{F(v_1)}{2}v + \left(1 - \frac{F(v_1)}{2}\right)\left(\bar{v}F(\bar{v}) + \int\limits_{\bar{v}}^{v} F(x)dx\right), & v \geq \bar{v}. \end{cases}$$

If instead she bids b_1, her payoff is

$$\pi_{b_1}(v) = F(v_1)v + \frac{1 - F(v_1)}{2}(v - b_1) + \frac{1 - F(v_1)}{2}v.$$

Equating $\pi_{b_1}(v_1)$ to $\pi_0(v_1)$ determines b_1.

Note that the slope of $\pi_0(v)$ is always lower than that of $\pi_{b_1}(v)$. This single crossing property, together with $\pi_{b_1}(v_1) = \pi_0(v_1)$ implies that a first round bidder will strictly prefer to bid zero rather than b_1 if her valuation is below v_1 and to bid b_1 rather than zero if her valuation is above v_1.

To complete the description of $b(v)$ I now have to specify the value of v_1. I do it by considering a particular deviation from $b(v)$ (namely, bidding above zero but below b_1) which must not be profitable for any valuation v.

If bidder 2 (of a particular type v) bids above zero but below b_1, then with probability $F(v_1)$ bidder 1 will bid zero and bidder 2 will win the object for free; however with probability $1 - F(v_1)$ bidder 1 will bid b_1 in which case not only will bidder 2 lose in the first round, but also bidder 3 will believe that bidder 2 is of type zero and will enter whenever bidder 3's own valuation is above c. Hence if valuation v of bidder 2 is below c, she will on average get $F(c)v$ in the second round, while if her valuation is above c, she will get on average

$$F(c)v + \int\limits_{c}^{v}[v - x]f(x)dx = F(c)c + \int\limits_{c}^{v} F(x)dx.$$

Her total payoff is therefore equal to

$$\pi_{<b_1}(v) = \begin{cases} F(v_1)v + (1 - F(v_1))F(c)v, & v \leq c \\ F(v_1)v + (1 - F(v_1))\left(F(c)c + \int\limits_{c}^{v} F(x)dx\right), & v \geq c. \end{cases}$$

Note that the actual bid does not matter, provided that it is above zero but below b_1.[9]

Proposition 2 $\left.\frac{\partial \bar{v}}{\partial v_1}\right|_{v_1=0} = \frac{1}{2}$.

Proof: By inspection.

Proposition 3 *The slope of $\pi_{<b_1}(v)$ is below that of $\pi_0(v)$ at $v = 0$ for small enough v_1.*

Proof: Both $\pi_0(v)$ and $\pi_{<b_1}(v)$ are linear in $v = 0$ for $v < c$. Their slopes equal

$$\pi_0' = \frac{F(v_1)}{2} + \left(1 - \frac{F(v_1)}{2}\right) F(\bar{v}),$$

and

$$\pi_{<b_1}' = F(v_1) + (1 - F(v_1))F(c).$$

Both these slopes tend to $F(c)$ as $v_1 \to 0$. Therefore it suffices to show that

$$\left.\frac{d\pi_0'}{dv_1}\right|_{v_1=0} > \left.\frac{d\pi_{<b_1}'}{dv_1}\right|_{v_1=0} \tag{2}$$

In view of Proposition 2 inequality (2) is equivalent to

$$\frac{f(0)}{2} - \frac{f(0)}{2}F(c) + \frac{f(c)}{2} > f(0) - f(0)F(c),$$

which follows immediately from $h(0) < h(c)$, QED.

Pick v_1 such that proposition 3 holds. Bidding above zero (and below b_1) is then not a profitable deviation for $v < v_1 < c$. Neither is such deviation profitable for $v > v_1$. To see this, note that $\pi_{b_1}(v_1) = \pi_0(v_1) > \pi_{<b_1}(v_1)$ and that

$$\pi_{b_1}'(v) = 1 > F(v_1) + (1 - F(v_1))F(v) = \pi_{<b_1}'(v),$$

so bidding below b_1 for $v > v_1$ is always inferior to bidding b_1.

Finally, I have to show that bidding above b_1 is not a profitable deviation either. This follows immediately from the following argument: both bidding b_1 and bidding above b_1

[9] Note that $\pi_{b^{(1)}(v)}(v) - \pi_{<b_1}(v)$ reaches its minimum at $v = 0$, so a bidder of type zero loses the least by deviating from $b^{(1)}(v)$. This is an explanation for assuming that an entrant who observes a deviation from $b^{(1)}$ believes that the deviant to has valuation zero.

result in getting the object with probability one, either in the first or in the second round (since bidder 3 never enters). However, if a bidder bids above b_1, she will end up paying b_1 whenever the valuation of her opponent is above v_1, while if she bids b_1 she will only have to pay b_1 in half of those cases. Therefore, it is always better to bid b_1 than to bid above b_1. Note also that once both bidders bid b_1 and the winner is determined at random, both bidders prefer to *lose* in the first round.

Therefore, as long as there exists $v_1 < c$ small enough for Proposition 3 to hold but large enough to satisfy $1 - E(v|v \geq v_1) < c$, there exists a step function equilibrium with a single step; such equilibrium requires substantial entry costs c. Note that there is a degree of freedom in choosing breakpoint v_1, so that a step function equilibrium is typically not unique; rather, there is a whole family of equilibria parametrized by their breakpoints.

I finish this section with an example; in the next section I construct a step function equilibrium for arbitrary positive c.

Example. Consider uniform distribution of valuations: $F(x) = x$ for $x \in [0,1]$. Let $c = \frac{1}{3}$ and pick $v_1 = c = \frac{1}{3}$. Then $\pi_{b_1}(v) = v - \frac{1}{3}b_1$ and $\pi_0(v) = \frac{1}{6}v + (1 - \frac{1}{6})\frac{1}{2}v$, which gives $b_1 = \frac{5}{12}$. Payoff from deviation is

$$\pi_{<b_1}(v) = \begin{cases} \frac{5}{9}v, & v \leq \frac{1}{3} \\ \frac{1}{27} + \frac{1}{3}v + \frac{1}{3}v^2, & v \geq \frac{1}{3}. \end{cases}$$

It is readily checked that $\pi_{<b_1}(v) < \pi_{b_1}(v)$ both at $v = 1$ and $v = \frac{1}{3}$, and, consequently,

$$b(v) = \begin{cases} 0, & 0 \leq v \leq \frac{1}{3}, \\ \frac{5}{12}, & \frac{1}{3} < v \leq 1, \end{cases}$$

is a symmetric equilibrium for $c = \frac{1}{3}$.

4 Small entry costs equilibrium.

In this section I finish up construction of equilibrium by showing that for any entry costs $c > 0$ there exists a step function which is an equilibrium first round bidding function. In particular, I will show how to construct points $0 = v_0 < v_1 < \cdots < v_{n+1} = 1$ and values $0 = b_0 < b_1 < \cdots < b_n$ in such a way that

$$b^{(n)}(v) = \begin{cases} b_k, & v_k \le v < v_{k+1}, \qquad k = 0, \ldots, n \end{cases}$$

is an equilibrium first round bidding function. I will choose breakpoints v_k in such a way that $v_{k+1} - v_k \le c$.

I specify bidder 3's off equilibrium beliefs as follows: if bidder 3 observes a bid between b_i and b_{i+1} he believes that his rival is of type v_i and enters whenever his valuation is above $v_i + c$.

Suppose that bidder 1 has valuation v and bidder 2 adheres to the above strategy. bidder 1 may choose any of the following strategies: bid b_k, or bid above b_k but below b_{k+1} (I use subscript $< b_{k+1}$ to denote this latter strategy). Her expected payoffs from following these strategies are, respectively,

$$\pi_{b_k}(v) = \begin{cases} \sum\limits_{i=0}^{k-1} (F(v_{i+1}) - F(v_i))(v - b_i) + \frac{F(v_{k+1}) - F(v_k)}{2}(v - b_k) + \left(1 - \frac{F(v_{k+1}) + F(v_k)}{2}\right) F(\bar{v}_k)v, & v \le \bar{v}_k, \\ \sum\limits_{i=0}^{k-1} (F(v_{i+1}) - F(v_i))(v - b_i) + \frac{F(v_{k+1}) - F(v_k)}{2}(v - b_k) + \left(1 - \frac{F(v_{k+1}) + F(v_k)}{2}\right) \left(F(\bar{v}_k)\bar{v}_k + \int\limits_{\bar{v}_k}^{v} F(x)dx\right) & v > \bar{v}_k \end{cases}$$

and

$$\pi_{<b_{k+1}}(v) = \begin{cases} \sum\limits_{i=0}^{k-1} (F(v_{i+1}) - F(v_i))(v - b_i) + (F(v_{k+1}) - F(v_k))(v - b_k) + (1 - F(v_{k+1}))F(v_k + c)v, & v \le v_k + c, \\ \sum\limits_{i=0}^{k-1} (F(v_{i+1}) - F(v_i))(v - b_i) + (F(v_{k+1}) - F(v_k))(v - b_k) + (1 - F(v_{k+1})) \left(F(v_k + c)(v_k + c) + \int\limits_{v_k + c}^{v} F(x)dx\right) & v > v_k + c \end{cases}$$

where

$$\bar{v}_k = c + E[x | v_k < x < v_{k+1}] = c + \frac{1}{F(v_{k+1}) - F(v_k)} \int\limits_{v_k}^{v_{k+1}} x f(x) dx \tag{3}$$

is the minimum valuation of bidder 3 which induces him to enter, conditional on him observing the price of b_k in the first round. Note that in equilibrium whenever bidder 3 enters he wins with probability one; this follows from the assumption that $v_{k+1} - v_k \le c$.

To make $b^{(n)}(v)$ symmetric equilibrium, I must ensure that it is optimal for a first round bidder to play $b^{(n)}(v)$ if she expects the other first round bidder to do so. I do it by choosing v_k and b_k appropriately in the iterative manner: choose v_1, then v_2 and so on to v_n; then choose b_k, $k = 1, \ldots, n$.

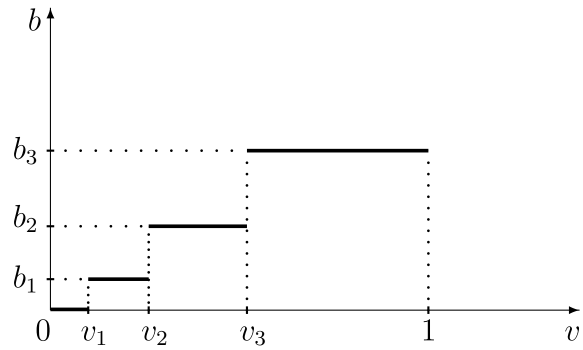

Figure 2: Equilibrium step bidding function $b^{(3)}(v)$.

Suppose that for all $i = 0, \ldots, k$ breakpoints v_i are already chosen in such a way that bidding b_i at valuation $v \in [v_i, v_{i+1}]$ is better than bidding b_j for $j \neq i$ and bidding anything between b_0 and b_1, between b_1 and b_2 and so on to between b_{k-1} and b_k.

If $1 - E[x|v_{k-1} < x \leq v_k] \geq c$, the construction is over: pick $n = k$. Clearly bidding above b_k is inferior to bidding b_k: both strategies imply winning the item for sure (bidder 3 does not enter) but the first implies higher expected price.

If $1 - E[x|v_{k-1} < x \leq v_k] < c$ proceed to choosing $v_{k+1} > v_k$ in such a way that bidding b_k is superior to bidding slightly above b_k for $v \in [v_k, v_{k+1}]$.

To show that it is always possible to choose v_{k+1} in this way, note that as $v_{k+1} \to v_k$ and $v \in [v_k, v_{k+1}]$ both $\pi_{b_k}(v)$ and $\pi_{<b_{k+1}}(v)$ have the same limit. Note also that for $v \in [v_k, v_{k+1}]$ both $\pi_{b_k}(v)$ and $\pi_{<b_{k+1}}(v)$ are linear in v. Therefore, to complete the proof it suffices to show that for small enough $v_{k+1} - v_k$ two inequalities are satisfied: $\pi_{b_k}(v_k) > \pi_{<b_{k+1}}(v_k)$ and $\pi'_{b_k}(v_k) > \pi'_{<b_{k+1}}(v_k)$. This is done in the following series of propositions.

Proposition 4 $\left.\frac{\partial \bar{v}_k}{\partial v_{k+1}}\right|_{v_{k+1}=v_k} = \frac{1}{2}$.

Proof: By inspection.

Proposition 5 *If $k > 0$, then $\pi_{b_k}(v_k) > \pi_{<b_{k+1}}(v_k)$ if v_{k+1} is sufficiently close to v_k.*

Proof: Note that

$$\pi_{b_k}(v_k) - \pi_{<b_{k+1}}(v_k) = -\frac{F(v_{k+1}) - F(v_k)}{2}(v_k - b_k) +$$

$$\left(1 - \frac{F(v_{k+1}) + F(v_k)}{2}\right) F(\bar{v}_k)v_k - (1 - F(v_{k+1}))F(v_k + c)v_k.$$

This expression equals zero at $v_{k+1} = v_k$. Hence, to prove that it is positive at v_{k+1} close enough to v_k it suffices to show that its derivative with respect to v_{k+1} is positive at $v_{k+1} = v_k$. In view of Proposition 4 this derivative equals

$$\frac{\partial}{\partial v_{k+1}} \left(\pi_{b_k}(v_k) - \pi_{<b_{k+1}}(v_k)\right)\bigg|_{v_{k+1}=v_k} = -\frac{f(v_k)}{2}(v_k - b_k) +$$

$$(1 - F(v_k))v_k \frac{f(v_k + c)}{2} - \frac{f(v_k)}{2} F(v_k + c)v_k + f(v_k)F(v_k + c)v_k \geq$$

$$\frac{v_k}{2}\left[-f(v_k) + (1 - F(v_k))f(v_k + c) + F(v_k + c)f(v_k)\right] =$$

$$\frac{v_k(1 - F(v_k))(1 - F(v_k + c))}{2}\left[h(v_k + c) - h(v_k)\right] > 0, \tag{4}$$

QED.

Proposition 6 *The slope of $\pi_{b_k}(v)$ is higher than that of $\pi_{<b_{k+1}}(v_k)$ at $v = v_k$ if v_{k+1} is sufficiently close to v_k.*

Proof is similar to that of Proposition 3 and is omitted. As in the proof of Proposition 5, the key assumption is that hazard rate $h(x)$ is increasing in x.

Therefore, it is always possible to choose v_{k+1} in such a way that bidding above b_k is inferior to bidding b_k for $v \in [v_k; v_{k+1}]$. Moreover, the following proposition shows that the iterative process of choosing breakpoints v_k will stop after finitely many steps:

Proposition 7 *For any $c > 0$ there exists $\varepsilon > 0$ for which v_k can be chosen in such a way that $v_{k+1} - v_k > \varepsilon$, $k = 1, \ldots, n$*

Proof builds on that of Proposition 5. Note that, since by assumption there exists a constant $m > 0$ such that $h'(x) > m$ for any $x \in [0, 1]$, the righthandside of expression (4) is uniformly bounded away from zero. The assertion then follows immediately from the assumption that $F(x)$ is twice continuously differentiable on compact set $[0, 1]$, QED.

Once breakpoints v_k are chosen, equilibrium bids b_k are determined by equations $\pi_{b_{k+1}}(v_{k+1}) = \pi_{b_k}(v_{k+1})$. These equations, together with the observation that the slope of $\pi_{b_{k+1}}(v)$ is higher than that of $\pi_{b_k}(v)$, prove that bidding b_k is superior to bidding b_{k+1} at $v < v_{k+1}$ and bidding b_{k+1} is superior to bidding b_k at $v > v_{k+1}$; in particular, this implies that $b_{k+1} > b_k$. Finally, note that the slope of $\pi_{<b_{k+1}}(v)$ is lower than that of $\pi_{b_{k+1}}(v)$ for $v > v_{k+1}$ and is higher than that of $\pi_{b_k}(v)$ for $v < v_{k+1}$. Therefore, since $\pi_{<b_{k+1}}(v_{k+1}) < \pi_{b_k}(v_{k+1}) = \pi_{b_{k+1}}(v_{k+1})$, deviating from $b^{(n)}(v)$ is never optimal. The construction is complete.

5 Uncertain entry.

In this section I extend the analysis to the case of uncertain entry. I now introduce a probability q that bidder 3 is present and able to compete (if he chooses to). With the complementary probability $1-q$ entry is impossible for exogenous reasons, regardless of the first round bidders' strategies. I assume, however, that bidders 1 and 2 only know q but not whether bidder 3 is actually present.

The following lemma characterizes equilibria in this game.

Lemma 2 *Consider a step function*

$$b^(n)(v) = \begin{cases} b_k, \quad v_k \le v < v_{k+1}, \qquad k = 0, \ldots, n \end{cases}$$

that is an equilibrium bidding function for $q = 1$ *(i.e., when the potential entrant is present for sure). Then function*

$$b^(n)(q, v) = \begin{cases} q \cdot b_k, \quad v_k \le v < v_{k+1}, \qquad k = 0, \ldots, n \end{cases}$$

is an equilibrium bidding function for a given $q \in [0, 1]$.

Proof: It is straightforward to compute the profit functions for both equilibrium bidding $b^{(n)}(q, v)$ and any deviation from it. The only term that changes in the expressions, compared to the case of $q = 1$ studied above, is the expected payoff in round two, which is now a weighted average of what it was before and the valuation of the incumbent (corresponding to the case when the entry does not happen for exogenous reasons). We have

$$\pi_{q\cdot b_k}(v) = \begin{cases} \sum_{i=0}^{k-1}(F(v_{i+1}) - F(v_i))(v - qb_i) + \frac{F(v_{k+1}) - F(v_k)}{2}(v - qb_k) \\ \qquad + \left(1 - \frac{F(v_{k+1}) + F(v_k)}{2}\right)((1-q)v + qvF(\bar{v}_k)), & v \le \bar{v}_k, \\ \sum_{i=0}^{k-1}(F(v_{i+1}) - F(v_i))(v - qb_i) + \frac{F(v_{k+1}) - F(v_k)}{2}(v - qb_k) \\ \qquad + \left(1 - \frac{F(v_{k+1}) + F(v_k)}{2}\right)\left((1-q)v + q\left(\bar{v_k}F(\bar{v}_k) + \int\limits_{\bar{v}_k}^{v} F(x)dx\right)\right) & v < \bar{v}_k, \end{cases}$$

and

$$\pi_{<q\cdot b_{k+1}}(v) = \begin{cases} \sum_{i=0}^{k-1}(F(v_{i+1}) - F(v_i))(v - qb_i) + (F(v_{k+1}) - F(v_k))(v - qb_k) \\ \qquad + (1 - F(v_{k+1}))((1-q) + qvc), & v \le v_k + c, \\ \sum_{i=0}^{k-1}(F(v_{i+1}) - F(v_i))(v - qb_i) + (F(v_{k+1}) - F(v_k))(v - qb_k) \\ \qquad + (1 - F(v_{k+1}))\left((1-q) + q\left(v_k + c)F(v_k + c) + \int\limits_{v_k+c}^{v} F(x)dx\right)\right), & v > v_k + c. \end{cases}$$

By observation, all inequalities that support $b^{(n)}(q, v)$ as an equilibrium follow from respective inequalities for $b^{(n)}(v)$, QED.

Lemma 2 shows that even if the entry is uncertain, the equilibrium as a step function still exists. Moreover, there is a simple characterization of its parameters: the break points v_i are the same, while the equilibrium bids are proportional to the probability that entry is possible. The intuition is that if the entry is less probable, the incumbents will be reluctant to take risk of winning the item in the first round by bidding high, since their hope is now stronger that the entrant will not be around and they will get the item for free in the second round. That is why the equilibrium bids are decreasing when the probability of no entry (equal to $(1-q)$) increases. In particular, if $q = 0$ equilibrium bids are identically equal to zero – if there is no entry threat, the first-round bidders can get one unit each at zero price.

6 Multiple first round bidders.

In this section I extend the analysis to the case of more than two bidders in the first round (and no exogenous barriers to entry). Assume that there are two rounds of ascending price auction with identical items for sale one in each round and $n + 1$ bidders, first n of which are present for both rounds and the $(n+1)$th one can only enter to the second round, if he chooses to do so, at costs c. Each bidder wants at most one item and the winner of the first round leaves as she gets it. All bidders' valuations are independently drawn from the same distribution on $[0, 1]$. I am looking for a perfect Bayesian equilibrium in this new game.

For this scenario it is important to know what exactly the entrant observes. Note that in the case of two first-round bidders, all the available information relevant for the entry decision is contained in the final price of the first round; it fully reflects the strategy of the only entrant's opponent. Moreover, in the case of two first round bidders the ascending price auction is isomorphic to the second price sealed bid auction. Here I assume that bidder $n+1$ observes the entire outcome of the first round, i.e., who dropped out at what price.

In case of multiple first round bidders, a bidding strategy in the first round not only specifies the price at which to drop out if nobody else dropped out before, but also how to react to the other bidders dropping out (and that is where the ascending price auction is different from the second price auction). The final price (at which the last bidder drops out) does not convey all available information about the strategy of the first round bidders, because it does not show who dropped out at what price before. Therefore, I assume that the entrant not only observes the outcome (the final price and the identity of the winner) of the first round of the auction, but also the bidding process (who dropped out at what price).

A symmetric perfect Bayesian equilibrium in weakly undominated strategies that I am looking for will therefore consist of:

- A 'first drop out' function $d_1(v)$ specifying the price at which a first round bidder of type v drops out, provided that nobody dropped out before;
- A 'second drop out' function $d_2(v, d_1)$ that specifies the price at which a first round bidder drops out if the only other bidder who dropped out before did it at price d_1; and so on, to

- A ‘last drop out’ function $d_{n-1}(v, d_1, d_2, \ldots, d_{n-2})$ that specifies the price at which a first round bidder drops out if there is only one other bidder left;
- Beliefs of bidder $n+1$ about valuations of the other bidders as a function of $v, d_1, \ldots, d_{n-1}$;
- An entry decision of bidder $n+1$ as a function of his beliefs.

As before, once the entry decision is made and the game proceeds to the second round, the only weakly undominated strategies that remain to the bidders are to drop out at their valuations.

Lemma 3 *A symmetric perfect Bayesian equilibrium in weakly undominated strategies exists, in which*

- $d_1(v) = v$,
- $d_2(v, d_1) = v$ *and so on to* $d_{n-2}(v, d_1, \ldots, d_{n-3}) = v$,
- $d_{n-1}(v, d_1, d_2, \ldots, d_{n-2})$ *is a step function with its breakpoints and step bids being functions of* d_{n-2}.

In the equilibrium described by Lemma 3 all first round bidders except for two last ones drop out at their true valuations and the two who remain play strategies similar to those of the model with two first round bidders.

Proof: Clearly none of the first $n-2$ bidders can gain by deviating from the suggested strategies – none of them can win either of the two items except at price that exceeds their valuations. Suppose now that the price reached d_{n-2} in the first round and there are two bidders left. Each of them believes that the other is following the strategies suggested above, so she updates the distribution of his rival to the same distribution with cdf $F(v)/F(d_{n-2})$ with support $[d_{n-2}, 1]$. With probability $F(d_{n-2})$ the valuation of bidder $n+1$ will not exceed d_{n-2} and he will not enter. With the complementary probability $1 - F(d_{n-2})$ his valuation will be at least d_{n-2}, in which case the game will be equivalent to the one with two first round, for which a step function equilibrium was constructed above. Therefore, the entire game, once all but two first round bidders have dropped out, is equivalent to the game with two incumbents and uncertain entry discussed in previous section, so Lemma 2 applies. This completes the proof.

7 Multiple Entrants.

Another natural generalization of the model involves multiple potential entrants. Assume again that there are only two bidders in the first round but now there are $N \geq 2$ entrants with valuations independent from each others and from those of the first round bidders who may choose to enter. I want to study whether step equilibrium bidding functions exist for the first round bidders.

Assume that entrants infer, upon observation of the first round price, that the remaining bidder's valuation is confined between v_k and v_{k+1}. I calculate (uniform) entry threshold value $\bar{v}_k \geq v_{k+1}$. As long as all but one potential entrants enter if and only if their valuations are above $\bar{v}_k$ the expected payoff to the last remaining entrant with valuation $v \leq \bar{v}_k$ equals $F^{N-1}(\bar{v}_k)(v - E[u|v_k \leq u < v_{k+1}])$. He will choose to enter if and only if this expected payoff is at least c. Symmetric equilibrium value of $\bar{v}_k$ is thus determined from equation

$$F^{N-1}(\bar{v}_k)(\bar{v}_k - E[u|v_k \leq u < v_{k+1}]) = c.$$

Likewise, if an incumbent deviates and bids above b_k but below b_{k+1} and thus, by assumption, signals type v_k, the entry threshold $\hat{v}_k$ is determined from equation

$$F^{N-1}(\hat{v}_k)(\hat{v}_k - v_k) = c.$$

To determine whether a step function equilibrium of the type constructed in Section 4 exists, one must take the derivatives with respect to v_{k+1} of the slopes of $\pi_{b_k}(v)$ and $\pi_{<b_{k+1}}(v)$ at $v_{k+1} = v_k$. It is easy to verify that inequality

$$\left.\frac{\partial \pi'_{b_k}(v)}{\partial v_{k+1}}\right|_{v_{k+1}=v_k} \geq \left.\frac{\partial \pi'_{<b_{k+1}}(v)}{\partial v_{k+1}}\right|_{v_{k+1}=v_k} \tag{5}$$

is equivalent to

$$\frac{h(v_k)}{h(\hat{v}_k)} \leq \frac{NF^{N-1}(\hat{v}_k)}{1+F(\hat{v}_k)+\cdots+F^{N-1}(\hat{v}_k)},$$

which clearly can not hold, at least for small c. Therefore, the equilibrium construction of Section 4 does not work for multiple potential entrants.

However, there are two ways to construct a step function equilibrium for the case of multiple potential entrants. The first is to alter off-equilibrium beliefs of the entrants. Note that so far I assumed that a first round bidder who bids strictly between b_k and b_{k+1} is believed to have valuation v_k; and alternative assumption may be that any deviant is believed to have valuation zero. It is easy to see that this alternative specification will make deviation strictly less attractive for an first round bidder, except if her valuation equals zero. It can be shown that under this specification of beliefs and an additional assumption $f(0) = 0$ a step function equilibrium exists (even without monotonicity assumption on hazard rate $h(x)$).

Another way to restore a step function equilibrium in the case of multiple entrants, maintaining the same assumptions on beliefs as in Section 4, is to assume stochastic number of potential entrants, unknown to each of them (and to first round bidders). Assume for example that the number of potential entrants that are around is distributed geometrically with parameter p,[10] so that the probability that there are exactly l potential entrants equals $p_l = p(1-p)^l$. Denoting by $\bar{v}'_k$ the derivative of the equilibrium threshold level

$$\left.\frac{\partial \bar{v}_k}{\partial v_{k+1}}\right|_{v_{k+1}=v_k} = \frac{1}{2}\frac{p}{p-c(1-p)f(\hat{v}_k)},$$

one can rewrite inequality (5) as

$$\frac{f(v_k)}{2}\left(1-\frac{p}{1-(1-p)F(\hat{v}_k)}\right)+\frac{p(1-p)f(\hat{v}_k)\bar{v}'_k}{[1-(1-p)F(\hat{v}_k)]^2}(1-F(v_k)) \geq f(v_k)\left(1-\frac{p}{1-(1-p)F(\hat{v}_k)}\right),$$

or, equivalently,

$$\frac{1}{2}h(v_k)[1-(1-p)F(\hat{v}_k)] \leq p\bar{v}'_k h(\hat{v}_k),$$

[10] Geometric distribution is chosen for convenience, as its conditional probability of having $i+1$ entrants given that there is at least one entrant (the probability that each potential entrant takes into consideration) equals unconditional probability of having i entrants. A step function equilibrium can be similarly constructed for Poisson distribution with sufficiently high parameter λ.

which holds as long as p is large enough. Applying construction similar to that in Section 4, it is easy to verify that for any positive c there exists $p < 1$ such that a step function equilibrium exists when the number of potential entrants is distributed exponentially with parameter p.

Therefore, a step function equilibrium construction can be expanded to the case of multiple potential entrants in two cases: to the case when $f(0) = 0$ and entrants have most optimistic beliefs about an incumbent who deviates and to the case when the number of entrants is stochastic with sufficiently low expected value.

8 Conclusion.

This paper studies how the entry deterrence motives affects bidding behavior in a sequential ascending price auction. The equilibrium that I construct – with bidding strategies being step-functions – suggests that bidders would be conscious about the potential for future entry and that they will want to at least partially conceal information about their private valuations in an effort to deter entry. As a result, the outcome of the auction is generally not efficient: the two items do not necessarily go to the bidders who value them the most. There are two sources of inefficiency, both caused by the coarseness of the information communicated through the step-function strategies. One source is that the potential entrant may not enter even though he has a higher valuation than the bidder that dropped out in the first round. The other source of inefficiency is that both first-round bidders may drop out at the same time in which case the item may be assigned to the first-round bidder who values it the lower.

These conclusions resemble in some respects those in Bhattacharyya [3]. Bhattacharyya studies a two-person, two-stage auction in which entry is endogenous in the first stage: first one bidder bids and then the other bidder decides whether to enter. If the other bidder enters, the auction proceeds to the second stage, which is an ascending auction, else the first bidder gets the item at the price he bid. Assuming that the second bidder doesn't enter if he is indifferent (which is a crucial assumption), there exists a non-trivial first-round bid (equal to half the true valuation of the first bidder when the distribution is uniform). This result is similar to mine in that the first bidder behaves strategically to deter entry. However, the setting is strategically very different from mine, since in an ascending auction the winning bidder only pays the price at which her opponent drops out, not the price at which she is prepared to drop out herself. This opens the door for bluffing, which indeed occurs in the equilibrium of my game, but not in that of Bhattacharyya's.

It is interesting to contrast my findings with those of von der Fehr [9]. Although his setting is remarkably similar to mine, it differs in two respect: whether bidders who do not participate in the first round can observe its outcome and whether bidders have to incur entry costs only once as they prepare the bid or every time they participate in an auction. While these differences appear minor, they change the results dramatically: in his model there exists an equilibrium first round bidding strategy that is strictly monotone in valuation, whereas I show that such strategy can not exist in my model. Furthermore, he shows that in his setup bidding strategies can not be constant over an interval, while I derive equilibrium bidding functions which are step functions.

The resulting family of step function equilibria also resemble partition equilibria in Crawford and Sobel [5]. In their paper they consider a wide class of games where an informed party (sender) sends as signal to an uniformed party (receiver) about sender's type which is of relevance to the payoffs of both. They find that under some assumptions on the structure of the payoffs the optimal signal for the sender consists of a subset of the range of types to which his type belongs. However, they assume away any uncertainty about the receiver's own type, which precludes their result from being at least directly applicable to the context of private value auctions. In their model equilibria are also multiple; they can, however, rank them in terms of coarseness of the signal. In my model equilibria typically can not be ranked in a similar way, since not only there typically exist equilibria with different number of steps in the bidding function, but also there is some leeway in the way that break points are chosen.

Many questions remain to be studied in sequential auctions with potential entry in later rounds. First, I do not know whether the family of step function equilibria that I have identified exhaust the class of all equilibria. Second, it would be of interest to analyze the optimal entry cost that an seller should charge (and whether later entry should be more or less expensive than early entry). More generally, the question of an optimal auction design, including optimal information transmission (what should potential entrants be allowed to know), remains open.

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