Tournaments with reserve performance

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Abstract

We study tournaments where winning a rank-dependent prize requires passing a reserve—a minimum performance standard. Agents' performance is determined by effort and noise. For log-concave noise distributions the optimal reserve is at the modal performance, and the optimal prize scheme is winner-take-all. In contrast, for log-convex noise distributions the optimal reserve is at the lower bound of the distribution of performance, which is passed with probability one in equilibrium, and it is optimal to award equal prizes to all qualifying agents. These pay schemes are optimal in a general class of symmetric monotone contracts that may depend on cardinal performance.

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1 Introduction

In many tournaments, simply performing better than others is not sufficient to win. There is often a minimum standard of performance to be passed, or a fixed challenge to be completed, in order to secure a prize. The Bréant Prize was offered in 1858 by the French Academy of Sciences for a cure for cholera (Crosland and Gálvez, 1989). In 1919, Raymond Orteig offered a prize to the first person who could fly nonstop from New York to Paris or the other way around (Bak, 2011). While these challenges are fixed and indivisible, in other settings the tournament organizer can be more flexible in choosing the level of difficulty. The Netflix Prize was set up in 2006 for the best algorithm predicting user ratings of films, under the condition that it had to be at least 10% more accurate than Netflix's existing algorithm at the time.[1] The 2004 DARPA Grand Challenge was a competition for the fastest driverless car to complete a 142-mile course (Hooper, 2004).

In this paper, we study how a minimum performance standard, which we call reserve performance, or simply a *reserve*, should be set by a tournament organizer. We utilize a rank-order tournament model à la Lazear and Rosen (1981) where players' performance is their effort distorted by idiosyncratic additive noise, and focus on risk-neutral, homogeneous players. While reserve price is a standard design tool in private value auctions, where its role is to filter out low valuation bidders (Myerson, 1981; Riley and Samuelson, 1981), it is not clear why a reserve may be useful in a symmetric tournament setting where players' types are the same and publicly known, and winning in equilibrium is determined entirely by luck. Yet, we show that by carefully choosing a reserve, it is possible to boost players' performance. Moreover, by combining reserve with prize allocation, rank-order tournaments achieve optimality in a much wider class of monotone, symmetric pay schemes that can be based on cardinal performance.

We find that the optimal tournament design depends critically on the properties of noise. When the distribution of noise is log-concave, as is the case for most popular distributions such as normal or logistic, it is optimal to set the reserve performance at the equilibrium effort plus the *mode* of the noise distribution, and to award a single prize. In this case, the optimal reserve mimics the modal competitor. Intuitively, in a winner-take-all (WTA) tournament, each player's marginal incentives in the symmetric equilibrium are determined by the likelihood that his noise realization coincides with the maximum between (i) the largest realization of noise among all other players and (ii) the *threshold level of noise*, defined as the reserve performance less the equilibrium effort. Therefore,

[1] See https://www.netflixprize.com/rules.html.

setting the threshold level of noise at the mode maximizes marginal incentives. It is, of course, possible that none of the players reaches the reserve performance, in which case nobody receives the prize. The Bréant Prize was never awarded, and the best-performing vehicle completed only 11 miles out of 142 in the 2004 DARPA Grand Challenge.

For a log-convex distribution of noise, e.g., the Pareto distribution, the optimal pay scheme is quite different. The reserve still mimics the modal competitor; however, since the density of noise is decreasing in this case, the mode is at the lower bound of the support, and hence exerting the equilibrium effort alone *guarantees* passing the reserve. Furthermore, it is optimal to share the prize equally among all the players who pass the reserve.[2] In equilibrium, there is effectively no competition among the players, and the tournament can be replaced by individual bonuses which are paid to all who qualify. Alternatively, this prize scheme can be implemented by randomly awarding one prize—the entire budget—to one of the players who passed the reserve.

It has long been argued (see, e.g., the discussion in the original article by Lazear and Rosen, 1981) that an important advantage of tournament pay schemes is that they only rely on *ordinal* performance comparisons, as opposed to standard moral hazard contracts that are based on cardinal information.[3] Therefore, tournaments can be used in settings where cardinal performance measurements are unreliable and cannot be contracted on, but, generally speaking, the principal should be able to do better when cardinal information is available. Surprisingly, we show that when noise is either log-concave or log-convex, the principal *cannot* do better than using the corresponding optimal tournament with reserve. In other words, the optimal tournament-with-reserve contracts we identify are optimal in a much larger class of pay schemes. This result is especially important in the log-convex case because the optimal moral hazard contract even for a single agent is unknown when the distribution of noise is not log-concave and hence the monotone likelihood ratio property (MLRP) does not hold. The result that, in equilibrium, the optimal tournament contract for log-convex noise is an individual bonus scheme, effectively provides a solution to this problem for our setting.[4]

[2] Even though all the players pass the reserve with probability one in equilibrium, the tournament prize schedule that implements this equilibrium is *conditional*, in the sense that the prize budget is fully shared only among the players who pass the reserve. More generally, in conditional prize schemes prizes can depend on the number of players with performance above the reserve. The simpler case of *unconditional* prizes is studied in Section 4.3.

[3] Strictly speaking, reserve performance is a cardinal benchmark. Yet, *comparing* performance to a particular benchmark is akin to comparing performance across agents, and is still informationally less demanding than measuring performance on a cardinal scale.

[4] Methodologically, we obtain our optimal pay scheme results without relying on the first-order ap-

We extend our results in several dimensions. We show that in both the log-concave and log-convex cases individual effort always decreases in the number of players—which does not hold without a reserve—while aggregate effort always increases if the marginal cost of effort is convex. We also consider arbitrary (unimodal) noise distributions that are not necessarily log-concave or log-convex, and arbitrary prize schedules that are not necessarily optimal. Under an additional monotonicity restriction on prize schedules, we show that the optimal threshold level of noise is always at or to the right of the mode, i.e., the optimal reserve is weakly higher than the modal performance. We also show that the equilibrium effort is decreasing in the number of players for a wide class of prize schedules and reserves. Finally, we obtain a tighter characterization of optimal tournament design when prizes are restricted to being *unconditional*, i.e., independent of the number of players who pass the reserve (see fn. 2). In this case, the optimal threshold noise is always at the mode, and optimal prizes are fully characterized for any (unimodal) noise distribution.

Related literature Starting with Lazear and Rosen (1981), a number of authors investigated the problem of optimal prize allocation in rank-order tournaments. For symmetric, unimodal noise distributions, Krishna and Morgan (1998) show the optimality of WTA in small tournaments (up to four risk-neutral players or up to three risk-averse players). Kalra and Shi (2001) and Terwiesch and Xu (2008) show that WTA is optimal for some particular log-concave distributions of noise. Schweinzer and Segev (2012) show the optimality of WTA for Tullock contests.[5] Akerlof and Holden (2012) and Ales, Cho and Körpeoğlu (2017) make somewhat different modeling assumptions and do not state most results in terms of the primitives; the latter paper, however, shows the crucial role of log-concavity of the noise distribution for the optimality of the WTA prize scheme. Drugov and Ryvkin (2020*b*) and Drugov and Ryvkin (2021) extend and generalize many of the previous results for the risk-neutral and risk-averse case, respectively. Yet, while already Lazear and Rosen (1981) mentioned competing against a "fixed standard," i.e., a reserve performance, as a way to obtain the first-best effort level, this literature has so far ignored

proach. Instead, we identify an upper bound on the symmetric equilibrium effort across all feasible pay schemes (ordinal, and then also cardinal), and then show that the corresponding tournament with reserve achieves that upper bound.

[5] The multi-prize nested Tullock contest can be equivalently represented as a rank-order tournament with the Gumbel distribution of noise, which is log-concave (Jia, 2008; Fu and Lu, 2012; Ryvkin and Drugov, 2020).

the analysis of optimal reserve performance and associated prize schemes.[6]

Kirkegaard (2021) considers the optimal tournament design problem using the standard moral hazard approach. He allows for general performance functions and asymmetric players but—as is common in the literature—imposes the monotone likelihood ratio property (MLRP) and the convexity of the distribution function condition (CDFC). The focus of his paper is on the design of biased contests when players are asymmetric. In contrast, we do not rely on the first-order approach in the construction of optimal contracts, which allows us to go beyond MLRP (log-concave noise distributions in our setup). We demonstrate that the optimal structure of tournament pay schemes is dramatically different for log-concave versus log-convex noise distributions. In addition, we assume symmetric players, and thus unbiased tournaments.

Reserve prices have been studied extensively in the auction literature starting at least with Myerson (1981) and Riley and Samuelson (1981), but the role of a reserve price in auctions is different. In a standard independent private value auction setting, types are privately observed before bidding. The optimal reserve price excludes low types as long as the effect of the FOSD shift in the distribution of types outweighs the loss of competition due to a reduced number of bidders. In a (symmetric) tournament, we can think of noise realizations as "types," except they are determined after bidding, i.e., a reserve performance excludes low types *ex post*. Therefore, there is no reduction in competition due to a reduction in the number of bidders, but rather there is some loss in marginal incentives because a player may not get a prize even when he is highly ranked if he cannot pass the reserve.

Finally, this paper is related to the literature on contracting under moral hazard. Since the classical papers of Rogerson (1985) and Jewitt (1988), the monotone likelihood ratio property (MLRP)—which is equivalent to log-concavity of the noise distribution in our setting—has been identified as necessary for the first-order approach. There has been some progress recently in analyzing settings where MLRP does not hold (e.g. Poblete and Spulber, 2012; Kadan and Swinkels, 2013; Kirkegaard, 2017). Still, the optimal contract when the noise is not log-concave is still unknown—even in the single agent case.

The rest of the paper is organized as follows. Section 2 sets up the model, and Section

[6] In the logit random utility model, which is formally related to the Tullock contest, the outside option—effectively, a reserve—has been introduced a long time ago (see, e.g. Anderson, De Palma and Thisse, 1992). Yet, its optimality has not been studied because it is treated as a parameter. Also, log-concavity (or a slightly weaker condition of $-1/(n+1)$-concavity, see Caplin and Nalebuff, 1991) has to be assumed for technical reasons in that model.

3 presents our main results. Extensions are considered in Section 4. Section 5 concludes. All missing proofs are collected in the Appendix.

2 Model setup

2.1 Preliminaries

There are $n \geq 2$ symmetric, risk-neutral players (we refer to a player individually as "he"), indexed by $i = 1, \dots, n$, who simultaneously and independently choose effort levels $e_i \in \mathbb{R}_+$. Player i's performance is stochastic and given by $Y_i = e_i + X_i$, where shocks X_i are i.i.d. across players, with an absolutely continuous cumulative density function (cdf) $F(\cdot)$ and probability density function (pdf) $f(\cdot)$ defined on an interval support $\mathcal{X} = [\underline{x}, \overline{x}]$ (where $\underline{x} < \overline{x}$ can be finite or infinite). We assume that f is unimodal with a mode x_m, continuous, piece-wise differentiable, and satisfies $f(\overline{x}) = 0$.[7] The cost of effort e_i to player i is $c(e_i)$, where $c(\cdot)$ is C^1, strictly increasing and strictly convex on $[0, \bar{e}]$, C^3 on $(0, \bar{e}]$, and satisfies $c(0) = c'(0) = 0$. Here, $\bar{e} = c^{-1}(1)$ is the maximum undominated effort in a contest with unit prize.

A principal ("she") sets a reserve $\rho \in \mathbb{R}$ and commits to a family of prize schedules $\mathbf{v} = (\mathbf{v}^1, \dots, \mathbf{v}^n)$, where $\mathbf{v}^s = (v_1^s, \dots, v_s^s)$ is a vector of prizes in the event s players pass the reserve, $s = 1, \dots, n$. Vectors $\mathbf{v}^s$ are nonnegative, monotone, and satisfy a budget constraint with total prize money normalized to one: $v_1^s \geq \ldots \geq v_s^s \geq 0$, $\sum_{r=1}^{s} v_r^s \leq 1$, $s = 1, \dots, n$. We will refer to prize schedules $\mathbf{v}$ satisfying these conditions as *feasible* and use $\mathcal{V}$ to denote the set of such prize schedules. Combinations of a reserve and a prize schedule, $(\rho, \mathbf{v})$, will be referred to as (tournament-with-reserve) *pay schemes*.

The principal observes whether or not each player's performance passes the reserve,[8] as well as the ranking of performance above the reserve. If none of the players passes the reserve, no prizes are awarded; otherwise, prize v_r^s is awarded to the player with performance ranked r among the s players above the reserve. That is, the player with the highest performance gets v_1^s, with the second highest—v_2^s, etc.[9]

[7] The last condition holds automatically when $\overline{x} = +\infty$. For a finite $\overline{x}$, it is adopted mainly for the ease of exposition, and is not required for many of our results when f is log-concave. In particular, our results hold when f is uniform. Note, however, that we do *not* assume $f(\underline{x}) = 0$, which would exclude many important cases, e.g., log-convex distributions.

[8] We use the terms "pass" or "above" in the weak sense. Meeting the reserve is considered passing, but happens with probability zero for an atomless F.

[9] Ties are broken randomly, but occur with probability zero for an atomless $F(\cdot)$.

Note that the prize schemes we consider are *conditional*, in the sense that the allocation of prizes may depend on the number of qualifying players, s. This is the most flexible approach, and it makes sense because the principal observes s and can commit to all possible scenarios. However, in some situations, due to institutional constraints, prize schedules have to be *unconditional*, i.e., such that v_r^s is independent of s for all $s \geq r$. The restriction to unconditional prizes simplifies the analysis substantially and allows for a more universal characterization of the optimal reserve. We consider unconditional prize schedules in detail in Section 4.3.

2.2 The principal's problem

The principal's objective is to maximize total effort, $\sum_{i=1}^{n} e_i$, which in our setting is equivalent to maximizing total expected performance, $\mathbb{E}(\sum_{i=1}^{n} Y_i)$. Furthermore, we restrict attention to the implementation of a symmetric, pure strategy equilibrium where all players $i = 1, \ldots, n$ choose the same effort $e_i = e^*$. For a fixed number of players, the principal's objective is then also equivalent to maximizing individual effort, e^*, as well as the highest expected performance, $\mathbb{E}(\max\{Y_1, \ldots, Y_n\})$.[10]

Suppose all players other than one indicative player choose effort e^*, and the indicative player chooses some deviation effort e. The probability that $s-1$ players, not including the indicative player, pass the reserve is $\gamma_{s-1}(\rho - e^*)$, where

$$\gamma_{s-1}(a) = \binom{n-1}{s-1} F(a)^{n-s}[1-F(a)]^{s-1}, \quad a \in \mathcal{X}.$$

Note that, in equilibrium, any reserve $\rho < e^* + \underline{x}$ is passed with probability one, while any reserve $\rho > e^* + \overline{x}$ is never passed. Therefore, without loss we can restrict attention to reserves such that $\rho - e^* \in \mathcal{X}$. Variable $a = \rho - e^*$ can be interpreted as the threshold level of noise that a player would need in order to pass the reserve in a symmetric equilibrium.

[10]Such objectives are relevant in situations where the principal values the prize money little, or not at all, relative to effort or output. In organizations, budgets are often allocated to divisions to be spent for a specific purpose, and any residual funds are swept away at the end of the budgeting period. Suppose, for example, that a division manager receives an allocation that can only be spent on year-end bonuses; yet, the manager's own performance evaluation is determined by the total output of her subordinates. In this case, the division manager would only be interested in motivating employees, and would not benefit at all if none of them receives a bonus. Similarly, a research funding agency may be interested in boosting research output and innovation regardless of whether grants are eventually awarded or prizes are paid out. In an educational setting, instructors value students' effort regardless of the grades they eventually assign, and those grades are costless.

The expected payoff of the indicative player can be written as

$$\pi(e, e^*) = \sum_{s=1}^{n} \gamma_{s-1}(\rho - e^*) \sum_{r=1}^{s} p^{(r,s)}(e, e^*) v_r^s - c(e), \tag{1}$$

where

$$p^{(r,s)}(e, e^*) = \binom{s-1}{r-1} \int_{[\rho-e,\infty)\cap\mathcal{X}} \tilde{F}(e - e^* + x)^{s-r} [1 - \tilde{F}(e - e^* + x)]^{r-1} dF(x) \tag{2}$$

is the probability that the indicative player is ranked r among s players above the reserve, conditional on there being $s - 1 \geq r - 1$ other players above the reserve. Here, $\tilde{F}(x) = \frac{F(x)-F(a)}{1-F(a)} \mathbb{1}_{\{x \geq a\}\cap\mathcal{X}}$ is the updated cdf of noise for those players.

We look for a candidate symmetric equilibrium as a solution to the symmetrized first-order condition (FOC), $\pi_e(e^*, e^*) = 0$. Notice that $\gamma_{s-1}(\rho - e^*)$ is independent of e; therefore, the symmetric FOC takes the form

$$g(\rho - e^*; \mathbf{v}) = c'(e^*), \quad g(a; \mathbf{v}) = \sum_{s=1}^{n} \gamma_{s-1}(a) \sum_{r=1}^{s} \tilde{\beta}_{r,s}(a) v_r^s, \tag{3}$$

where $\tilde{\beta}_{r,s}(a) = p_e^{(r,s)}(e^*, e^*)$ is

$$\tilde{\beta}_{r,s}(a) = \binom{s-1}{r-1} \int_a^{\overline{x}} \tilde{F}(x)^{s-r-1} [1 - \tilde{F}(x)]^{r-2} [s - r - (s-1)\tilde{F}(x)] \tilde{f}(x) dF(x) + f(a) \mathbb{1}_{r=s}. \tag{4}$$

In deriving this expression, we assumed that the lower limit of integration in (2) is $\rho - e \geq \underline{x}$, which holds in equilibrium.

Coefficient $\tilde{\beta}_{r,s}(a)$ represents the marginal effect of effort, in equilibrium, on the probability of being ranked r and passing the reserve $\rho = e^* + a$, conditional on $s - 1$ other players passing it as well. It can be shown that for any s, $\sum_{r=1}^{s} \tilde{\beta}_{r,s}(a) = f(a)$. Indeed, the sum of these coefficients is the marginal effect of effort on the probability of being ranked s *or better*, i.e., effectively, on the probability of simply passing the reserve, which is $\frac{\partial}{\partial e}[1 - F(\rho - e)]_{e=e^*} = f(a)$.

It is clear that (3) is a necessary condition for a pure strategy symmetric equilibrium but it is, in general, not sufficient. However, we can postpone the discussion of the equilibrium existence. The approach we take instead is, first, to consider the principal's problem *as if* Eq. (3) determines the equilibrium, and then impose additional restrictions on the primitives and verify that the optimal payment scheme indeed induces the equilibrium.[11] Formally, we consider the problem

$$\max_{(\rho,\mathbf{v})} e^* \quad \text{s.t. } g(\rho - e^*; \mathbf{v}) = c'(e^*), \tag{5}$$

where the maximization is over all feasible tournament-with-reserve pay schemes.

3 Optimal pay schemes

3.1 Optimal tournaments with reserve

In general, problem (5) is rather complex because it requires joint optimization over the reserve and prize schedules. However, as we show in this section, in two prominent special cases optimal prizes are independent of the reserve and can be identified separately. The result relies on the following lemma.

Lemma 1 *For any $a \in \mathcal{X}$, $s \in \{1, \ldots, n\}$,*
(i) If f is log-concave, then $\tilde{\beta}_{r,s}(a)$ is decreasing in r;
(ii) If f is log-convex, then $\tilde{\beta}_{r,s}(a)$ is increasing in r.

Lemma 1 is based on the following representation of coefficients $\tilde{\beta}_{r,s}(a)$, obtained from (4) via integration by parts (for details, see Appendix A):

$$\tilde{\beta}_{r,s}(a) = -\frac{1 - F(a)}{s}\mathbb{E}(\lambda(\tilde{X}_{(s-r+1:s)})).$$

Here, $\lambda(x) = \frac{f'(x)}{f(x)}$ is the log-derivative of the pdf and $\tilde{X}_{(s-r+1:s)}$ is the order statistic of the updated distribution of noise. The latter is decreasing in r in the sense of first-order stochastic dominance (FOSD). The result then follows because $\lambda(x)$ is decreasing (resp. increasing) in x for f log-concave (resp. log-convex).

Recall that $c'(\cdot)$ is a strictly increasing function. The constraint in problem (5), therefore, implies that the highest effort e^* that can possibly be reached is given by $c'^{-1}(g_m)$, where $g_m = \max_{a\in\mathcal{X},\mathbf{v}\in\mathcal{V}} g(a; \mathbf{v})$ if the maximum exists. The following proposition—our

[11] Thus, effectively, we consider a relaxed problem and then formulate conditions under which its solution also solves the original problem of interest, see Lemmas A1 and A2 in the Appendix.

first main result—shows that in each of the two cases of Lemma 1 this maximum is well-defined and induced by a particular combination of prizes and reserve.[12]

Proposition 1 *(i) If f is log-concave, then the winner-take-all prize schedule, $v_r^s = \mathbb{1}_{r=1}$, and reserve $\rho^* = c'^{-1}(g_{\text{WTA}}(x_m)) + x_m$, are optimal. The resulting effort, $e^* = c'^{-1}(g_{\text{WTA}}(x_m))$, is the unique symmetric equilibrium in the tournament. Here,*

$$g_{\text{WTA}}(a) = (n-1)\int_a^{\overline{x}} F(x)^{n-2} f(x) dF(x) + F(a)^{n-1} f(a). \tag{6}$$

(ii) If f is log-convex, then equal prize sharing among all players who pass the reserve, $v_r^s = \frac{1}{s}$, and reserve $\rho^* = c'^{-1}(\frac{f(\underline{x})}{n}) + \underline{x}$, are optimal. The resulting effort, $e^* = c'^{-1}(\frac{f(\underline{x})}{n})$, is the unique symmetric equilibrium in the tournament.

The intuition for Proposition 1 is as follows. The introduction of a reserve creates two effects. First, the reserve acts as an additional artificial competitor with deterministic performance. In order to maximize its impact, the additional player's performance should be set at the most likely level—the equilibrium effort plus the mode of the noise distribution. The second effect is due to *ex post* prize sharing, which arises because, in general, prize schedules are conditional, i.e., prizes may depend on the number of players who pass the reserve. Depending on a prize schedule, a player who passes the reserve may prefer that either fewer or more other players pass it along with him. For example, consider a contest of three players and prize schedule $\mathbf{v} = ((1), (1, 0), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}))$. That is, if one or two players pass the reserve, the winner gets the entire prize, but if three players pass, the prize is divided equally among all of them. In this case, conditional on being ranked first and passing the reserve, a player would prefer a higher reserve so that fewer others could pass; but conditional on being ranked second or third and passing the reserve, a player would prefer a lower reserve so that all three players pass. The overall effect combines all these incentives and, depending on a prize schedule, can go in either direction.

For part (i) of the proposition, when the distribution of noise is log-concave, Lemma 1 implies that the winner-take-all (WTA) schedule is optimal for any reserve. By construction, WTA is an unconditional schedule; therefore, the *ex post* prize sharing effect

[12] The proof of Proposition 1 contains Lemmas A1 and A2 which provide technical conditions that ensure that the proposed symmetric equilibrium exists and is unique in each case. In essence, we require that the distribution of noise be sufficiently dispersed, but do not restrict its shape beyond mild regularity. For that reason, and not to distract the reader, we do not include these conditions in the statements of the propositions in this section. For more details, see the discussions at the end of Section 3.2 and after the proof of Lemma A2 in Appendix A.

is absent, and the optimal reserve is at the modal performance.[13] For part (ii), when the distribution of noise is log-convex, Lemma 1 implies equal prize sharing is optimal for any reserve. The *ex post* prize sharing effect then calls for a higher optimal reserve because more passing players means a smaller prize to each of them—as in the example above. However, log-convex densities are decreasing fast, and, when the reserve is increased beyond the modal performance, the loss of incentives from shifting the artificial competitor is not compensated by the positive effect of a higher expected prize due to *ex post* prize sharing.[14]

Proposition 1 is illustrated in Figure 1. We consider tournaments of four players with noise distributed as Gumbel(0,1), $f_{\text{Gumbel}(0,1)}(x) = \exp(-x - \exp(-x))$ (the top panel), and Pareto(1,1), $f_{\text{Pareto}(1,1)}(x) = \frac{1}{x^2}\mathbb{1}_{x\geq 1}$ (the bottom panel). Each panel shows function $g(a; \mathbf{v})$, which determines the equilibrium effort with reserve $\rho = e^* + a$, for all possible prize schemes of the form $[k_1, k_2, k_3, k_4]$, where k_s is the number of equal prizes awarded if s players pass the reserve.[15] The cases of WTA, $[1, 1, 1, 1]$, and maximum prize sharing, $[1, 2, 3, 4]$, are highlighted. The Gumbel distribution is log-concave, with a mode at zero; hence Proposition 1(i) states that the optimal pay scheme is WTA with the threshold level of noise $a^* = 0$. As seen from the top panel in Figure 1, this is indeed the case. Moreover, consistent with Lemma 1, the WTA scheme is optimal for any value of a, not just the optimal one, whereas maximum prize sharing is the worst scheme for any a. In contrast, the Pareto distribution is log-convex, and Proposition 1(ii) implies maximum prize sharing and reserve at the lower bound are optimal. Consistent with Lemma 1, this scheme is optimal for all a, and the WTA scheme yields the lowest effort for all a. At low reserves (a approaching $\underline{x}$), the curves are grouped into "bunches" converging to four distinct levels of $g(a; \mathbf{v})$. From top to bottom, these correspond to the values of $k_4 = 1$ through 4 for Gumbel, and $k_4 = 4$ through 1 for Pareto. Indeed, for $a \to \underline{x}$, all players pass the reserve with probability one in equilibrium; therefore, k_4 is the only relevant prize allocation rule. For Gumbel, WTA is optimal and prize sharing is detrimental for effort, and the opposite holds for Pareto. As expected, effort converges to zero under full prize sharing ($k_4 = 4$) for Gumbel.

As noted above, Figure 1 shows that, depending on the distribution of noise, the WTA

[13] This result generalizes to any unconditional prize schedule, see Proposition 7 in Section 4.3.

[14] Importantly, log-convex densities cannot have an interior mode. In order to fully illustrate the interplay between the two effects, one has to consider more general noise distributions that are neither log-concave nor log-convex. In Section 4.2, we consider such distributions and provide further examples.

[15] Proposition 5 in Section 4.2 shows that the optimal prize scheme has this form for any noise distribution.

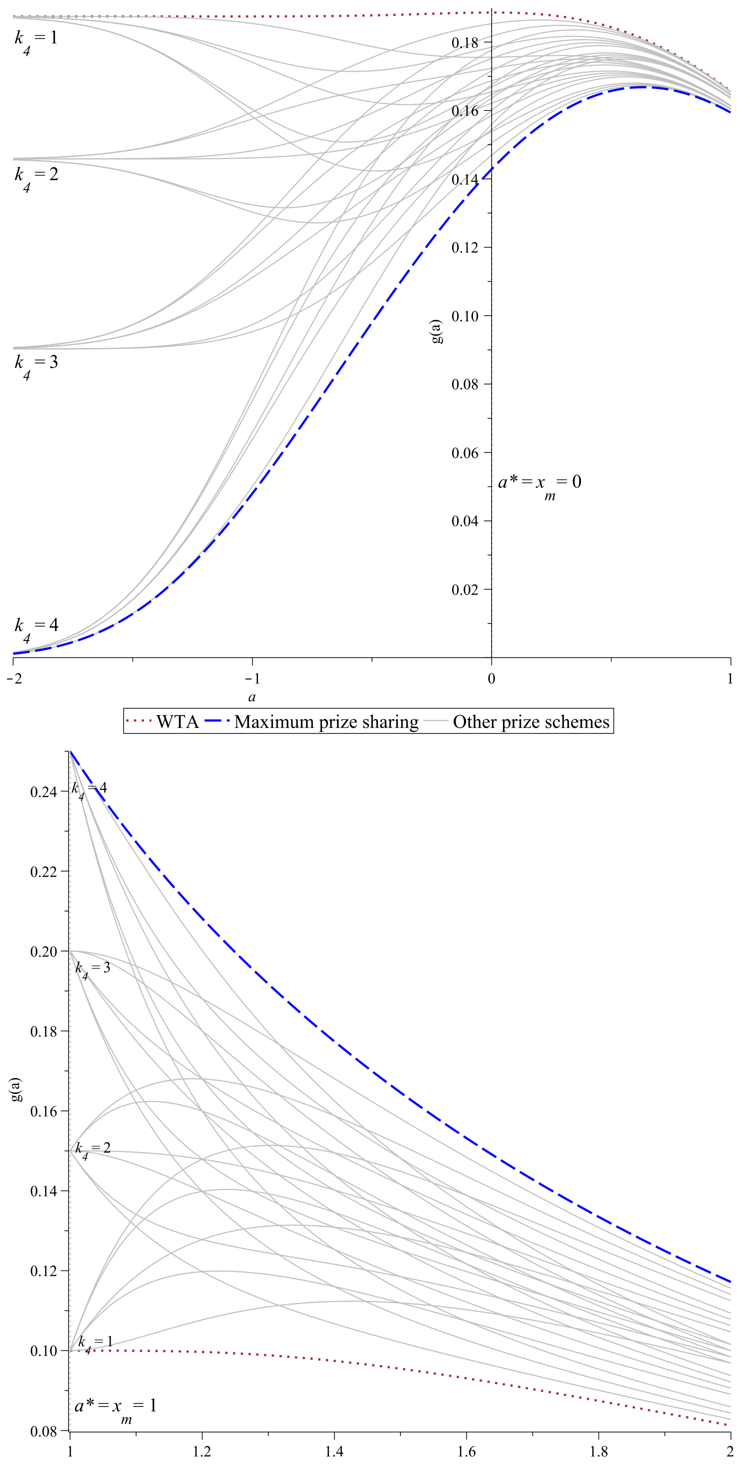

Figure 1: $g(a;\mathbf{v})$ for Gumbel(0,1) (top) and Pareto(1,1) (bottom) distributions, $n = 4$ and various prize schemes. The curves show $g(a;\mathbf{v})$ for all possible prize schemes of the form $[k_1, k_2, k_3, k_4]$, where $k_s \leq s$ represents the number of equal prizes awarded when s players pass the reserve.

and maximum prize sharing can be both the best or the *worst* possible prize schemes. Hence, having at least some idea about the shape of the distribution of performance shocks is critical. Using, say, the WTA scheme and hoping that it will do fine even when it is not optimal may have dramatic consequences for performance. Another interesting feature of Figure 1 is that some $g(a;\mathbf{v})$ curves are not unimodal in a. Those curves correspond to prize schedules (v_r^s) with nonmonotonicities in k_s; that is, cases such as $[1,2,2,1]$ where a smaller number of prizes are awarded when more players pass the reserve.

3.2 Optimal cardinal pay schemes

The tournament-with-reserve pay schemes identified in Proposition 1 are optimal in a much wider class of pay schemes where compensation can be conditioned on agents' cardinal outputs. To see this, let $\mathbf{w}:\mathbb{R}^n \to \mathbb{R}_+^n$ denote a general cardinal pay scheme in which $w_i(\mathbf{y})$—player i's compensation—may depend on the entire vector of outputs $\mathbf{y}=(y_1,\ldots,y_n)$. We consider pay schemes satisfying the following properties for all $\mathbf{y}\in\mathbb{R}^n$ and $i\in\{1,\ldots,n\}$.

Assumption 1 *(a) Symmetry:* $w_i(y_1,\ldots,y_n)=w_{\tau(i)}(y_{\tau(1)},\ldots,y_{\tau(n)})$ *for any permutation* τ *of indices* $\{1,\ldots,n\}$*;*

(b) Monotonicity: $w_i(\mathbf{y})$ *is increasing in* y_i*;*

(c) Budget constraint: $\sum_{i=1}^n w_i(\mathbf{y})\le 1$.

Proposition 2 *The optimal tournaments with reserve—WTA and fully equitable prize sharing for* f *log-concave and log-convex, respectively—are optimal in the class of pay schemes satisfying Assumption 1.*

Proposition 2 is our second main result. It shows that the restriction to tournaments with reserve is without loss of generality if one considers arbitrary pay schemes based on fully observable cardinal outputs, including individual contracts as well as arbitrarily complex relative performance schemes. An obvious advantage of tournament schemes is in that they rely only on ordinal output comparisons.[16]

The equilibrium existence conditions underlying Propositions 1 and 2 are stated and discussed as part of the proof of Proposition 1 in the Appendix. They ensure that, in

[16] It may be argued that in the presence of a cardinal reserve a tournament pay scheme ceases to be fully ordinal. However, it is still ordinal in the sense that the only additional assessment the principal needs to make is whether or not output *passes* the reserve—a qualitative assessment, which is arguably not more demanding than deciding whether one player's output is higher than another player's output. This argument fits the interpretation of reserve as an additional, virtual "player" in the tournament.

each of the two cases, (i) the FOC (3) has a unique solution, $e^* > 0$; (ii) this solution is incentive-compatible, i.e., it yields a global maximum to $\pi(e, e^*)$ over e; and (iii) that it is individually rational, i.e., $\pi(e^*, e^*) \geq 0$. Ultimately, the conditions amount to requiring that the distribution of noise be sufficiently dispersed and the cost of effort be uniformly strictly convex. Such conditions are typical for (pure) equilibrium existence in noisy tournament models.[17] Importantly, they do not restrict the *shape* of the distribution of noise as do the MLRP or CDFC conditions in the contract literature.

4 Applications and extensions

4.1 The effect of the number of players

It is of interest to explore how the optimal reserve and the resulting equilibrium effort change with the number of players. The result is given by the following proposition.

Proposition 3 *Suppose $f(\cdot)$ is log-concave or log-convex. Then the optimal equilibrium effort, e^*, and optimal reserve, ρ^*, are decreasing in n.*

Recall that for both log-concave and log-convex noise distributions, we can write the optimal reserve as $\rho^* = e^* + x_m$, where $x_m = \underline{x}$ in the latter case. Thus, the dependence of ρ^* on n follows that of the equilibrium effort. The optimal choice of reserve is such that the relevant part of the pdf of noise is decreasing. Indeed, players whose noise realizations fall in the range where the pdf is increasing do not pass the reserve in equilibrium. In WTA tournaments without reserve, the dependence of e^* on n follows the shape of the pdf (Ryvkin and Drugov, 2020); therefore, e^* and ρ^* are decreasing in n. When f is log-convex, the optimal pay scheme is equal prize sharing, which in equilibrium turns into an individual bonus with prize $\frac{1}{n}$; hence, e^* and ρ^* decrease with n as well.

It is intuitive that the optimal reserve decreases with the number of players since boosting competition becomes less important. Yet, interestingly, the optimal threshold level of noise, $a^* = \rho^* - e^*$, is independent of n; that is, even though the reserve becomes more forgiving in larger tournaments, the equilibrium probability of passing the reserve remains the same. Therefore, if the noise distribution is symmetric and log-concave then

[17] In Tullock contests where the probability of player i winning as a function of the vector of players' efforts is modeled as as $\frac{e_i^\zeta}{\sum_{j=1}^n e_j^\zeta}$, the corresponding condition is an upper bound on the discriminatory power ζ, whose inverse is a measure of dispersion of the Gumbel distribution in the additive noise representation.

in equilibrium each player passes the optimal reserve with probability $\frac{1}{2}$ for any number of players.

For aggregate effort, $E^* = ne^*$, the dependence on n is determined by two effects: The direct linear effect, which is always positive, and the indirect effect through the adjustment of e^*, which is always negative due to Proposition 3. Yet, a general characterization can be obtained under additional restrictions on the cost function.

Proposition 4 *(i) If f is log-concave and $c''' \geq 0$, then E^* increases in n.*

(ii) If f is log-convex, then E^* increases in n when $c''' \geq 0$ and decreases in n when $c''' \leq 0$.

For WTA tournaments without reserve, Ryvkin and Drugov (2020) show that the dependence of aggregate effort on n follows the shape of the hazard (or failure) rate of noise when the cost of effort is quadratic. Log-concave distributions have an increasing hazard rate; therefore, E^* increases in n when costs are quadratic, i.e., when E^* is proportional to $ng_{\mathrm{WTA}}(x_m)$. Then, for $c''' \geq 0$, $E^* = nc'^{-1}(g_{\mathrm{WTA}}(x_m))$ also increases in n because c'^{-1} is concave. This result holds for any reserve, not just the optimal one.

When f is log-convex, the result follows directly from the representation of aggregate effort as $E^* = nc'^{-1}(\frac{f(x)}{n})$. The dependence on n is determined by whether c'^{-1} is concave or convex.

4.2 Optimal prize schemes in the general case

The analysis so far has been restricted to log-concave and log-convex noise distributions, for which we showed that the optimal reserve corresponds to modal performance, $a^* = x_m$, and the optimal allocation of prizes awards some number k_s of *equal* prizes for each number of qualified players s. In the log-concave case $k_s = 1$ and is independent of s, and in the log-convex case $k_s = s$.

In this section, we extend the analysis to more general unimodal noise distributions that are neither log-concave nor log-convex. We show that optimal pay schemes have a similar structure: There exists an optimal threshold level of noise, $a^* \in \mathcal{X}$, such that reserve $\rho^* = e^* + a^*$ is optimal, and the optimal allocation of prizes involves k_s *equal* prizes awarded for each s. However, in general, a^* is not necessarily at the mode of f, and k_s can be anywhere between 1 and s, depending on the details of the distribution of noise.

As seen from the analysis in Section 3.1, the optimal combination of threshold noise a and prize schedule $\mathbf{v}$ is a solution to $\max_{a\in\mathcal{X},\mathbf{v}\in\mathcal{V}} g(a;\mathbf{v})$. Function $g(a;\mathbf{v})$, defined by

(3), is continuous on $\mathcal{X} \times \mathcal{V}$, and the domain $\mathcal{X} \times \mathcal{V}$ is compact if we treat $\mathcal{X}$ as a subset of extended reals. Therefore, by the Extreme Value Theorem there exists a pair $(a^*; \mathbf{v}^*)$ (where $a^* = \underline{x} = -\infty$ is possible) solving this problem.[18]

Function $g(a; \mathbf{v})$ is linear in prizes; therefore, for a given a^*, the optimal allocation of prizes solves a linear programming problem. Moreover, $g(a; \mathbf{v})$ is additive separable in s, and hence, for each s, $\mathbf{v}^s$ solves

$$\max_{\mathbf{v}^s} \sum_{r=1}^{s} \tilde{\beta}_{r,s}(a^*) v_r^s \quad \text{s.t. } v_1^s \geq \ldots \geq v_s^s \geq 0, \quad \sum_{r=1}^{s} v_r^s \leq 1. \tag{7}$$

Using the approach developed by Drugov and Ryvkin (2020*b*), we obtain the following result.

Proposition 5 *Suppose $a^* \in \mathcal{X}$ is optimal and the equilibrium existence conditions are satisfied. The optimal allocation of prizes for each s is a two-prize schedule $\mathbf{v}^s = (\frac{1}{k_s}, \ldots, \frac{1}{k_s}, 0, \ldots, 0)$ awarding equal prizes to the top k_s players above the reserve, where*

$$k_s \in \arg \max_{k=1,\ldots,s} \frac{1}{k} \sum_{r=1}^{k} \tilde{\beta}_{r,s}(a^*), \quad s = 1, \ldots, n. \tag{8}$$

Proposition 5 shows that optimal prize schedules generically have the structure $[k_1, \ldots, k_n]$ introduced in the discussion of Figure 1 in Section 3.1.[19] Figure 2 provides an illustration with noise distributed as Burr(3,1), $f_{\text{Burr}(3,1)}(x) = \frac{3x}{(x^3+1)^2} \mathbb{1}_{x \geq 0}$, which has a unimodal hazard rate and is neither log-concave nor log-convex. Similar to Figure 1, we consider tournaments of four players and all possible prize schemes of the form $[k_1, k_2, k_3, k_4]$. In this case, the optimal prize scheme is $[1, 1, 2, 2]$, that is, one prize is given when one or two players pass the reserve and two prizes are given when three or four players pass.

We now turn to the analysis of the location of a^*. As discussed after Proposition 1, the reserve has two effects on players' incentives. The first effect is the introduction of an artificial extra player, whose impact is maximized when the reserve is set at the modal performance. The second effect—*ex post* prize sharing—is based on the fact that under any conditional prize schedule the prize of each player who passed the reserve depends on the number of others passing along with him. This effect can go in either direction depending on the prize schedule. However, the player who is ranked first (and

[18] It is easy to see that $a^* = \overline{x}$ is impossible because $f(\overline{x}) = 0$ and hence $\tilde{\beta}_{k,s}(\overline{x}) = 0$ and $g(\overline{x}; \mathbf{v}) = 0$ for any $\mathbf{v}$.

[19] The same structure is optimal for any reserve, not necessarily the optimal one.

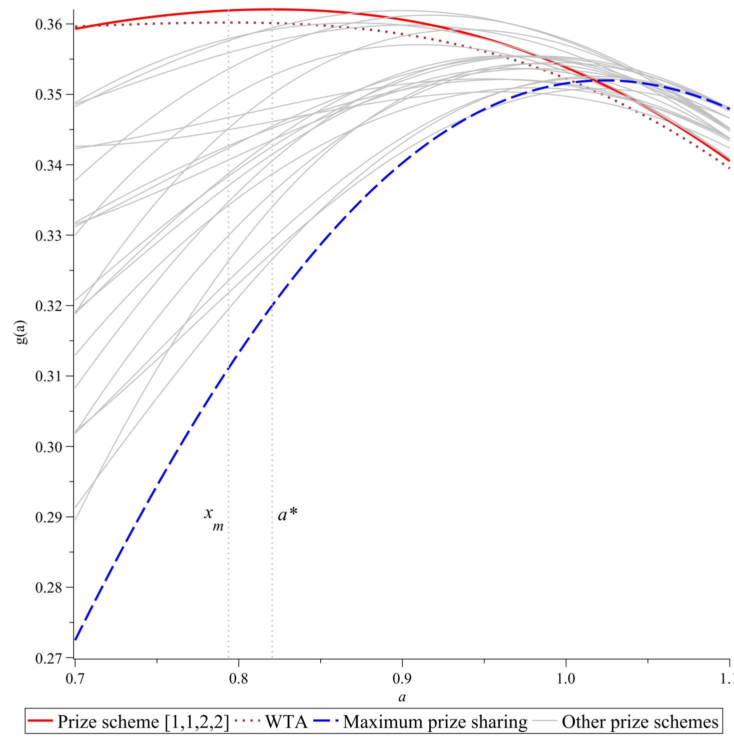

Figure 2: $g(a;\mathbf{v})$ for Burr(3,1) distribution with $f_{\text{Burr}(3,1)}(x) = \frac{3x}{(x^3+1)^2}\mathbb{1}_{x\geq 0}$, $n = 4$, and various prize schemes. The curves show $g(a;\mathbf{v})$ for all possible prize schemes of the form $[k_1, k_2, k_3, k_4]$, where $k_s \leq s$ represents the number of equal prizes awarded when s players pass the reserve.

is above the reserve) always prefers no other players to pass—unless the prize schedule is WTA, in which case he is indifferent—meaning he prefers the reserve to be higher in general. Hence, the effect pushing the optimal reserve higher is always present when a prize schedule is truly conditional, i.e., when different numbers of prizes are awarded at least for some values of s. In order to exclude the opposite effect, i.e., the possibility that a player prefers more players to pass, which would drive the optimal reserve down, it is sufficient to assume that prizes v_r^s are decreasing in s. The result is given by the following lemma.

Lemma 2 *For any feasible prize schedule* $\mathbf{v}$ *such that* v_r^s *is decreasing in* s*,* $g(a;\mathbf{v})$ *is maximized at* $a \geq x_m$*. Moreover, if there exist* r *and* $s' > s$ *such that* $v_r^{s'} < v_r^s$ *and* $f'(x_m) = 0$*, the inequality is strict.*

Optimal prize schemes of the form $[k_1, \ldots, k_n]$, as specified in Proposition 5, *and* with v_r^s decreasing in s, form the class of *K-max schedules* that assign up to K equal prizes for each s. Formally, schedule $\mathbf{v}$ is a K-max schedule if there is a $K \in \{1, \ldots, n\}$ such that $v_r^s = \frac{1}{\min\{K,s\}} \mathbb{1}_{r \leq \min\{K,s\}}$. For example, $\mathbf{v} = ((1), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}, 0))$ is the 2-max prize schedule for $n = 3$.[20] Note that in Figure 2 $a^* \approx 0.82$, which is above the mode of the Burr(3,1) distribution $x_m = 2^{-\frac{1}{3}} \approx 0.79$. Yet, the optimal scheme $[1, 1, 2, 2]$ is not a K-max schedule. This shows that the requirement of v_r^s decreasing in s is only sufficient but not necessary.

It can also be of interest how the equilibrium effort, e^*, changes with the number of players, n, for more general noise distributions and prize schedules. For some $n' > n$, consider prize schedules $\mathbf{v} = (\mathbf{v}^1, \ldots, \mathbf{v}^n)$ and $\mathbf{v}' = (\mathbf{v}^1, \ldots, \mathbf{v}^{n'})$. Note that $\mathbf{v}$ assigns the same prizes as $\mathbf{v}'$ for up to n players passing the reserve. We will refer to $\mathbf{v}$ as a *truncated version* of $\mathbf{v}'$. Alternatively, we can think of $\mathbf{v}'$ as an *expansion* of $\mathbf{v}$.

Proposition 6 *Suppose $\mathbf{v}'$ satisfies the condition of Lemma 2 and $\mathbf{v}$ is a truncated version of $\mathbf{v}'$. Suppose further that reserves are chosen optimally, and let e^* and $e^{*\prime}$ denote the corresponding equilibrium efforts. Then $e^{*\prime} \leq e^*$.*

The proof of Proposition 6 relies on a more general result, proved in the Appendix, that $g(a; \mathbf{v}') \leq g(a; \mathbf{v})$ for any fixed $a \geq x_m$. The result then follows if the reserve (and hence a) is chosen optimally in each case, with the help of Lemma 2.

4.3 Unconditional prizes

In Section 3 we consider fully flexible pay schemes where prizes can be conditioned on the number of agents who pass the reserve. However, sometimes the designer cannot commit to a pay scheme so complex, or is institutionally restricted to a single prize schedule. For example, in college admissions, "prizes," i.e., whether someone is admitted or not, as well as scholarships, are fixed and do not depend on the size of the entering class. The same holds for hiring and promotion tournaments in organizations. Innovation tournaments in crowdsourcing platforms such as TopCoder or the XPRIZE Foundation, typically fix a prize schedule in advance and do not award more money to winners if fewer high-quality projects are submitted.[21]

[20]The set of feasible prize schemes $\mathbf{v}$ with v_r^s decreasing in s is much larger—when non-optimal prize schemes are allowed—and includes schedules with unequal prizes, for example, $\mathbf{v} = ((1), (0.6, 0.4), (0.5, 0.3, 0.2))$ for $n = 3$.

[21]Another factor to consider is the possibility of collusion. When the prize budget is fully shared among agents who pass the reserve, a collusion scheme where all but one agent drop out of competition, and the single remaining agent then shares the prize with everyone, can be profitable for the agents. A fixed prize schedule would eliminate incentives to collude.

In this section, we consider such a scenario, where the principal has to commit to a single, *unconditional* prize schedule $\mathbf{v} = ((v_1), (v_1, v_2), \ldots, (v_1, \ldots, v_n))$ where prizes do not depend on the number of agents who pass the reserve. That is, an agent ranked r receives prize v_r if he passes the bar regardless of how many others also pass. As before, feasible prizes are monotone, nonnegative, and satisfy a budget constraint: $v_1 \geq \ldots \geq v_n \geq 0$, $\sum_{r=1}^{n} v_r \leq 1$. The following lemma provides a sharp characterization of the optimal reserve that holds for any unimodal noise distribution.

Lemma 3 *For any feasible unconditional prize schedule* $\mathbf{v}$*,* $g(a; \mathbf{v})$ *is maximized at* $a = x_m$*.*

As shown in the proof of Lemma 3, for unconditional prizes function $g(a; \mathbf{v})$ in the FOC (3) simplifies to $g(a; \mathbf{v}) = \sum_{r=1}^{n} \beta_{r,n}(a) v_r$, where

$$\beta_{r,n}(a) = -\binom{n-1}{r-1} \int_a^{\overline{x}} F(x)^{n-r}[1 - F(x)]^{r-1} f'(x) dx \tag{9}$$

is maximized at $a = x_m$. This implies the optimal reserve is $\rho^* = e^* + x_m$ as in the log-concave case in Section 3, although here $f(\cdot)$ is not necessarily log-concave and WTA is not necessarily optimal. Let $\hat{\beta}_{r,n} = \beta_{r,n}(x_m)$ and $\hat{B}_{r,n} = \sum_{k=1}^{r} \hat{\beta}_{k,n}$. The following proposition, whose proof is identical to that of Proposition 5, characterizes the optimal allocation of prizes given the optimal reserve.

Proposition 7 *Suppose* f *is unimodal. The optimal unconditional allocation of prizes is a two-prize schedule with* $v_1 = \ldots = v_{r^*} = \frac{1}{r^*}$ *and* $v_{r^*+1} = \ldots = v_n = 0$*, where*

$$r^* \in \arg\max_{r=1,\ldots,n} \frac{\hat{B}_{r,n}}{r}. \tag{10}$$

The optimal reserve is $\rho^* = e^* + x_m$*, where the symmetric equilibrium effort is* $e^* = c'^{-1}(g(\frac{\hat{B}_{r^*,n}}{r^*}; \mathbf{v}^*))$*.*

As seen from (9), all coefficients $\hat{\beta}_{r,n}$ are positive. Moreover,

$$\hat{\beta}_{r,n} = -\frac{1}{n} \int_{x_m}^{\overline{x}} \lambda(x) dF_{(n+1-r:n)}(x).$$

When $f(\cdot)$ is log-concave for $x \geq x_m$, $\lambda(x)$ is decreasing; therefore, $\hat{\beta}_{r,n}$ is decreasing in r, and WTA is optimal. When $f(\cdot)$ is log-convex for $x \geq x_m$, $\lambda(x)$ is increasing; therefore,

$\hat{\beta}_{r,n}$ is increasing in r. In this case, everyone who passes the reserve gets the same prize $\frac{1}{n}$, including the player ranked n if all of them pass. These results are consistent with those in Section 3.

However, since for unconditional prizes the optimal reserve and equilibrium effort are related as $\rho^* = e^* + x_m$ for any feasible prize schedule, we have a full characterization of optimal prizes for noise distributions going beyond log-concave and log-convex. Generically, it is optimal to promise up to r^* equal prizes of size $\frac{1}{r^*}$ each, with r^* given by (10). These prizes are awarded to $\min\{r^*, s\}$ top players, where s is the number of players who pass the reserve.

This prize structure is similar to the one obtained by Drugov and Ryvkin (2020*b*) for tournaments without a reserve. The crucial difference from Drugov and Ryvkin (2020*b*) is that $r^* = n$—everyone, including the player ranked n, receiving a prize—may be optimal, due to the presence of a reserve, which provides incentives beyond competition. In this case, which holds, e.g., when the distribution of noise is log-convex, the optimal tournament effectively becomes an individual bonus scheme.

We can also explore how aggregate effort $E^* = ne^*$ depends on n in this case. The result is given by the following proposition.

Proposition 8 *Suppose $c''' \geq 0$ and unconditional prizes are chosen optimally. Then aggregate effort $E^* = ne^*$ increases in n for any reserve.*

5 Conclusion

Reserve performance plays an important role in tournament design. In various settings with competitive incentives, such as innovation contests, promotions, or university admissions, it is in the principal's interest to not only reward the best performer(s) but also to ensure that the performance of those selected exceeds a minimum performance standard.

This paper is the first to study the role of reserve performance in rank-order tournaments. We show that such a reserve can significantly boost effort. More importantly, together with prize allocation, reserves achieve optimality in a much wider class of monotone, symmetric pay schemes that can be based on cardinal performance.

The optimal combination of reserve and prize allocation is determined by the properties of noise. For light-tailed noise (log-concave distributions), the optimal tournament scheme is highly competitive: The entire prize is awarded to the top player who also has to surpass the modal performance. In contrast, for extremely heavy-tailed noise (log-convex distributions) it is optimal to essentially shut down the competition and award a prize with probability one to everyone who exerts the equilibrium effort. The resulting individual bonus scheme represents an optimal contract in the absence of the MLRP condition.

Our finding that both reserve and prize allocation are important tournament design instruments is consistent with stylized facts. Indeed, there is significant variation in the number of prizes awarded and threshold performance requirements in various contest settings. For example, in the challenges mentioned in the opening paragraph of the Introduction the prize structure is winner-take-all, but the 2007 DARPA Grand Challenge had three prizes, and the French Academy of Sciences had been giving multiple subsidiary prizes "around" the area of the main prize. Multiple prizes are naturally awarded in other contexts such as grant applications, university admissions, bonuses, and promotions in organizations.

Settings with light-tailed and heavy-tailed shocks can be broadly characterized as those where luck plays a relatively smaller and larger role, respectively, in tournament outcomes. Our results imply that it is not a good idea to use tournaments under heavy-tailed shocks, at least if the goal is to motivate effort. A better approach is to impose a mild performance requirement and use individual bonuses. One can also consider a setting with heterogeneous agents where the principal's goal is to select the best employee. Heavy-tailed shocks make performance-based selection inefficient because winning, especially in large tournaments, is entirely due to luck and is not informative about ability.

References

Akerlof, Robert J., and Richard T. Holden. 2012. "The nature of tournaments." *Economic Theory*, 51(2): 289–313.

Ales, Laurence, Soo-Haeng Cho, and Ersin Körpeoğlu. 2017. "Optimal award scheme in innovation tournaments." *Operations Research*, 65(3): 693–702.

Anderson, Simon P., André De Palma, and Jacques-François Thisse. 1992. *Discrete Choice Theory of Product Differentiation.* Cambridge, Mass.:MIT Press.

Bak, Richard. 2011. *The Big Jump: Lindbergh and the Great Atlantic Air Race.* Wiley.

Caplin, Andrew, and Barry Nalebuff. 1991. "Aggregation and imperfect competition: On the existence of equilibrium." *Econometrica*, 59(1): 25–59.

Crosland, Maurice, and Antonio Gálvez. 1989. "The emergence of research grants within the prize system of the French Academy of Sciences, 1795-1914." *Social Studies of Science*, 19(1): 71–100.

Drugov, Mikhail, and Dmitry Ryvkin. 2020*a*. "How noise affects effort in tournaments." *Journal of Economic Theory*, 188: 105065.

Drugov, Mikhail, and Dmitry Ryvkin. 2020*b*. "Tournament rewards and heavy tails." *Journal of Economic Theory*, 190: 105116.

Drugov, Mikhail, and Dmitry Ryvkin. 2021. "Optimal prizes in tournaments under nonseparable preferences." *Working paper*. https://drive.google.com/file/d/1kyK4irxlq7a0EM6mihOcU98e2sxKwlDm/view.

Fu, Qiang, and Jingfeng Lu. 2012. "Micro foundations of multi-prize lottery contests: a perspective of noisy performance ranking." *Social Choice and Welfare*, 38(3): 497–517.

Hooper, Joseph. 2004. "From Darpa Grand Challenge 2004: DARPA's Debacle in the Desert." *Popular Science*, June, 4. https://www.popsci.com/scitech/article/2004-06/darpa-grand-challenge-2004darpas-debacle-desert/.

Jewitt, Ian. 1988. "Justifying the First-Order Approach to Principal-Agent Problems." *Econometrica*, 56(5): 1177–1190.

Jia, Hao. 2008. "A stochastic derivation of the ratio form of contest success functions." *Public Choice*, 135(3-4): 125–130.

Kadan, Ohad, and Jeroen M. Swinkels. 2013. "On the moral hazard problem without the first-order approach." *Journal of Economic Theory*, 148(6): 2313–2343.

Kalra, Ajay, and Mengze Shi. 2001. "Designing optimal sales contests: A theoretical perspective." *Marketing Science*, 20(2): 170–193.

Kirkegaard, René. 2017. "A unifying approach to incentive compatibility in moral hazard problems." *Theoretical Economics*, 12(1): 25–51.

Kirkegaard, René. 2021. "Contest design with stochastic performance." *American Economic Journal: Microeconomics*, forthcoming.

Krishna, Vijay, and John Morgan. 1998. "The winner-take-all principle in small tournaments." *Advances in Applied Microeconomics*, 7: 61–74.

Lazear, Edward P., and Sherwin Rosen. 1981. "Rank-order tournaments as optimum labor contracts." *Journal of Political Economy*, 89(5): 841–864.

Myerson, Roger B. 1981. "Optimal auction design." *Mathematics of Operations Research*, 6(1): 58–73.

Poblete, Joaquín, and Daniel Spulber. 2012. "The form of incentive contracts: agency with moral hazard, risk neutrality, and limited liability." *RAND Journal of Economics*, 43(2): 215–234.

Riley, John G., and William F. Samuelson. 1981. "Optimal auctions." *American Economic Review*, 71(3): 381–392.

Rogerson, William P. 1985. "The First-Order Approach to Principal-Agent Problems." *Econometrica*, 53(6): 1357–1367.

Ryvkin, Dmitry, and Mikhail Drugov. 2020. "The shape of luck and competition in winner-take-all tournaments." *Theoretical Economics*, 15(4): 1587–1626.

Schöttner, Anja. 2008. "Fixed-prize tournaments versus first-price auctions in innovation contests." *Economic Theory*, 35(1): 57–71.

Schweinzer, Paul, and Ella Segev. 2012. "The optimal prize structure of symmetric Tullock contests." *Public Choice*, 153(1-2): 69–82.

Terwiesch, Christian, and Yi Xu. 2008. "Innovation contests, open innovation, and multiagent problem solving." *Management Science*, 54(9): 1529–1543.

A Proofs

Proof of Lemma 1

Note that (4) can be written as

$$\tilde{\beta}_{r,s}(a) = \binom{s-1}{r-1} \int_a^{\overline{x}} \frac{\partial}{\partial x} \left[\tilde{F}(x)^{s-r}[1-\tilde{F}(x)]^{r-1}\right] f(x)dx + f(a)\mathbb{1}_{r=s}.$$

Integrating by parts, obtain:

$$
\begin{aligned}
\tilde{\beta}_{r,s}(a) &= \binom{s-1}{r-1}\left[\tilde{F}(x)^{s-r}[1-\tilde{F}(x)]^{r-1}f(x)\Big|_a^{\overline{x}} - \int_a^{\overline{x}} \tilde{F}(x)^{s-r}[1-\tilde{F}(x)]^{r-1}f'(x)dx\right] \\
&+ f(a)\mathbb{1}_{r=s} = -\binom{s-1}{r-1}\int_a^{\overline{x}} \tilde{F}(x)^{s-r}[1-\tilde{F}(x)]^{r-1}f'(x)dx \qquad (11)\\
&= -(1-F(a))\binom{s-1}{r-1}\int_a^{\overline{x}} \tilde{F}(x)^{s-r}[1-\tilde{F}(x)]^{r-1}\tilde{f}(x)\frac{f'(x)}{f(x)}dx \\
&= -\frac{1-F(a)}{s}\int_a^{\overline{x}} \lambda(x)\tilde{f}_{(s-r+1:s)}(x)dx.
\end{aligned}
$$

Here, $\lambda(x) = \frac{f'(x)}{f(x)}$, and $\tilde{f}_{(s-r+1:s)}$ is the pdf of the $(s-r+1)$-th order statistic among s i.i.d. draws from $\tilde{F}$. We used that $f(\overline{x}) = 0$. The factor $1-F(a)$ emerges because the updated pdf of noise for players who pass the reserve is $\tilde{f}(x) = \frac{f(x)}{1-F(a)}$.

The results for both parts follow immediately because f log-concave (resp. log-convex) implies $\lambda(x)$ decreasing (resp. increasing), and the $(s-r+1)$-th order statistics are FOSD-decreasing in r. ■

Proof of Proposition 1

Part (i): Suppose f is log-concave and hence, by Lemma 1(i), $\tilde{\beta}_{k,s}(a)$ is decreasing in k. In this case, for any given a, the optimal prize schedule is winner-take-all (WTA), with $v_k^s = \mathbb{1}_{k=1}$ for each s. This gives $g(a;\mathbf{v}) = g_{\text{WTA}}(a)$, where

$$
\begin{aligned}
g_{\text{WTA}}(a) &= \sum_{s=1}^{n}\gamma_{s-1}(a)\tilde{\beta}_{1,s}(a) \\
&= \sum_{s=1}^{n}\binom{n-1}{s-1}F(a)^{n-s}[1-F(a)]^{s-1}\left[(s-1)\int_a^{\overline{x}}\tilde{F}(x)^{s-2}\tilde{f}(x)dF(x) + f(a)\mathbb{1}_{s=1}\right] \\
&= \sum_{s=1}^{n}\frac{(n-1)!}{(n-s)!(s-2)!}F(a)^{n-s}\int_a^{\overline{x}}[F(x)-F(a)]^{s-2}f(x)dF(x) + F(a)^{n-1}f(a) \\
&= (n-1)\int_a^{\overline{x}}F(x)^{n-2}f(x)dF(x) + F(a)^{n-1}f(a).
\end{aligned}
$$

Differentiating with respect to a, we obtain

$$g'_{\text{WTA}}(a) = -(n-1)F(a)^{n-2}f(a)^2 + (n-1)F(a)^{n-2}f(a)^2 + F(a)^{n-1}f'(a) = F(a)^{n-1}f'(a).$$

Therefore, if f is unimodal so is $g_{\text{WTA}}(a)$, with the same mode, $a = x_m$, and maximum

$g_m = g_{\text{WTA}}(x_m)$. The corresponding solution to (5) is $e^* = c'^{-1}(g_{\text{WTA}}(x_m))$, and it satisfies the equation with reserve $\rho^* = e^* + x_m$.

We will now identify sufficient conditions for this solution to implement a unique symmetric, pure strategy equilibrium for the players. The result is summarized in the following lemma.

Lemma A1 *Suppose*

(a) $f(x)$ is uniformly bounded: $f_m = \sup_{x\in\mathcal{X}} f(x) < \infty$;

(b) $f'(x)$ is uniformly bounded above and below: $f'_{\max} = \sup_{x\in\mathcal{X}} f'(x) < \infty$, $f'_{\min} = \inf_{x\in\mathcal{X}} f'(x) > -\infty$;

(c) $c''(e)$ is bounded away from zero: $c_0 = \inf_{e\in[0,\bar{e}]} c''(e) > 0$, $\bar{e} = c^{-1}(1)$;

(d) $-f'_{\min} < c_0$, $c'^{-1}(g_{\text{WTA}}(x_m)) + x_m < \overline{x}$, and $f_m < c'(\bar{e})$;

(e) $f_m[(n-1)f_m + \max\{0, f'_{\max}\}] < c_0$;

(f) $\frac{1-F(x_m)^n}{n} \geq c(c'^{-1}(f_m))$.

Then e^* is the unique symmetric equilibrium under the optimal pay scheme in the log-concave case.

Proof of Lemma A1 The proof proceeds in three steps: (1) We show that condition (d) implies that (3) has exactly one solution e^*. (2) We show that condition (e) implies that $\pi(e, e^*)$ is globally strictly concave in e, and hence $e = e^*$ is incentive-compatible. Finally, (3) we show that condition (f) implies $\pi(e^*, e^*) \geq 0$.

(1) Letting $S(e) = g_{\text{WTA}}(\rho - e) - c'(e)$, obtain

$$S'(e) = -g'_{\text{WTA}}(\rho - e) - c''(e) = -F(\rho - e)^{n-1} f'(\rho - e) - c''(e).$$

Recall that f is interior unimodal and hence $f'_{\min} < 0$. This implies $S'(e) \leq -f'_{\min} - c_0 < 0$ from the first part of (d). Thus, (3) has at most one solution. To ensure that a solution in $(0, \bar{e})$ exists for $\rho = \rho^*$, note that $S(0) = g_{\text{WTA}}(\rho^*) > 0$ due to the second condition in (d), whereas $S(\bar{e}) = g_{\text{WTA}}(\rho^* - \bar{e}) - c'(\bar{e}) \leq g_m - c'(\bar{e}) < 0$ because

$$g_m = \int_{x_m}^{\overline{x}} f(x) dF(x)^{n-1} + F(x_m)^{n-1} f_m \leq f_m[1 - F(x_m)^{n-1}] + F(x_m)^{n-1} f_m = f_m < c'(\bar{e}),$$

where the last inequality is the third part of (d).

(2) For the WTA prize schedule with reserve $\rho^* = e^* + x_m$, payoff (1) takes the form

$$\begin{aligned}
\pi(e, e^*) &= \sum_{s=1}^{n} \binom{n-1}{s-1} F(x_m)^{n-s} \int_{x_m}^{\overline{x}} [F(e - e^* + x) - F(x_m)]^{s-1} dF(x) - c(e) \\
&= \sum_{s=0}^{n-1} \binom{n-1}{s} F(x_m)^{n-1-s} \int_{x_m}^{\overline{x}} [F(e - e^* + x) - F(x_m)]^{s} dF(x) - c(e) \\
&= \int_{x_m}^{\overline{x}} F(e - e^* + x)^{n-1} dF(x) - c(e). \qquad (12)
\end{aligned}$$

This gives

$$\pi_e(e, e^*) = (n-1) \int_{x_m}^{\overline{x}} F(e - e^* + x)^{n-2} f(e - e^* + x) dF(x) - c'(e)$$

and

$$\begin{aligned}
\pi_{ee}(e, e^*) &= (n-1)(n-2) \int_{x_m}^{\overline{x}} F(e - e^* + x)^{n-3} f(e - e^* + x)^2 f(x) dx \\
&+ (n-1) \int_{x_m}^{\overline{x}} F(e - e^* + x)^{n-2} f'(e - e^* + x) f(x) dx - c''(e) \\
&= (n-1) \int_{x_m}^{\overline{x}} f(e - e^* + x) f(x) dF(e - e^* + x)^{n-2} \\
&+ \int_{x_m}^{\overline{x}} f'(e - e^* + x) f(x) dF(e - e^* + x)^{n-1} - c''(e) \\
&\leq (n-1) f_m^2 [F(e - e^* + \overline{x})^{n-2} - F(e - e^* + x_m)^{n-2}] \\
&+ \max\{f'_{\max}, 0\} f_m [F(e - e^* + \overline{x})^{n-1} - F(e - e^* + x_m)^{n-1}] - c_0 \\
&\leq (n-1) f_m^2 + \max\{f'_{\max}, 0\} f_m - c_0 < 0,
\end{aligned}$$

where the last inequality is due to (e).

(3) From (12), the symmetric equilibrium payoff can be written as

$$\pi(e^*, e^*) = \int_{x_m}^{\overline{x}} F(x)^{n-1} dF(x) - c(e^*) = \frac{1 - F(x_m)^n}{n} - c(e^*).$$

Recall that $c'(e^*) = g_m \leq f_m$, implying $e^* \leq c'^{-1}(f_m)$ and hence

$$\pi(e^*, e^*) \geq \frac{1 - F(x_m)^n}{n} - c(c'^{-1}(f_m)) \geq 0,$$

where the last inequality is due to (f). ■

Part (ii): Suppose now that f is log-convex and $f(\overline{x}) = 0$. It follows that f is decreasing, and hence its mode is at $x_m = \underline{x}$. From Lemma 1(ii), $\tilde{\beta}_{k,s}(a)$ is increasing in k; therefore, for each s the optimal (feasible) pay scheme shares the prize equally among all players above the reserve, $v_k^s = \frac{1}{s}$. This gives $g(a; \mathbf{v}) = g_{\mathrm{ES}}(a)$, where, using representation (11),

$$
\begin{aligned}
g_{\mathrm{ES}}(a) &= \sum_{s=1}^{n} \gamma_{s-1}(a) \frac{1}{s} \sum_{k=1}^{s} \tilde{\beta}_{k,s}(a) \\
&= -\sum_{s=1}^{n} \gamma_{s-1}(a) \frac{1}{s} \sum_{k=1}^{s} \binom{s-1}{k-1} \int_a^{\overline{x}} \tilde{F}(x)^{s-k} [1 - \tilde{F}(x)]^{k-1} f'(x) dx \\
&= \sum_{s=1}^{n} \binom{n-1}{s-1} F(a)^{n-s} [1 - F(a)]^{s-1} f(a) \frac{1}{s} = f(a) \sum_{s=1}^{n} \frac{(n-1)!}{s!(n-s)!} F(a)^{n-s} [1 - F(a)]^{s-1} \\
&= \frac{f(a)}{n[1 - F(a)]} \sum_{s=1}^{n} \binom{n}{s} F(a)^{n-s} [1 - F(a)]^{s} = \frac{f(a)[1 - F(a)^n]}{n[1 - F(a)]}.
\end{aligned}
$$

When f is log-convex, it also has a decreasing failure (or hazard) rate $h(x) = \frac{f(x)}{1-F(x)}$, which implies $g_{\mathrm{ES}}(a)$ is decreasing and $a = \underline{x}$ is optimal. Thus, the maximum of $g_{\mathrm{ES}}(a)$ is $g_m = \frac{f(\underline{x})}{n}$. The corresponding optimal effort e^* and reserve ρ^* are as in part (ii) of the proposition. Similar to the proof of part (i), we will now formulate sufficient conditions for the equilibrium existence.

Lemma A2 *Suppose conditions (a)-(c) from Lemma A1 are satisfied and*

(d') $-f'_{\min} + f(\underline{x})^2 < c_0$, $c'^{-1}(\frac{f(\underline{x})}{n}) + \underline{x} < \overline{x}$, and $f(\underline{x}) < nc'(\bar{e})$;

(f') $\frac{1}{n} \geq c(c'^{-1}(\frac{f(\underline{x})}{n}))$.

Then e^* is the unique symmetric equilibrium under the optimal pay scheme in the log-convex case.

Proof of Lemma A2 The proof follows the same three steps as the proof for part (i).

(1) Let $S(e) = g_{\mathrm{ES}}(\rho - e) - c(e)$. Then

$$
\begin{aligned}
S'(e) = &-\frac{f'(\rho - e)[1 - F(\rho - e)^n]}{n[1 - F(\rho - e)]} + \frac{f(\rho - e)^2 F(\rho - e)^{n-1}}{1 - F(\rho - e)} \\
&- \frac{f(\rho - e)^2 [1 - F(\rho - e)^n]}{n[1 - F(\rho - e)]^2} - c''(e) \leq -f'_{\min} + f(\underline{x})^2 - c_0 < 0.
\end{aligned}
$$

The last inequality follows from the first condition in (d'). To obtain the upper bound for the first term in $S'(e)$, we used that, for any $x \in \mathcal{X}$, $F(x) \leq 1$ and hence

$$\frac{1-F(x)^n}{n[1-F(x)]} = \frac{1}{n}[1 + F(x) + \ldots + F(x)^{n-1}] \leq 1.$$

For the second term, note that $h(x) = \frac{f(x)}{1-F(x)}$ is a decreasing function; therefore,

$$\frac{f(x)^2 F(x)^{n-1}}{1-F(x)} = h(x)f(x)F(x)^{n-1} \leq h(\underline{x})f(\underline{x}) = f(\underline{x})^2.$$

The third term in $S'(e)$ is negative and can be dropped.

Thus, the FOC $g_{\mathrm{ES}}(\rho - e) = c'(e)$ can have at most one solution. To ensure that a solution exists for $\rho = \rho^*$, note that $S(0) = g_{\mathrm{ES}}(\rho^*) > 0$ due to the second condition in (d'), whereas $S(\bar{e}) = g_{\mathrm{ES}}(\rho^* - \bar{e}) - c'(\bar{e}) \leq \frac{f(\underline{x})}{n} - c'(\bar{e}) < 0$ due to the third condition in (d').

(2) For the pay scheme with equal prize sharing and reserve $\rho^* = e^* + \underline{x}$, payoff (1) is simply

$$\pi(e, e^*) = \frac{1}{n}F(e - e^* + \underline{x}) - c(e).$$

Indeed, if $n-1$ players choose effort e^*, the indicative player will pass the reserve with probability $F(e - e^* + \underline{x})$ and receive the prize $\frac{1}{n}$ if he does. This gives $\pi_e(e, e^*) = \frac{1}{n}f(e - e^* + \underline{x}) - c'(e)$ and

$$\pi_{ee}(e, e^*) = \frac{1}{n}f'(e - e^* + \underline{x}) - c''(e) \leq -\frac{f'_{\min}}{n} - c_0 < 0,$$

where the last inequality follows from the first condition in (d').

(3) The symmetric equilibrium payoff is

$$\pi(e^*, e^*) = \frac{1}{n} - c(e^*) = \frac{1}{n} - c\left(c'^{-1}\left(\frac{f(\underline{x})}{n}\right)\right) \geq 0$$

due to (f'). ■

Regularity conditions Conditions (a) and (b) in Lemma A1 are relatively mild regularity conditions for the pdf of noise $f(\cdot)$. Condition (c) is somewhat restrictive in that it requires the second derivative of the cost function to be bounded away from zero.[22] It is

[22]For a similar restriction, see, e.g., Schöttner (2008).

satisfied by the widely used quadratic function as well as, more generally, by all functions of the form $c(e) = \frac{c_0 e^2}{2} + \kappa(e)$, where $\kappa(\cdot)$ is convex. It is also satisfied by functions $c(e) = c_0 e^{\varkappa}$ for $\varkappa \in (1, 2]$. It is, however, not satisfied by the same functions with $\varkappa > 2$.

Dispersion Conditions (d)-(f) of Lemma A1, and parallel conditions (d') and (f') of Lemma A2, can be written explicitly in terms of the dispersion of the distribution of noise. To this end, consider a parameterized family of distributions with cdf $F(x;\sigma) = F(\frac{x}{\sigma};1)$ and pdf $f(x;\sigma) = \frac{1}{\sigma} f(\frac{x}{\sigma};1)$, where $\sigma > 0$ is a scale parameter. Let f_{m1}, $f'_{\min 1}$ and $f'_{\max 1}$ denote the bounds for the "unit scale" version of the distribution. Then for any $\sigma > 0$ the bounds in conditions (d)-(f), (d') and (f') can be written as $f_m = \frac{1}{\sigma} f_{m1}$, $f'_{\min} = \frac{1}{\sigma^2} f'_{\min 1}$ and $f'_{\max} = \frac{1}{\sigma^2} f'_{\max 1}$. It is then clear that for any combination of other parameters fixed, a sufficiently large $\sigma > 0$ can be found such that all the conditions are satisfied. The scale parameter σ is a measure of the dispersion of the distribution.[23] In most standard cases (although not in general; see Drugov and Ryvkin (2020*a*)), such as the uniform, normal, Gumbel or Pareto distribution, it is proportional to the standard deviation and also orders distributions in terms of variance and second-order stochastic dominance. ■

Proof of Proposition 2

In any symmetric pure strategy equilibrium where all players exert the same effort $e > 0$, a symmetric FOC is satisfied. Without loss, consider player 1 and assume that all players other than 1 choose effort e. Player 1's payoff from effort e_1 can be written as[24]

$$\begin{aligned}
\pi_1(e_1, e) &= \int dF(x_1) \dots \int dF(x_n) w_1(e_1 + x_1, e + x_2, \dots, e + x_n) - c(e_1) \\
&= \int_{\underline{x}+e_1}^{\overline{x}+e_1} dq_1 \int dF(x_2) \dots \int dF(x_n) w_1(q_1, e + x_2, \dots, e + x_n) f(q_1 - e_1) - c(e_1).
\end{aligned}$$

Next, we differentiate π_1 with respect to e_1, set $e_1 = e$, and change the first variable of integration back to x_1. The result is

$$\begin{aligned}
&\int dF(x_2) \dots \int dF(x_n) \left[w_1(e + \overline{x}, e + x_2, \dots, e + x_n) f(\overline{x}) - w_1(e + \underline{x}, e + x_2, \dots, e + x_n) f(\underline{x}) \right] \\
&- \int dx_1 \int dF(x_2) \dots \int dF(x_n) w_1(e + x_1, \dots, e + x_n) f'(x_1) = c'(e). \qquad (13)
\end{aligned}$$

[23]More precisely, $F(x;\sigma')$ dominates $F(x;\sigma)$ in the *dispersive order* for $\sigma' < \sigma$.

[24]The change of variable $x_1 \rightarrowtail q_1 = x_1 + e_1$ is needed in order to shift the dependence on e_1 to the distribution of noise because w_1 may be discontinuous.

The left-hand side of (13) can be increased by dropping the negative term with $f(\underline{x})$ and shifting the lower limit of integration over x_1 to x_m. Using the assumption that $f(\overline{x}) = 0$, we obtain

$$c'(e) \leq R(e; \mathbf{w}) \equiv -\int_{x_m}^{\overline{x}} dF(x_1) \int dF(x_2) \ldots \int dF(x_n) w_1(e + x_1, \ldots, e + x_n) \lambda(x_1). \quad (14)$$

Here, $\lambda(x) = \frac{f'(x)}{f(x)}$. The following lemma shows that function $R(e; \mathbf{w})$ can be uniformly bounded above by the already familiar bounds in the two cases of Proposition 1.

Lemma A3 *For any pay scheme* $\mathbf{w}$ *satisfying Assumption 1,*

(i) If f is log-concave, then $R(e; \mathbf{w}) \leq g_{\mathrm{WTA}}(x_m)$;

(ii) If f is log-convex, then $R(e; \mathbf{w}) \leq \frac{f(\underline{x})}{n}$.

Proof of Lemma A3

Part (i) Let $D_1 = \{\mathbf{x} \in \mathcal{X}^n : x_1 > x_m > x_2, \ldots, x_n\}$ be the set of noise realizations such that agent 1 is the only one above the bar. Then we can write

$$R(e; \mathbf{w}) = \int_{D_1} w_1 \lambda(x_1) dF(x_1) \ldots dF(x_n) + \int_{\mathcal{X}^n \setminus D_1} w_1 \lambda(x_1) dF(x_1) \ldots dF(x_n), \quad (15)$$

where for brevity we omit the arguments of w_1. In the first term in (15), setting $w_1 = 1$ will produce an upper bound

$$\begin{aligned}
&- \int_{D_1} f'(x_1) dx_1 dF(x_2) \ldots dF(x_n) = - \int_{x_m}^{\overline{x}} df(x_1) \int_{\underline{x}}^{x_m} dF(x_2) \ldots \int_{\underline{x}}^{x_m} dF(x_n) \\
&= f(x_m) F(x_m)^{n-1}.
\end{aligned}$$

Next, we show that the second term in (15) is weakly increased by replacing w_1 with 1 for $x_1 > \max\{x_2, \ldots, x_n\}$ and zero otherwise.

Suppose $x_2 = \max\{x_2, \ldots, x_n\}$. Then we can split the domain of integration into $(\mathcal{X}^n \setminus D_1) \cap \{x_1 > x_2\}$ and $(\mathcal{X}^n \setminus D_1) \cap \{x_1 < x_2\}$. In the latter integral, we relabel variables x_1 and x_2 so that $w_1(q_1, q_2, q_3 \ldots, q_n)$ turns into $w_1(q_2, q_1, q_3 \ldots, q_n)$, which, by the symmetry property (a), is equal to $w_2(q_1, q_2, q_3 \ldots, q_n)$. Additionally, $f'(x_1) f(x_2)$ turns into $f'(x_2) f(x_1)$, keeping everything else intact. Further, we use the budget constraint property (c) to replace w_2 with $1 - w_1$, which increases the sum. The resulting upper bound is

$$\int_{x_1>x_2>x_m} w_1\lambda(x_1)dF(x_1)\dots dF(x_n) + \int_{x_1>x_2>x_m} (1-w_1)\lambda(x_2)dF(x_1)\dots dF(x_n)$$
$$= \int_{x_1>x_2>x_m} \lambda(x_2)dF(x_1)\dots dF(x_n) + \int_{x_1>x_2>x_m} w_1[\lambda(x_1)-\lambda(x_2)]dF(x_1)\dots dF(x_n).$$

Here, for brevity we omitted the remaining characterization of the integration domains and the arguments of w_1. Since f is log-concave, $\lambda(x)$ is increasing, and hence replacing w_1 with 1 in the second term will increase the expression further. Using integration by parts, we can transform the result as

$$\int_{x_1>x_2>x_m} \lambda(x_1)dF(x_1)\dots dF(x_n) = -\int_{x_1>x_2>\max\{x_m,x_3,\dots,x_n\}} f'(x_1)f(x_2)\dots f(x_n)dx_1\dots dx_n$$
$$= -\int_{x_m}^{\overline{x}} df(x_1)\int_{x_m}^{x_1} dF(x_2)\int_{\underline{x}}^{x_2} dF(x_3)\dots\int_{\underline{x}}^{x_2} dF(x_n) = -\int_{x_m}^{\overline{x}} df(x_1)\int_{x_m}^{x_1} dF(x_2)F(x_2)^{n-2}$$
$$= -f(x_1)\int_{x_m}^{x_1} dF(x_2)F(x_2)^{n-2}\Big|_{x_m}^{\overline{x}} + \int_{x_m}^{\overline{x}} f(x_1)^2F(x_1)^{n-2}dx_1 = \int_{x>x_m} F(x)^{n-2}f(x)^2dx.$$

Recall that we arbitrarily chose x_2 to be the maximum in $\{x_2,\dots,x_n\}$. There are $n-1$ elements in this set, all of which are equally likely to be the maximum, and all of these cases are equivalent from player 1's perspective. Thus, the expression above must be multiplied by $n-1$, which gives

$$R(e;\mathbf{w}) \le f(x_m)F(x_m)^{n-1} + (n-1)\int_{x>x_m} F(x)^{n-2}f(x)^2dx = g_{\text{WTA}}(x_m).$$

Part (ii) When f is log-convex, it is also convex. In conjunction with the assumption that $f(\overline{x})=0$ this implies f is decreasing and $x_m=\underline{x}$. Assuming $x_2=\max\{x_2,\dots,x_n\}$ and following the same sequence of steps as in the proof of part (i), we obtain the bound

$$\int_{x_1>x_2} \lambda(x_2)dF(x_1)\dots dF(x_n) + \int_{x_1>x_2} w_1[\lambda(x_1)-\lambda(x_2)]dF(x_1)\dots dF(x_n)$$

The log-convexity of f implies that $\lambda(x)$ is decreasing, and the second term is maximized by setting $w_1=\frac{1}{n}$—the lowest value of w_1 that preserves symmetry and monotonicity of the pay scheme. Then (14) gives

$$R(e;\mathbf{w}) \le -\frac{1}{n}\int f'(x_1)f(x_2)\dots f(x_n)dx_1\dots dx_n = \frac{1}{n}f(\underline{x}).$$

■

Since the bounds on $R(e;\mathbf{w})$ in Lemma A3 are reached by the corresponding optimal tournaments with reserve, the result follows. ■

Proof of Proposition 3

Suppose first that f is log-concave. Equation (6) can be written in the form

$$g_{\text{WTA}}(a) = \int \hat{f}(x;a)dF(x)^{n-1}, \quad \hat{f}(x;a) = \begin{cases} f(a), & x \le a \\ f(x), & x > a \end{cases} \tag{16}$$

By construction, $\hat{f}(x;x_m)$ is decreasing in x. It then follows from (16) that $g_{\text{WTA}}(x_m)$ is decreasing in n, and hence so are e^* and ρ^*.

For f log-convex, the result follows directly from part (ii) of Proposition 1. ■

Proof of Proposition 4

Part (i): For f log-concave, we have $E^* = nc'^{-1}(g_{\text{WTA}}(x_m))$. Suppose first that $c(e)$ is quadratic, in which case c'^{-1} is linear and E^* is proportional to $ng_{\text{WTA}}(x_m)$. Using representation (16) from the proof of Proposition 3, we can write

$$ng_{\text{WTA}}(x_m) = n(n-1)\int \hat{f}(x;x_m)F(x)^{n-2}f(x)dx = \int \frac{\hat{f}(x;x_m)}{1-F(x)}f_{(n-1:n)}(x)dx,$$

where $f_{(n-1:n)}(x) = n(n-1)F(x)^{n-2}[1-F(x)]f(x)$ is the pdf of $X_{(n-1:n)}$—the second highest order statistic among n i.i.d. draws of X.

Since f is log-concave, the distribution of noise has an increasing hazard rate $h(x) = \frac{f(x)}{1-F(x)}$. This implies function $\frac{\hat{f}(x;x_m)}{1-F(x)}$ is also increasing. The result then follows because $X_{(n-1:n)}$ is FOSD-increasing in n.

For a more general cost function with $c''' \ge 0$, c' is convex and hence c'^{-1} is concave. Therefore, if E^* is increasing in n when c'^{-1} is linear, it will be increasing in n in this case as well.

Part (ii): For f log-convex, $E^* = nc'^{-1}(\frac{f(\underline{x})}{n})$, and hence the effect of n depends on whether c'^{-1} is concave or convex, or, equivalently, on whether c' is convex or concave, which produces the result. ■

Proof of Proposition 5

Fix an $s \in \{1, \ldots, n\}$ and consider problem (7). Using a technique similar to Drugov and Ryvkin (2020*b*), introduce $\tilde{B}_{r,s} = \sum_{k=1}^{r} \tilde{\beta}_{k,s}(a^*)$ and let $d_r^s = v_r^s - v_{r+1}^s$ ($r \leq s$), where for convenience we define $v_{s+1}^s = 0$. In the new variables $\mathbf{d}^s = (d_1^s, \ldots, d_s^s)$ the optimization problem becomes

$$\max_{\mathbf{d}^s} \sum_{r=1}^{s} \tilde{B}_{r,s} d_r^s \quad \text{s.t.} \quad d_1^s, \ldots, d_s^s \geq 0, \quad \sum_{r=1}^{s} r d_r^s = 1.$$

This problem, generically, has a corner solution with $rd_r^s = 1$ for some $r = k_s$ and $d_r^s = 0$ for all $r \neq k_s$. Such solutions correspond to two-prize schedules with $v_1^s = \ldots = v_{k_s}^s = \frac{1}{k_s}$ and $v_{k_s+1} = \ldots = v_s = 0$, where k_s is given by (8). ■

Proof of Lemma 2

Using (11) from the proof of Lemma 1, obtain

$$\begin{aligned} g(a;\mathbf{v}) &= \sum_{s=1}^{n} \gamma_{s-1}(a) \sum_{k=1}^{s} \tilde{\beta}_{k,s}(a) v_k^s \\ &= -\sum_{s=1}^{n} \binom{n-1}{s-1} F(a)^{n-s} [1-F(a)]^{s-1} \sum_{k=1}^{s} v_k^s \binom{s-1}{k-1} \int_a^{\overline{x}} \tilde{F}(x)^{s-k} [1-\tilde{F}(x)]^{k-1} f'(x) dx \\ &= -\sum_{s=1}^{n} \binom{n-1}{s-1} F(a)^{n-s} \sum_{k=1}^{s} v_k^s \binom{s-1}{k-1} \int_a^{\overline{x}} [F(x)-F(a)]^{s-k} [1-F(x)]^{k-1} f'(x) dx. \end{aligned} \tag{17}$$

Changing the order of summation, $\sum_{s=1}^{n}\sum_{k=1}^{s} \to \sum_{k=1}^{n}\sum_{s=k}^{n}$, using the identity $\binom{n-1}{s-1}\binom{s-1}{k-1} = \binom{n-1}{k-1}\binom{n-k}{s-k}$, and changing the second variable of summation to $l = s-k$, we further rewrite $g(a;\mathbf{v})$ as

$$\begin{aligned} g(a;\mathbf{v}) &= -\sum_{k=1}^{n} \binom{n-1}{k-1} \sum_{l=0}^{n-k} \binom{n-k}{l} v_k^{k+l} F(a)^{n-k-l} \int_a^{\overline{x}} [F(x)-F(a)]^{l} [1-F(x)]^{k-1} f'(x) dx \\ &= -\sum_{k=1}^{n} \binom{n-1}{k-1} \int_a^{\overline{x}} F(x)^{n-k} [1-F(x)]^{k-1} N_k(x,a) f'(x) dx, \end{aligned} \tag{18}$$

where

$$N_k(x,a) = \sum_{l=0}^{n-k} \binom{n-k}{l} \left(\frac{F(a)}{F(x)}\right)^{n-k-l} \left(1 - \frac{F(a)}{F(x)}\right)^{l} v_k^{k+l}.$$

We will now show that $N_k(x,a)$ is increasing in a. For brevity, let $\xi = 1 - \frac{F(a)}{F(x)}$ and consider

$$M(\xi) = \sum_{l=0}^{n-k} \binom{n-k}{l} \xi^l (1-\xi)^{n-k-l} v_k^{k+l}.$$

Assuming $\xi \in (0,1)$ and differentiating with respect to ξ, obtain

$$\begin{aligned}
M'(\xi) &= \sum_{l=0}^{n-k} \binom{n-k}{l} \xi^{l-1}(1-\xi)^{n-k-1-l}[l(1-\xi) - (n-k-l)\xi] v_k^{k+l} \\
&= \frac{1}{\xi(1-\xi)} \sum_{l=0}^{n-k} \binom{n-k}{l} \xi^l (1-\xi)^{n-k-l}[l - (n-k)\xi] v_k^{k+l} \\
&= \frac{1}{\xi(1-\xi)} \left[\mathbb{E}(L v_k^{L+k}) - \mathbb{E}(L)\mathbb{E}(v_k^{L+k})\right] = \frac{1}{\xi(1-\xi)} \mathrm{Cov}(L, v_k^{k+L}) \le 0,
\end{aligned}$$

where $L \sim \text{Binomial}(n-k, \xi)$, and the inequality follows from v_k^s decreasing in s. Since ξ is decreasing in a, the result follows.

Then, for any $a \le x_m$, we can split the integral in (18) to obtain

$$\begin{aligned}
g(a; \mathbf{v}) = \sum_{k=1}^{n} \binom{n-1}{k-1} \Bigg[&- \int_a^{x_m} F(x)^{n-k}[1-F(x)]^{k-1} N_k(x,a) f'(x) dx \\
&+ \int_{x_m}^{\overline{x}} F(x)^{n-k}[1-F(x)]^{k-1} N_k(x,a) |f'(x)| dx \Bigg] \le g(x_m; \mathbf{v}).
\end{aligned}$$

The inequality follows because the first term is negative and the second term is increasing in a.

For the second part of the lemma, note that the condition that $v_k^{s'} < v_k^s$ for some k and $s' > s$, means that the prize schedule is not unconditional. This implies $\mathrm{Cov}(L, v_k^{k+L}) < 0$ in the expression for $M'(\xi)$ above and hence $N_k(x,a)$ is strictly decreasing in a for $a \ne x$. It is sufficient to show that $g_a(x_m; \mathbf{v}) > 0$. Differentiating (18) and setting $a = x_m$ obtain

$$\begin{aligned}
g_a(x_m; \mathbf{v}) = \sum_{k=1}^{n} \binom{n-1}{k-1} \Bigg[& F(x_m)^{n-k}[1-F(x_m)]^{k-1} v_k^k f'(x_m) \\
&- \int_{x_m}^{\overline{x}} F(x)^{n-k}[1-F(x)]^{k-1} \frac{\partial N_k(x,x_m)}{\partial a} f'(x) dx \Bigg] > 0,
\end{aligned}$$

where the inequality follows because $f'(x_m) = 0$ and the second term is positive. ■

Proof of Proposition 6

We start with the following lemma that provides the more general result mentioned after the proposition.

Lemma A4 *Suppose $\mathbf{v}'$ satisfies the condition of Lemma 2 and $\mathbf{v}$ is a truncated version of $\mathbf{v}'$. Then for any $a \geq x_m$, $g(a;\mathbf{v}') \leq g(a;\mathbf{v})$.*

Proof of Lemma A4

We use representation (18) of $g(a;\mathbf{v})$ from the proof of Lemma 2. It is positive because $a \geq x_m$, and can be written as an integral of $\mathbb{E}(N_{K+1}(x,a))$, with $K \sim \text{Binomial}(n-1, 1-F(x))$. Thus, in order to show that this term is decreasing in n, it is sufficient to show that $N_k(x,a)$ is decreasing in k and in n. The effect of n is clear immediately because we can write $N_k(x,a) = \mathbb{E}(v_k^{k+L})$, with $L \sim \text{Binomial}(n-k,\xi)$, and v_k^{k+l} is decreasing in l. To see the effect of k, let $m = n-k$ and consider the difference $N_{k-1}(x,a) - N_k(x,a)$, for $k \geq 2$, in the form

$$\begin{aligned}
&N_{k-1}(x,a) - N_k(x,a) = N_{n-m-1}(x,a) - N_{n-m}(x,a) \\
&= \sum_{l=0}^{m+1} \binom{m+1}{l} \xi^l (1-\xi)^{m+1-l} v_{n-m-1}^{n-m-1+l} - \sum_{l=0}^{m} \binom{m}{l} \xi^l (1-\xi)^{m-l} v_{n-m}^{n-m+l}.
\end{aligned}$$

Using the identity $\binom{m+1}{l} = \binom{m}{l-1} + \binom{m}{l}$, we rewrite the first term as

$$\begin{aligned}
&\sum_{l=0}^{m+1} \binom{m}{l-1} \xi^l (1-\xi)^{m+1-l} v_{n-m-1}^{n-m-1+l} + \sum_{l=0}^{m+1} \binom{m}{l} \xi^l (1-\xi)^{m+1-l} v_{n-m-1}^{n-m-1+l} \\
&= \sum_{l=0}^{m} \binom{m}{l} \xi^{l+1} (1-\xi)^{m-l} v_{n-m-1}^{n-m+l} + \sum_{l=0}^{m} \binom{m}{l} \xi^l (1-\xi)^{m+1-l} v_{n-m-1}^{n-m-1+l},
\end{aligned}$$

and obtain

$$\begin{aligned}
&N_{k-1}(x,a) - N_k(x,a) \\
&= \sum_{l=0}^{m} \binom{m}{l} \xi^l (1-\xi)^{m-l} [\xi v_{n-m-1}^{n-m+l} + (1-\xi) v_{n-m-1}^{n-m-1+l} - v_{n-m}^{n-m+l}] \geq 0,
\end{aligned}$$

where the inequality follows because $v_{n-m-1}^{n-m+l} \geq v_{n-m}^{n-m+l}$ and $v_{n-m-1}^{n-m-1+l} \geq v_{n-m}^{n-m+l}$. ■

To prove the proposition, let a^* and $a^{*\prime}$ denote the corresponding optimal threshold levels of noise. Then $a^*, a^{*\prime} \geq x_m$ from Lemma 2. Moreover, $c'(e^*) = g(a^*;\mathbf{v}) \geq g(a^{*\prime};\mathbf{v}) \geq$

$g(a^{*\prime};\mathbf{v}^\prime) = c^\prime(e^{*\prime})$, where the first inequality is due to the optimality of a^* under $\mathbf{v}$ and the second one is due to Lemma A4. ■

Proof of Lemma 3

The result follows from representations (17) and (18) of $g(a;\mathbf{v})$ in the proof of Lemma 2. The expression simplifies because $v_k^s = v_k$ is independent of s, which gives $N_k(x,a) = v_k$ and

$$g(a;\mathbf{v}) = -\sum_{k=1}^{n} \binom{n-1}{k-1} v_k \int_a^{\overline{x}} F(x)^{n-k}[1-F(x)]^{k-1} f^\prime(x)dx.$$

This is maximized at $a = x_m$, and the result follows. ■

Proof of Proposition 8

For quadratic costs, aggregate effort is proportional to $ng(x_m;\mathbf{v}^*) = n\frac{\hat{B}_{r^*,n}}{r^*}$, where r^* is the optimal number of prizes. We have

$$\hat{B}_{r,n}(a) = \sum_{k=1}^{r} \beta_{r,n}(a) = r\binom{n-1}{r} \int_a^{\overline{x}} F(x)^{n-r-1}[1-F(x)]^{r-1} f(x)dF(x) + f(a)\mathbb{1}_{r=n},$$

which gives

$$\frac{\hat{B}_{r,n}(a)}{r} = \frac{1}{n}\int_a^{\overline{x}} h(x)dF_{n-r:n}(x) + \frac{f(a)}{n}\mathbb{1}_{r=n},$$

and hence

$$ng(a;\mathbf{v}^*) = \int_a^{\overline{x}} h(x)dF_{n-r^*:n}(x) + f(a)\mathbb{1}_{r^*=n}.$$

Here, $h(x) = \frac{f(x)}{1-F(x)}$ is the hazard rate of noise. We need to show that $ng(a;\mathbf{v}^*)$ is increasing in n. It is sufficient to show that when n becomes $n+1$, the expression can be increased by either keeping r^* unchanged or replacing it with r^*+1. Thus, it is sufficient to show that either

$$\int_a^{\overline{x}} h(x)dF_{n+1-r^*:n+1}(x) + f(a)\mathbb{1}_{r^*=n+1} \geq \int_a^{\overline{x}} h(x)dF_{n-r^*:n}(x) + f(a)\mathbb{1}_{r^*=n} \tag{19}$$

or

$$\int_a^{\overline{x}} h(x)dF_{n-r^*:n+1}(x) + f(a)\mathbb{1}_{r^*=n} \geq \int_a^{\overline{x}} h(x)dF_{n-r^*:n}(x) + f(a)\mathbb{1}_{r^*=n},$$

(or both). The latter inequality is equivalent to

$$\int_a^{\overline{x}} h(x)dF_{n-r^*:n+1}(x) \geq \int_a^{\overline{x}} h(x)dF_{n-r^*:n}(x).$$

If it holds, we are done. Suppose it does not, i.e., assume that

$$\int_a^{\overline{x}} h(x)dF_{n-r^*:n+1}(x) < \int_a^{\overline{x}} h(x)dF_{n-r^*:n}(x).$$

We will now show that (19) holds. First, note that if $r^* = n$ then $ng(a;\mathbf{v}^*) = f(a)$, i.e., aggregate effort is independent of n and we are done. Suppose $r^* < n$. In this case the terms with the indicator functions in (19) are zero. Subtracting $\int_a^{\overline{x}} h(x)dF_{n-r^*:n+1}(x)$ from both sides of (19), we obtain

$$\int_a^{\overline{x}} h(x)[f_{n+1-r^*:n+1}(x) - f_{n-r^*:n+1}(x)]dx \geq \int_a^{\overline{x}} h(x)[f_{n-r^*:n}(x) - f_{n-r^*:n+1}(x)]dx,$$

where

$$\begin{aligned}
&f_{n+1-r^*:n+1}(x) - f_{n-r^*:n+1}(x)\\
&= \frac{(n+1)!}{(n-r^*)!r^*!}F(x)^{n-r^*}[1-F(x)]^{r^*}f(x) - \frac{(n+1)!}{(n-r^*-1)!(r^*+1)!}F(x)^{n-r^*-1}[1-F(x)]^{r^*+1}f(x)\\
&= \frac{(n+1)!}{(n-r^*)!(r^*+1)!}F(x)^{n-r^*-1}[1-F(x)]^{r^*}[(r^*+1)F(x) - (n-r^*)(1-F(x))]f(x)\\
&= \frac{(n+1)!}{(n-r^*)!(r^*+1)!}F(x)^{n-r^*-1}[1-F(x)]^{r^*}[(n+1)F(x) - (n-r^*)]f(x),
\end{aligned}$$

and

$$\begin{aligned}
&f_{n-r^*:n}(x) - f_{n-r^*:n+1}(x)\\
&= \frac{n!}{(n-r^*-1)!r^*!}F(x)^{n-r^*-1}[1-F(x)]^{r^*}f(x) - \frac{(n+1)!}{(n-r^*-1)!(r^*+1)!}F(x)^{n-r^*-1}[1-F(x)]^{r^*+1}f(x)\\
&= \frac{n!}{(n-r^*-1)!(r^*+1)!}F(x)^{n-r^*-1}[1-F(x)]^{r^*}[(r^*+1)F(x) - (n+1)(1-F(x))]f(x)\\
&= \frac{n!}{(n-r^*-1)!(r^*+1)!}F(x)^{n-r^*-1}[1-F(x)]^{r^*}[(n+r^*+2)F(x) - (n+1)]f(x)\\
&= \frac{(n+1)!}{(n-r^*)!(r^*+1)!}F(x)^{n-r^*-1}[1-F(x)]^{r^*}\left[\left(n-r^*+\frac{(r^*+1)(n-r^*)}{n+1}\right)F(x) - (n-r^*)\right]f(x).
\end{aligned}$$

It is, therefore, sufficient to show that

$$n+1 \geq n-r^* + \frac{(r^*+1)(n-r^*)}{n+1},$$

which holds because

$$\frac{(r^* + 1)(n - r^*)}{n + 1} \leq r^* + 1.$$

Thus, we showed that $ng(a; \mathbf{v}^*)$ is increasing in n for any a, which implies $nc'^{-1}(g(a; \mathbf{v}^*))$ is increasing in n for $c''' \geq 0$, i.e., aggregate effort is increasing in n when (unconditional) prizes are chosen optimally for any reserve, including the optimal reserve with $a = x_m$. ■